UCMERCED MESA LAB

Fractional Order Modeling

A Tutorial Introduction and An Application in Characterizing Complex Relaxation Processes

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Sept. 14, 2012. Friday 12:00-13:20

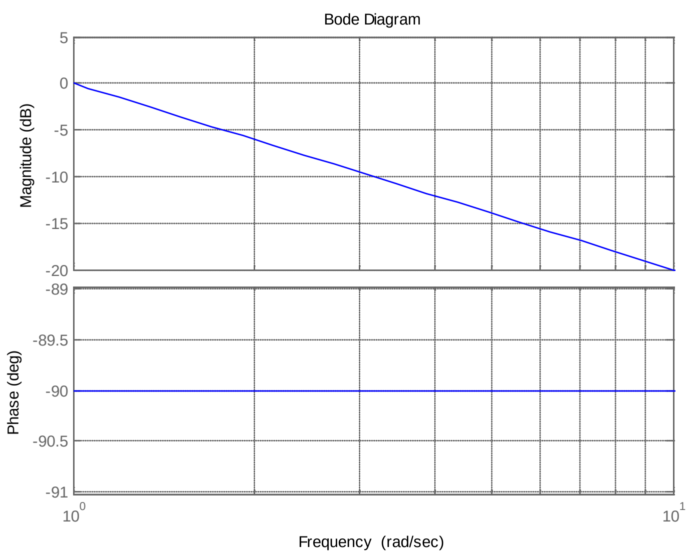
COB 267

Outline

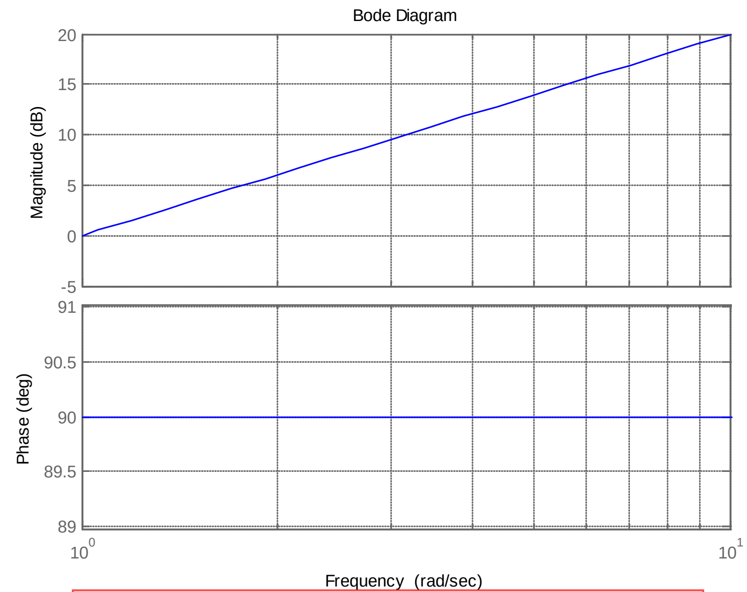
- What is Fractional Calculus and Fractional Order Modeling/Controls
- A Worked Out Example on Fractional Order Modeling (FOM)
- Summary of Benefits of Using FOM
- An Application: Parameter-Distributed and DO FOM of Complex Relaxation Processes

Fractional (Noninteger)(order) operator

- First order differentiator: s
- First order integrator: $1/s$



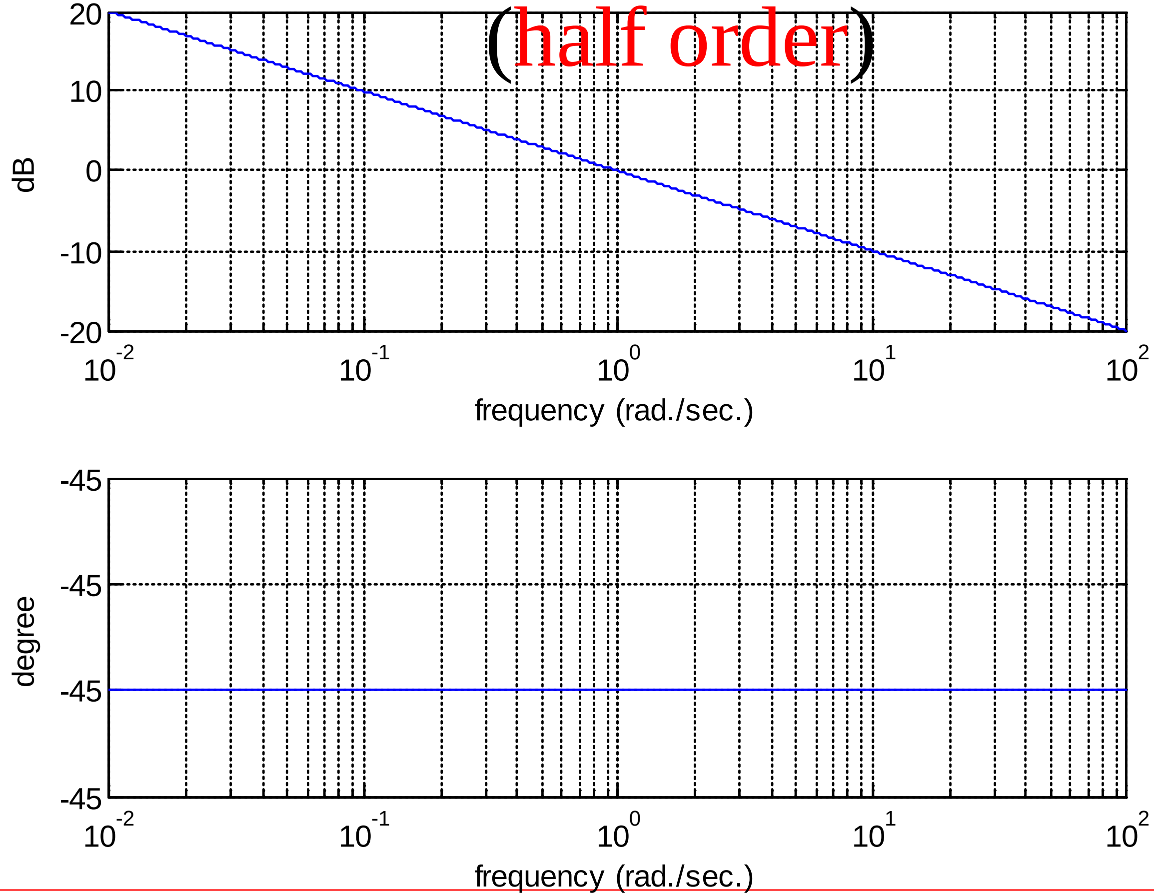
sys1=tf([1],[1,0]);figure;bode(sys1);grid on;



sys1=tf([1],[1,0]);figure;bode(1/sys1);grid on;

What is s^{α} when α is a *non-integer*?

Fractional Order Integrator (half order)



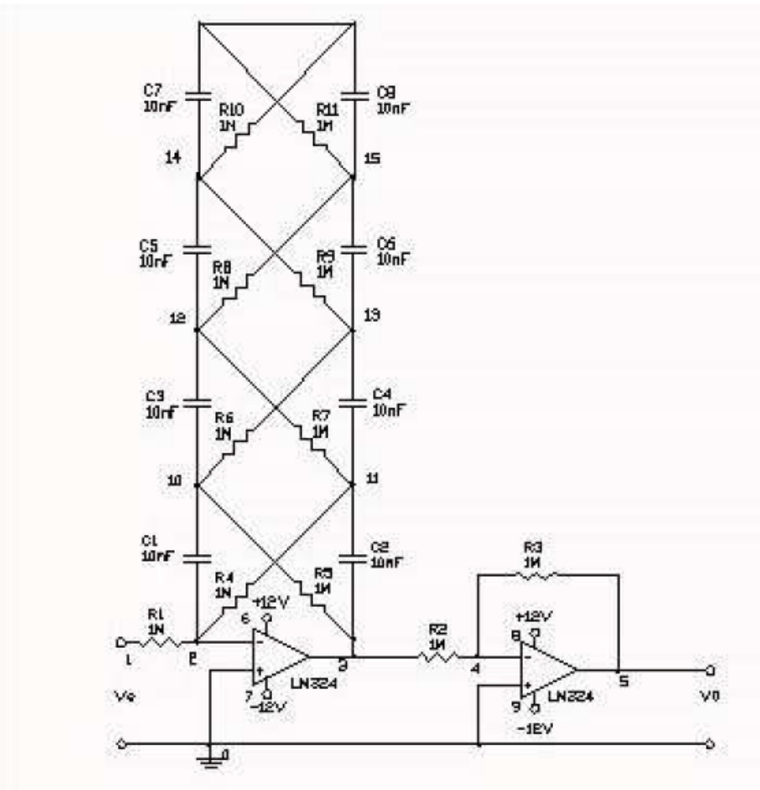
$$G(s) = \frac{1}{\sqrt{s}}$$

Legal in MATLAB

and everywhere?

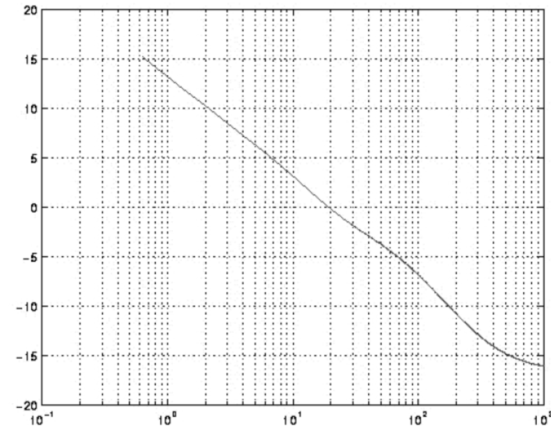
```
a=-0.5;w=logspace(-2,2,1000);fi=(j.*w).^a;
figure;subplot(2,1,1);semilogx(w,20*log10(abs(fi)));
xlabel('frequency (rad./sec.)');ylabel('dB');grid on
subplot(2,1,2);semilogx(w,180*angle(fi)/pi);
xlabel('frequency (rad./sec.)');ylabel('degree');grid on
```

Possible? Possible! Legal!!

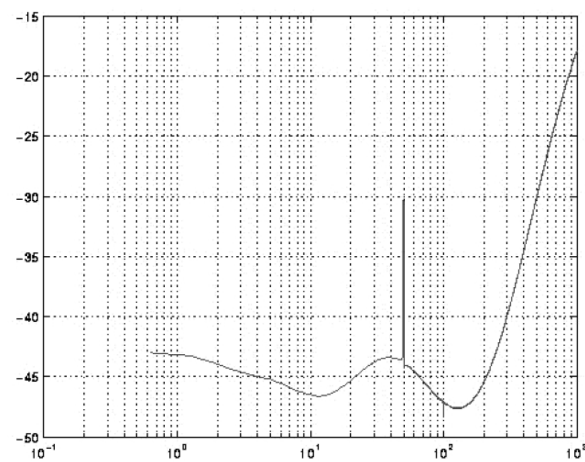


Analog $1/\sqrt{s}$ using op-amps.

Magnitude plot (dB vs. rad./sec.)



Phase plot (deg. vs. rad./sec.



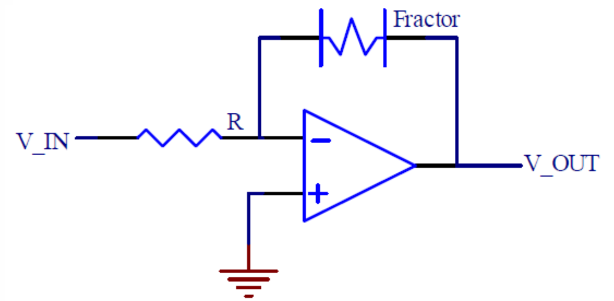
I. Petras, I. Podlubny, P. O'Leary, L. Dorcak, and Vinagre B. "**Analogue Realization of Fractional Order Controllers**". FBERG, Technical University of Kosice, Kosice, Slovak, **ISBN 8070996277** edition, 2002.

