General Robustness Analysis and Robust Fractional-Order PD Controller Design for Fractional-Order Plants

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General Robustness Analysis and Robust Fractional-Order PD Controller Design for Fractional-Order Plants

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Abstract: Most of the existing controller tuning methods are based on accurate system model and sensitive to some inevitable uncertainties and unmeasurable disturbance. Aiming at this problem, a thorough robustness analysis on a typical kind of fractional-order delay system has been made in this paper. A kind of robust fractional order proportional and derivative controller is proposed based on phase and gain margins. The tuning methods are demonstrated under different circumstance, namely there is gain variation, time constant variation, order variation or even multiple parameters variations in system transfer function. Simulation results show that the closed-loop control system with the proposed controller can achieve both robustness and satisfactory dynamic performance, and outperform the conventional PID controller in all cases.

1 Introduction

Most real objects and dynamic process models are essentially non-integer-order (fractional-order, FO) ones [1–5], but some of them may only contain little fractionality which could be approximated by integer-order (IO) plants. Most of the time before, the reason of using IO models was the lack of effective solution algorithms for FO differential equations [3]. However, this will not be a problem anymore since the spring up of approximation methods for fractional derivative and integral operator. From then on, the three hundreds years old topic, FO calculus, has shown its extraordinary talents in many fields including science and engineering aspects [2, 6–11]. Among the booming development of FO calculus related research, controller design based on fractional derivatives and integrals has achieved several remarkable work [2, 6, 10–13]. FOPID controller proposed by Podlubny which is an extension of IOPID controller is one of the most mentionable work in this field [6]. It made the derivative and integral orders not confine in the domain of integer, so that the frequency response of controlled system would be adjusted more continuously and flexibly. There have been several pioneer work which showed that the FOPID controller outperformed conventional PID controller by theoretical and practical proving [3, 12, 14–16]. It has also been verified that the great flexibility of FO operators would help in achieving better performance and robustness in comparison with traditional PID controller [12].

There are two key points in controller design, optimal dynamic control performance and robustness. The former one has always been the first factor to be taken into account in research field, because it is more visualized and compelling. However, the latter one holds the same or even more significant position in practice on account of some inevitable model uncertainties and environmental disturbance. It has been shown that some controllers were extremely fragile that the control performance would be affected by vanishingly small perturbance [17]. Only being optimal in dynamic performance demand is far from satisfactory because there is always an implicit assumption in these controller design methods that the plants are accurate and the controllers are implemented exactly. Actually, this requirement is very hard to reach in real practice situations, so that controllers should be able to tolerate some uncertainties in control loop.

Focusing on robustness analysis, gain and phase margins have always served as important indicators [18, 19]. Robust PID controllers based on gain and phase margins measure were developed for stable processes in [18, 19] and unstable processes in [20, 21]. However, as the development of FOPID controllers, the robust PID/PI/PD FO controllers have presented more satisfactory performance compared with conventional ones used on both FO plants and IO plants [11, 22–26]. For example, the Crone controller aiming at gain, parameters variations and plant uncertainties were put forward in [22, 23] by Oustaloup, et.al.; Luo and Chen proposed a FO [PD] controller which could tolerate gain variation for a class of FO systems [12], and they also designed a FOPD controller for a special type of motion control system [27]; a kind of FOPI controller tuning algorithm for robustness to plant uncertainties was given in [24] by Monje, et.al.; The robust controllability of interval FO linear time invariant systems was discussed in [25] and verified by illustrative examples; exploration on robustness of time constant was proposed by Jin, Chen and Xue [26]. However, most of the above mentioned work were focused merely on system model gain variation or only for certain IO plants. But real system plants may contain uncertainties in different parameters at the same time and not confine in IO plants, so that controller which has the ability of tolerating various parameters uncertainties is in high demand.

In this paper, we made a thorough robustness analysis of a typical fractional-order delay system, discussed about how to design a FOPD controller which is robust to gain variation, time constant variation, order variation, or even robust to multiple parameters variations. Each of the above mentioned cases have a simulation example to show its advantageous robustness to system parameter variation compared with the conventional PID controller based on ITAE index.

The rest of this paper is organized as follows: In section 2, the basic robustness theory is given as well as the formations of controlled plant and proposed FOPD controller; The detailed tuning rules of the proposed controller aiming at gain variation, time constant variation, order variation, and multiple parameter variation are presented respectively in section 3, 4, 5, 6 with simulation verification; Finally, the conclusions are made in section 7.

2 FOPD Controller and General Specifications

2.1 The Fractional-Order Plant Considered

In this paper, we discuss a general FO controlled plant which has been widely used in several research areas, such as biological material, cells, tissues and so on [12, 26, 28]. We have also added a delay term to the model to make it more universal, and it could be set to zero if not needed. The FO transfer function has the following form:

$$P(s) = \frac{K}{s(Ts^{\alpha}+1)}e^{-Ls}, \tag{1}$$

where, K is the plant gain and could be normalize to 1 without loss of generality, because it could be incorporated in the proportional gain K_p of the system controller [12]; α is a fractional order which could be set into an arbitrary; T and L are time constant and delay of the controlled system respectively.

2.2 The Fractional-Order PD Controller

The FOPD controller proposed in this paper has the most general form which has been used by numerous researchers. It is also easy to realize and implement in both simulation and practice. It is depicted as:

$$C(s) = K_p + K_d s^{\mu}, \mu \in (0, 2), \tag{2}$$

in which, K_p, K_d are proportional and derivative coefficients severally, and μ is the derivative order of the controller. Since there is already an integrator in the controlled system, so an FOPD controller is preferred in this paper. FOPD controller is a branch of FOPID, it also has the advantages of tuning flexibly and has less parameters to tune.

