Università degli Studi di Brescia
Dipartimento di Ingegneria dell'Informazione

Robust Fractional Control

An overview of the research activity carried on at the University of Brescia

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Why Fractional Control?

extra degrees of freedom and capability of modeling a wider range of dynamics

PUSH AHEAD THE FUNDAMENTAL ROBUSTNESS/PERFORMANCE TRADE-OFF!

Warning!

The design is much more complex

Outline

Fractional-order proportional-integral-derivative controller

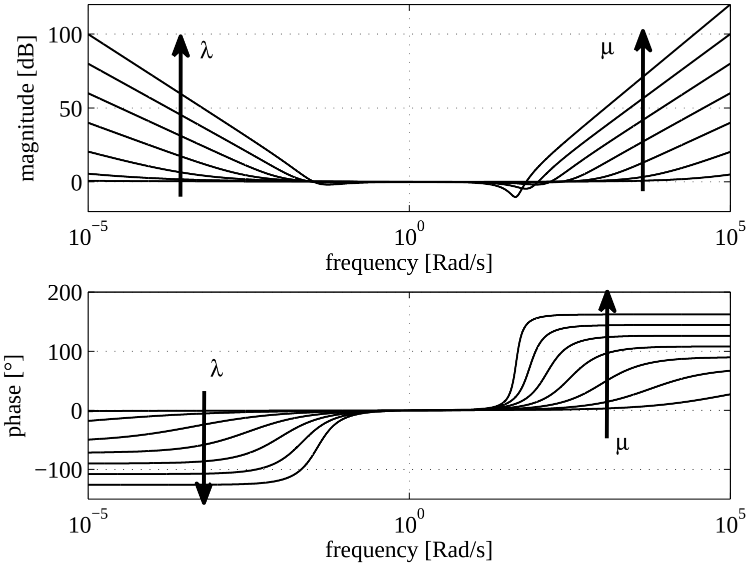
Fractional-Order Proportional-Integral-Derivative (FOPID) controllers are the natural generalization of standard PID controllers

FOPID controller

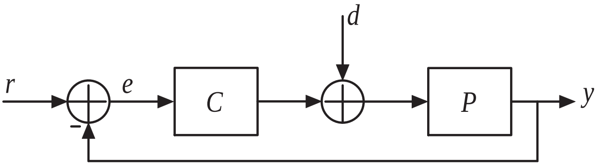
$$C(s) = K_p \frac{T_i s^{\lambda} + 1}{T_i s^{\lambda}} (T_d s^{\mu} + 1)$$

- λ and μ are the non integer orders of the integral and derivative terms
- they have 5 parameters instead of three
- they are more flexible: through the exponents a continuous regulations of the slope is possible...

FOPID controller



Tuning rules: problem formulation



Task

- set-point step following
- load disturbance step rejection

Dynamics

- Integral Plus Dead Time (IPDT) process $P(s) = \frac{K}{s}e^{-Ls}$
- First-Order Plus Dead Time (FOPDT) process $P(s) = \frac{K}{Ts+1}e^{-Ls}$
- Unstable First-Order Plus Dead Time (UFOPDT) process $P(s) = \frac{K}{1-Ts}e^{-Ls}$

Tuning rules: optimization function and constraints

In order to get optimal tuning rules the integrated absolute error (IAE) has been minimized.

$$IAE = \int_0^{\infty} |e(t)|dt = \int_0^{\infty} |r(t) - y(t)|dt,$$

Maximum Sensitivity M_s

$$M_s = \max_{\omega \in [0, +\infty)} \frac{1}{|1 + C(s)P(s)|}$$

M_s represents also the inverse of the minimum distance of the Nyquist plot from the critical point