Fractor: Analogue device

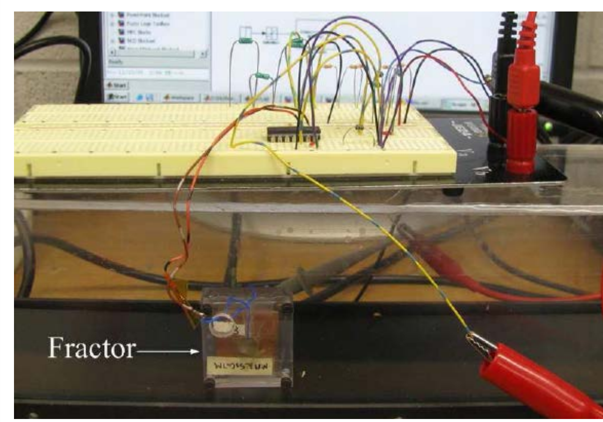
NSF SBIR 0538866

Fractional Calculus Day at USU, April 19, 2005

Photo credit: Igor Podlubny



$$G(s) = -\frac{K}{R(s\tau)^{\lambda}}$$



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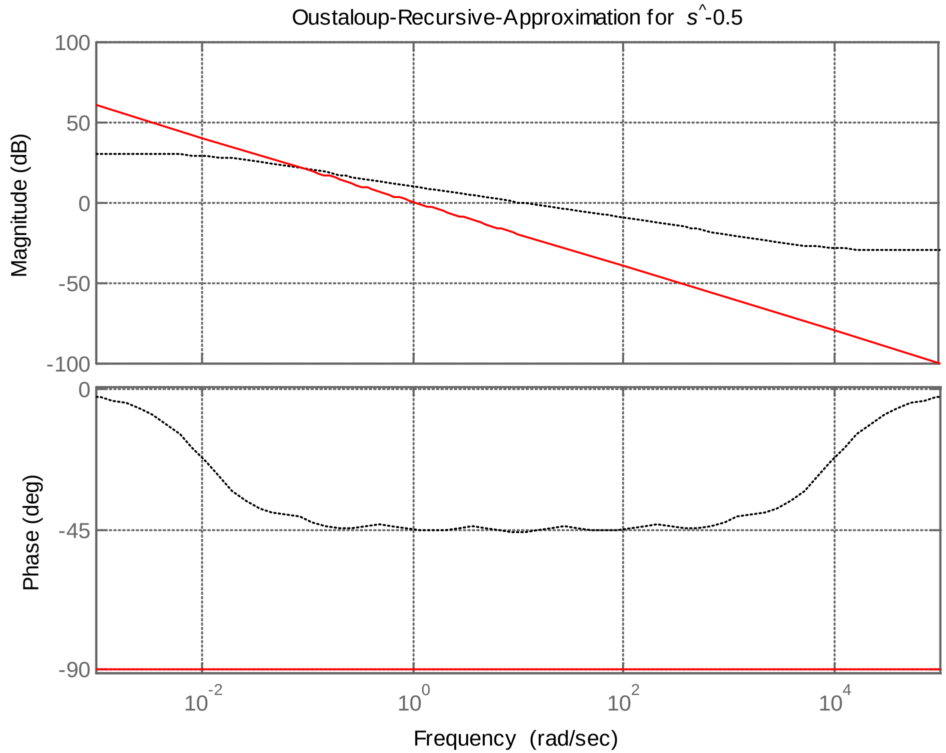
Oustaloup's Recursive Approximation for fractional order differentiators/integrator

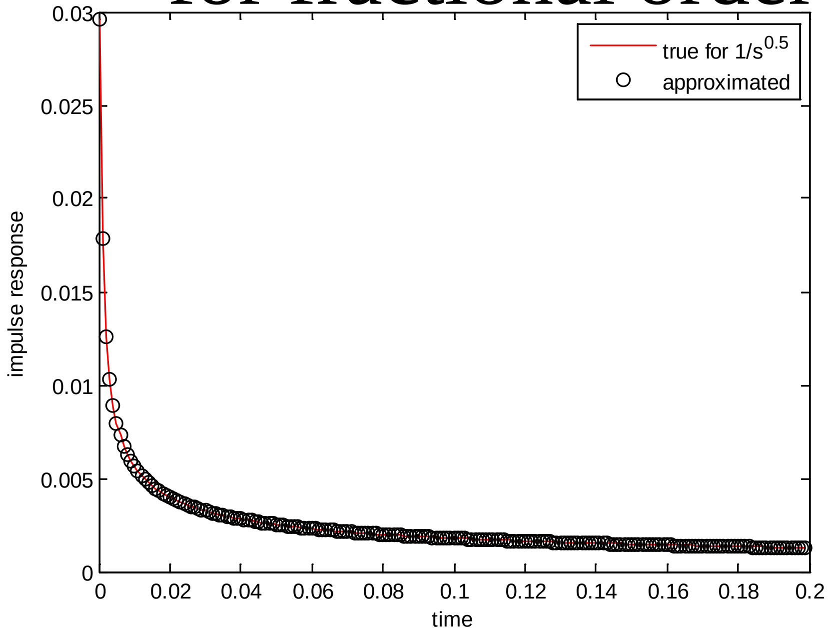


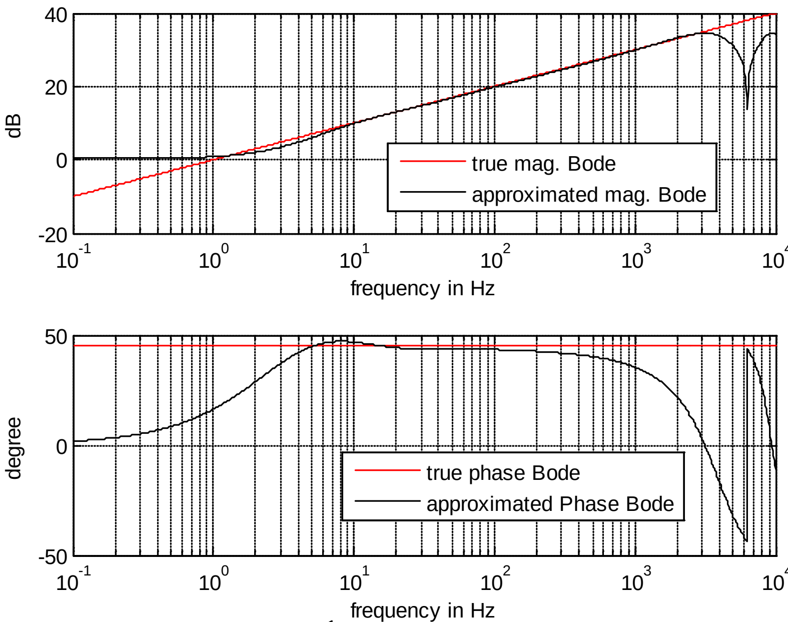

```
w_L=0.1;w_H=1000; r=-0.5; figure; N=3; sys1=tf(1,[1,0]);
sys_N_tf=ora_foc(r,N,w_L,w_H);bode(sys_N_tf,'k:',sys1,'r-');grid on;
title(['Oustaloup-Recursive-Approximation for {\it s}^{\^}',num2str(r)]);
```

$$G(s)=\frac{1}{s^{\gamma}}\approx\frac{B(s)}{A(s)}$$

http://www.mathworks.com/matlabcentral/fileexchange/3802-oustaloup-recursive-approximation-for-fractional-order-differentiators

Chen's impulse response invariant discretization for fractional order differentiators/integrator





irid_fod(0.5,.001,7)

$$G(s) = \frac{1}{s^{\gamma}} \approx \frac{B(z^{-1})}{A(z^{-1})}$$

http://www.mathworks.com/matlabcentral/fileexchange/21342-impulse-response-invariant-discretization-of-fractional-order-integratorsdifferentiators

What is s^α when α is a *non-integer* ?

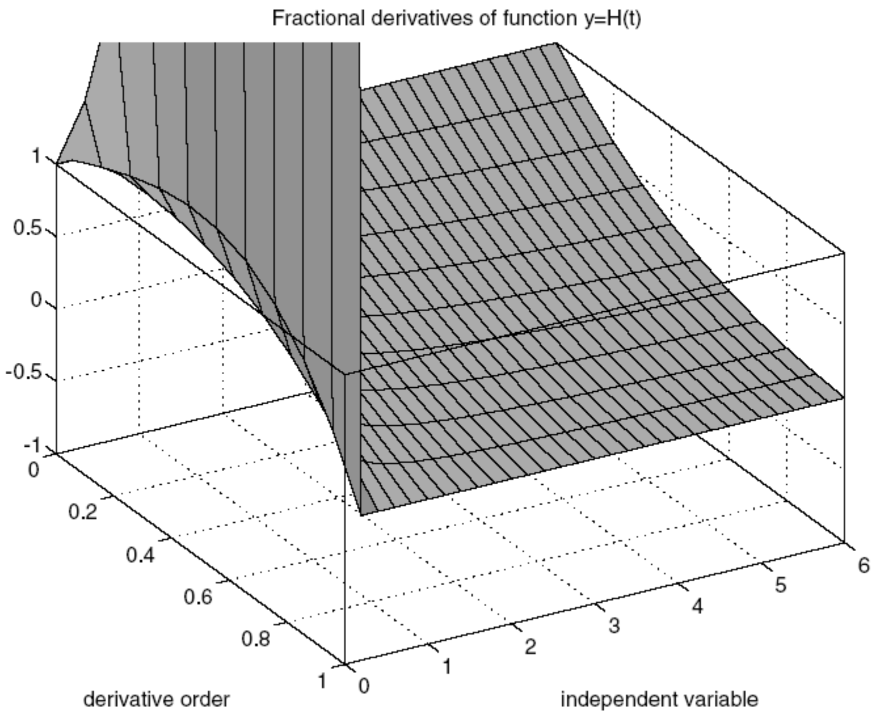
Operator $_a\mathrm{D}_t^\alpha$

A generalization of differential and integral operators:

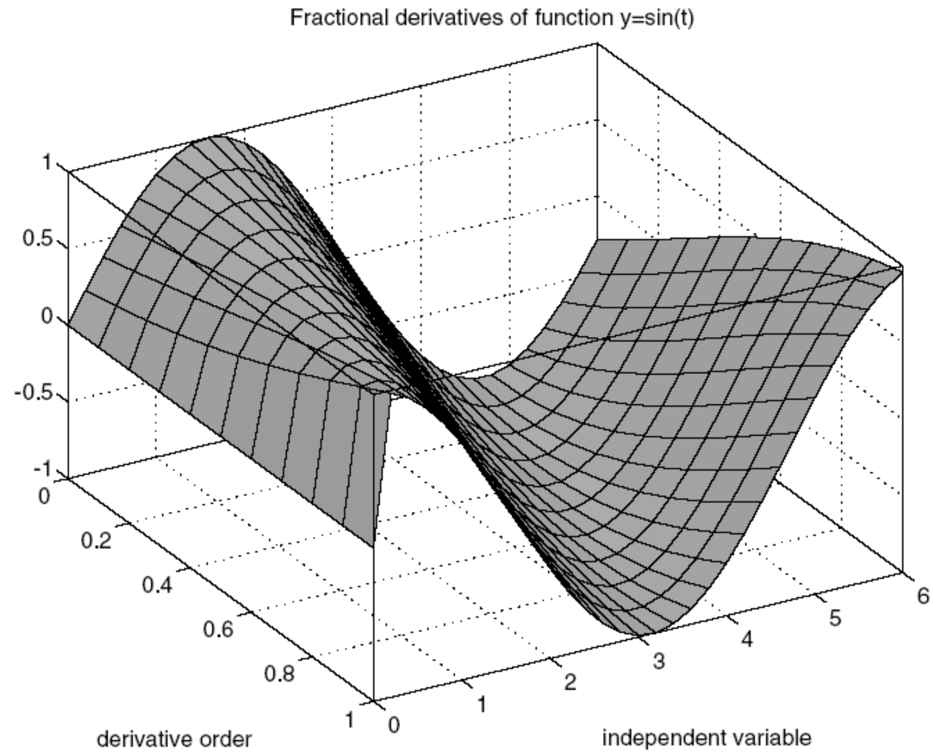
$$ {}_a\mathrm{D}_t^\alpha = \begin{cases} \mathrm{d}^\alpha/\mathrm{d}t^\alpha & \mathbb{R}(\alpha) > 0, \\ 1 & \mathbb{R}(\alpha) = 0, \\ \int_a^t (\mathrm{d}\tau)^{-\alpha} & \mathbb{R}(\alpha) < 0. \end{cases} \quad (7) $$

There are two commonly used definitions for the general fractional order differentiation and integral, i.e., the **Grünwald-Letnikov definition** and the **Riemann-Liouville definition**.