2.3 Specifications for Robust Controller Design

From the above two subsections which introduced controlled plant form $P(s)$ and controller form $C(s)$, the system open loop transfer function could be obtained by:

$$G(s) = C(s)P(s). \tag{3}$$

The most classic and reliable specifications for robust controller tuning are gain and phase margins. According to the basic definition of gain crossover frequency and phase margin, the proposed robust FOPD controller should satisfy the following three specifications:

1> Phase margin specification

$$Arg[G(j\omega)]|_{\omega=\omega_c} = Arg[C(j\omega)P(j\omega)]|_{\omega=\omega_c} = -\pi + \phi_m, \tag{4}$$

where, ω_c is the interested gain crossover frequency and ϕ_m is the required phase margin. Note that equation (4) is a definition which has been used as the controller design specification here.

2> Gain crossover frequency specification

Corresponding to the interested gain crossover frequency point, the amplitude of the open-loop transfer function should be zero in logarithmic frequency domain, which means the amplitude should be 1 at the selected gain crossover frequency:

$$|G(j\omega)||_{\omega=\omega_c} = |C(j\omega)P(j\omega)||_{\omega=\omega_c} = 1. \tag{5}$$

3> Robustness to certain parameters variations.

The robustness specification aiming at different parameters under different circumstance will be given in the following sections.

3 Robust FOPD tuning specification to gain variation

3.1 Preliminary

First, we substitute equations (1) (2) into equation (3) and obtained:

$$G(s) = \frac{K_p + K_d s^{\mu}}{s(Ts^{\alpha}+1)}e^{-Ls}. \tag{6}$$

Then time domain transfer functions in equations (1) (2) should be turned into frequency domain with application of Euler formula $j^{\alpha} = (e^{j\frac{\pi}{2}})^{\alpha} = \cos(\frac{\alpha\pi}{2}) + jsin(\frac{\alpha\pi}{2})$:

$$P(j\omega) = \frac{1}{(j\omega)(T(j\omega)^{\alpha}+1)}e^{-Lj\omega}. \tag{7}$$

$$\begin{aligned} C(j\omega) &= K_p + K_d(j\omega)^{\mu} \\ &= K_p + K_d\omega^{\mu}e^{\frac{\mu\pi}{2}j} \\ &= K_p + K_d\omega^{\mu}\cos(\frac{\mu\pi}{2}) + K_d\omega^{\mu}sin(\frac{\mu\pi}{2})j. \end{aligned} \tag{8}$$

$$|P(j\omega)| = \frac{1}{\sqrt{(T\omega^{1+\alpha}sin(\frac{\alpha\pi}{2}))^2 + (\omega + T\omega^{1+\alpha}cos(\frac{\alpha\pi}{2}))^2}} = \frac{1}{\sqrt{N}}, \tag{9}$$

where, N is a substituted symbol which make the following derivation brief and easy to be understood.

$$Arg[P(j\omega)] = -tan^{-1}\frac{T\omega^{\alpha}\sin(\frac{\alpha\pi}{2})}{1+T\omega^{\alpha}cos(\frac{\alpha\pi}{2})} - \frac{\pi}{2} - L\omega. \tag{10}$$

$$|C(j\omega)| = \sqrt{(K_d\omega^{\mu}sin(\frac{\mu\pi}{2}))^2 + (K_p + K_d\omega^{\mu}cos(\frac{\mu\pi}{2}))^2}, \tag{11}$$

$$\begin{aligned} Arg[C(j\omega)] &= \tan^{-1}(\frac{K_d\omega^{\mu}sin(\frac{\mu\pi}{2})}{K_p + K_d\omega^{\mu}\cos(\frac{\mu\pi}{2})}) \\ &= tan^{-1}(\frac{\sin(\frac{(1-\mu)\pi}{2}) + \frac{K_d\omega^{\mu}}{K_p}}{cos(\frac{(1-\mu)\pi}{2})}) - \frac{(1-\mu)\pi}{2}. \end{aligned} \tag{12}$$

So, substitute equations (9), (10), (11), (12) into equtions (4) and (5) as:

$$|G(j\omega)| = \frac{\sqrt{(K_d\omega^{\mu}sin(\frac{\mu\pi}{2}))^2 + (K_p + K_d\omega^{\mu}cos(\frac{\mu\pi}{2}))^2}}{N} = 1. \tag{13}$$

$$\begin{aligned} Arg[G(j\omega)] &= tan^{-1}(\frac{K_d\omega^{\mu}sin(\frac{\mu\pi}{2})}{K_p + K_d\omega^{\mu}\cos(\frac{\mu\pi}{2})}) \\ &-tan^{-1}\frac{T\omega^{\alpha}\sin(\frac{\alpha\pi}{2})}{1+T\omega^{\alpha}cos(\frac{\alpha\pi}{2})} - \frac{\pi}{2} - L\omega = -\pi + \phi_m. \end{aligned} \tag{14}$$

3.2 Robust Controller Tuning and Numerical Computation Process w.r.t. Gain Variation

It has already been discussed in [29] that if the bode plot is flat at the selected crossover frequency, the system will be robust to gain changes, and the overshoots of corresponding step responses will keep almost the same, which is called 'Flat Phase' specification. Therefore, the proposed FOPD controller should also satisfy this specification as:

$$\left.\frac{d(Arg[G(j\omega)])}{d\omega}\right|_{\omega=\omega_c} = 0. \tag{15}$$

According to equation (14), equation (15) could be further expanded by:

$$\frac{d(Arg[G(j\omega_c)])}{d\omega} = \frac{\mu{\omega_c}^{\mu-1}(\frac{K_d}{K_p})\cos(\frac{(1-\mu)\pi}{2})}{(\cos(\frac{(1-\mu)\pi}{2}))^2 + (sin(\frac{(1-\mu)\pi}{2}) + (\frac{K_d}{K_p})\omega_c{}^{\mu})^2}$$
$$-\frac{\alpha T{\omega_c}^{\alpha-1}\sin(\frac{\alpha\pi}{2})}{(T{\omega_c}^{\alpha}sin(\frac{\alpha\pi}{2}))^2 + (1+T{\omega_c}^{\alpha}\cos(\frac{\alpha\pi}{2}))^2} - L = 0. \tag{16}$$