- $M_s = 1.4$ robust tuning
- $M_s = 2.0$ aggressive tuning

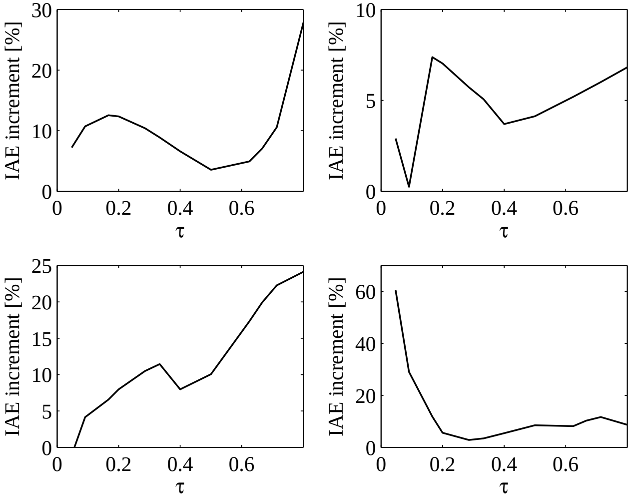
The optimization process has been numerically solved for different normalized dead time $\frac{L}{T}$

The results have been interpolated to obtain general tuning rules

The optimal IAE have been interpolated to obtain performance assessment rules

For the sake of comparison tuning rules have been developed also for integer PID controllers

Results for FOPDT



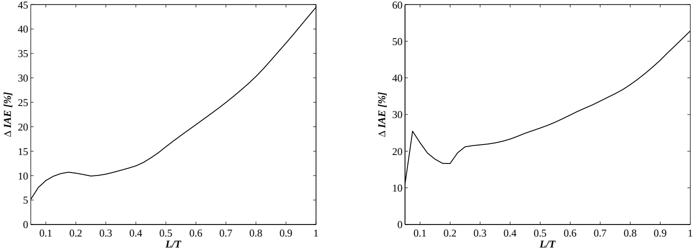
Top-left: set-point with $M_s = 1.4$*. Top-right: set-point with* $M_s = 2.0$*. Bottom-left: load disturbance with* $M_s = 1.4$*. Bottom-right: load disturbance with* $M_s = 2.0$*.*

Integral

M_S	1.4 sp	2.0 sp	1.4 ld	2.0 ld
ΔIAE [%]	17.2	6.34	19.1	22.7

The optimization can be performed just once!

Unstable



Left: set-point. Right: load disturbance

Fractional controllers always perform better than their integer counterparts!

Selected publications

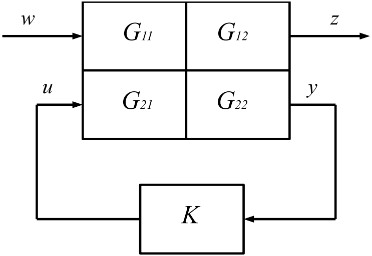
- F. Padula and A. Visioli. *Tuning rules for optimal PID and fractional-order PID controllers.* Journal of Process Control, 21(1):69-81, 2011.

- F. Padula and A. Visioli. *Optimal tuning rules for proportional-integral-derivative and fractional-order proportional-integral-derivative controllers for integral and unstable processes.* IET Control Theory and Applications, 6(6):776-786, 2012.

- F. Padula and A. Visioli. *Set-point weight tuning rules for fractional-order PID controllers.* Asian Journal of Control, 15(4):1-13, 2013.

Agenda

The standard $\mathcal{H}_\infty$ control problem



Let $\mathcal{C}$ be the set of stabilizing controllers

Problem 1

$$\min_{K \in \mathcal{C}} \|T_{zw}\|_\infty$$

where

$$T_{zw} = G_{11} + G_{12}K(1 - G_{22}K)^{-1}G_{21}$$

A Model-matching problem

Theorem: Youla parametrization

The set $\mathcal{C}$ of all stabilizing controllers K is:

$$\mathcal{C} = \left\{ \frac{X + MQ}{Y - NQ} \ : \ Q \in \mathcal{H}_\infty \right\}$$

where $P = MN^{-1}$ and M, N, X, Y satisfy the Bezout identity $NX + MY = 1$

using the previous result

$$z = (T_1 - QT_2)w$$

Problem 2

Find $Q \in \mathcal{H}_\infty$ such that the model-matching error $\|T_1 - QT_2\|_\infty$ is minimized, where both T_1 and T_2 are in $\mathcal{FRH}_\infty$

using inner-outer factorization

$$\|T_1 - T_2Q\|_\infty = \|R - X\|_\infty$$

where

$$R = \in \mathcal{FRL}_\infty, \ X = \in \mathcal{H}_\infty$$

Nehari's theorem

There exists a closest $\mathcal{H}_\infty$-matrix X to a given $\mathcal{L}_\infty$-matrix R, and $\|R - X\| = \|\Gamma_R\|$, where Γ_R is the Hankel operator with symbol R

