External boundary regional controllability for nonlocal diffusion systems involving the fractional Laplacian ⋆

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Abstract: The goal of this paper is to investigate regional exact controllability from the exterior of the nonlocal diffusion system governed by parabolic partial differential equations (PDEs) with the fractional Laplacian. For this purpose, we first explore an explicit expression of solutions to the system. Use this, some equivalent conditions to achieve regional exact controllability of the considered systems are given. Then, we propose an approach on the minimum energy control problem using the Hilbert uniqueness method (HUM). It is presented that the minimum control input can be explicitly given with respect to the subregion, the actuators structure and the spectral theory of fractional Laplacian under zero Dirichlet exterior boundary conditions. An example is finally included to illustrate our theoretical results.

Keywords: Regional exact controllability; Nonlocal diffusion systems; Fractional Laplacian; Minimum energy control; Hilbert uniqueness method.

1. INTRODUCTION

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Given $T < \infty$, the system we are concerned with is a nonlocal diffusion system governed by the following parabolic PDEs with the fractional Laplacian on a bounded domain:

$$\begin{cases} y_t(x,t) + (-\triangle)^s y(x,t) = 0 \text{ in } \Omega \times (0,T), \\ y(x,t) = Bu(t) \text{ in } (\mathbf{R}^n \backslash \Omega) \times (0,T), \\ y(x,0) = y_0(x) \text{ in } \Omega, \end{cases} \tag{1}$$

where $y(x,t)$ is the state to be controlled, $(-\triangle)^s$ with $s \in (0,1]$ denotes the fractional Laplace operator to be specified later and

$$Bu(t) = g^T(x)u(t) = \sum_{i=1}^{p} \Upsilon_{D_i} g_i(x) u_i(t) \tag{2}$$

denotes the control function localized in some nonempty subregion of $\mathbf{R}^n \backslash \Omega$ depending on the number and structure of actuators. More precisely, $u = [u_1, u_2, \cdots, u_p]^T \in L^2(0,T;\mathbf{R}^p)$ is the control input which is provided by p actuators, $g(x) = \left[\Upsilon_{D_1} g_1(x), \Upsilon_{D_2} g_2(x), \cdots, \Upsilon_{D_p} g_p(x)\right]^T$ denotes the actuators' spatial distribution with $g_i \in L^2(\mathbf{R}^n \backslash \Omega)$ and Υ_D is the characteristic function of the set $D \subseteq (\mathbf{R}^n \backslash \Omega)$. It is worth pointing out that this $Bu(t)$ is a finite-dimensional controller and may be unbounded (see, e.g., Sakawa (1974); Demetriou et al. (2010); Ge et al. (2016b, 2017a, 2018); Curtain and Zwart (2012)).

It is well known that operator $(-\triangle)^s$ with $0 < s < 1$ is a non-local pseudo-differential operator defined by a singular integral (see monographs Pozrikidis (2016); Ge and Chen (2019); Chen and Holm (2018) for example). By Warma (2019), only knowing $y(x,t)$ at the boundary $\partial\Omega$ is not enough to know $y(x,t)$ on all $\Omega \subseteq \mathbf{R}^n$ and besides, the Dirichlet problem

$$\begin{cases} (-\triangle)^s \phi = 0 \text{ in } \Omega, \; s \in (0,1), \\ \phi = Bu(t) \text{ on } \partial\Omega \end{cases} \tag{3}$$

is not well-posed. These yield that the actuators $Bu(t)$ cannot be localized on some subset of the boundary $\partial\Omega$, i.e., the second equation in (1) cannot be replaced by $y(x,t) = Bu(t)$ in $\partial\Omega \times (0,T)$. Moreover, since the zero exterior Dirichlet problem for fractional Laplacian given in (1) is well-posed, we conclude that the formulation presented in system (1) to replace the classical boundary control problems of PDEs is right. For more knowledge on the property of fractional Laplacian, we refer the reader to Ros-Oton and Serra (2014); Vázquez (2012, 2014); Chen et al. (2017) and the references therein.