Then, assume that

$$B_z = \frac{K_d}{K_p}, \tag{17}$$

$$A_3 = \frac{\alpha T{\omega_c}^{\alpha-1}\sin(\frac{\alpha\pi}{2})}{(T{\omega_c}^{\alpha}sin(\frac{\alpha\pi}{2}))^2 + (1+T{\omega_c}^{\alpha}\cos(\frac{\alpha\pi}{2}))^2} + L, \tag{18}$$

note that B_z should be positive since the system should maintain stable during control process. Then, equation (16) could be turned into a simplified format as:

$$A_3{\omega_c}^{2\mu}B_z^2 + (2A_3{\omega_c}^{\mu}sin(\frac{(1-\mu)\pi}{2})$$
$$-\mu{\omega_c}^{\mu-1}cos(\frac{(1-\mu)\pi}{2}))B_z + A_3 = 0. \tag{19}$$

And equation (14) could be simplified by using equation (17) as:

$$Arg[G(j\omega)] = tan^{-1}(\frac{B_z{\omega_c}^{\mu}sin(\frac{\mu\pi}{2})}{1+B_z\omega_c{}^{\mu}\cos(\frac{\mu\pi}{2})})$$
$$-tan^{-1}\frac{T{\omega_c}^{\alpha}\sin(\frac{\alpha\pi}{2})}{1+T{\omega_c}^{\alpha}cos(\frac{\alpha\pi}{2})} - \frac{\pi}{2} - L\omega_c = -\pi + \phi_m. \tag{20}$$

So far, equations (19) and (20) have only two unknown parameters μ, B_z, therefore, we could solve these two equations to achieve the value of μ, B_z, and then K_p, K_d could be obtained by equations (13) and (17).

3.3 Simulation Illustration w.r.t. Gain Variation

Theoretically, equations (19) and (20) could be solved jointly to obtain the analytical values of μ, B_z. However, the process of solving these two nonlinear equations will be very complicated and make little sense, so that we choose using the graphical method which has already been used in [12, 15, 26, 27] and proved to be a pretty efficient way to get numerical values of the parameters to be achieved. In this subsection, the plant parameters T, α, L in equation (1) are chosen as $2, 0.5$ and 0.2, respectively. The interested specifications are set as $\omega_c = 1(rad/s), \phi_m = 70°$. In accordance with equations (19) and (20), the relationship graphic lines of μ, B_z could be obtained as shown in Fig. 1. Clearly, the intersection point of these two lines ($\mu = 1.115, B_z = 0.29$) is the numerical parameter values we are looking for. Then, $K_p = 2.8265, K_d = 0.8197$ are achieved by equations (13) and (17). The bode plot of the designed open-loop system are shown in Fig. 2. It is verified that the interested specification $\omega_c = 1(rad/s), \phi_m = 70°$ are fulfilled, and the 'Flat Phase' character has been reached at the interest crossover gain frequency. The simulation of fractional-order systems can be referred to [30].

An optimal PID controller tuned based on ITAE index is also employed here to make comparison. For fair comparison, the system controlled by PID controller is also required to have the same phase margin and similar ω_c with the system controlled by FOPD controller

The parameters of the conventional PID controller are $K_p = 6.994, K_i = 0.1876, K_d = 2.6929, \omega_c = 2.3(rad/s)$.

The step responses of the system controlled by the proposed robust FOPD controller and PID controller are shown in Fig. 3 with $\pm 20\%$ gain variation (the desired value is 100%, +20% is 120%, −20% is 80%). The performance of the responses do not have much difference, however, the overshoots of the step responses remain almost the same under gain changes in Fig. 3(a), but they change more in Fig. 3(b).

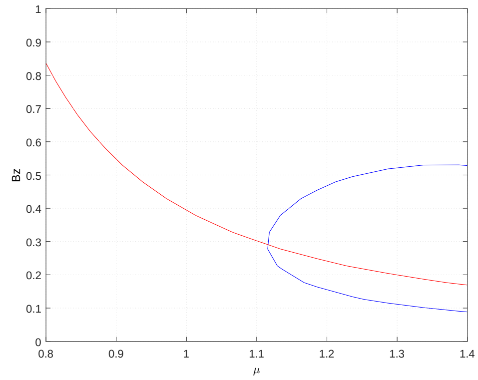

Fig. 1: Numerical solution plot of μ, B_z w.r.t. gain variation

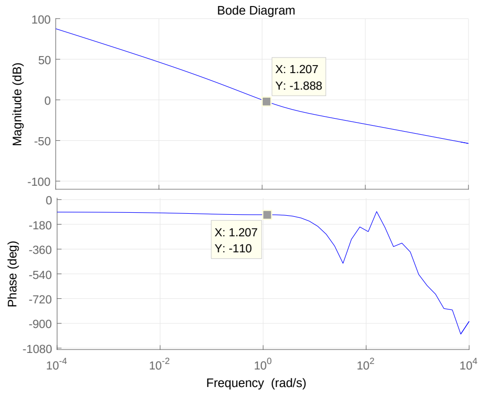

Fig. 2: Bode plot w.r.t. gain variation

4 Robust FOPD tuning specification to time constant (T) variation

The inaccuracy in system model will not only appear in gain but also appear in other parameters. Therefore, merely robust to gain change is far from satisfactory controller tuning algorithm. We discussed FOPD tuning specifications w.r.t. gain variation in last section, and in this section, the proposed FOPD controller would have the ability to tolerate both gain and time constant changes. However, the time constant changes will also bring about changes in crossover frequency [26].

