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Bifurcation Analysis of a Vibro-Impact Viscoelastic Oscillator with Fractional Derivative Element

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To the best of authors' knowledge, little work has been focused on the noisy vibro-impact systems with fractional derivative element. In this paper, stochastic bifurcation of a vibro-impact oscillator with fractional derivative element and a viscoelastic term under Gaussian white noise excitation is investigated. First, the viscoelastic force is approximately replaced by damping force and stiffness force. Thus the original oscillator is converted to an equivalent oscillator without a viscoelastic term. Second, the nonsmooth transformation is introduced to remove the discontinuity of the vibro-impact oscillator. Third, the stochastic averaging method is utilized to obtain analytical solutions of which the effectiveness can be verified by numerical solutions. We also find that the viscoelastic parameters, fractional coefficient and fractional derivative order can induce stochastic bifurcation.

Keywords: Stochastic bifurcation; vibro-impact; viscoelastic oscillator; fractional derivative element.

1. Introduction

Fractional calculus developing in parallel with the classical calculus has received considerable attention in recent decades. Many authors have focused on the dynamical response of fractional systems. Huang extended the stochastic averaging method to discuss the stochastic response and stability of fractional nonlinear systems subject to Gaussian white noise excitations [Huang & Jin, 2009]. Chen put forward an innovative bifurcation control method based upon the fractional-order feedback controller to control the stochastic jump bifurcation

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of noisy Duffing oscillator. Deng and Zhu developed a stochastic averaging method for quasi-Hamiltonian systems under fractional Gaussian noise [Deng & Zhu, 2016]. Malara and Spanos addressed the problem of determining the nonlinear response of a fractional plate driven by random loads [Malara & Spanos, 2018]. Colinas-Armijo and Di Paola used two methods to evaluate the response of a linear viscoelastic material modeled by the fractional Maxwell model and subject to a Gaussian stochastic temperature process [Colinas-Armijo *et al.*, 2018]. Kougioumtzoglou proposed a new multiple-input/single-output system identification approach for parameter identification of fractional oscillators subject to incomplete nonstationary data [Kougioumtzoglou *et al.*, 2017]. Ma and Li established a local fractional center manifold for a finite-dimensional fractional ordinary differential system [Ma & Li, 2016]. In addition, comprehensive review papers have been completed by Rossikhin [Rossikhin & Shitikova, 1997, 2010], Machado [Machado *et al.*, 2011] and Chen [Chen *et al.*, 2009; Li *et al.*, 2017], respectively.

Vibro-impact systems, as a class of discontinuous and strongly nonlinear systems, can exhibit complicated dynamical behaviors [Luo *et al.*, 2006; Di Bernardo *et al.*, 2003]. Zhu discussed the stochastic response of a vibro-impact Duffing system under external Poisson impulses [Rossikhin, 2015]. Xu explored the stochastic response of an inelastic vibro-impact system under Gaussian white noise with the help of equivalent nonlinearization technique [Xu *et al.*, 2014]. Iourtchenko and Song considered the stochastic vibro-impact systems with one or two rigid barrier(s) by numerical simulation [Iourtchenko & Song, 2006]. Kumar investigated the stochastic bifurcations of a Duffing–van der Pol oscillator under random excitations [Kumar *et al.*, 2016a, 2016b, 2017]. Nguyen developed a mathematical model of vibro-impact mobile system to predict the progression rate of the system [Nguyen *et al.*, 2017]. Feng studied the chaotic saddles of a nonlinear vibro-impact system using the bisection procedure and an improved stagger-and-step method [Feng *et al.*, 2009]. Two nice overviews of vibro-impact dynamics have been presented by Namachchivaya [Namachchivaya & Park, 2005] and Dimentberg [Dimentberg & Iourtchenko, 2004].

To the authors' knowledge, little work was focused on the dynamical systems with fractional derivative elements. In this paper, we carry out a bifurcation analysis of a vibro-impact viscoelastic oscillator with fractional derivative element under Gaussian white noise excitation.

2. Model and Its Simplification

We consider the vibro-impact viscoelastic system with fractional derivative element under Gaussian white noise excitation in the following form

$$\begin{aligned} &\ddot{x} + Z + \varepsilon\beta_1 D^\alpha x + \varepsilon\beta_2 f(x,\dot{x})\dot{x} + \omega_0^2 x \\ &\quad = \varepsilon^{1/2}\xi(t), \quad x > 0, \\ &\dot{x}_+ = -r\dot{x}_-, \quad x = 0. \end{aligned} \tag{1}$$

The variables ε, β_1, β_2 and ω_0 are system parameters; $0 < r \le 1$ is the coefficient of restitution factor; $\dot{x}_-$ and $\dot{x}_+$ are the velocities just before and after the impact, respectively. $\xi(t)$ is Gaussian white noise with zero mean and auto-correlation $E[\xi(t)\xi(t+\tau)] = 2D\delta(\tau)$. There are many definitions for fractional derivatives [Li & Zeng, 2015; Ma & Li, 2017, 2018], in this paper, $D^\alpha x$ is defined as follows

$$D^\alpha x = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{x(t-u)}{u^{\lambda_1}}du, \quad (0 < \alpha \le 1). \tag{2}$$