Since fractional Laplacian can efficiently describe the processes with interactions between two domains arising at a distance, i.e., long range interaction, system (1) is a typical model of anomalous diffusion governing the stopped $\alpha-$stable Lévy motion, and has been utilized to characterize anomalous transport in many diverse disciplines (see, e.g., Rudolf (2000); Lischke et al. (2018); Ge and Chen (2019)). Typical examples include the cooling/heating process for steel and glass manufacturing in Meyer and Philip (2005), the modeling for distribution of suspended sediment of unsteady flows in Li et al. (2019), or the

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application to the atmospheric motion for large-scale flow in Bourgeois and Beale (1994), etc.

Notice that controllability for infinite-dimensional system is one of the major concerns in modern control theory. However, as stated in El Jai and Pritchard (1988), there exist many systems in practical that are not controllable on the whole domain but may be regionally controllable on some critical subregions. This is the concept of regional controllability, which has been investigated in El Jai and Pritchard (1988); El Jai et al. (1995) for conventional parabolic PDEs and in Ge et al. (2016a,c, 2017b) for time-fractional diffusion systems. Based on these, in this paper, we consider the regional exact controllability problems of system (1). More precisely, let $L^2(\Omega)$ be the usual square integrable function space endowed with the norm $\|\cdot\|_{L^2(\Omega)}$ and the inner product $(\cdot,\cdot)_{L^2(\Omega)}$. Given any subregion $\omega \subseteq \Omega$ with positive Lebesgue measure, we say that system (1) is regionally exactly controllable in $L^2(\omega)$ at time T if for any target function $y_T \in L^2(\omega)$, there exists a control function $u \in L^2(0,T;\mathbf{R}^p)$ such that the corresponding unique mild solution y satisfies

$$\chi_\omega y(\cdot,T,u) = y_T. \tag{4}$$

Here $y(x,t,u)$ is the solution of system (1) driven by the control input u and $\chi_\omega : L^2(\Omega) \to L^2(\omega)$ denotes the projection operator given by $\chi_\omega y = y|_\omega$. It yields that the investigation on regional exact controllability would allow for a reduction in the number of physical actuators and be possible to reduce the computational complexity to some extent. This is appealing in real applications. Furthermore, this paper also concerns the minimum energy control problem for regional exact controllability of system (1) with the controller designed from the exterior of the bounded domain Ω. To the best of our knowledge, no result is available on this topic.

For this purpose, we first explore an explicit representation of solutions to system (1). This, together with the definition of regional exact controllability and the operator theory, we give some equivalent conditions to ensure that the considered system is regionally exactly controllable. Use this, an approach on how to explicitly solve the minimum energy control problem with respect to the subregion, the actuators structure and the spectral theory of fractional Laplacian is then presented.

The rest of this paper is organized as follows. Some basic results to be used are presented in the next section. In Section 3, we give our main results on regional exact controllability of the studied system and the approach to explicitly solving the corresponding minimum energy control problem. An example is worked out at last.

2. PRELIMINARIES

Let $H_0^s(\Omega) = \{\phi \in H^s(\mathbf{R}^n) : \phi = 0 \text{ in } \mathbf{R}^n\backslash\Omega\}$, where $H^s(\mathbf{R}^n)$ is the fractional Sobolev space defined in Section 3 of Di Nezza et al. (2012). From Abdellaoui et al. (2016), it follows that $H_0^s(\Omega)$ is a Hilbert space endowed with the norm

$$\|\phi\|_{H_0^s(\Omega)} = \left(\int_\Omega\int_\Omega \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy\right)^{1/2}. \tag{5}$$