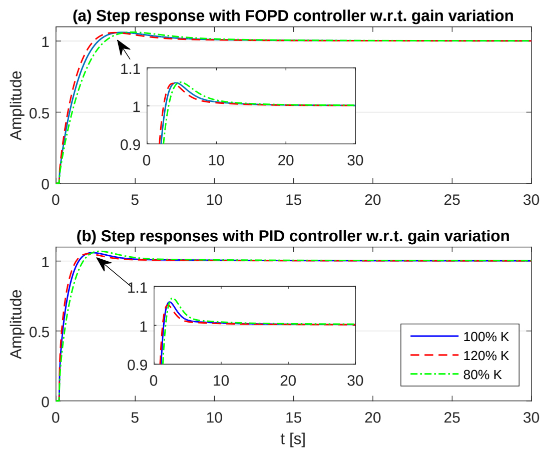

Fig. 3: Step responses with FOPD and PID controllers w.r.t. gain variation

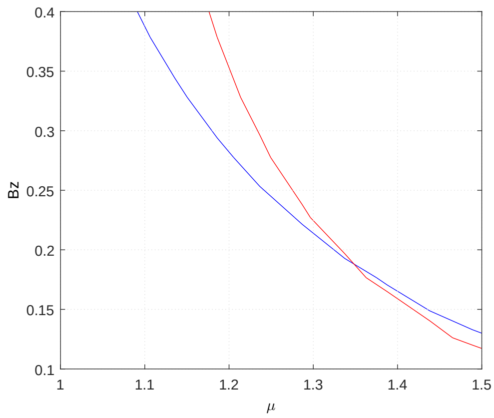

Fig. 4: Numerical solution plot of μ, B_z w.r.t. T variation

4.1 Robust Controller Tuning and Numerical Computation Process w.r.t. Time Constant (T) Variation

Different from gain variation, the proposed FOPD controller should be robust to both changes in gain and time constant, so the following specification should be fulfilled to reach the robust requirement compared with equation (16):

$$\left.\frac{\partial\,|G(j\omega)|}{\partial\omega}\right|_{(\omega_c,T_0)}\Delta\omega+\left.\frac{\partial\,|G(j\omega)|}{\partial T}\right|_{(\omega_c,T_0)}\Delta T=0,$$
$$\left.\frac{\partial Arg[G(j\omega)]}{\partial\omega}\right|_{(\omega_c,T_0)}\Delta\omega+\left.\frac{\partial Arg[G(j\omega)]}{\partial T}\right|_{(\omega_c,T_0)}\Delta T=0. \quad (21)$$

And if the robust controller is asked to satisfy both equations in (21), it is equal to fulfil the equation as:

$$\left.\frac{\frac{\partial|G(j\omega)|}{\partial\omega}}{\frac{\partial|G(j\omega)|}{\partial T}}\right|_{(\omega_c,T_0)}=\left.\frac{\frac{\partial Arg[G(j\omega)]}{\partial\omega}}{\frac{\partial Arg[G(j\omega)]}{\partial T}}\right|_{(\omega_c,T_0)}=-\frac{\Delta T}{\Delta\omega}, \quad (22)$$

where, $\partial Arg[G(j\omega)]/\partial\omega$ has already been achieved in equation (16). And,

$$\frac{\partial Arg[G(j\omega)]}{\partial T}=-\frac{\omega^\alpha\sin(\frac{\alpha\pi}{2})}{(1+T\omega^\alpha\cos(\frac{\alpha\pi}{2}))^2+(T\omega^\alpha sin(\frac{\alpha\pi}{2}))^2}, \quad (23)$$

$$\frac{\partial\,|G(j\omega)|}{\partial\omega}=\frac{1}{2}K_p(\frac{D}{N})^{-\frac{1}{2}}\cdot$$
$$(\frac{\left[2B_z^2\mu\omega^{2\mu-1}+2B_z\mu\omega^{\mu-1}\cos(\frac{\mu\pi}{2})\right]N-D(\frac{\partial N}{\partial\omega})}{N^2}), \quad (24)$$

where, $D=1+B_z^2\omega^{2\mu}+2B_z\omega^\mu\cos(\frac{\mu\pi}{2})$, and $(\partial N/\partial\omega)=T^2(2+2\alpha)\omega^{2\alpha+1}+2\omega+2T\cos(\frac{\alpha\pi}{2})\omega^{\alpha+1}(2+\alpha)$, and

$$\frac{\partial\,|G(j\omega)|}{\partial T}=\frac{1}{2}K_p(\frac{D}{N})^{-\frac{1}{2}}-D(\frac{\partial N}{\partial T}), \quad (25)$$

where,$(\partial N)/(\partial T)=2\omega^{2+2\alpha}T+2\omega^{2+\alpha}\cos(\frac{\alpha\pi}{2})$.

Therefore, substitute equations (16), (23), (24), (25) into equation (22) obtained:

$$(2M_1\mu\omega^{2\mu-1}-M_2\omega^{2\mu})B_z^2+(cos(\frac{(1-\mu)\pi}{2})\mu\omega^{\mu-1}$$
$$+2\mu M_1\omega^{\mu-1}cos(\frac{\mu\pi}{2})-2M_2\omega^\mu cos(\frac{\mu\pi}{2}))B_z-M_2=0, \quad (26)$$

where, $M_1=(N(\partial Arg[G(j\omega)]/\partial T))/(\partial N/\partial T)$ and $M_2=((\partial Arg[G(j\omega)]/\partial T)\cdot(\partial N/\partial\omega))/(\partial N/\partial T)+A_3$.

Equations (20) and (26) also have only two unknown parameters μ, B_z whose numerical solutions could be obtained by graphical methods, and then K_p, K_d could be obtained from equations (13) and (17).

4.2 Simulation Illustration w.r.t. Time Constant Variation

In this case, the plant parameters in equation (1) are $T=0.1, \alpha=0.5, L=0.1$, and the interest specifications are $\omega_c=5(rad/s), \phi_m=70°$. According to the graphical plot of the intersection of the relationship lines of B_z, μ based on equations (20) and (26) in Fig. 4, the proposed FOPD controller parameters are got as $K_p=2.9695, K_d=0.5403, \mu=1.3596$. The bode plot of the system controlled by FOPD controller is shown in Fig. 5, and it satisfies the required specifications at interest crossover frequency.