R can be factorized as $R = R_1 + R_2$ with $R_1 \in \mathcal{RL}_\infty$ (integer!) unstable and analytic in the left half plane (antistable) and $R_2 \in \mathcal{H}_\infty$ and it holds that $\Gamma_R = \Gamma_{R_1}$

R_1 is integer, thus Γ_R has finite rank and can be computed by means of known techniques
It can be shown that the optimal model-matching error is integer and real-rational...

...this is a Nevanlinna-Pick optimal interpolation problem!

Each RHP zero of T_2 plays the role of an interpolation constraint to avoid internal instability: $Q \in \mathcal{H}_\infty$, no zero/pole cancelations in the RHP

Theorem

Consider the model-matching problem, the optimal model matching error is an all-pass in $\mathcal{RH}_\infty$ whose coefficients are completely determined by the interpolation constraints

$$E^o(z_i) = T_1(z_i) \; i = 1, \ldots, n$$
$$\left.\frac{\mathrm{d}^k E^o(s)}{\mathrm{d}s^k}\right|_{s=z_i} = \left.\frac{\mathrm{d}^k T_1(s)}{\mathrm{d}s^k}\right|_{s=z_i} \quad k = 1, \ldots, m_i - 1; \; i = 1, \ldots, n$$

being m_i the multiplicity of the ith RHP zero of T_2 and E^o the optimal model-matching error

The optimal interpolation error is integer and real-rational!

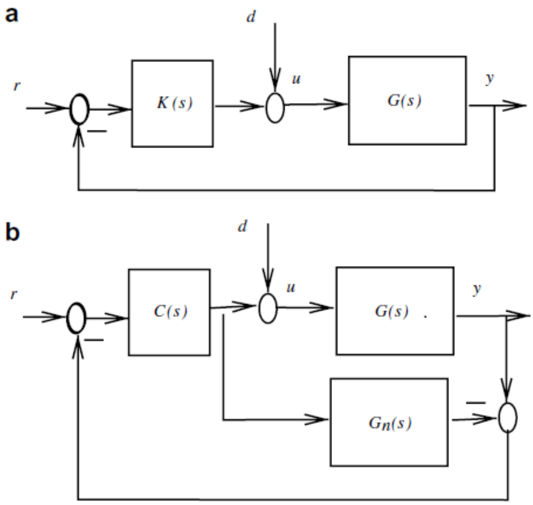
The optimal Youla parameter Q is $\mathcal{FRH}_\infty$

The optimal controller is a fractional real-rational function

Selected publications

F. Padula, S. Alcantara, R. Vilanova, and A. Visioli. *$\mathcal{H}_\infty$ control of fractional linear systems.* Automatica 49(17):2276-2280, 2013.

Agenda



a) feedback configuration, b) IMC configuration

Using the IMC controller $C(s)$ the closed loop transfer function has a very simple expression:

$$T(s) = G_n(s)C(s)$$

the equivalent feedback controller can be easily recovered by means of:

$$K(s) = \frac{C(s)}{1 - C(s)G_n(s)}$$

Weighted model-matching problem

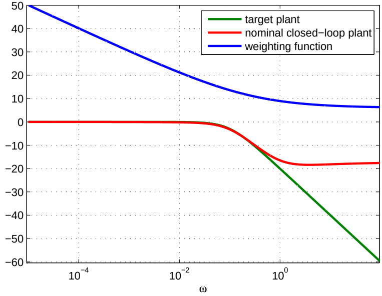
By means of the IMC controller we can set up a *weighted* model-matching problem:

$$C^o(s) := \min_{C(s)} \| W(s)(M(s) - C(s)G_n(s)) \|_\infty$$

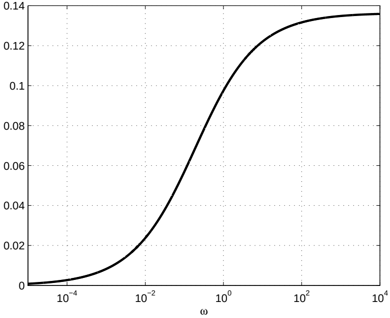
Find $C(s)$ such as the ∞-norm of the *weighted* difference between the nominal closed-loop transfer function and the desired closed-loop transfer function is minimized

The role of the weighting function is of main concern: it allows the user to give more importance to certain frequency ranges (typically low frequencies) and less importance to other frequency ranges (typically high frequency).