Example: Heaviside's unit step

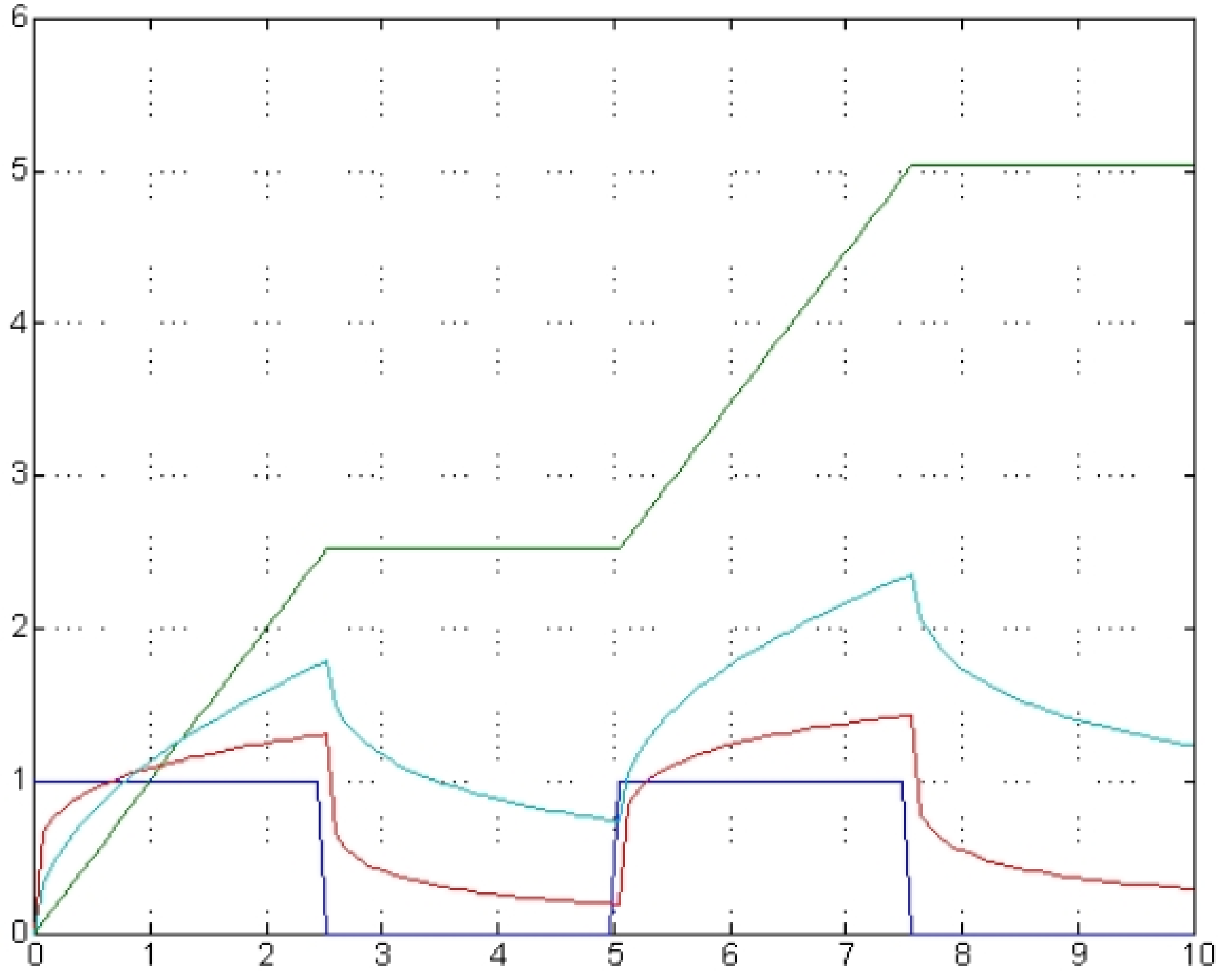


Example: $\sin(t)$



Slide credit: Igor Podlubny

Fractional derivatives of ramp function.

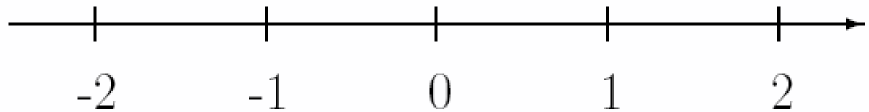


Anything surprising so far?

Quite intuitive in fact.

For example,

... from integer to non-integer ...



$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n}$$

$$x^n = e^{n \ln x}$$

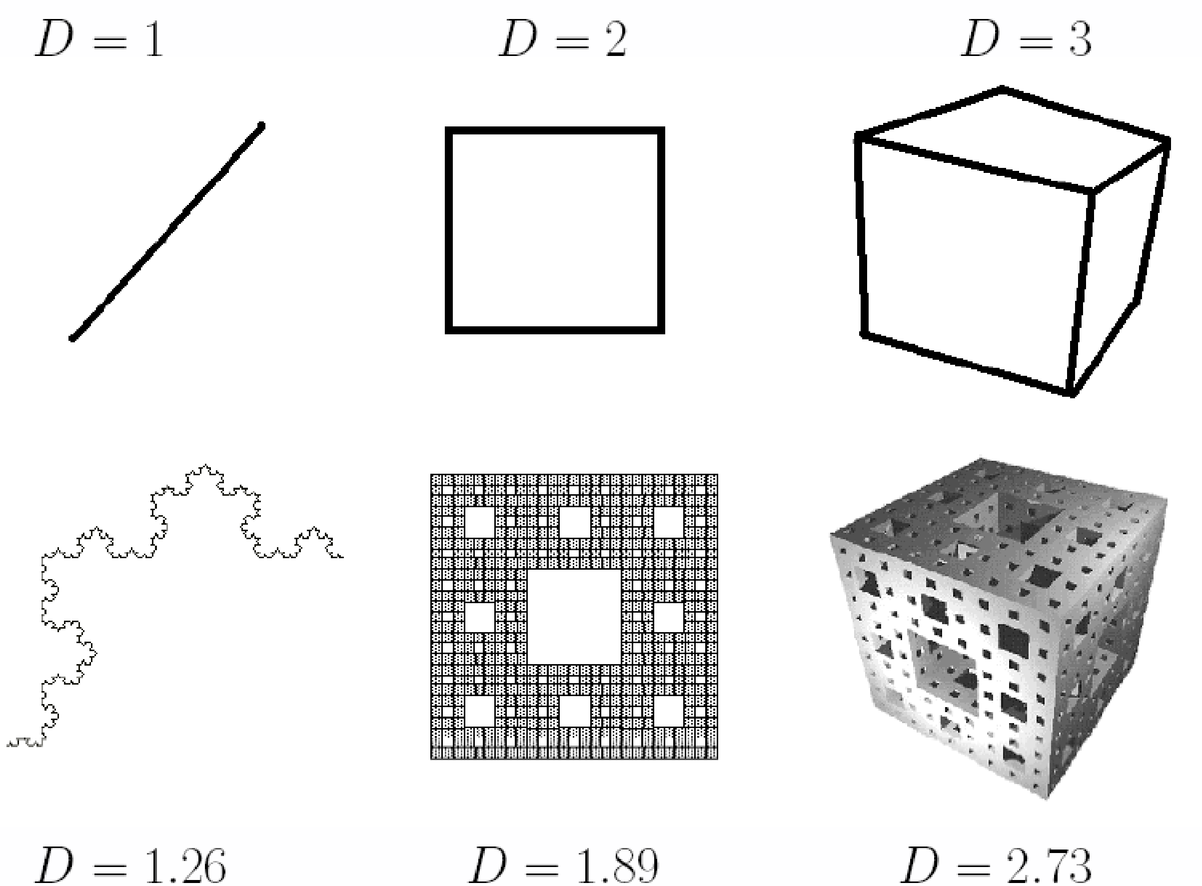
$$n! = 1 \cdot 2 \cdot 3 \cdot \ \ldots \cdot (n-1) \cdot n,$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0,$$

$$\Gamma(n+1) = 1 \cdot 2 \cdot 3 \cdot \ \ldots \cdot n = n!$$

Slide credit: Igor Podlubny

. . . from integer to non-integer . . .



Slide credit: Igor Podlubny

Interpolation of operations

$$f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \ldots$$

$$f, \quad \int f(t)dt, \quad \int dt \int f(t)dt, \quad \int dt \int dt \int f(t)dt, \quad \ldots$$

$$\ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots$$

Slide credit: Igor Podlubny

"Fractional Order Thinking" or, "In Between Thinking"

- For example
  - Between integers there are non-integers;
  - Between logic 0 and logic 1, there is the "**fuzzy logic**";
  - Between integer order splines, there are "**fractional order splines**"
  - Between integer high order moments, there are **noninteger order moments (e.g. FLOS)**
  - Between "integer dimensions", there are **fractal dimensions**
  - Fractional Fourier transform (FrFT) – in-between time-n-freq.
  - Non-Integer order calculus (fractional order calculus – abuse of terminology.) (FOC)

Fractional Calculus was born in 1695

$$\frac{d^n f}{dt^n}$$

What if the order will be $n = \frac{1}{2}$?

It will lead to a paradox, from which one day useful consequences will be drawn.