Then, given $\phi \in H_0^s(\Omega)$, the fractional Laplacian $(-\triangle)^s : H_0^s(\Omega) \to H^{-s}(\Omega)$ to be used throughout this paper is defined as

$$(-\triangle)^s\phi(x) = (-\triangle)^s\widetilde{\phi}(x),\ s \in (0,1], \tag{6}$$

where

$$\widetilde{\phi}(x) = \begin{cases} \phi(x), & x \in \Omega, \\ 0, & x \in \mathbf{R}^n\backslash\Omega \end{cases} \tag{7}$$

is the extension of ϕ. Moreover, by Warma (2019) and Kwaśnicki (2012), we get that eigenvalue-eigenfunction pair $(\lambda_{n,s},\xi_{n,s})$ of operator $(-\triangle)^s$, $s \in (0,1]$ under the boundary condition $\phi(x) = 0$, $x \in \mathbf{R}^n\backslash\Omega$ satisfies

(1) $\lambda_{n,s} \le \lambda_n^s$ holds true for all n, where λ_n denotes the eigenvalue of $-\Delta$ under the Dirichlet boundary conditions. In particular, when $\Omega = (-1,1)$, the Theorem 1 of Kwaśnicki (2012) yields that
$$\lambda_{n,s} = \left(\frac{n\pi}{2} - \frac{(1-s)\pi}{4}\right)^{2s} + O\left(\frac{1}{n}\right) < \left(\frac{n^2\pi^2}{4}\right)^s = \lambda_n^s.$$
(2) $\{\xi_{n,s}\}_{n\ge1}$ forms a Riesz basic of $L^2(\Omega)$.

With these, the domain of operator $(-\triangle)^s$ can be given by $\mathcal{D}((-\triangle)^s) = \left\{\phi \in H_0^s(\Omega) : \sum_{n=1}^{\infty}\lambda_{n,s}^2(\phi,\xi_{n,s})_{L^2(\Omega)}^2 < \infty\right\}$ and for $\phi \in \mathcal{D}((-\triangle)^s)$, we have

$$(-\triangle)^s\phi(x) = \sum_{n=1}^{\infty}\lambda_{n,s}(\phi,\xi_{n,s})_{L^2(\Omega)}\xi_{n,s}(x). \tag{8}$$

This yields the following continuous embeddings

$$C_0^\infty(\Omega) \hookrightarrow \mathcal{D}((-\triangle)^s) \hookrightarrow H_0^s(\Omega) \hookrightarrow L^2(\Omega), \tag{9}$$

where $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω.

In addition, consider two dual systems:

$$\begin{cases} (-\triangle)^s\phi(x) = 0 \text{ in } \Omega, \\ \phi(x) = h(x) \text{ in } (\mathbf{R}^n\backslash\Omega), \end{cases} \tag{10}$$

$$\begin{cases} (-\triangle)^s\varphi(x) = f(x) \text{ in } \Omega, \\ \varphi(x) = 0 \text{ in } (\mathbf{R}^n\backslash\Omega), \end{cases} \tag{11}$$

we introduce a nonlocal normal operator $\mathcal{N}_s$ given by

$$\mathcal{N}_s\phi(x) := C_{n,s}\int_\Omega \frac{\phi(x)-\phi(y)}{|x-y|^{n+2s}}dy,\ x \in (\mathbf{R}^n\backslash\Omega) \tag{12}$$

with $C_{n,s} = \frac{s4^s\Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)}$ and get the following lemmas.

Lemma 1. [Warma (2019)] The operator $\mathcal{N}_s$ maps $H^s(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n\backslash\Omega)$ and moreover, $\mathcal{N}_s : \mathcal{D}((-\triangle)^s) \subseteq H^s(\mathbf{R}^n) \to L^2(\mathbf{R}^n\backslash\Omega)$ is bounded.