Weighted model-matching problem



nominal closed-loop frequency response (red), target frequency response (green), weighting function (blue)



model-matching error

The model mismatch is bigger at those frequencies where the weighting function is smaller

Weighted model-matching problem

The functions involved in the model-matching problem are chosen as follows:

1. Process model:

$$G_{n_t}(s) = \frac{K}{1 + Ts^{\alpha}} \mathrm{e}^{-Ls}$$

2. Nominal process transfer function

$$G_n(s) = K \frac{1 - Ls}{1 + Ts^{\alpha}}$$

3. Target closed-loop transfer function

$$M(s) = \frac{1}{1 + T_m s^{\lambda}}$$

4. Weighting function

$$W(s) = \frac{1 + zs^{\mu}}{s^{\mu}}$$

where T_m, λ, z and μ are parameters to be selected

Suboptimal FOPID controller

the equivalent feedback controller is

$$K^o(s) = \frac{1}{K}\frac{1+Ts^\alpha}{\frac{\rho}{\gamma}s^\mu + T_m s^\lambda + T_m(z+\frac{\rho}{\gamma})s^{\lambda+\mu}} \times \left(1+\frac{\rho T_m}{\gamma L^\mu}s^\mu\right)\left(1+\frac{-\sum_{k=m}^{n-1} L^{\frac{k}{n}}s^{\frac{k}{n}} + \sum_{k=n}^{m-1} L^{\frac{k}{n}}s^{\frac{k}{n}}}{\sum_{k=0}^{n-1} L^{\frac{k}{n}}s^{\frac{k}{n}}}\right)$$

it has the same low-frequency behavior of a filtered FOPID controller, neglecting the last term (that for low frequencies tends to one) a suboptimal controller is obtained:

$$\tilde{K}(s) = \frac{1}{K(\frac{\rho}{\gamma}+T_m)}\frac{(1+Ts^\alpha)(1+T_m\frac{L^\mu+z}{L^\mu+T_m}s^\mu)}{s^\mu(1+T_m\frac{\frac{\rho}{\gamma}+z}{\frac{\rho}{\gamma}+T_m}s^\mu)}$$

It is a filtered FOPID controller in series form

When $\mu = 1$, $\tilde{K}(s) = K^o(s)$

In any case the suboptimal controller stabilizes the nominal system!

Robust stability

Assume that the process belongs to a family $\mathfrak{F}$ defined as:

$$\mathfrak{F} = \{G(s) = G_{n_t}(s)(1 + \Delta_m(s)) : |\Delta_m(j\omega)| < |\Gamma(j\omega)|\}$$

$\Delta_m(s) = (G(s) - G_{n_t}(s))/G_{n_t}(s)$ is the uncertainty description

$\Gamma(j\omega)$ is a frequency dependent function that upper bounds the system uncertainty.

Robust stability condition:

$$\|\Gamma(s)T_n(s)\|_\infty < 1$$

where $T_n(s)$ is the nominal closed-loop transfer function.
Sufficient condition:

$$|T_n(j\omega)| < |1/\Gamma(j\omega)|$$

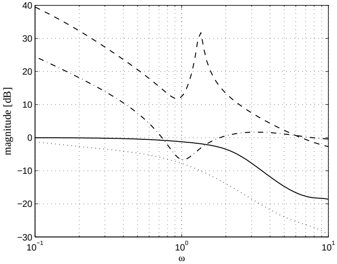
The right hand side of this inequality is usually a low-pass transfer function. **It defines a robust stability boundary.**

Based on robustness and desired bandwidth tuning guidelines have been provided.