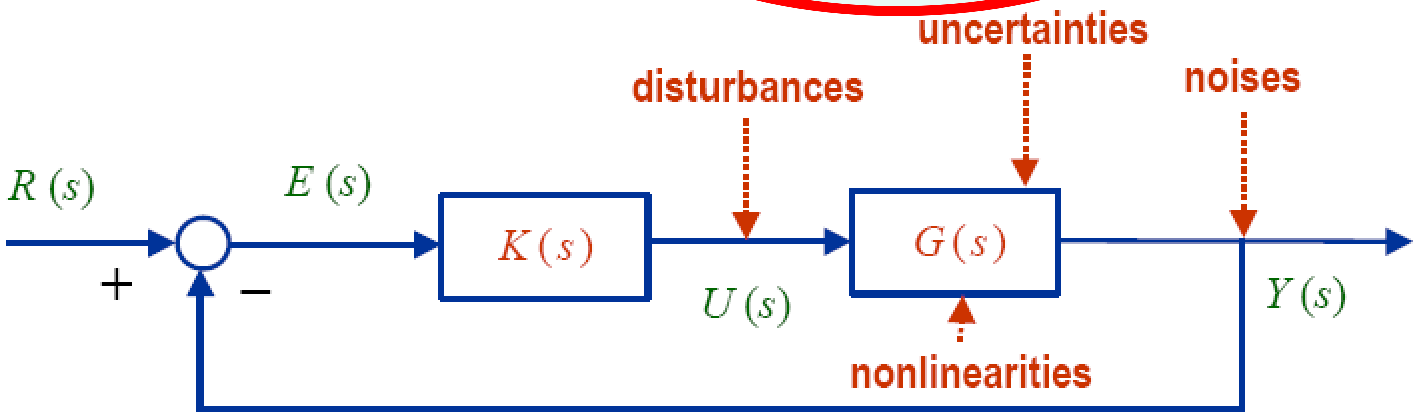
G.F.A. de L'Hôpital (1661–1704)

G.W. Leibniz (1646–1716)

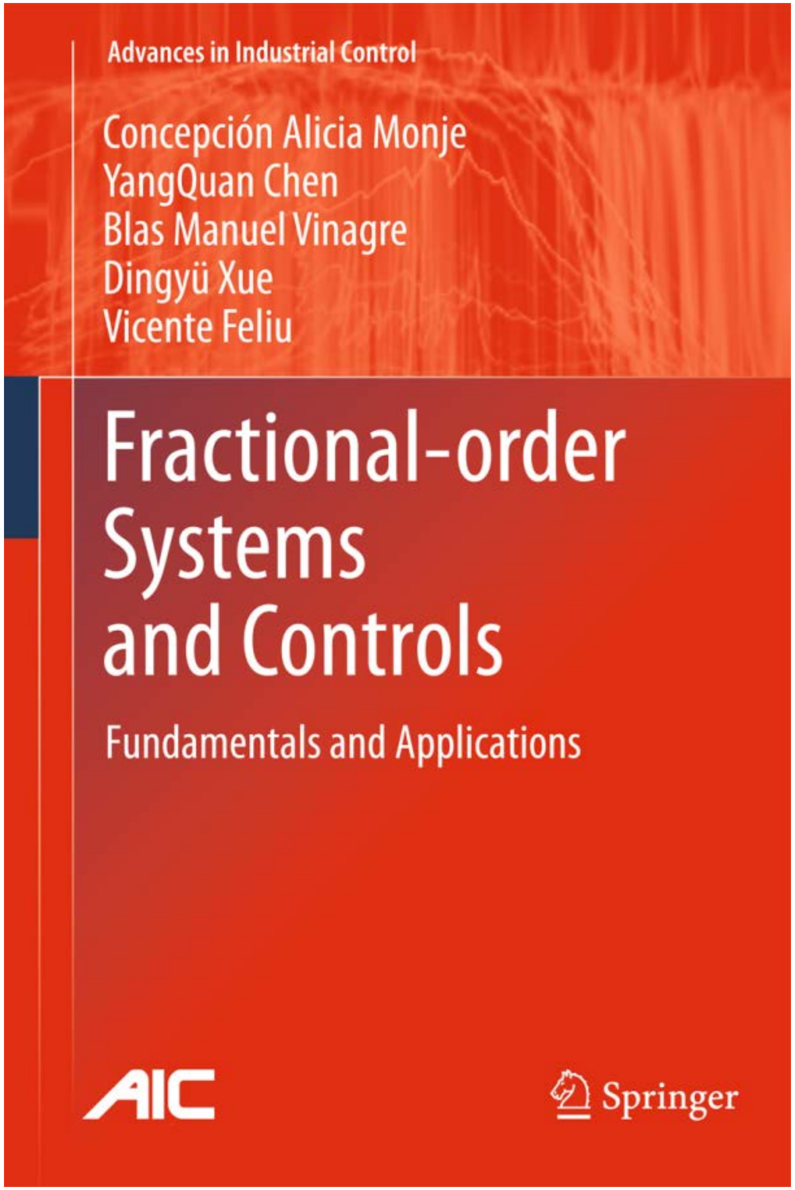
Slide credit: Igor Podlubny

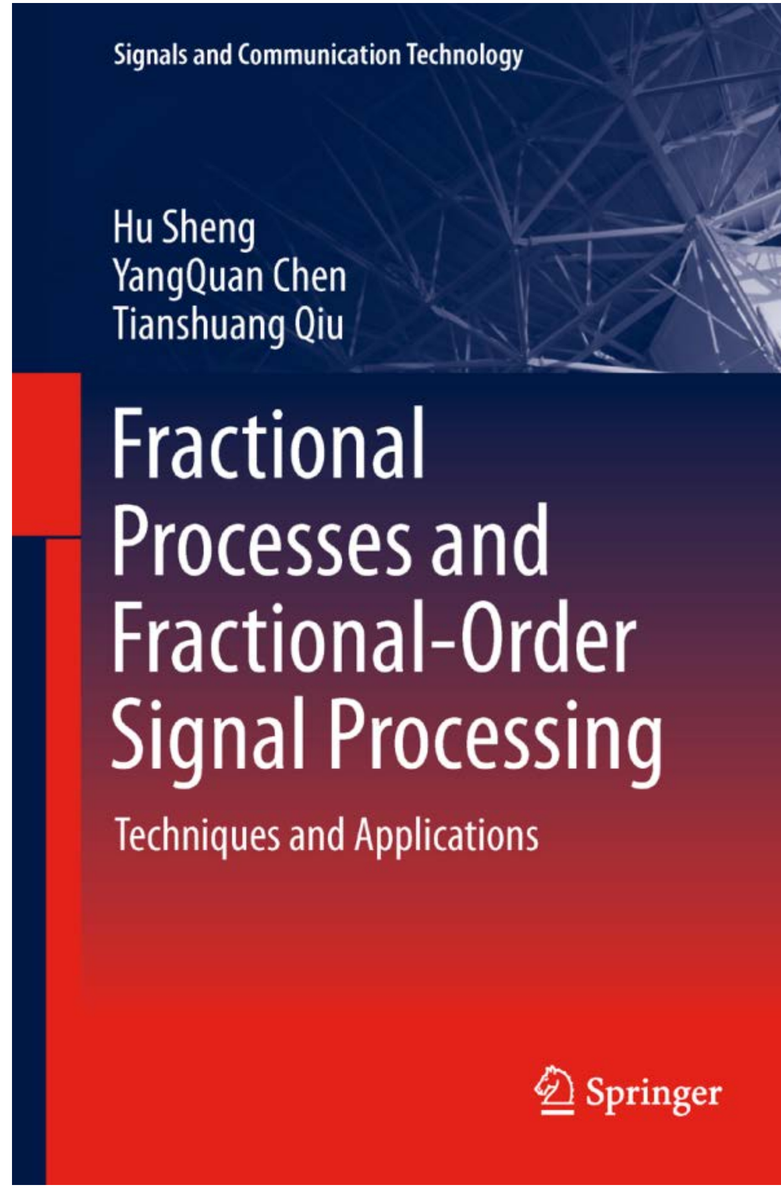
FOMs and Fractional Order Controls

- IO Controller + IO Plant
- FO Controller + IO Plant
- FO Controller + FO Plant
- IO Controller + FO Plant



Concepcin A. Monje, YangQuan Chen, Blas Vinagre, Dingyu Xue and Vicente Feliu (2010). "Fractional Order Systems and Controls - Fundamentals and Applications." Advanced Industrial Control Series, Springer-Verlag, www.springer.com/engineering/book/978-1-84996-334-3 (2010), 415 p. 223 ill.19 in color.





2001-2010

2005-2011

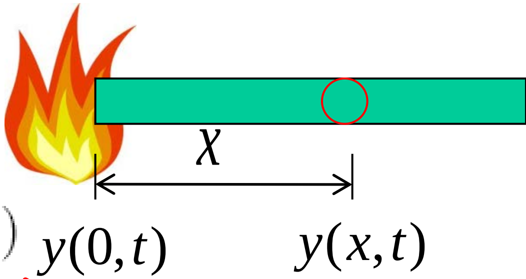
Outline

- **What is Fractional Calculus and Fractional Order Modeling/Controls**
- **A Worked Out Example on Fractional Order Modeling (FOM)**
- Summary of Benefits of Using FOM
- An Application: Parameter-Distributed and DO FOM of Complex Relaxation Processes

Modeling: heat transfer

$$\frac{\partial^2 y(x,t)}{\partial x^2} = k^2 \frac{\partial y(x,t)}{\partial t},$$

$$(t > 0, \quad 0 < x < \infty)$$



Boundary condition: $y(0,t) = m(t)$

$y(x,0) = 0$ initial condition

$\left|\lim_{x\to\infty} y(x,t)\right| < \infty$ Physical limit

Transfer function:

$$\frac{\mathrm{d}^2 Y(x,s)}{\mathrm{d}x^2} = k^2 s Y(x,s)$$

$$Q(0,s) = M(s)$$

$$\left|\lim_{x\to\infty} Y(x,s)\right| < \infty$$

$$Y(x,s) = A(s)e^{-kx\sqrt{s}} + B(s)e^{kx\sqrt{s}}$$

$$\begin{aligned} A(s) &= Y(0,s) = M(s) \\ B(s) &= 0 \end{aligned}$$

$$Y(x,s) = M(s)e^{-kx\sqrt{s}}$$

$$G(s) = \frac{Y(x,s)}{M(s)} = e^{-kx\sqrt{s}}$$

think about transfer function $e^{-\sqrt{s}}$!

Irrational Transfer Function.

Taylor series expansion: polynomial of half order integrators!!

Ideal physical plant model:

$$G_p(s) = e^{-\sqrt{s}}$$

First Order Plus Time Delay (FOPTD) Model:

$$G_{IO}(s) = \frac{K_1}{T_1 s + 1} e^{-L_1 s}$$

Time Delay with Single Fractional Pole Model:

$$G_{FO}(s) = \frac{K_2}{T_2 s^{0.5} + 1} e^{-L_2 s}$$

All models are wrong but some are useful.

George E. P. Box

All models are wrong but some are dangerous ...

Leonard A. Smith

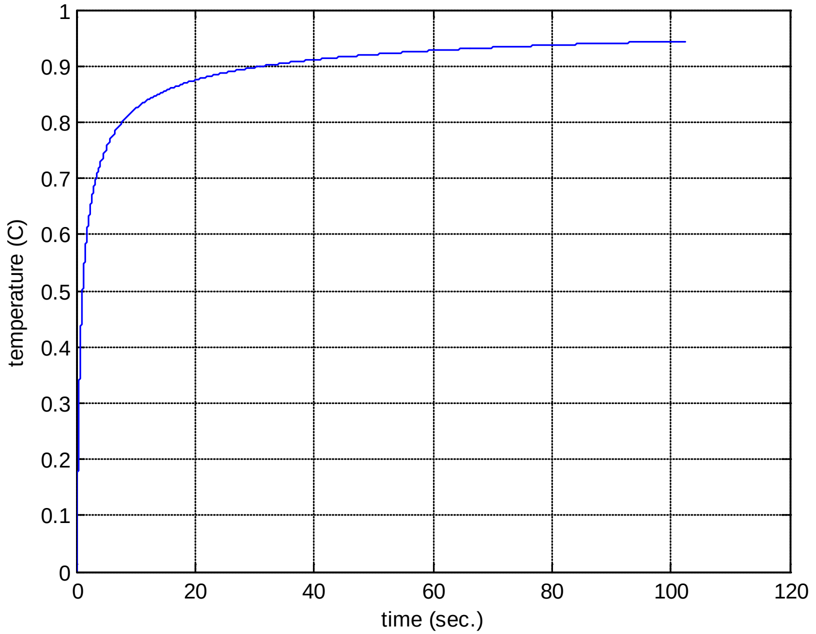
Step response of the "Ideal Plant"

$$y(0,t)=m(t)=1u(t), M(s)=\frac{1}{s}$$

$$Y(x,s)\big|_{x=1}=G(x,s)\big|_{x=1}M(s)=G_p(s)M(s)=\frac{1}{s}e^{-\sqrt{s}}$$

So, "Reaction-Curve" or Step response of the "Ideal Plant"