The following viscoelastic model of Z can be expressed as:

$$Z = \int_0^t h(t-s)x(s)ds,$$

where $h(t)$ is the relaxation function which has the following form

$$\begin{aligned} h(t) &= \sum_{i=1}^n \beta_i \exp\left(-\frac{t}{\lambda_i}\right) \\ &= \beta \exp\left(-\frac{t}{\lambda}\right). \end{aligned}$$

Without loss of generality, let $i = 1$. β_i and λ_i are the general elastic modulus and the relaxation time, respectively. Then

$$Z = \int_0^t \beta \exp\left(-\frac{t-s}{\lambda}\right) x(s)ds.$$

According to [Zhu & Cai, 2011; Ling *et al.*, 2011], the viscoelastic force Z can be replaced by the conservative force and the damping force:

$$Z = \int_0^t \beta \exp\left(-\frac{t-s}{\lambda}\right) x(s)ds = \frac{\lambda\beta}{1+\lambda^2\overline{\omega}^2}(x - \lambda\dot{x}) = \kappa_1 x - \kappa_2 \dot{x}. \quad (3)$$

Substituting Eq. (3) into the original system (1), yields

$$\ddot{x} + \varepsilon\beta_1 D^\alpha x + [\varepsilon\beta_2 f(x,\dot{x}) - \kappa_2]\dot{x} + \omega_1^2 x = \varepsilon^{1/2}\xi(t), \quad x > 0, \quad (4)$$
$$\dot{x}_+ = -r\dot{x}_-, \quad x = 0,$$

where

$$\omega_1^2 = \omega_0^2 + \kappa_1.$$

The following transformation [Dimentberg & Iourtchenko, 2004; Zhuravlev, 1976; Feng *et al.*, 2008] is introduced to remove the discontinuity in Eq. (4)

$$x = x_1 = |y|, \quad \dot{x} = x_2 = \dot{y}\,\mathrm{sgn}(y), \quad \ddot{x} = \ddot{y}\,\mathrm{sgn}(y). \quad (5)$$

Substituting Eq. (5) into Eq. (4) leads to the following equations:

$$\ddot{y}\,\mathrm{sgn}(y) + \varepsilon\beta_1 D^\alpha(|y|) + [\varepsilon\beta_2 f(|y|, \dot{y}\,\mathrm{sgn}(y)) - \kappa_2]\dot{y}\,\mathrm{sgn}(y) + \omega_1^2|y| = \varepsilon^{1/2}\xi(t), \quad t \neq t_*, \quad (6a)$$

$$\dot{y}_+ = r\dot{y}_-, \quad t = t_*, \quad (6b)$$

in which $y(t_*) = 0$.

After multiplying Eq. (6a) by $\mathrm{sgn}(y)$, we get the following formulas:

$$\ddot{y} + \varepsilon\beta_1\,\mathrm{sgn}(y) D^\alpha(|y|) + [\varepsilon\beta_2 f(|y|, \dot{y}\,\mathrm{sgn}(y)) - \kappa_2]\dot{y} + \omega_1^2 y = \varepsilon^{1/2}\xi(t)\mathrm{sgn}(y), \quad t \neq t_*, \quad (7a)$$

$$\dot{y}_+ = r\dot{y}_-, \quad t = t_*. \quad (7b)$$

Then, according to [Feng *et al.*, 2009] we can obtain the equivalent equation without impact term of the original vibro-impact oscillator (1):

$$\ddot{y} + \varepsilon\beta_1\,\mathrm{sgn}(y) D^\alpha(|y|) + [\varepsilon\beta_2 f(|y|, \dot{y}\,\mathrm{sgn}(y)) - \kappa_2]\dot{y} + (1-r)\dot{y}|\dot{y}|\delta(y) + \omega_1^2 y = \varepsilon^{1/2}\xi(t)\,\mathrm{sgn}(y). \quad (8)$$

3. Stochastic Averaging Procedure

Introduce the following transformation [Huang & Jin, 2009]

$$y(t) = A(t)\cos\Psi(t), \quad \dot{y}(t) = -A(t)\omega_1 \sin\Psi(t), \quad \Psi(t) = \omega_1 t + \Phi, \quad (9)$$

where A, Ψ, Φ are random processes. The equations for the variables A and Φ are

$$\frac{dA}{dt} = \varepsilon F_{11}(A,\Phi) + \varepsilon F_{12}(A,\Phi) + \varepsilon^{1/2} G_{11}(A,\Phi)\xi(t), \quad (10)$$

$$\frac{d\Phi}{dt} = \varepsilon F_{21}(A,\Phi) + \varepsilon F_{22}(A,\Phi) + \varepsilon^{1/2} G_{21}(A,\Phi)\xi(t), \quad (11)$$