Example

$$G(s) = \frac{1}{s^2 + 0.4s + 1} e^{-0.6s}$$

$$G_{n_t,F}(s) = \frac{1}{1.05s^{1.702} + 1} e^{-0.74s} \qquad G_{n_t,I}(s) = \frac{1}{0.566s + 1} e^{-0.9s}$$



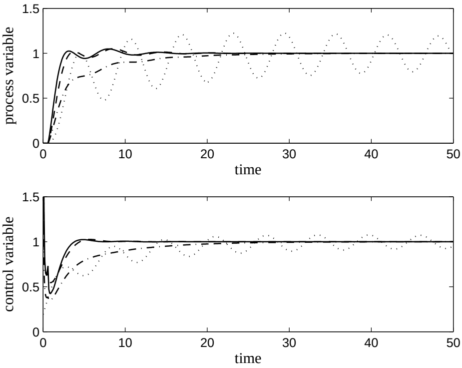
T_m has been fixed to 1.5 ($\mu = 1$)

In the fractional case z can be reduced to 0 preserving robust stability. The selection of z can be done just to speed up or slow down the system response

In the integer case it is necessary to set $z = 10$ to achieve robust stability

Example

Step responses



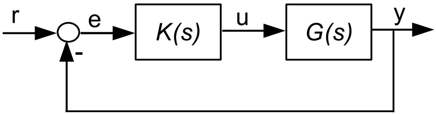
Step-responses with $T_m = 1.5L$ and $\mu = 1$: integer model (dotted line $z = 10$) and the fractional model for different values of z (dash-dot line $z = 10$, dashed line $z = 1$ and solid line $z = 0.1$)

Selected publications

F. Padula, R. Vilanova, and A. Visioli. $\mathcal{H}_\infty$ *optimization based fractional-order PID controllers design.* International journal of Robust and Nonlinear Control (in press, available online).

Agenda

Command signal design



where $G(s) = \bar{G}(s)e^{-Ls}$ and $T(s) = \frac{K(s)G(s)}{1+K(s)G(s)}$

Find a command signal such that a smooth transition of the output between 0 and 1 is obtained within in finite amount of time τ satisfying a set of constraints on the control variable and its derivatives

Given a sufficiently smooth desired output $\bar{y}(\cdot;\tau)$ find the command signal $r(\cdot;\tau)$ such that, for the $\tau-$parameterized couple $r(\cdot;\tau), \bar{y}(\cdot;\tau)$, it holds that

$$\mathcal{L}[\bar{y}(t-L;\tau)] = T(s)\mathcal{L}[r(t;\tau))]$$

Moreover, satisfy

$$|D^i u(t;\tau)| < u_M^i, \ \forall t > 0, \ i = 0, 1, \ldots, I$$

Desired output, τ-parameterized *transition polynomial* $\bar{y}(t;\tau) \in C^{(n)}$

$$\bar{y}(t;\tau) := \begin{cases} 0 & \text{if} \quad t < 0 \\ \frac{(2n+1)!}{n!\tau^{2n+1}} \sum_{r=0}^{n} \frac{(-1)^{n-r}\tau^{r} t^{2n-r+1}}{r!(n-r)!(2n-r+1)} & \text{if} \quad 0 \leq t \leq \tau \\ 1 & \text{if} \quad t > \tau \end{cases}$$

- smooth, through the parameter *n* the regularity of the transition polynomial can be arbitrarily selected;
- monotonic;
- finite time transition.

Dynamic inversion

Consider the transfer function $H(s)$ of Σ:

$$H(s) = \frac{b(s)}{a(s)} = \frac{\sum_{k=0}^{m} b_k s^{k\nu}}{s^{p\nu} + \sum_{k=0}^{p-1} a_k s^{k\nu}}$$

where ν is the commensurate order.

$\rho = (p-m)\nu$ the relative order of Σ.