Comparison of unit step responses of system with proposed FOPD controller and conventional PID controller ($K_p=2.5329, K_i=0.5136, K_d=0.5528, \omega_c=4.6(rad/s)$) are provided in Fig. 6 respectively w.r.t. time constant T variation from 80% to 120%. As expectation, Fig. 6(a) shows advantages in both robustness and settling time compared with Fig. 6(b), the step responses have almost no change when time constant varies from 80% to 120%.

5 Robust FOPD Tuning Specification to Fractional Order (α) variation

5.1 Robust Controller Tuning and Numerical Computation Process w.r.t. Fractional Order (α) Variation

There is another important parameter in the controlled plant in equation (1), the fractional order α, and we will discuss the FOPD controller design specifications aiming at this parameter in this section. It has to be noted that α affects a lot in bode plot shaping as well as other control performance. Almost the same as the specifications listed in the last section, apart from the basic limitations in equations (4) and (5), another specification aiming at α change would be:

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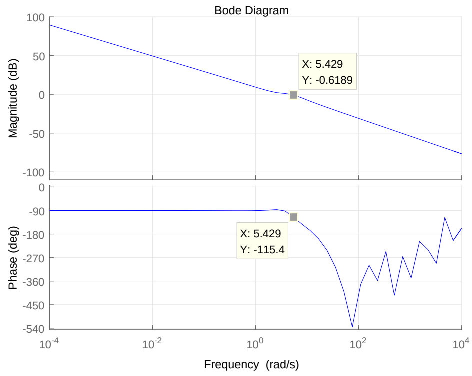

Fig. 5: Bode plot w.r.t. T variation

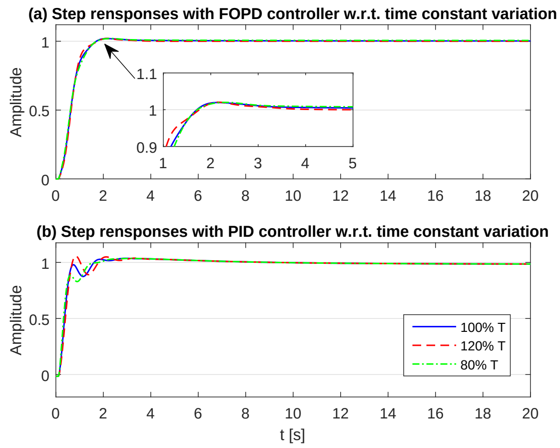

Fig. 6: Step responses with FOPD and PID controllers w.r.t. T variation

$$\left.\frac{\partial\left|G(j\omega)\right|}{\partial\omega}\right|_{(\omega_c,\alpha_0)}\Delta\omega + \left.\frac{\partial\left|G(j\omega)\right|}{\partial\alpha}\right|_{(\omega_c,\alpha_0)}\Delta\alpha = 0,$$
$$\left.\frac{\partial Arg[G(j\omega)]}{\partial\omega}\right|_{(\omega_c,\alpha_0)}\Delta\omega + \left.\frac{\partial Arg[G(j\omega)]}{\partial\alpha}\right|_{(\omega_c,\alpha_0)}\Delta\alpha = 0. \quad (27)$$

And similarly, it is equal to:

$$\left.\frac{\frac{\partial|G(j\omega)|}{\partial\omega}}{\frac{\partial|G(j\omega)|}{\partial\alpha}}\right|_{(\omega_c,\alpha_0)} = \left.\frac{\frac{\partial Arg[G(j\omega)]}{\partial\omega}}{\frac{\partial Arg[G(j\omega)]}{\partial\alpha}}\right|_{(\omega_c,\alpha_0)} = -\frac{\Delta\alpha}{\Delta\omega}, \quad (28)$$

where, $\partial Arg[G(j\omega)]/\partial\omega$ and $\partial\left|G(j\omega)\right|/\partial\omega$ have been given in equations (16) and (24),

$$\frac{\partial\left|G(j\omega)\right|}{\partial\alpha} = \frac{1}{2}K_p(\frac{D}{N})^{-\frac{1}{2}}(\frac{-D(\frac{\partial N}{\partial\alpha})}{N^2}), \quad (29)$$

where, $\partial N/\partial\alpha = 2T^2\omega^{2+2\alpha}\ln(\omega) + 2\omega^{2+\alpha}\ln(\omega)T\cos(\frac{\alpha\pi}{2}) - \pi\omega^{2+\alpha}T\sin(\frac{\alpha\pi}{2})$.

$$\frac{\partial Arg[G(j\omega)]}{\partial\alpha} = -\frac{T\omega^\alpha\ln(\omega)\sin(\frac{\alpha\pi}{2}) + T\omega^\alpha(\frac{\pi}{2})\cos(\frac{\alpha\pi}{2}) + (\frac{\alpha}{2})T^2\omega^{2\alpha}}{(T\omega^\alpha\sin(\frac{\alpha\pi}{2}))^2 + (1+T\omega^\alpha cos(\frac{\alpha\pi}{2}))^2}. \quad (30)$$

Similarly, substitute equations (16), (24), (29), (30) into equation (28) and we got:

$$(2Q_1\mu\omega^{2\mu-1} - Q_2\omega^{2\mu})B_z^2 + (cos(\frac{(1-\mu)\pi}{2})\mu\omega^{\mu-1}$$
$$+2\mu Q_1\omega^{\mu-1}\cos(\frac{\mu\pi}{2}) - 2Q_2\omega^\mu\cos(\frac{\mu\pi}{2}))B_z - Q_2 = 0, \quad (31)$$

where, $Q_1 = (N(\partial Arg[G(j\omega)]/\partial\alpha))/(\partial N/\partial\alpha)$ and $Q_2 = ((\partial Arg[G(j\omega)]/\partial\alpha)\cdot(\partial N/\partial\omega))/(\partial N/\partial\alpha)) + A_3$. Clearly, equations (20) and (31) could get graphical plot of relationships of μ, B_z , and equations (13) and (17) will work in getting the values of K_p, K_d.