where

$$\varepsilon F_{11} = \frac{\varepsilon\beta_1 \sin\Psi}{\omega_1}\mathrm{sgn}(A\cos\Psi) D^\alpha(|A\cos\Psi|),$$
$$\varepsilon F_{12} = -A\sin^2\Psi[\varepsilon\beta_2 f(|A\cos\Psi|, -A\omega_1\sin\Psi) - \kappa_2 + (1-r)|-A\omega_1\sin\Psi|\delta(A\cos\Psi)],$$
$$\varepsilon F_{21} = \frac{\varepsilon\beta_1\cos\Psi}{A\omega_1} D^\alpha(A\cos\Psi),$$
$$\varepsilon F_{22} = -\sin\Psi\cos\Psi[\varepsilon\beta_2 f(|A\cos\Psi|, -A\omega_1\sin\Psi) - \kappa_2 + (1-r)|-A\omega_1\sin\Psi|\delta(A\cos\Psi)],$$
$$G_1 = -\frac{\sin\Psi}{\omega_1}\mathrm{sgn}(A\cos\Psi),$$
$$G_2 = -\frac{\cos\Psi}{A\omega_1}\mathrm{sgn}(A\cos\Psi).$$

The averaged Itô equation for $A(t)$ is of the form

$$dA = m(A)dt + \sigma(A)dB(t), \quad (12)$$

where

$$m(A) = \varepsilon\left\langle F_{11} + F_{12} + D\frac{\partial G_{11}}{\partial A}G_{11} + D\frac{\partial G_{11}}{\partial \Phi}G_{21}\right\rangle_{\Psi}, \tag{13}$$

$$\sigma^2(A) = \varepsilon\langle 2DG_{11}^2\rangle_{\Psi}. \tag{14}$$

To obtain the explicit expression, the key step is to simplify the first term of Eq. (13). According to the definition Eq. (2),

$$\langle\varepsilon F_{11}\rangle_{\Psi} = \left\langle \frac{\varepsilon\beta_1 A}{\omega_1\Gamma(1-\alpha)}\operatorname{sgn}(A\cos\Psi)\sin\Psi \times \frac{d}{dt}\int_0^t \frac{|\cos(\Psi-\omega_1 u)|}{u^{\alpha}}du\right\rangle_{\Psi}. \tag{15}$$

Then the cosine function in Eq. (15) can be replaced by the following Fourier series

$$|\cos\theta| = \frac{2}{\pi} + \sum_{n=1}^{\infty} B_n\cos(2n\theta), \tag{16}$$

where

$$B_n = \frac{4}{\pi}\frac{(-1)^n}{1-4n^2}.$$

According to Eqs. (15) and (16) and [Yurchenko *et al.*, 2017; Yang *et al.*, 2018]

$$\langle\varepsilon F_{11}\rangle_{\Psi} = -\frac{32\varepsilon\beta_1 A}{\pi^2\omega_1}\sin\frac{\alpha\pi}{2}\sum_{n=1}^{\infty}\frac{n^2(2n\omega_1)^{\alpha-1}}{(1-4n^2)^2} \approx -\frac{32\varepsilon\beta_1 A}{\pi^2\omega_1}\sin\frac{\alpha\pi}{2}\sum_{n=1}^{15}\frac{n^2(2n\omega_1)^{\alpha-1}}{(1-4n^2)^2}.$$

The Fokker–Planck–Kolmogorov (FPK) equation corresponds to Eq. (12) as given by

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial A}[m(A)p] + \frac{1}{2}\frac{\partial^2}{\partial A^2}[\sigma^2(A)p]. \tag{17}$$

The boundary conditions for Eq. (17) are

$$p = c,\quad c\in(-\infty,+\infty)\quad \text{as } A = 0,$$
$$p\to 0,\quad \frac{\partial p}{\partial A}\to 0,\quad A\to\infty.$$

With the help of the aforementioned boundary conditions, the stationary solution of Eq. (17) is expected to be

$$p(A) = \frac{C}{\sigma^2(A)}\exp\left[\int_0^A \frac{2m(s)}{\sigma^2(s)}ds\right], \tag{18}$$

in which C is the normalization constant. The joint stationary PDF of the original displacement and velocity $p(x_1, x_2)$ and corresponding marginal stationary PDFs $p(x_1)$ and $p(x_2)$ can be obtained according to Eq. (18) and [Huang & Jin, 2009; Yang *et al.*, 2015; Yang *et al.*, 2017].

4. Example

To assess the accuracy of the proposed method, the following oscillator is considered.

$$\ddot{x} + a_1 D^{\alpha}x + Z + (-b_1 + b_2x^2 + b_3\dot{x}^2)\dot{x} + \omega_0^2 x = \xi(t),\quad x > 0,$$
$$\dot{x}_+ = -r\dot{x}_-,\quad x = 0, \tag{19}$$

where a_1, b_1, b_2, b_3 and ω_0 are constants, $\xi(t)$ is Gaussian white noise with intensity 2D. The corresponding equivalent stochastic oscillator without impact term is of the following form based on what has been discussed in Sec. 2:

$$\ddot{y} + a_1\operatorname{sgn}(y)D^{\alpha}(|y|) + [-b_4 + b_2y^2 + b_3\dot{y}^2 + (1-r)|\dot{y}|\delta(y)]\dot{y} + \omega_1^2 y = \operatorname{sgn}(y)\xi(t), \tag{20}$$

where

$$b_4 = b_1 + \kappa_2,$$

$$\omega_1^2 = \frac{\lambda^2\omega_0^2 - 1 + \sqrt{(1+\lambda^2\omega_0^2)^2 + 4\beta\lambda^3}}{2\lambda^2}.$$