$$y(t)=L^{-1}[\frac{1}{s}e^{-\sqrt{s}}]$$



Magic code to do $y(t) = L^{-1}[\frac{1}{s}e^{-\sqrt{s}}]$

```
% step response of normalized 1D heat equation when x=1
clear all;close all; alpha=.5; Ts=0.1;
F= @(s)  exp(-s.^alpha)./s;
%---------------------------------------------------------------
alfa=0; M=1024; P=20; Er=1e-10; tm=M*Ts; wmax0=2*pi/Ts/2; L = M;
Taxis=[0:L-1]*Ts;  n=1:L-1; n=n*Ts ;
N=2*M; qd=2*P+1; t=linspace(0,tm,M); NT=2*tm*N/(N-2); omega=2*pi/NT;
c=alfa-log(Er)/NT; s=c-i*omega*(0:N+qd-1);
Fsc=feval(F,s); ft=fft(Fsc(1:N)); ft=ft(1:M);
q=Fsc(N+2:N+qd)./Fsc(N+1:N+qd-1); d=zeros(1,qd); e=d;
    d(1)=Fsc(N+1); d(2)=-q(1); z=exp(-i*omega*t);
    for r=2:2:qd-1; w=qd-r; e(1:w)=q(2:w+1)-q(1:w)+e(2:w+1); d(r+1)=e(1);
        if r>2; q(1:w-1)=q(2:w).*e(2:w)./e(1:w-1); d(r)=-q(1);
        end
    end
    A2=zeros(1,M); B2=ones(1,M); A1=d(1)*B2; B1=B2;
    for n=2:qd
    A=A1+d(n)*z.*A2; B=B1+d(n)*z.*B2;A2=A1; B2=B1; A1=A; B1=B;
    end
ht=exp(c*t)/NT.*(2*real(ft+A./B)-Fsc(1));
%---------------------------------------------------------------
figure;tt=0:(length(ht)-1);tt=tt*Ts;plot(tt,ht);
xlabel('time (sec.)');ylabel('temperature (C)');grid on
```

Application of numerical inverse Laplace transform algorithms in fractional calculus
Journal of the Franklin Institute, *Volume 348, Issue 2*, *March 2011*, *Pages 315-330*
Hu Sheng, Yan Li, YangQuan Chen http://dx.doi.org/10.1016/j.jfranklin.2010.11.009 (Check ref [8])

So, let us do fitting!

Ideal physical plant model: $G_p(s) = e^{-\sqrt{s}}$

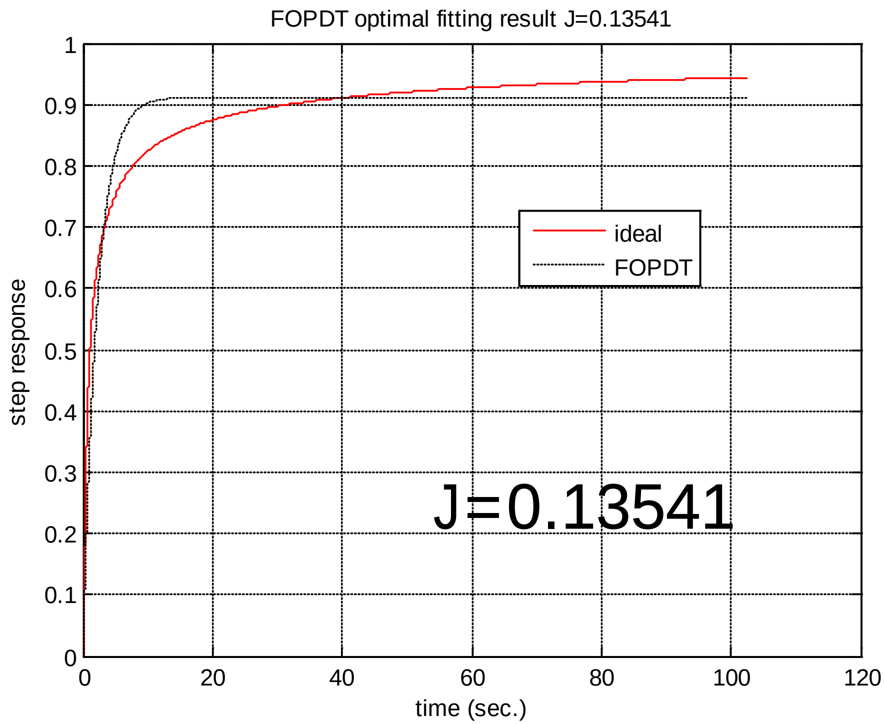
First Order Plus Time Delay (FOPTD) Model:

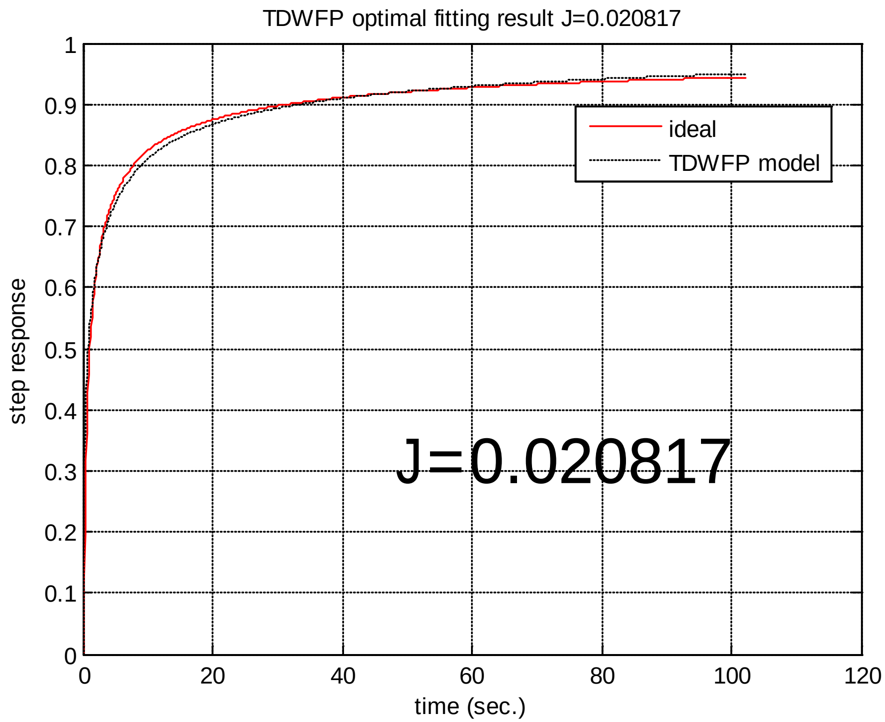
$$G_{IO}(s) = \frac{K_1}{T_1 s + 1} e^{-L_1 s}$$

Time Delay with Single Fractional Pole Model:

$$G_{FO}(s) = \frac{K_2}{T_2 s^{0.5} + 1} e^{-L_2 s}$$

All models are wrong but some are useful. **George E. P. Box**





K1	T1	L1
0.9120	2.2393	0

K2	T2	L2
1.0197	1.2312	0.0001

Fitting code for $G_{IO}(s)=\frac{K_1}{T_1 s+1}e^{-L_1 s}$

```
% Ts: sampling period; ht: step response (from NILT numerical inverse
% Laplace transform)
% previously we got Ts and ht array (reaction curve)
options=optimset('TolX',1e-10,'TolFun',1e-10);
Tic;[x,FVAL,EXITFLAG] =fminsearch(@(x) fopdtfit(x,ht,Ts),[1,1,0],options);toc
% May need to wait half minute
K1=x(1);T1=x(2);L1=x(3);T=(0:length(ht)-1)*Ts;if L1<0; L1=0; end
sysfoptd=tf([K1],[T1,1],'iodelay',L1);
y=step(sysfoptd,T);plot(T,ht,'r',t,y,'k:');grid on;
title(['FOPDT optimal fitting result J=',num2str(FVAL)]);
xlabel('time (sec.)');ylabel('step response'); legend('ideal', 'FOPDT')
```

```
% fitting using FOPTD model - integral of error square (ISE)
function [J]=foptdfit(x,y0,Ts);
K1=x(1);T1=x(2);L1=x(3);T=(0:length(y0)-1)*Ts;if L1<0; L1=0; end
sysfoptd=tf([K1],[T1,1],'iodelay',L1);
y=step(sysfoptd,T);
J=(y'-y0)*(y-y0')*Ts;
```

Fitting code for $G_{FO}(s) = \frac{K_2}{T_2 s^{0.5}+1} e^{-L_2 s}$

```
options=optimset('TolX',1e-10,'TolFun',1e-10);
Tic;[x,FVAL,EXITFLAG] =fminsearch(@(x) tdwfpfit(x,ht,Ts),[1,2,0],options);toc
% May need to wait 1000+ seconds!
K1=x(1);T1=x(2);L1=x(3);Np=length(ht);T=(0:Np-1)*Ts;if L1<0; L1=0; end
y=mlf(0.5,1.5,-T.^0.5/T1);y=(K1/T1)*(T.^0.5) .* y;
Nstep=floor(L1/Ts);
y1=zeros(size(y));y1(Nstep+1:Np)=y(1:Np-Nstep);
y=y1;plot(T,ht,'r',t,y,'k:');grid on;
title(['TDWFP optimal fitting result J=',num2str(FVAL)]);
xlabel('time (sec.)');ylabel('step response'); legend('ideal', 'TDWFP model')
```

```
% fitting using TDWFP model - integral of error square (ISE)
function [J]=tdwfpfit(x,y0,Ts);
K1=x(1);T1=x(2);L1=x(3);Np=length(y0);T=(0:Np-1)*Ts;if L1<0; L1=0; end
y=mlf(0.5,1.5,-T.^0.5/T1);y=(K1/T1)*(T.^0.5) .* y;
Nstep=floor(L1/Ts);y1=zeros(size(y));y1(Nstep+1:Np)=y(1:Np-Nstep);
J=(y1-y0)*(y1-y0)'*Ts;
% get MLF.m from
% www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-function
```

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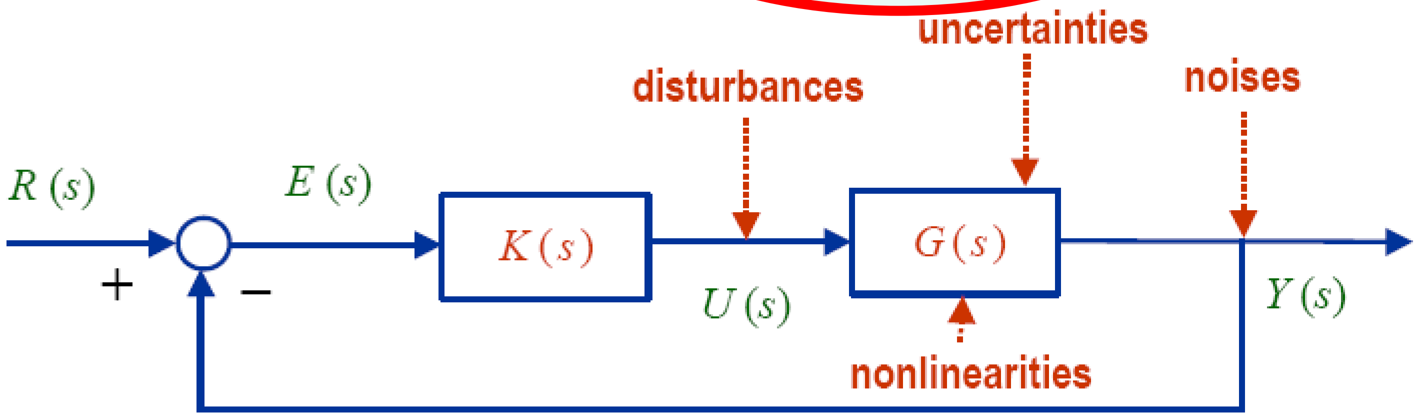
Benefits of FOM

- Captures (more) physics $G_p(s) = e^{-\sqrt{s}} \rightarrow G_{FO}(s) = \frac{K_2}{T_2 s^{0.5} + 1} e^{-L_2 s}$
- Reaction curve fitting: **Better than the best** FOPDT model $G_{IO}(s) = \frac{K_1}{T_1 s + 1} e^{-L_1 s}$
- Could be a nice starting point for better controller design?