Define the set of all the cause/effect pairs associated with Σ:

$$\mathcal{B} \quad := \left\{ (u(\cdot), y(\cdot)) \in P_c \times P_c : \sum_{k=0}^{m} b_k D^{k\nu} u = D^{p\nu} y + \sum_{k=0}^{p-1} a_k D^{k\nu} y \right\}$$

Consider the system $H(s)$, given the desired transition polynomial (*i.e*, $C^{(k)}$ for some $k \in \mathbb{N}$) $\bar{y}(t;\tau)$, find the input $u(t;\tau)$ such as the $\tau-$parameterized couple $(u(\cdot;\tau), \bar{y}(\cdot;\tau)) \in \mathcal{B}$, $\bar{y}(0;\tau) = 0$ and $\bar{y}(t;\tau) = 1 \; \forall t \geq \tau$,
moreover satisfy

$$|D^i u(t;\tau^*)| < u_M^i, \; \forall t > 0 \; i = 0, 1, \ldots, I$$

Dynamic inversion

Using Laplace transform, the inversion is algebraic in the frequency domain:

$$U(s;\tau) = H^{-1}(s)\bar{Y}(s;\tau)$$

Σ is assumed to be commensurate here thus the following techniques apply:

- Polynomial division
- Partial fraction expansion

and the inverse system can be decomposed as follows

$$H^{-1}(s) = \gamma_{n-m}s^{\rho} + \gamma_{n-m-1}s^{\rho-\nu} + \cdots + \gamma_1 s^{\nu} + \gamma_0 + H_0(s)$$

$H_0(s)$, *zero dynamics* of Σ, strictly proper

$$H_0(s) = \sum_{i=1}^{m} \frac{g_i}{(s^{\nu} - \lambda_i)^{k_i+1}}$$

Inverse transforming...

...the zero dynamics is the summation of Mittag-Leffler functions

$$\eta_0(t) = \sum_{i=1}^{m} \frac{g_i}{k_i!} \varepsilon_{k_i}(t, \lambda_i; \nu, \nu) = \sum_{i=1}^{m} \frac{g_i}{k_i!} t^{k_i\nu+\nu-1} \frac{d_i^k}{d(\lambda_i t^\nu)^{k_i}} E_{\nu,\nu}(\lambda t^\nu)$$

Proposition

If $n > [\rho] + 1 + l$, for τ sufficiently large then

$$\begin{aligned} u(t;\tau) &= \gamma_{n-m} D^\rho \bar{y}(t;\tau) + \gamma_{n-m-1} D^{\rho-\nu} \bar{y}(t;\tau) + \cdots \\ &+ \gamma_1 D^\nu \bar{y}(t;\tau) + \gamma_0 \bar{y}(t;\tau) + \int_0^t \eta_0(t-\xi)\bar{y}(\xi;\tau) d\xi \end{aligned}$$

The convolution integral becomes

$$\begin{aligned} &\int_0^t \eta_0(t-\xi) y(\xi;\tau) d\xi = \sum_{i=1}^m \frac{g_i}{k_i!} \left[\frac{(2n+1)!}{n!\tau^{2n+1}} \sum_{r=0}^n \frac{(-1)^{n-r}\tau^r}{r!(n-r)!(2n-r+1)} (2n-r+1)! \right. \\ &\times [\varepsilon_{k_i}(t, \lambda_i; \nu, 2n-r+2+\nu) \\ &- \begin{cases} 0 & \text{if} \quad 0 \le t \le \tau \\ \sum_{j=0}^{2n-r+1} \binom{2n-r+1}{j} (2n-r+1-j)! \tau^j & \\ \times \varepsilon_{k_i}(t-\tau, \lambda_i; \nu, 2n-r+2-j+\nu) & \text{if} \quad t > \tau \end{cases} \Bigg] \\ &+ \left. \begin{cases} 0 & \text{if} \quad 0 \le t \le \tau \\ \varepsilon_{k_i}(t-\tau, \lambda_i; \nu, 1+\nu) & \text{if} \quad t > \tau \end{cases} \right] \end{aligned}$$

Command signal synthesis, again

The open loop system is first inverted obtaining $r_{ol}(t;\tau)$ via dynamic inversion of the delay-free open loop transfer function

$$K(s)\bar{G}(s)$$

a delayed correction term is then added to avoid the delayed feedback effect

$$r_c(t;\tau) = \bar{y}(t-L;\tau)$$

finally, the command signal is computed

$$r(t;\tau) = r_{ol}(t;\tau) + r_c(t;\tau)$$

under the existence condition

$$n \geq [\rho_{\bar{G}}] + 1 + l$$

Selected publications

F. Padula and A. Visioli. *Inversion-based feedforward and reference signal design for fractional constrained control systems.* Automatica. 50(8):2169-2178, 2014.