The averaged drift and diffusion coefficients are

$$m(A) = -\frac{1}{8}b_2A^3 - \frac{3}{8}b_3\omega_1^2A^3 + b_5A + \frac{D}{2A\omega_1^2}, \tag{21}$$

$$\sigma^2(A) = \frac{D}{\omega_1^2}, \tag{22}$$

in which

$$b_5 = \frac{b_4}{2} - \frac{(1-r)\omega_1}{\pi} - \frac{32a_1}{\pi^2\omega_1}\sin\frac{\alpha\pi}{2}\sum_{n=1}^{15}\frac{n^2(2n\omega_1)^{\alpha-1}}{(1-4n^2)^2}.$$

So, the stationary solution of oscillator (20) is of the form

$$p(A) = \frac{C\omega_1^2 A}{D} \exp\left[\frac{b_5\omega_1^2}{D}A^2 - \left(\frac{b_2\omega_1^2}{16D} + \frac{3b_3\omega_1^4}{16D}\right)A^4\right],$$

$$p(x_1, x_2) = \frac{C\omega_1}{\pi D} \exp\left[\frac{b_5\omega_1^2}{D}\left(x_1^2 + \frac{x_2^2}{\omega_1^2}\right) - \left(\frac{b_2\omega_1^2}{16D} + \frac{3b_3\omega_1^4}{16D}\right)\left(x_1^2 + \frac{x_2^2}{\omega_1^2}\right)^2\right].$$

4.1. *Effectiveness of the proposed approach*

In this section, different levels of control parameters b_2, b_3 and noise intensity D are considered to verify the reliability and accuracy of the proposed technique. System parameters are fixed to be $b_1 = 0.09$, $a_1 = -0.01$, $\alpha = 0.5$, $\lambda = 1.0$, $\beta = -0.01$, $\omega_0 = 1.0$, $r = 0.95$. The blue solid lines are theoretical predictions while discrete dots are numerical results in the following figures, respectively.

First, noise intensity $D = 0.16$ is fixed. Different levels of control parameters b_2 and b_3 are considered then. Figure 1 gives the theoretical and numerical results of the probability density functions of amplitude, displacement and velocity for different control parameter b_2, Fig. 2 gives the results for different parameter b_3. It can be seen that the theoretical results agree well with those from numerical results. It is also shown that the effects of these two parameters b_2 and b_3 on the system response are the same.

Then, to scrutinize the effect of the noise intensity D, here, control parameters $b_2 = 0.09$ and $b_3 = 0.09$ are fixed. Figure 3 gives the theoretical and numerical results of the probability density functions of amplitude, displacement and velocity for different noise intensity D. It can also be found that the theoretical results and numerical results are coincident. So, the reliability and accuracy of the proposed technique are verified by Figs. 1–3.

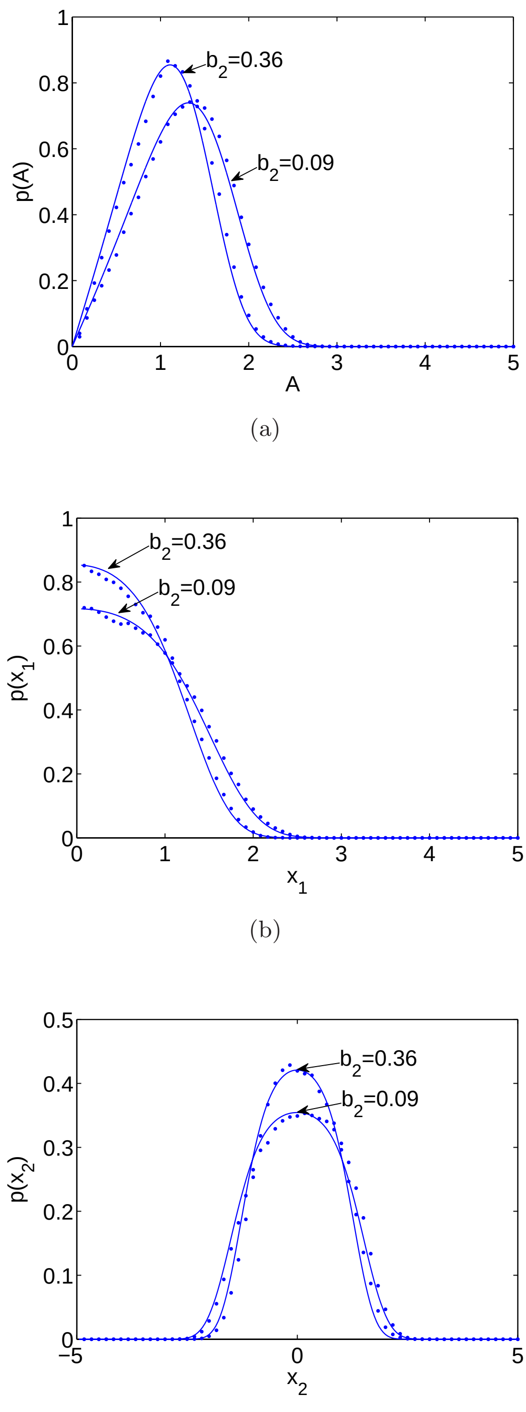

Fig. 1. Probability density functions of amplitude, displacement and velocity for different control parameter b_2.