Lemma 2. [Warma (2019)] Given $h \in H^s(\mathbf{R}^n\backslash\Omega)$, suppose that $\Phi_h \in H^s(\mathbf{R}^n)$ is the unique mild solution of system (10). Then, for any $f \in L^2(\Omega)$, it leads to

$$\int_\Omega \Phi_h(x)f(x)dx + \int_{\mathbf{R}^n\backslash\Omega} h(x)\mathcal{N}_s\Psi_f(x)dx = 0, \tag{13}$$

where $\Psi_f \in H_0^s(\Omega)$ represents the unique mild solution of (11). Moreover, if one takes $f(x) = \lambda_{n,s}\xi_{n,s}(x)$, we have $\Psi_f(x) = \xi_{n,s}(x)$ and then

$$\lambda_{n,s}\int_\Omega \Phi_h(x)\xi_{n,s}(x)dx+\int_{\mathbf{R}^n\backslash\Omega} h(x)\mathcal{N}_s\xi_{n,s}(x)dx=0.$$

Lemma 3. A function $y\in C\left(0,T;H^s(\mathbf{R}^n)\right)$ is said to be the unique mild solution of system (1) if it satisfies

$$\begin{aligned} y(x,t)&=\sum_{n=1}^{\infty}e^{-\lambda_{n,s}t}(y_0,\xi_{n,s})_{L^2(\Omega)}\xi_{n,s}(x)\\ &-\sum_{n=1}^{\infty}\int_0^t\frac{(Bu(\tau),\mathcal{N}_s\xi_{n,s})_{L^2(\mathbf{R}^n\backslash\Omega)}}{e^{\lambda_{n,s}(t-\tau)}}d\tau\xi_{n,s}(x). \end{aligned}\tag{14}$$

Proof. Let $\Phi_u(x,t)$ be the unique solution of system

$$\begin{cases}(-\triangle)^s\phi(x,t)=0 \text{ in } \Omega\times(0,T),\\ \phi(x,t)=Bu(t) \text{ in } (\mathbf{R}^n\backslash\Omega)\times(0,T)\end{cases}\tag{15}$$

and let $\omega(t,x)$ solve

$$\begin{cases}\omega_t(x,t)+(-\triangle)^s\omega(x,t)=-\Phi_{ut}(x,t) \;\text{ in } \Omega\times(0,T),\\ \omega(x,t)=0 \;\text{ in } (\mathbf{R}^n\backslash\Omega)\times(0,T),\\ \omega(x,0)=y_0(x) \;\text{ in } \Omega.\end{cases}\tag{16}$$

Then, one has

$$\begin{aligned}&(\omega(x,t)+\Phi_u(x,t))_t+(-\triangle)^s\left(\omega(x,t)+\Phi_u(x,t)\right)\\ &=\Phi_{ut}(x,t)-\Phi_{ut}(x,t)+(-\triangle)^s\Phi_u(x,t)=0\end{aligned}\tag{17}$$

and moreover, $\omega(x,t)+\Phi_u(x,t)=Bu(t)$ in $(\mathbf{R}^n\backslash\Omega)\times(0,T)$, $\omega(x,0)+\Phi_u(x,0)=y_0(x)$. Therefore, $y(x,t)=\omega(x,t)+\Phi_u(x,t)$ solves the problem (1).

In addition, by the Lemma 1 of Ge and Chen (2019), system (16) has a unique solution given by

$$\begin{aligned}\omega(x,t)&=\sum_{n=1}^{\infty}e^{-\lambda_{n,s}t}(y_0,\xi_{n,s})_{L^2(\Omega)}\xi_{n,s}(x)\\ &-\sum_{n=1}^{\infty}\int_0^t e^{-\lambda_{n,s}(t-\tau)}(\Phi_{ut}(x,t),\xi_{n,s})_{L^2(\Omega)}d\tau\xi_{n,s}(x).\end{aligned}\tag{18}$$