Agenda

Problem formulation

So far we have introduced two results:

- robust controller design
- command signal synthesis

now we want to put the previous results together.
Consider the family of plants

$$\mathfrak{F} = \Big\{ \tilde{G}(s) = \frac{\bar{K}}{\tilde{T}s^{\tilde{\lambda}} + 1} e^{-\tilde{L}s} : \tilde{K} \in [K_{min}, K_{max}],$$
$$\tilde{T} \in [T_{min}, T_{max}], \tilde{\lambda} \in [\lambda_{min}, \lambda_{max}], \tilde{L} \in [L_{min}, L_{max}] \Big\}$$

and the nominal system

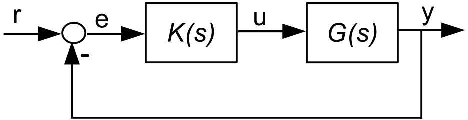
$$G(s) = \frac{K}{Ts^{\lambda} + 1} e^{-Ls}$$

whose parameters are the mean values of the corresponding uncertainty intervals

Define extremal system $G_i(s)$ $i = 1, \ldots, 16$ for the family $\mathfrak{F}$ each system obtained with any possible combination of the extremal values of the uncertainty intervals.

Problem formulation

Given a unity feedback loop



We want to design a controller $K(s)$ and a command signal $r(t)$ to satisfy:

- robustness
- control variable limitation
- overshoot limitation
- settling time minimization

for the whole family of plants $\mathfrak{F}$

Optimal feedback/feedforward control

define the worst-case settling time (at a given percentage)

$$t_{s,wc}(\tau, T_m, z) := \max_{i=1,\ldots,16} t_{s,i}(\tau, T_m, z),$$

$\{\tau, T_m, z\}$ is a set of tuning parameter

Min-max problem

$$\min_{\tau, T_m, z} t_{s,wc}(\tau, T_m, z)$$

subject to

1. (Robust stability) $\|\Gamma(s)T_n(s)\|_\infty < 1$;
2. (Maximum overshoot) $\max y_i(t;\tau, T_m, z) < y_f(1 + O_{max}),\ i = 1, \ldots, 16$;
3. (Maximum control variable) $\max |u_i(t;\tau, T_m, z)| < U_{max},\ i = i, \ldots, 16$;

where $(u_i(\cdot), y_i(\cdot))$ is the input-output couple for the ith extremal system, y_f is the set-point value, $O_{max} >$ is the maximum allowable overshoot and $U_{max} > 0$ the maximum acceptable control variable.

Optimal feedback/feedforward control

only a constraints on the control variable is imposed ($l = 0$)
the existence condition for both control signal and command signal reduces to

$$n \geq [\rho_{K\bar{G}}] + 1$$

moreover it can be shown

Lemma

There always exists a couple of parameters T_m, z such that the optimal controller $K^o(s)$ stabilizes the family $\mathfrak{F}$ provided that the parametric uncertainty over the process dc-gain K is lower than 1, *i.e.*,

$$\frac{K_{max} - K}{K} < 1$$

Theorem

The min-max problem is solvable provided that

$$\frac{K_{max} - K}{K} < 1$$

and

$$U_{max} > \frac{y_f}{G_i(0)}, i = 1, \ldots, 16$$

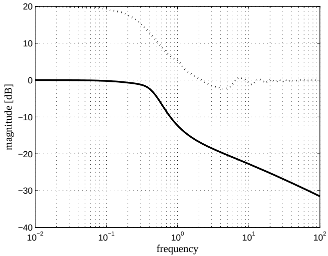
$$G(s) = \frac{1}{s^{1.5} + 1} e^{-s}$$

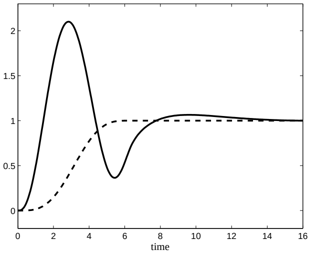
1. uncertainty of $\pm 10\%$ over the plant's parameters
2. settling time at 2%
3. unitary set-point value $y_f = 1$
4. maximum control variable of $U_{max} = 1.5$
5. maximum overshoot of $O_{max} = 0.2$