4.2. *Stochastic bifurcation analysis*

In this paper, stochastic bifurcation refers to stochastic P-bifurcation which occurs when the shape of the stationary probability density function changes from unimodal to bimodal. This section focuses on the analysis of stochastic bifurcation phenomenon induced by viscoelastic parameters, fractional coefficient and fractional derivative order.

First, we discuss the stochastic bifurcation induced by viscoelastic parameter λ. The system

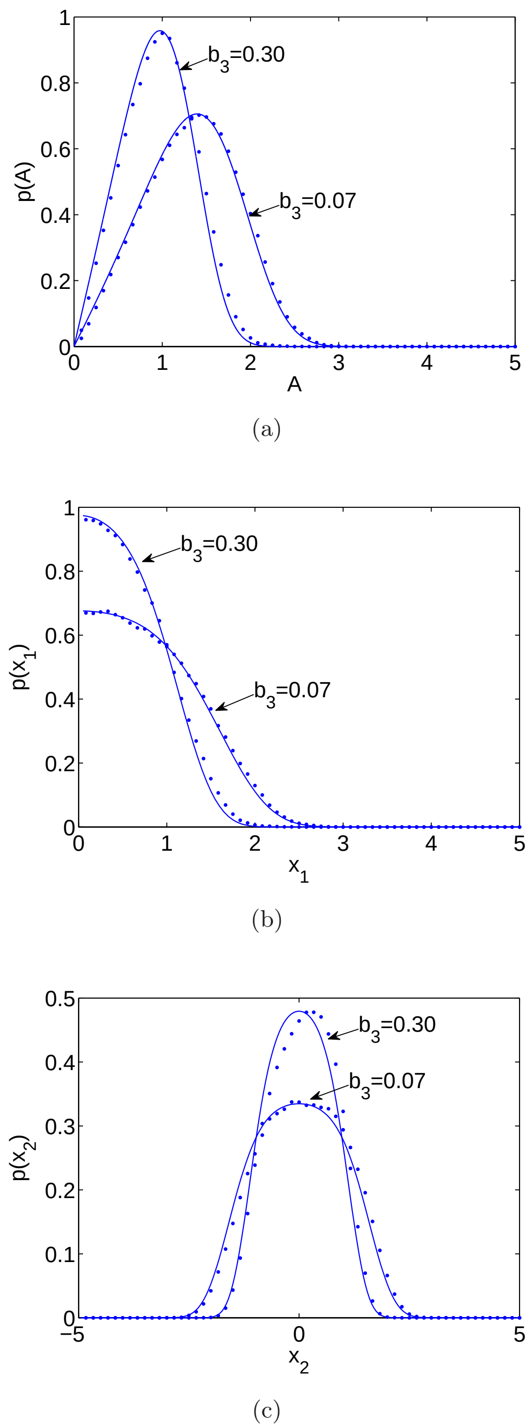

Fig. 2. Probability density functions of amplitude, displacement and velocity for different control parameter b_3.

parameters are taken to be $b_1 = 0.032$, $b_2 = 0.004$, $b_3 = 0.004$, $a_1 = -0.01$, $\alpha = 0.5$, $\beta = -0.03$, $\omega_0 = 1.0$, $D = 0.01$, $r = 0.955$. Figure 4 gives the joint stationary probability density $p(x_1, x_2)$ of displacement and velocity for different viscoelastic parameter λ. In order to better understand the stochastic bifurcation, the corresponding section graphs are presented in Fig. 5. An inspection of these two figures clearly indicates that at $\lambda = 0.9$, the joint stationary probability density has one peak and the corresponding section graph is unimodal. At $\lambda = 0.1$, the shape of the joint stationary probability density changes to crater-like