FOMs and Fractional Order Controls

- **IO Controller + IO Plant**
- **FO Controller + IO Plant**
- **FO Controller + FO Plant**
- **IO Controller + FO Plant**



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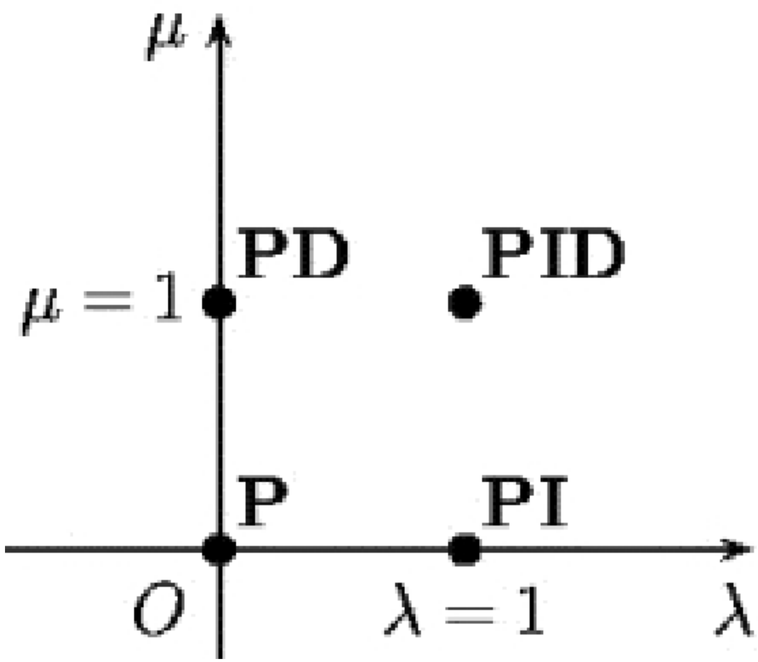
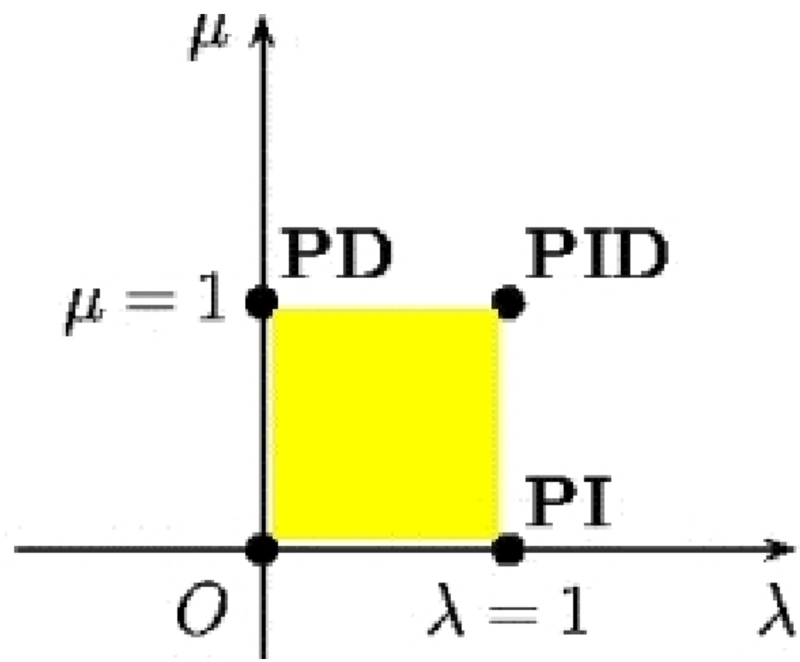
Fractional order PID control

- 90% are PI/PID type in industry. (**Ubiquitous**)

$$u(t) = K_p\left(e(t) + T_i \mathrm{D}_t^{-\lambda} e(t) + \frac{1}{T_d} \mathrm{D}_t^{\mu} e(t)\right). \qquad (\mathrm{D}_t^{(*)} \equiv {}_0\mathrm{D}_t^{(*)}).$$





Igor Podlubny. "Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers". IEEE Trans. Automatic Control,44(1): 208–214, 1999.

*YangQuan Chen, Dingyu Xue, and Huifang Dou. "**Fractional Calculus and Biomimetic Control**". IEEE Int. Conf. on Robotics and Biomimetics (RoBio04), August 22-25, 2004, Shengyang, China.*

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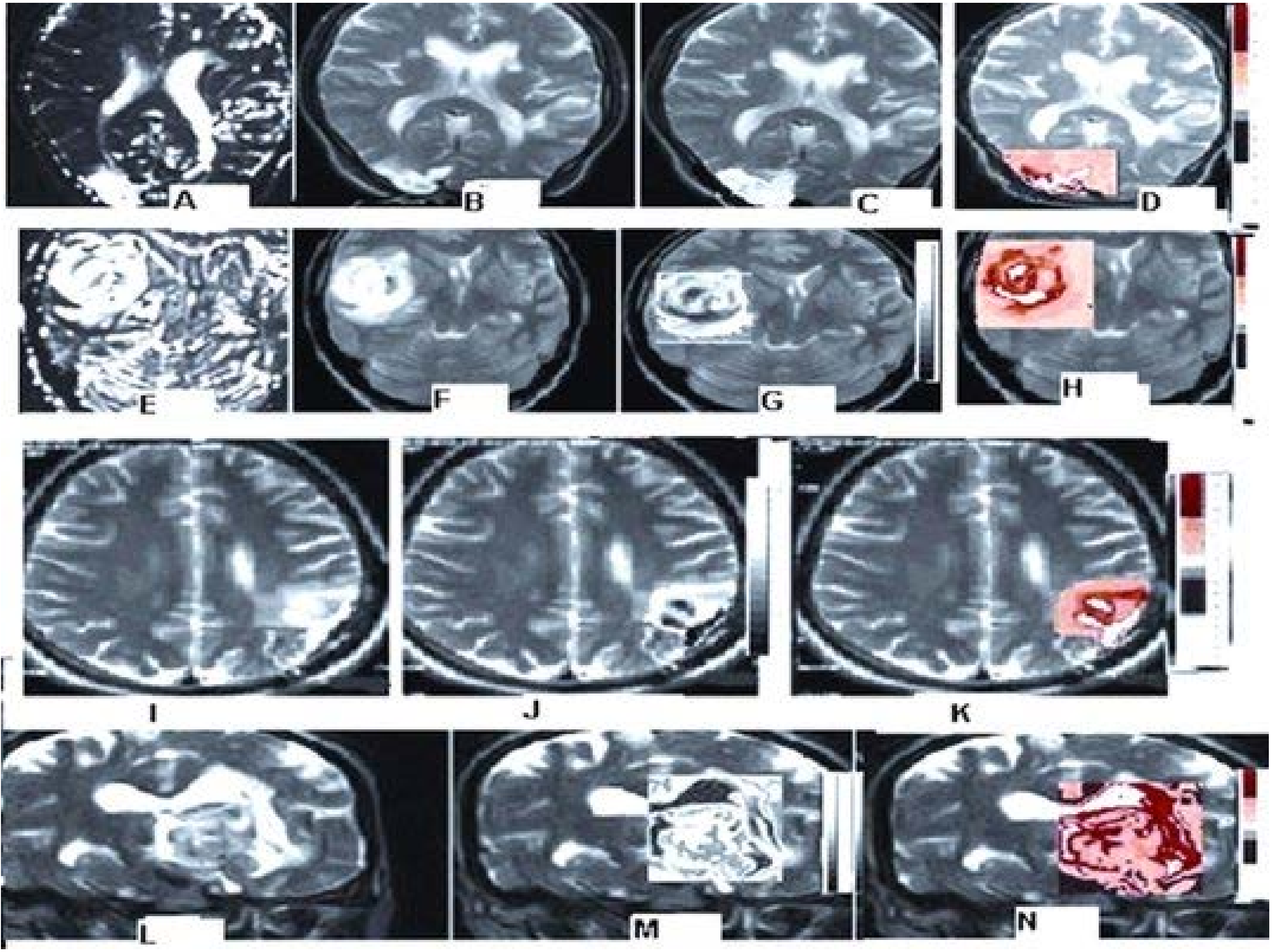
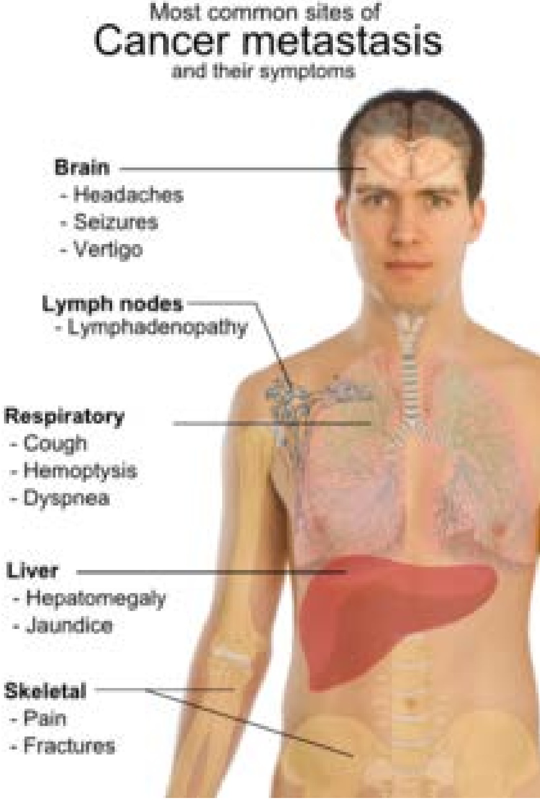
- **An Application: Parameter-Distributed and DO FOM of Complex Relaxation Processes**