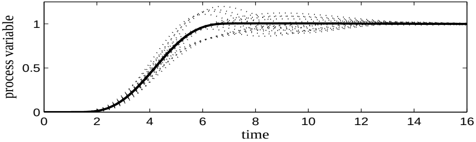
Solving procedure

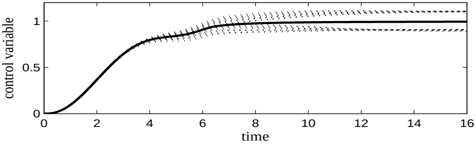
- Numerically compute the robust stability boundary by gridding the process uncertainty
- Obtain $\Gamma(j\omega)$ by upper bounding the computed uncertainties for each frequency
- Select $\beta = \max[1, \lambda] = 1.5$ and a transition polynomial with regularity $n = 3$ to satisfy the existence condition
- Numerically (genetic algorithm) solve the min-max problem

Example





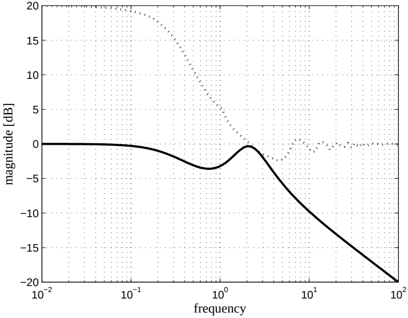


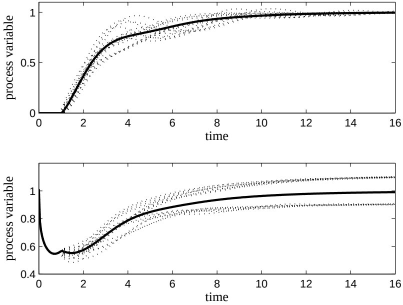


The obtained optimal parameters are $T_m = 3.5469$, $z = 3.4155$ and $\tau = 6.3122$

the optimal worst-case settling time is $t_{s,wc} = 13.42$

Results optimizing the settling time but using a step command signal...





The obtained optimal parameters are $T_m = 4.8747$, $z = 5.6705$ and $\tau = 9.7502$

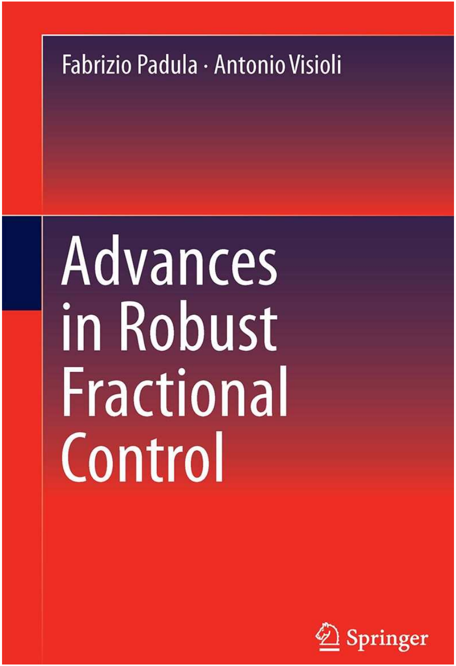
the optimal worst-case settling time is $t_{s,wc} = 18.11$

...the combined feedback/feedforward optimization performance improvements is 26 % !

Agenda

Conclusions: ROBUST PERMFORMANCE

- a complete set of constrained optimal time-scale invariant tuning rules for PID and FOPID controllers, for stable unstable and integral processes
- the solution of the scalar standard $\mathcal{H}_\infty$ control problem for fractional SISO LTI systems
- an $\mathcal{H}_\infty$ model-matching robust design methodology suitable for both monotonic and nonmonotonic dynamics
- the solution of a constrained optimal input-output dynamic inversion problem for fractional LTI systems
- an inversion-based feedforward signal design for fractional control loops
- a combined feedback/feedforward control design technique to cope with uncertainty in an effective way
- ...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