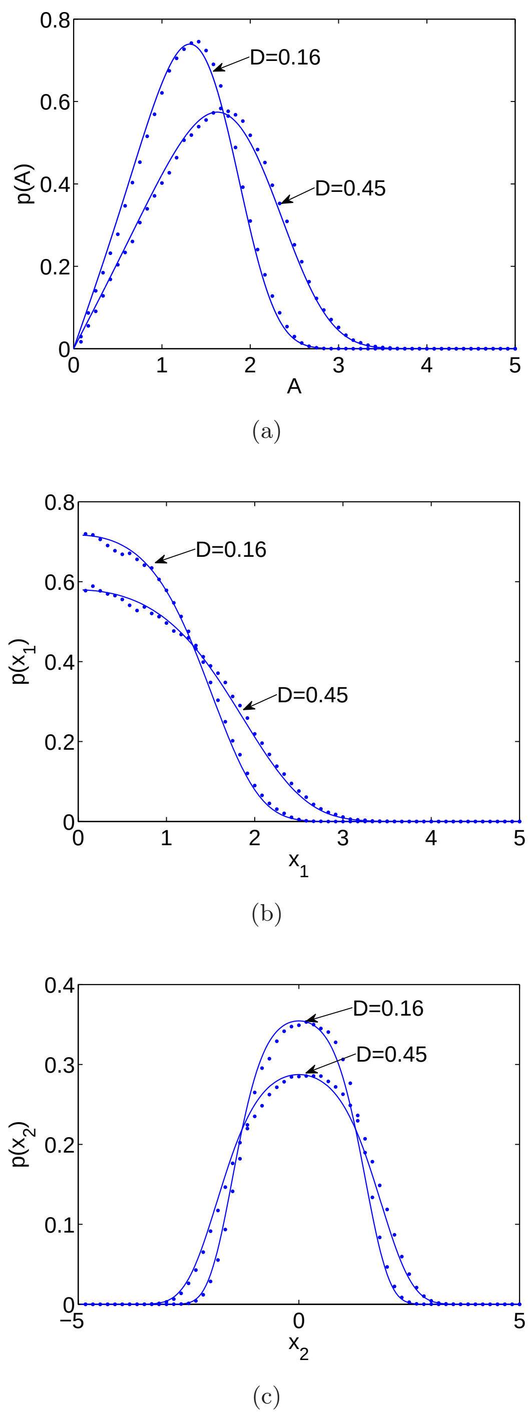

Fig. 3. Probability density functions of amplitude, displacement and velocity for different control parameter D.

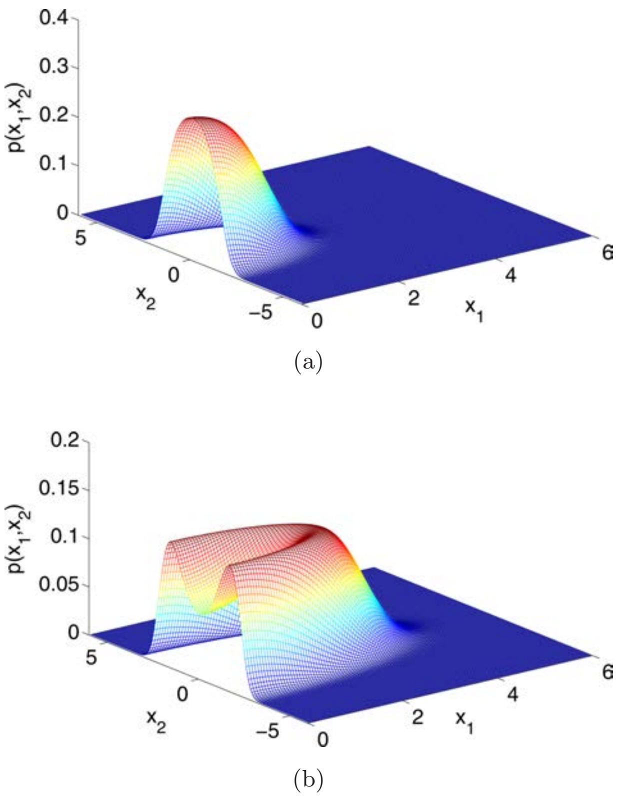

Fig. 4. The joint stationary probability density $p(x_1, x_2)$ of displacement and velocity for different λ. (a) $\lambda = 0.9$ and (b) $\lambda = 0.1$.

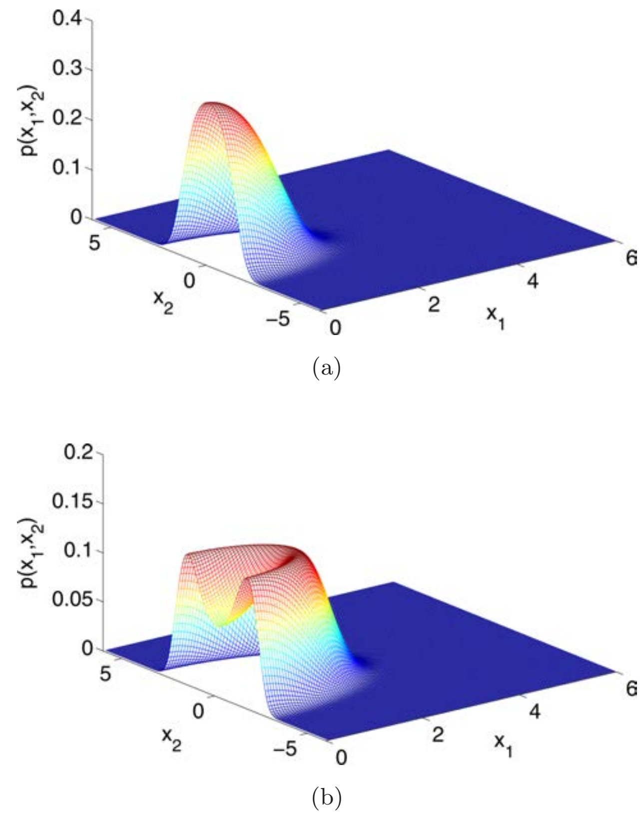

Fig. 6. The joint stationary probability density of displacement and velocity for different β. (a) $\beta = -0.03$ and (b) $\beta = -0.001$.

structure and the corresponding section graph is bimodal. This phenomenon indicates that stochastic P-bifurcation takes place as λ decreases from 0.90 to 0.10.