5.2 *Simulation Illustration w.r.t. Fractional Order (α) Variation*

The values of μ, B_z are also got from the graphical plots intersection shown in Fig. 7 according to equations (20) and (31) with plant model parameters $T=1, \alpha=1.5, L=0.1$ in equation (1). The selected crossover frequency is $\omega_c = 1(rad/s), \phi_m = 70^\circ$, and the proposed controller tuning parameters are $K_p = 1.5996, K_d = 1.2956, \mu = 1.6862$. The bode plot in Fig. 8 verifies the accuracy of the system controlled by FOPD controller. Compared with gain and time constant discussed in above sections, order α plays a more important role in system identification, and it affects a lot in the shape of system responses to different set-points. Therefore, we tried more variation conditions in this case and chose $\pm10\%, \pm20\%$ variation in α to test the robustness of the proposed controller in Fig. 9(a). Comparison is also made with traditional PID controller ($K_p = 0.8062, K_i = 0.2085, K_d = 1.3125, \omega_c = 0.82(rad/s)$) in Fig. 9(b). The robustness of the system controlled by FOPD controller in Fig. 9(a) is comparatively good with little change during α varied in different scopes, and the settling time is relatively smaller compared with that of PID controller. Nevertheless, the weak robustness of the system step responses with PID controller in Fig. 9(b) are hard to be accepted as well as the longer settling time. It is also seen that the bigger the parameter variation is, the more advantages the system responses controlled by the proposed FOPD controller will have. Therefore, robust FOPD controller still outperforms PID controller.

6 Robust FOPD Tuning Specification to Multiple Parameters (K, T, α) Variations

6.1 *Robust Controller Tuning and Numerical Computation Process w.r.t. Multiple Parameters (K, T, α) Variations*

In this section, we proposed a FOPD controller which could be robust to multiple parameters (K, T, α) variations have already been discussed above. Similar to the controller tuning specifications stated in the above sections, the controller should satisfy three limitations listed in section 2. But different from the other three kinds of robust FOPD controller, the robustness specification in this case would be more complicate since it cannot be integrated into one equation as equation (22) or equation (28). Both of the phase and margins functions have three terms as:

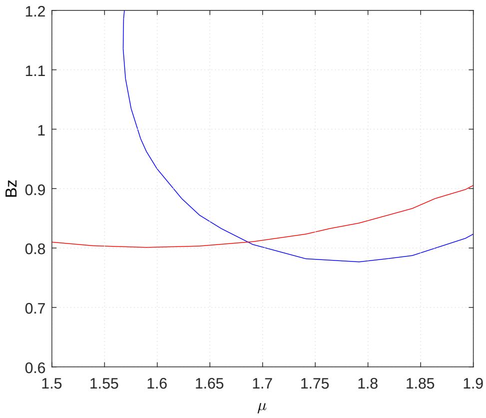

Fig. 7: Numerical solution plot of μ, B_z w.r.t. α variation

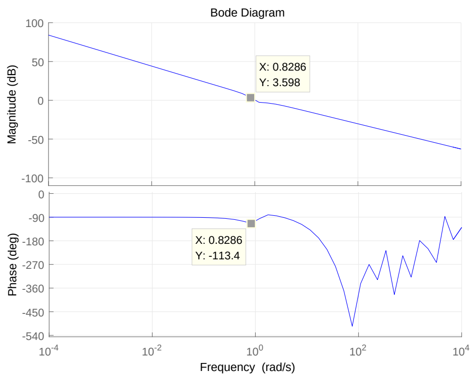

Fig. 8: Bode plot w.r.t. α variation

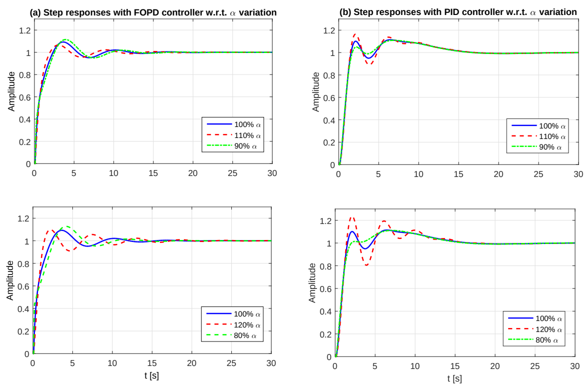

Fig. 9: Step responses with FOPD and PID controllers w.r.t. α variation

$$\frac{\partial\,|G(j\omega)|}{\partial\omega}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta\omega+\frac{\partial\,|G(j\omega)|}{\partial T}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta T$$
$$+\frac{\partial\,|G(j\omega)|}{\partial\alpha}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta\alpha=0,$$
$$\frac{\partial Arg[G(j\omega)]}{\partial\omega}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta\omega+\frac{\partial Arg[G(j\omega)]}{\partial T}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta T$$
$$+\frac{\partial Arg[G(j\omega)]}{\partial\alpha}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta\alpha=0. \quad (32)$$

Unfortunately, these two equations above are extremely complicated to solve and can't be transferred into a simplified version, even graphical plots can't guarantee to find a result. In this case, we tried to solve this question in an alternative way. Since both of the right sides of the equations above are zero, we turned each of the equations into two parts and solved them one by one like:

$$\frac{\partial\,|G(j\omega)|}{\partial\omega}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta\omega=0, \quad (33)$$

$$\frac{\partial Arg[G(j\omega)]}{\partial\omega}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta\omega=0, \quad (34)$$

$$\frac{\partial\,|G(j\omega)|}{\partial T}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta T+\frac{\partial\,|G(j\omega)|}{\partial\alpha}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta\alpha=0, \quad (35)$$

$$\frac{\partial Arg[G(j\omega)]}{\partial T}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta T+\frac{\partial Arg[G(j\omega)]}{\partial\alpha}\bigg|_{(\omega_c,\alpha_0,T_0)}\Delta\alpha=0. \quad (36)$$