Complex relaxation in NMR

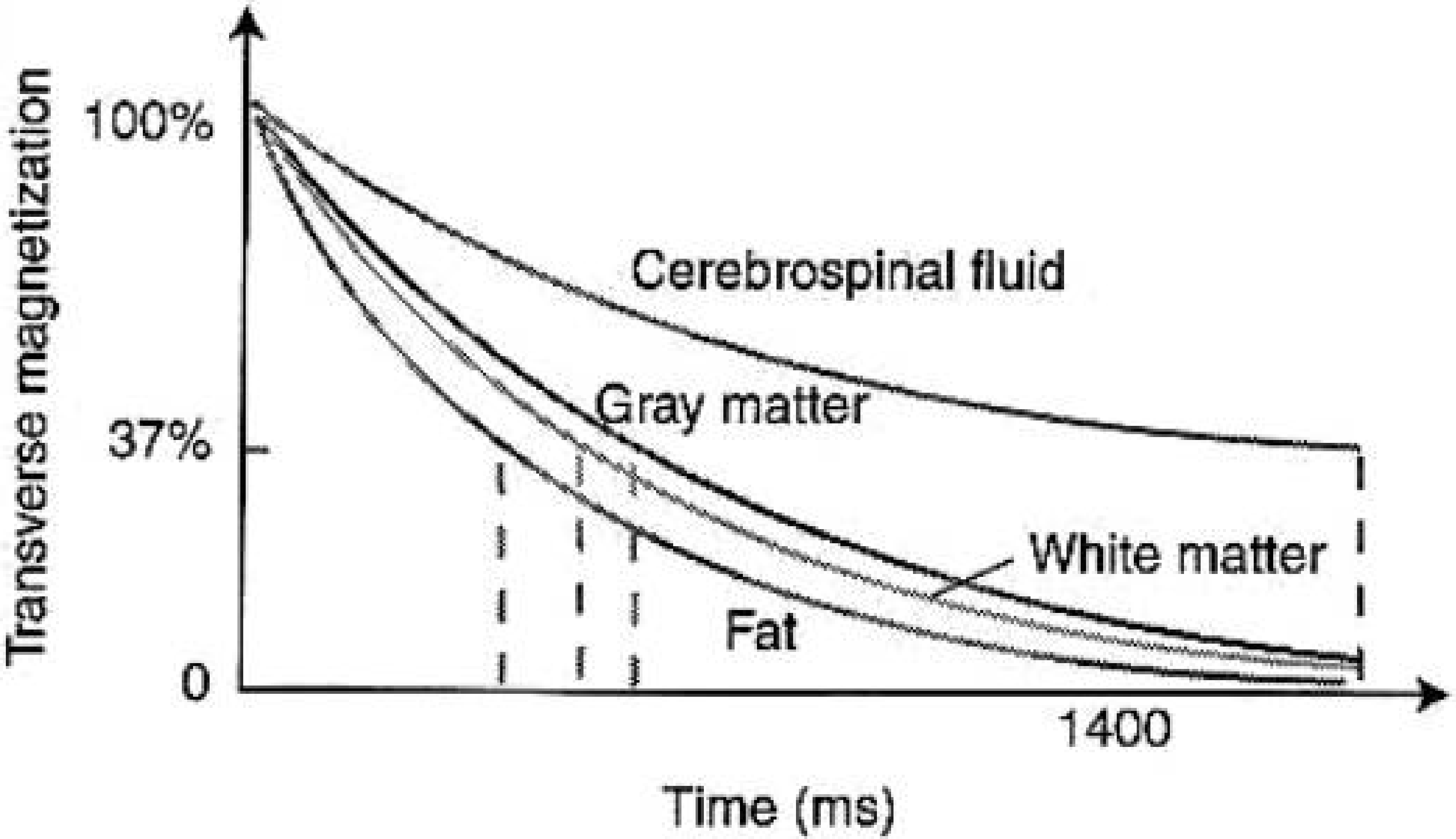


http://en.wikipedia.org/wiki/Metastasis



http://www.ispub.com/journal/the-internet-journal-of-radiology/volume-13-number-1/in-vivo-mr-measurement-of-refractive-index-relative-water-content-and-t2-relaxation-time-of-various-brain-lesions-with-clinical-application-to-discriminate-brain-lesions.article-g08.fs.jpg

T2 relaxation in NMR



http://hs.doversherborn.org/hs/bridgerj/DSHS/apphysics/NMR/T2.htm

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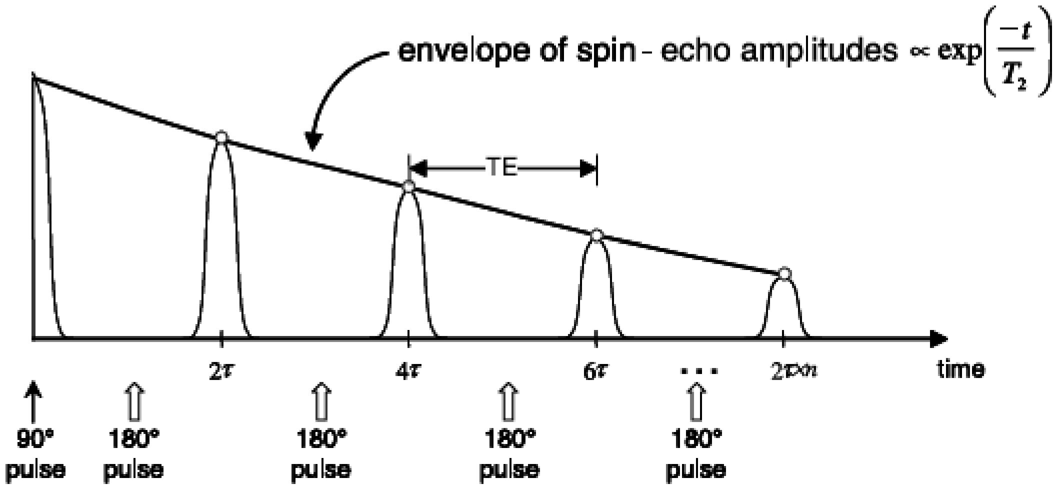
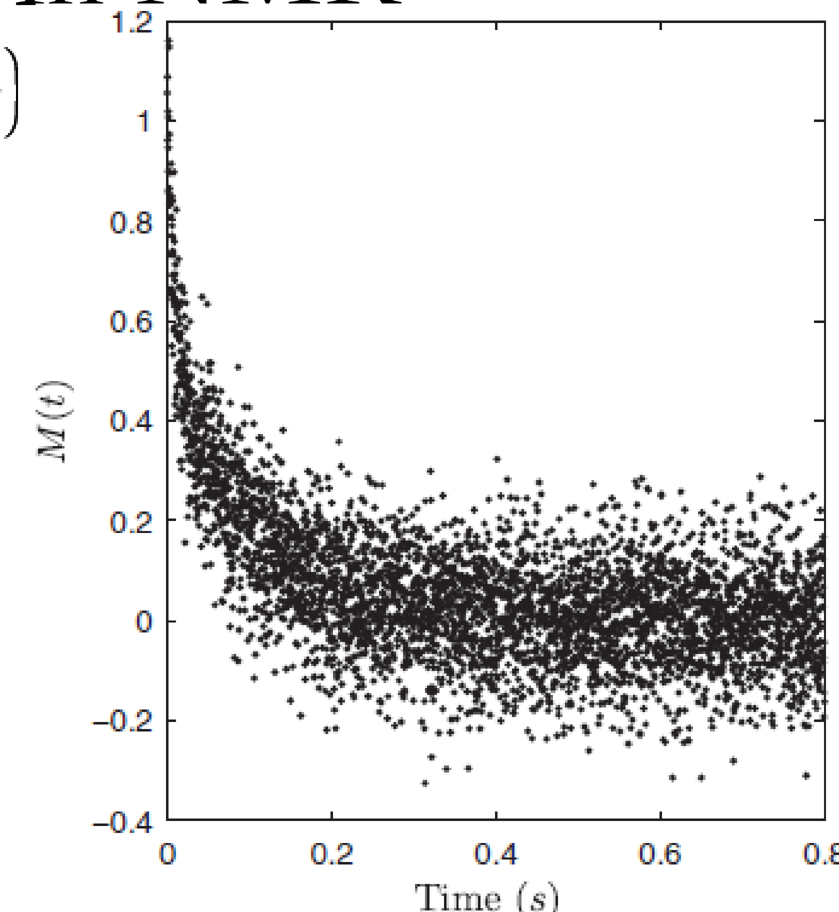
T2 relaxation in NMR

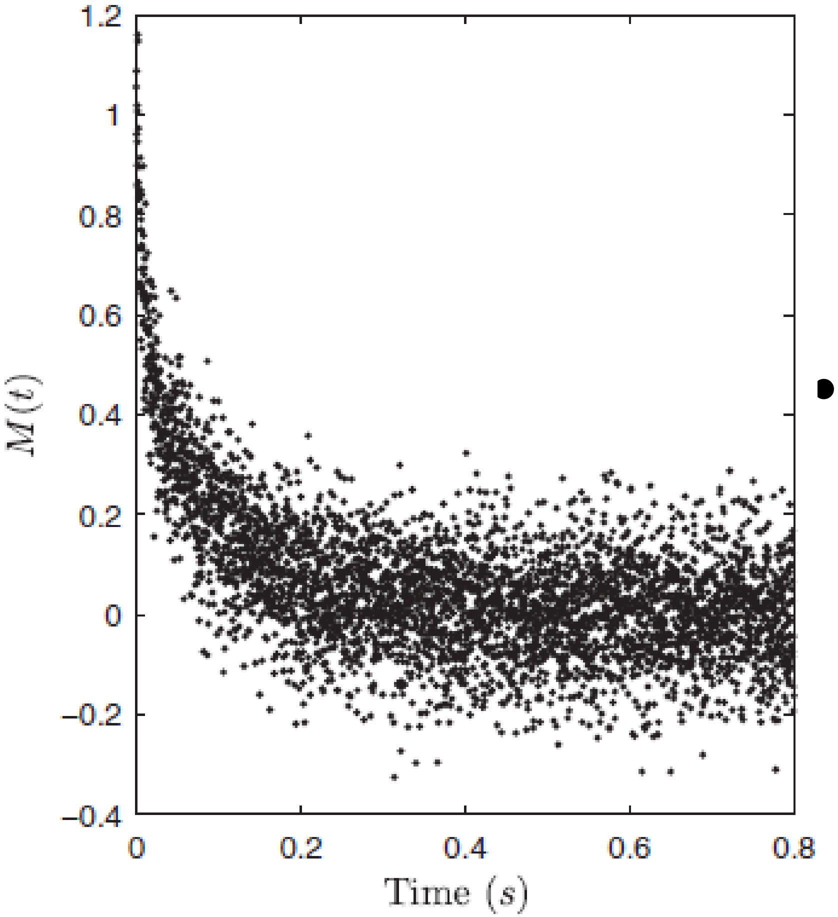




Carr–Purcell–Meiboom–Gill (*CPMG*) pulse sequence, as shown in Fig. 1, is widely used to measure spin–spin *relaxation* time *T2*

magnetization decay data *M(t) with poor SNR*

Complex relaxation: How to characterize or model it?



- Debye relaxation $\exp(-t/\tau)$

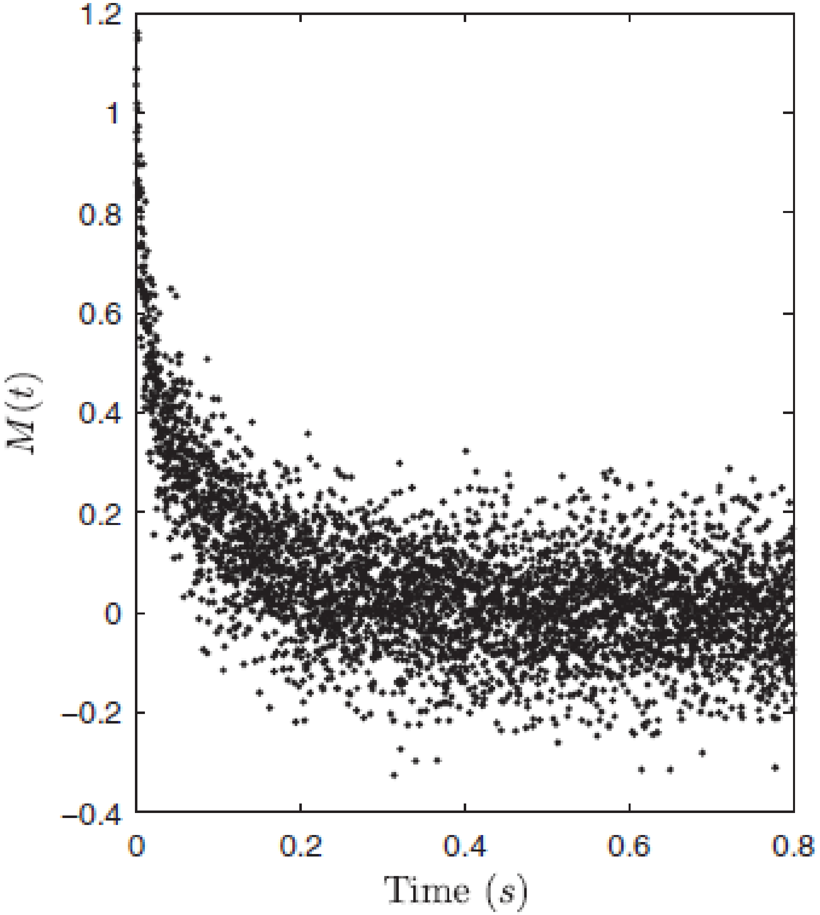
$$1/(1+\tau s)$$

- Distributed-parameter (infinite # of time constants)

$$\int_0^T \frac{f(\tau)}{\tau s+1} d\tau$$

(H. Fröhlich, 1949)

Complex relaxation: How to better characterize or model it?