Second, we explore the stochastic bifurcation induced by viscoelastic parameter β. The system parameters are taken to be $b_1 = 0.032$, $b_2 = 0.004$, $b_3 = 0.004$, $a_1 = -0.01$, $\alpha = 0.5$, $\lambda = 1.0$, $\omega_0 = 1.0$, $D = 0.01$, $r = 0.955$. Figure 6 gives the joint stationary probability density $p(x_1, x_2)$ for different viscoelastic parameter β. Figure 7 shows the corresponding section graphs. Based on the same analysis, stochastic bifurcation takes place as viscoelastic parameter β increases from -0.03 to -0.001.

Third, we explore the stochastic bifurcation induced by fractional coefficient a_1. The system

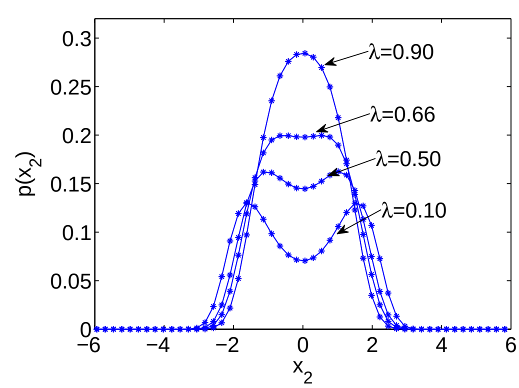

Fig. 5. Section graphs of stationary probability density $p(x_1, x_2)$ when $x_1 = 0.3$ for different λ.

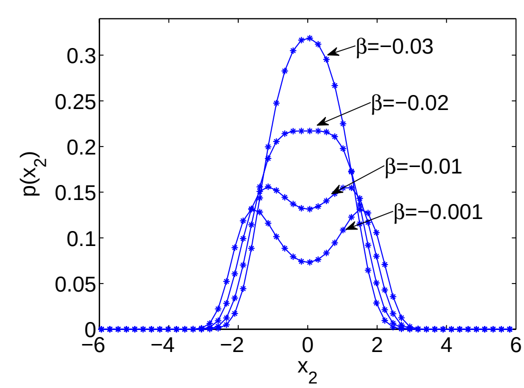

Fig. 7. Section graphs of stationary probability density $p(x_1, x_2)$ when $x_1 = 0.3$ for different β.

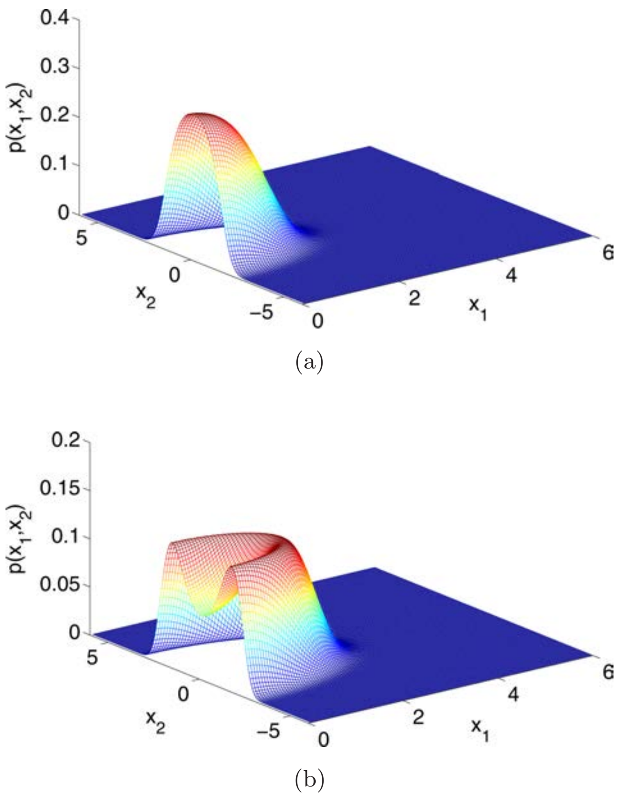

Fig. 8. The joint stationary probability density of displacement and velocity for different fractional coefficient a_1. (a) $a_1 = -0.001$ and (b) $a_1 = -0.03$.

parameters are taken to be $b_1 = 0.032$, $b_2 = 0.004$, $b_3 = 0.004$, $\alpha = 0.3$, $\lambda = 1.0$, $\beta = -0.01$, $\omega_0 = 1.0$, $D = 0.01$, $r = 0.952$. Figure 8 gives the joint stationary probability density $p(x_1, x_2)$ for different fractional coefficient a_1. Figure 9 shows the corresponding section graphs. Based on the same analysis, stochastic bifurcation takes place as fractional coefficient a_1 decreases from -0.001 to -0.03.