Since the second part of (18) satisfies

$$\begin{aligned}&\sum_{n=1}^{\infty}\int_0^t e^{-\lambda_{n,s}(t-\tau)}(\Phi_{ut}(x,t),\xi_{n,s})_{L^2(\Omega)}d\tau\xi_{n,s}(x)\\ &=\Phi_u(x,t)-\sum_{n=1}^{\infty}\int_0^t\frac{\lambda_{n,s}(\Phi_u(x,t),\xi_{n,s})_{L^2(\Omega)}}{e^{\lambda_{n,s}(t-\tau)}}d\tau\xi_{n,s}(x),\end{aligned}$$

by Lemma 2, it follows that

$$\begin{aligned}y(x,t)&=\omega(x,t)+\Phi_u(x,t)\\ &=\sum_{n=1}^{\infty}e^{-\lambda_{n,s}t}(y_0,\xi_{n,s})_{L^2(\Omega)}\xi_{n,s}(x)\\ &\quad-\sum_{n=1}^{\infty}\int_0^t\frac{(Bu(\tau),\mathcal{N}_s\xi_{n,s})_{L^2(\mathbf{R}^n\backslash\Omega)}}{e^{\lambda_{n,s}(t-\tau)}}d\tau\xi_{n,s}(x).\end{aligned}$$

This completes the proof.

3. MAIN RESULTS

This section aims to investigate the regional exact controllability problems of system (1) and then propose a direct approach on exploring the minimum energy control policy.

Consider the operator $D_T:L^2(0,T;\mathbf{R}^p)\to L^2(\Omega)$ given by

$$(D_Tu)(x)=\sum_{n=1}^{\infty}\int_0^T\frac{(Bu(\tau),\mathcal{N}_s\xi_{n,s})_{L^2(\mathbf{R}^n\backslash\Omega)}}{e^{\lambda_{n,s}(T-\tau)}}d\tau\xi_{n,s}(x)$$

for all $u\in L^2(0,T;\mathbf{R}^p)$, one has the following result.

Theorem 4. Given $T>0$, there is an equivalence among the following three properties:

$\langle1\rangle$ System (1) is regionally exactly controllable in $L^2(\omega)$ at time T;
$\langle2\rangle$ $Im(\chi_\omega D_T)=L^2(\omega)$;
$\langle3\rangle$ For any $\varphi\in L^2(\omega)$, a constant $\gamma>0$ can be found satisfying

$$\|\varphi\|_{L^2(\omega)}\le\gamma\|D_T^*\chi_\omega^*\varphi\|_{L^2(0,T;\mathbf{R}^p)}.\tag{19}$$

Here D_T^* denotes the adjoint operator of D_T and χ_ω^* is the adjoint operator of χ_ω given by

$$\chi_\omega^*y(x)=\begin{cases}y(x), & x\in\omega,\\ 0, & x\in\Omega\backslash\omega.\end{cases}\tag{20}$$

Proof. $\langle1\rangle\Leftrightarrow\langle2\rangle$: When $u\equiv0$ in system (1), by Lemma 3, the unique mild solution of system (1) satisfies

$$y_1(x,t)=\sum_{n=1}^{\infty}e^{-\lambda_{n,s}t}(y_0,\xi_{n,s})_{L^2(\Omega)}\xi_{n,s}(x).\tag{21}$$

For any target function $y_T\in L^2(\omega)$, since $y_1(\cdot,T)\in L^2(\Omega)$, we have $\chi_\omega y_1(\cdot,T)-y_T\in L^2(\omega)$. Then,

i) if $Im(\chi_\omega D_T)=L^2(\omega)$, we can find a $u\in L^2(0,T;\mathbf{R}^p)$ satisfying $\chi_\omega y_1(\cdot,T)-y_T=\chi_\omega D_Tu$. This implies

$$\chi_\omega y(\cdot,T,u)=y_T.\tag{22}$$

Here $y(x,T,u)=y_1(x,T)-(D_Tu)(x)$ denotes the solution of system (1) at time T under the control input u. Hence, system (1) is regionally exactly controllable in $L^2(\omega)$ at time T.

ii) On the contrary, based on above definition in Eq.(4), if system (1) is regionally exactly controllable in $L^2(\omega)$ at time T, we have

$$\{\chi_\omega y(x,T,u):u\in L^2\left(0,T;\mathbf{R}^p\right)\}=L^2(\omega).\tag{23}$$