Make clear of these four equations by merging the last two into one asïijŽ

$$\frac{\frac{\partial|G(j\omega)|}{\partial T}}{\frac{\partial|G(j\omega)|}{\partial\alpha}}\Bigg|_{(\omega_c,\alpha_0,T_0)}=\frac{\frac{\partial Arg[G(j\omega)]}{\partial T}}{\frac{\partial Arg[G(j\omega)]}{\partial\alpha}}\Bigg|_{(\omega_c,\alpha_0,T_0)}=-\frac{\Delta\alpha}{\Delta T}. \quad (37)$$

Note that all the terms of equation (37) has been given in equations (23), (25), (29), (30), and if ω_c,α_0,T_0 are fixed constants, there will be no unknown parameters in it. So that equation (37) would only be a test condition for parameters get for the proposed controller. To sum up, there are three limitations should be reached, equation (4), (33) and (34) respectively. The detailed formulation of equations (4) and (34) have already been given in equations (20) and (16), and equation (33) could be turned into:

$$(2\mu\omega^{2\mu-1}N-(\frac{\partial N}{\partial\omega})\omega^{2\mu})B_z^2+[2\mu\omega^{\mu-1}Ncos(\frac{\mu\pi}{2})$$
$$-2(\frac{\partial N}{\partial\omega})\omega^{\mu}cos(\frac{\mu\pi}{2})]B_z-(\frac{\partial N}{\partial\omega})=0, \quad (38)$$

where, $\partial N/\partial\omega$ has been given in equation (24). However, there are three equations to be solved, equations (16), (20) and (38), so that we turned ω_c into a tuning parameter as well, and under this circumstance, ω, B_z, μ are the parameters to be achieved. After the above derivation, this problem can be finally worked out by solving equations (16), (20), (38) by graphical method, and then, values of K_p, K_d will be achieved by solving equations (13), (17).

6.2 Simulation Illustration w.r.t. Multiple Parameters (K, T, α) Variations

Different from other cases, we have three equations in this case, and can't guarantee to have an intersection in a two dimensional plot. Therefore, we first made B_z in equation (38) explicit and substituted it into equation (20) , and got the graphical plot with two lines of relationship of ω_c, μ with the intersection ($\mu=1.47, \omega_c=2.81(rad/s)$) as shown in Fig. 10 with parameters setting as $T=0.5, \alpha=0.5, L=0.5, \phi_m=70^\circ$. The proposed parameters are got as $K_p=2.9777, K_d=1.39, \mu=1.297$. When these values are substituted back into equation (37) , they matched the equality. Fig. 11 shows the bode plot of the system controlled by the proposed controller, and it satisfies the required specifications at the crossover frequency got from the graphic plot in Fig. 10.

Compared the system step responses with proposed FOPD controller in Fig. 12(a) to that with PID controller ($K_p=3.3205, K_i=1.056, K_d=3.1369, \omega_c=2.56(rad/s)$) in Fig. 12(b), the former

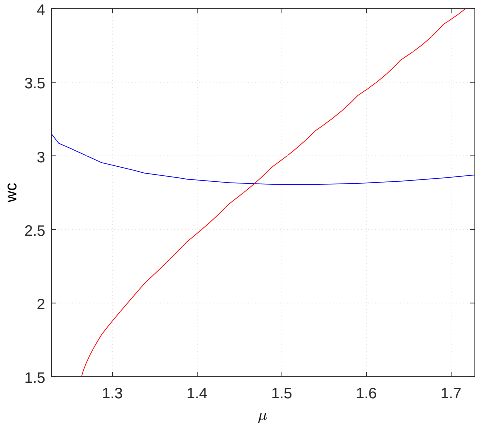

Fig. 10: Numerical solution plot of μ, B_z w.r.t. multiple parameter variation

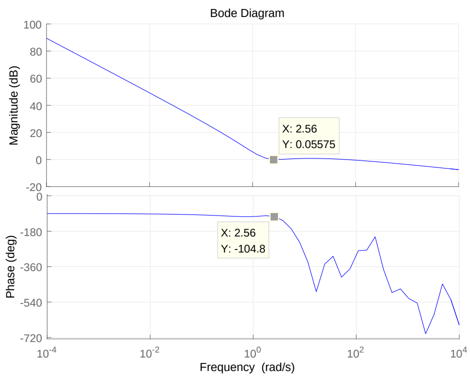

Fig. 11: Bode plot w.r.t. multiple parameter variation

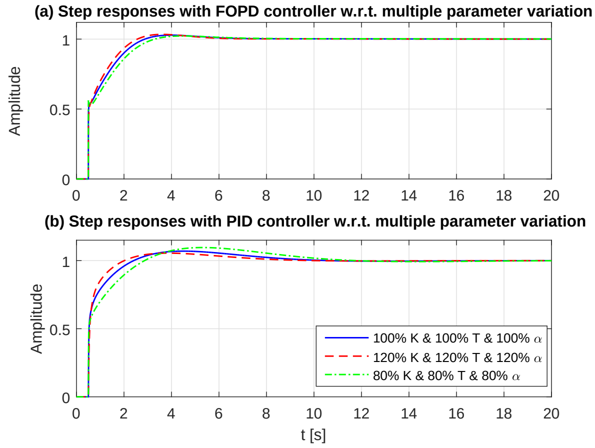

Fig. 12: Step responses with FOPD and PID controllers w.r.t. multiple parameter variation

one outperforms the latter one both in robustness and settling time. It means that the proposed controller still work well with multiple parameters changes in the system.

It has to be noted that all the FOPD controllers proposed under different circumstance meet control system stability limitation, and they could also be applied to IO system with α set to 1. Since delay term L will not exist if we do derivation of phase margin, so we did not discuss about the robustness w.r.t. delay in this paper, and it will be one of our further research topics on robustness control of FO plants.

7 Conclusion

This paper proposes a kind of FOPD controller tuning method which is robust to different parameter variation of a typical kind of fractional-order plant based on phase and gain margin specifications. Tuning algorithms under four different circumstance namely gain variation, time constant variation, order variation as well as multiple parameters variations have been given, and numerical results were got based on graphical method which made the controller parameters simply to achieve. The advantages in robustness as well as dynamic performance of the proposed FOPD controller are demonstrated from the comparison of the closed-loop step responses with conventional PID controller based on ITAE index. Furthermore, the FOPD controller will also work on integer-order plant by simply setting the order to an integer. Future research efforts may include the other kinds of robust controller e.g. FOPID design for general processes, the application and implementation problems of the proposed controller design algorithm in practical systems, determining the position and the width of the flat phase.