- Cole-Cole (1941)

$$1/\left(1+\tau s^{\alpha}\right)$$

- Distributed-parameter (infinite # time constants)

$$\int_0^{T} \frac{f(\tau)}{\tau s^{\alpha}+1} d\tau$$

(Hu, Li and Chen, IEEE CDC 2010)

More complex relaxation models

- Cole-Davidson

$$H_{\text{C-D}}(s)=\int_0^T \frac{f(\tau)}{(1+\tau s)^{\beta}}\,\mathrm{d}\tau$$

- Havriliak-Negami

$$H_{\text{H-N}}(s)=\int_0^T \frac{f(\tau)}{(1+\tau s^{\alpha})^{\beta}}\,\mathrm{d}\tau$$

- **Distributed-order case? Sure!**

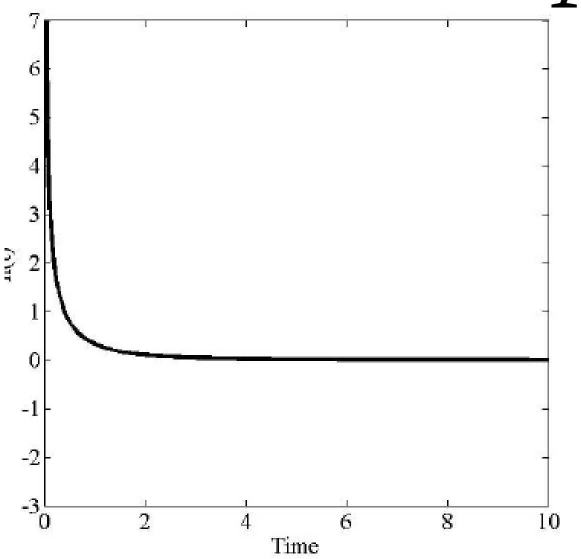
$$H(s)=\int_0^1 \frac{f(\gamma)}{\tau s^{\gamma}+1}\,\mathrm{d}\gamma$$

More complex relaxation models: **Why?**

- **More complexities captured**
- **More informing (useful) parameters extracted**
- **Closer to the nature of the *real* (fractional) dynamics**
- **Not much added computational overhead**

increase model complexities! When?

An illustration

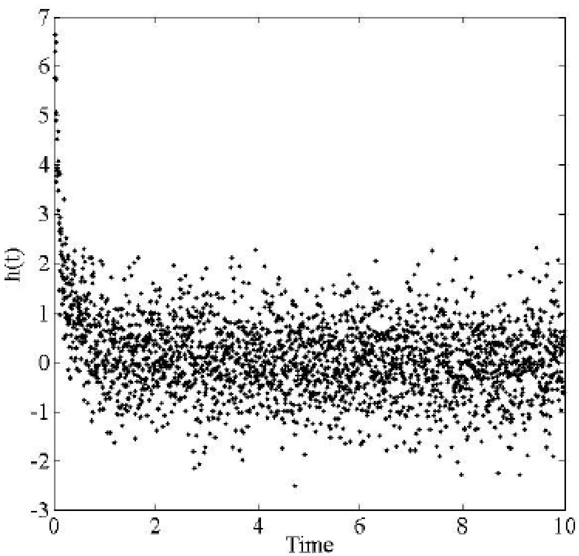


- Distributed-parameter (infinite time constants)

$$\int_0^T \frac{f(\tau)}{\tau s^{\alpha}+1}\,d\tau$$

- $\alpha = 0.75$, T=1 sec.

$$f(\tau) = 1 + 2\tau$$



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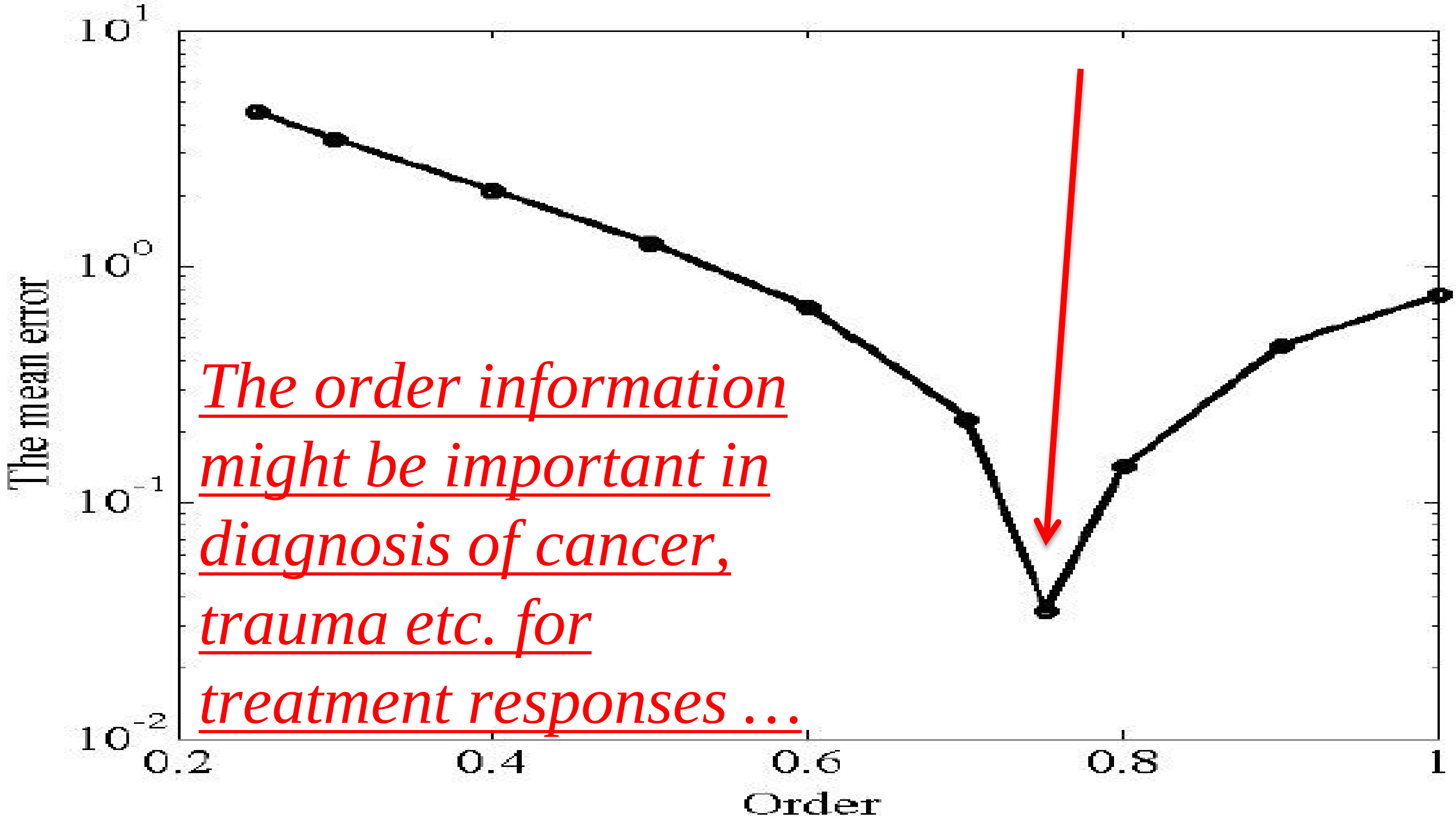
Scanning the “order” and fitting

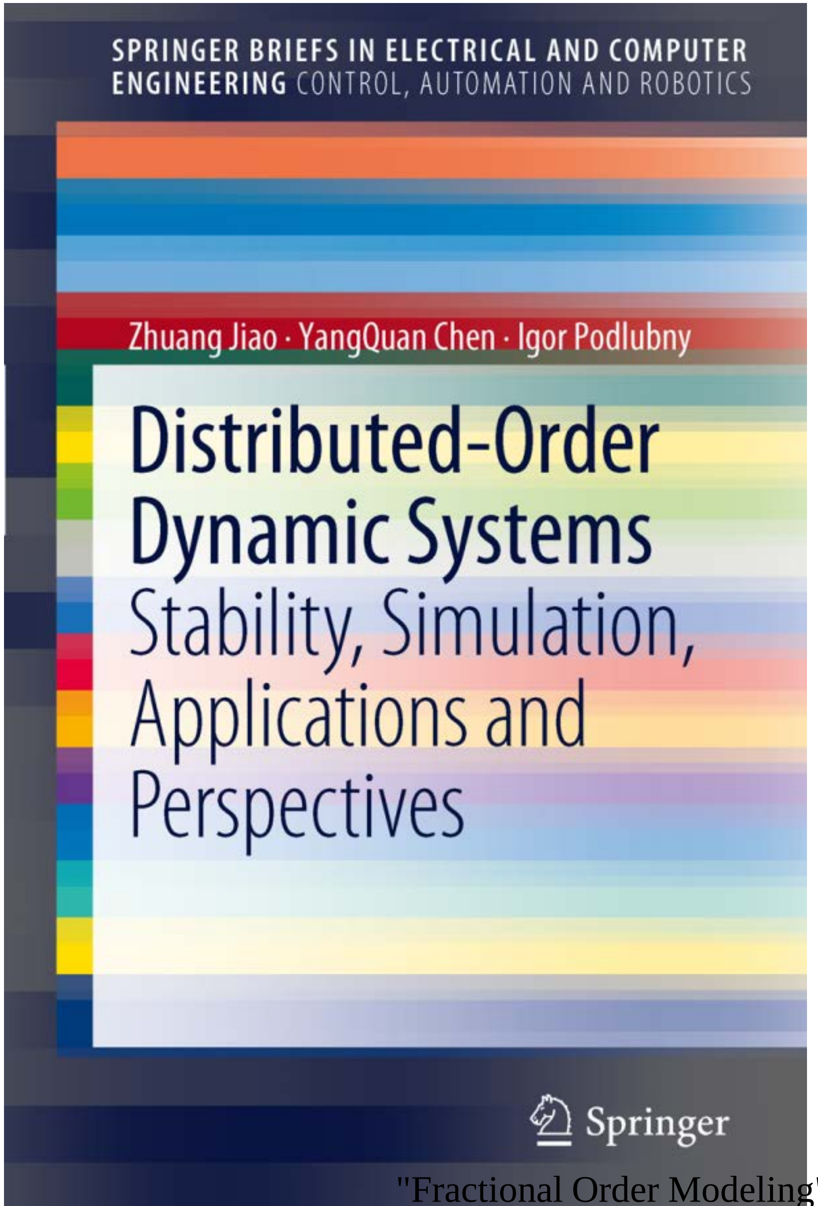


The order information might be important in diagnosis of cancer, trauma etc. for treatment responses …

A take home message

- Go "Fractional Order Models" if (note: not **iff**)
  - The signals are from a complex "system"
    - biological,
    - physiological,
    - behavioral,
    - social (crowds, community emergence, alcohol diffusion …)
    - man-made, (transportation/comm/power networks, etc.)
    - natural (geological, ecological, environmental etc.)
    - cosmological
    - etc.

Go "Fractional Order Models", too?



$$H(s) = \int_0^1 \frac{f(\gamma)}{\tau s^\gamma + 1} \mathrm{d}\gamma$$

Thank you for attending my talk!

For more information, check

http://mechatronics.ece.usu.edu/foc/afc

http://www.youtube.com/user/FractionalCalculus

coming soon

http://mechatronics.ucmerced.edu/