This is, for any $y_T\in L^2(\omega)$, a control input $u\in L^2(0,T;\mathbf{R}^p)$ can be found such that

$$\begin{aligned}0&=\chi_\omega y(\cdot,T,u)-y_T\\ &=\chi_\omega(y(\cdot,T,u)-y_1(\cdot,T))-(y_T-\chi_\omega y_1(\cdot,T)).\end{aligned}\tag{24}$$

Since $y_T-\chi_\omega y_1(\cdot,T)\in L^2(\omega)$ as a consequence of (21), we get that

$$\{\chi_\omega\left(y(\cdot,T,u)-y_1(\cdot,T)\right):u\in L^2\left(0,T;\mathbf{R}^p\right)\}=L^2(\omega),$$

i.e., $Im(\chi_\omega D_T)=L^2(\omega)$.

$\langle2\rangle\Leftrightarrow\langle3\rangle$: Consider the following general result presented in Pritchard and Wirth (1978):

Let E,F,G be reflexive Hilbert spaces and $d\in\mathcal{L}(E,G)$, $k\in\mathcal{L}(F,G)$. Then the following two properties are equivalent

1) $Im(d)\subseteq Im(k)$;
2) $\exists\;\gamma>0$ such that $\|d^*z^*\|_{E^*}\le\gamma\|k^*z^*\|_{F^*},\;\;\forall z^*\in G$,

choose $E=G=L^2(\omega)$, $F=L^2(0,b;\mathbf{R}^p)$, $d=Id_{L^2(\omega)}$ and $k=\chi_\omega D_T$, it yields that $L^2(\omega)\subseteq Im(\chi_\omega D_T)$ is equivalent to

$$\|\varphi\|_{L^2(\omega)}\leq\gamma\|D_T^*\chi_\omega^*\varphi\|_{L^2(0,b;\mathbf{R}^p)},\quad\forall\varphi\in L^2(\omega)\quad(25)$$

for some $\gamma>0$. Based on the definition of D_T, obviously, $Im(\chi_\omega D_T)\subseteq L^2(\omega)$. Then, we get the results and complete the proof.

Taking into account that condition (19) does not allow for the pointwise actuator since for such case, B is an unbounded operator. To this end, we suppose that

(A_1) B is densely defined and its adjoint operator B^* exists.

If B is bounded, obviously, (A_1) holds true. Furthermore, since $(-\Delta)^s$ is self-adjoint (see e.g.Xu (2018) and Guan (2006)), given $v\in L^2(\Omega)$, by $\langle D_Tu,v\rangle_{L^2(\Omega)}=\langle u,D_T^*v\rangle_{L^2(0,T;\mathbf{R}^p)}$, one has

$$(D_T^*v)(t)=B^*\sum_{n=1}^{\infty}e^{-\lambda_{n,s}(T-t)}\left(\xi_{n,s},v\right)_{L^2(\Omega)}\mathcal{N}_s\xi_{n,s},\quad(26)$$

where $\langle\cdot,\cdot\rangle_*$ denotes the dual pairing between space $*$ and its dual space.

In what follows, given any target function $y_T\in L^2(\omega)$, we focus on discussing the following minimum energy control problem for regional exact controllability of system (1):

$$\inf_u J(u)=\inf_u\left\{\frac{1}{2}\int_0^T\|u(t)\|^2_{\mathbf{R}^p}dt:u\in U_T\right\},\quad(27)$$

where $U_T=\{u\in L^2\left(0,T;\mathbf{R}^p\right):\chi_\omega y(x,T,u)=y_T\}$ is a nonempty closed convex set. The main tool to be used is the HUM. For more knowledge on this method, we refer the reader to Lions (1971) and Glowinski et al. (2008).