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9 References

1 Oustaloup A.: 'La dérivation non entière', (Hermes, 1995)
2 Podlubny I.: 'Fractional-order systems and fractional-order controllers,', *Institute of Experimental Physics, Slovak Academy of Sciences, Kosice*, 1994, **12**, pp. 1–18
3 Chen Y., Petráš I., Xue D.: 'Fractional order control - a tutorial', *Processdings of American Control Conference, 2009. ACC'09.*, 2009, pp. 1397–1411
4 Nakagawa M., Sorimachi K.: 'Basic characteristics of a fractance device', *IEICE Trans. on Fundamentals of Electronics, Communications and Computer Sciences*, 1992, **75**, pp. 1814–1819
5 Westerlund S., Ekstam L.: 'Capacitor theory', *IEEE Trans. on Dielectrics and Electrical Insulation* , 2008, **1**, pp. 826–839
6 Podlubny I.: 'Fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers', *IEEE Trans. on Automatic Control*, 1999, **44**, pp. 208–214
7 Zhang S., Yu Y., Wang H.: 'Mittag-Leffler stability of fractional-order Hopfield neural networks', *Nonlinear Analysis: Hybrid Systems*, 2015, **16**, pp. 104–121
8 Zhang S., Yu Y., Wang Q.: 'Stability analysis of fractional-order Hopfield neural networks with discontinuous activation functions', *Neurocomputing*, 2016, **171**, pp. 1075–1084
9 Wei Y., Chen Y., Cheng S., et al.: 'Completeness on the stability criterion of fractional order LTI systems',

Fractional Calculus and Applied Analysis, 2017, **20**, pp. 159–172
10 Oustaloup A., Sabatier J., Moreau X.: 'From fractal robustness to the CRONE approach ', *ESAIM*, 1998, **5**, pp. 177–192
11 Valério D., Costa J.S.: 'Time-domain implementatio of fractional order controllers ', *IEE Proceedings-Control Theory and Appl.*, 2005, **152**, pp. 539–552
12 Luo Y., Chen Y.: 'Fractional-order [proportional derivative] controller for robust motion control: Tuning procedure and validation ', *Proceedings of American Control Conference*, 2009, pp. 1412–1417
13 Liu L., Tian S., Xue D., Zhang T., and Chen Y.: 'Continous fractional-order Zero Phase Error Tracking Control', *ISA Transactions*, 2018, online, DOI:10.1016/j.isatra.2018.01.025
14 Čech M., Schlegel M.:'Generalized robust stability regions for fractional PID controllers', *Proceedings of IEEE International Conference on Industrial Technology* , 2013, pp. 76–81
15 Luo Y., Chen Y.: 'Stabilizing and robust fractional order PI controller synthesis for first order plus time delay systems ', *Automatica*, 2012, **48**, pp. 2159–2167
16 Liu L., Pan F., Xue D.: 'Variable-order fuzzy fractional PID controller', *ISA Trans.*, 2015, **55**, pp. 227–233
17 Keel L., Bhattacharyya S.P.: 'Robust, fragile, or optimal?', *IEEE Trans. on Automatic Control* , 1997, **42**, pp. 1098–1105
18 Ho W., Hang C., Zhou J.: 'Performance and gain and phase margins of well-known PI tuning formulas ', *IEEE Trans. on Control Systems Technology* , 1995, **3**, pp. 245–248
19 Ho W., Gan P., Tay E., Ang E.: 'Performance and gain and phase margins of well-known PID tuning formulas', *IEEE Trans. on Control Systems Technology*, 1996, **4**, pp. 473–477
20 Ho W., Xu W.: 'PID tuning for unstable processes based on gain and phase-margin specifications', *IEE Proceedings-Control Theory and Appli.*, 1998, **145**, pp. 392–396
21 Wang Y.G., Cai W.J.: 'Advanced proportional-integral-derivative tuning for integrating and unstable processes with gain and phase margin specifications', *Industrial & Engineering Chemistry Research*, 2002, **41**, pp. 2910–2914
22 Oustaloup A.:'Oustaloup A. Diversity and Non-Integer Differentiation for System Dynamics', (Wiley-ISTE, 2014)
23 Sabatier, J., Lanusse, P., Melchior, P., Oustaloup, A.:'Fractional Order Differentiation and Robust Control Design: CRONE, H-infinity and Motion Control', (Springer, 2015)
24 Monje C.A., Chen Y., Vinagre B.M., Xue D., Feliu-Batlle V.: 'Fractional-order systems and controls: fundamentals and applications', (Springer Science & Business Media, 2010)
25 Chen Y., Ahn H.S., Xue D.: 'Robust controllability of interval fractional order linear time invariant systems', *Signal Processing*, 2006, **86**, pp. 2794–2802
26 Jin Y., Chen Y., Xue D.: 'Time-constant robust analysis of a fractional order [proportional derivative] controller ', *IET Control Theory Appl.*, 2011, **5**, pp. 164–172
27 Luo Y., Chen Y., Pi Y.: 'Experimental study of fractional order proportional derivative controller synthesis for fractional order systems', *Mechatronics*, 2011, **21**, pp. 204–214
28 Magin R.L.: 'Fractional calculus in bioengineering, part 1 ', *Critical Reviews in Biomedical Engineering* , 2004, **32**
29 Chen Y. and Moore K.L.: 'Relay feedback tuning of robust PID controllers with iso-damping property', *IEEE Trans. on Systems, Man, and Cybernetics, Part B*, 2005, **35**, pp. 23–31
30 Xue D.: 'Fractional-order Control Systems: Fundamentals and Numerical Implementations', (deGruyter, Berlin, 2017)