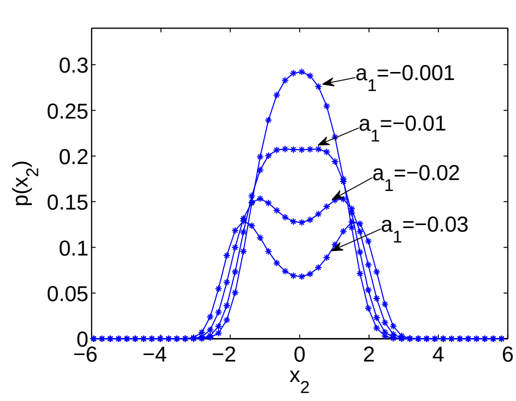

Fig. 9. Section graphs of stationary probability density $p(x_1, x_2)$ when $x_1 = 0.3$ for different a_1.

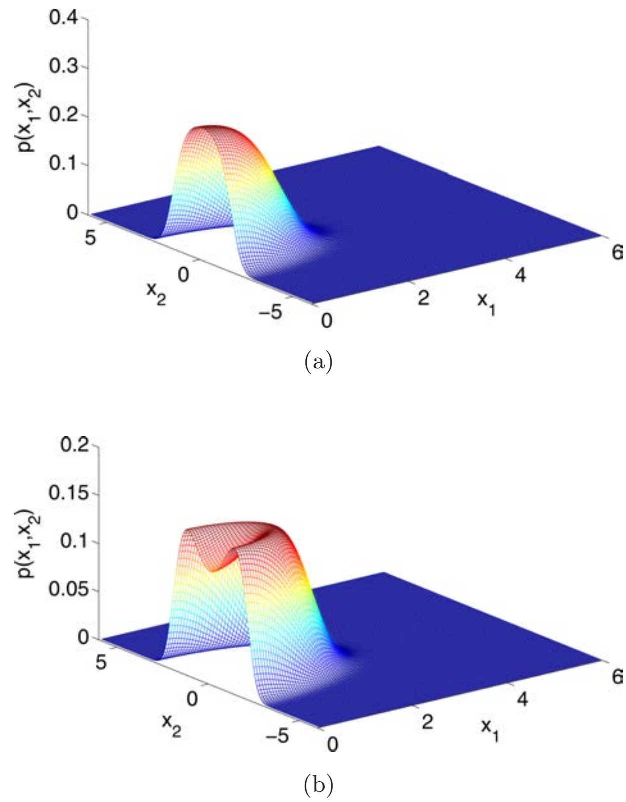

Fig. 10. The joint stationary probability density of displacement and velocity for different fractional order α. (a) $\alpha = 0.1$ and (b) $\alpha = 0.9$.

In the end, we explore the stochastic bifurcation induced by fractional derivative order α. The system parameters are taken to be $b_1 = 0.032$, $b_2 = 0.004$, $b_3 = 0.004$, $a_1 = -0.01$, $\lambda = 1.0$, $\beta = -0.01$, $\omega_0 = 1.0$, $D = 0.01$, $r = 0.952$. Figure 10 gives the joint stationary probability density $p(x_1, x_2)$ for different fractional derivative order α. Figure 11 shows the corresponding section graphs. Based on

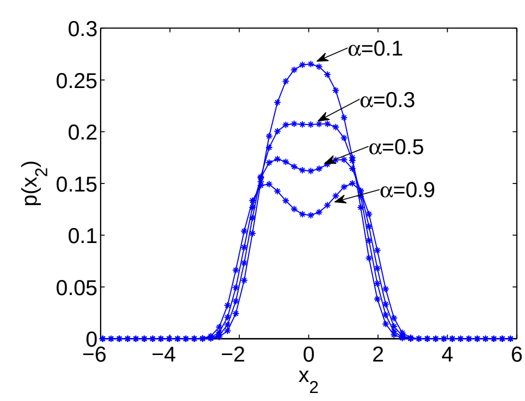

Fig. 11. Section graphs of stationary probability density $p(x_1, x_2)$ when $x_1 = 0.3$ for different fractional order α.

the same analysis, stochastic bifurcation takes place as fractional derivative order α increases from 0.1 to 0.9.

5. Conclusions

We discussed the stochastic bifurcation of a vibro-impact oscillator with fractional derivative element and a viscoelastic term under Gaussian white noise excitation. The original oscillator is converted to an equivalent oscillator without a viscoelastic term. The stochastic averaging method and the nonsmooth transformation are utilized to obtain the analytical solutions the effectiveness of which can be verified by numerical solutions. We also discussed the stochastic bifurcation phenomenon induced by the viscoelastic parameters, fractional coefficient and fractional derivative order.

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