Indeed, let

$$G=\left\{g\in L^2(\Omega):g=0\text{ in }\Omega\backslash\omega\right\}.\quad(28)$$

For any $g\in G$, consider the following adjoint system

$$\begin{cases}v_t(x,t)=(-\triangle)^s v(x,t)\ \text{ in }\Omega\times(0,T),\\ v(x,t)=0\text{ in }(\mathbf{R}^n\backslash\Omega)\times(0,T),\\ v(x,T)=g\text{ in }\Omega,\end{cases}\quad(29)$$

by the Lemma 1 of Ge and Chen (2019), its unique mild solution satisfies

$$v(x,t)=\sum_{n=1}^{\infty}e^{-\lambda_{n,s}(T-t)}\left(\xi_{n,s},g\right)_{L^2(\Omega)}\xi_{n,s}(x).\quad(30)$$

Define the following semi-norm on G

$$g\in G\to\|g\|^2_G=\int_0^T\|B^*\mathcal{N}_s v(\cdot,t)\|^2_{\mathbf{R}^p}dt,\quad(31)$$

the following lemma is obtained.

Lemma 5. If system (1) is regionally exactly controllable in $L^2(\omega)$ at time T, then (31) defines a norm on G.

Proof. If system (1) is regionally exactly controllable in $L^2(\omega)$ at time T, by $\langle 3\rangle$ of Theorem 4, one has

$$D_T^*g=0\Rightarrow g=0.\quad(32)$$

Moreover, since

$$B^*\mathcal{N}_s v(\cdot,t)=B^*\sum_{n=1}^{\infty}e^{-\lambda_{n,s}(T-t)}\left(\xi_{n,s},g\right)_{L^2(\Omega)}\mathcal{N}_s\xi_{n,s}$$

and $\|g\|^2_G=0\Leftrightarrow B^*\mathcal{N}_s v(\cdot,t)=0$, by Eq.(26), we get that

$$\|g\|_G=\sqrt{\int_0^T\|B^*\mathcal{N}_s v(\cdot,t)\|^2_{\mathbf{R}^p}dt}=0\Rightarrow g=0.\quad(33)$$

Therefore, $\|\cdot\|_G$ is a norm of space G and the proof is finished.

Furthermore, consider the following system

$$\begin{cases}\psi_t(x,t)+(-\triangle)^s\psi(x,t)=0\ \text{ in }\Omega\times(0,T),\\ \psi(x,t)=BB^*\mathcal{N}_s v(x,t)\text{ in }(\mathbf{R}^n\backslash\Omega)\times(0,T),\\ \psi(x,0)=0\text{ in }\Omega\end{cases}\quad(34)$$

and let $\Lambda_: G\to L^2(\omega)$ be

$$\Lambda g=-\chi_\omega\psi(\cdot,T).\quad(35)$$

Suppose that $\widetilde{\psi}(t)$ satisfies

$$\begin{cases}\widetilde{\psi}_t(x,t)+(-\triangle)^s\widetilde{\psi}(x,t)=0\ \text{ in }\Omega\times(0,T),\\ \widetilde{\psi}(x,t)=0\text{ in }(\mathbf{R}^n\backslash\Omega)\times(0,T),\\ \widetilde{\psi}(x,0)=y_0(x)\text{ in }\Omega.\end{cases}\quad(36)$$

For any target function $y_T\in L^2(\omega)$, it yields that the minimum energy control to achieve regional exact controllability of y_T at time T is equal to solving the equation

$$\Lambda g=y_T-\chi_\omega\widetilde{\psi}(x,T),\quad x\in\Omega.\quad(37)$$

Theorem 6. Given any target function $y_T\in L^2(\omega)$, if system (1) is regionally exactly controllable in $L^2(\omega)$ at time T under the control input $u^*(t)$ given by

$$\begin{aligned}u^*(t)&=B^*\mathcal{N}_s v(x,t)\\&=\left(B^*\sum_{n=1}^{\infty}e^{-\lambda_{n,s}(T-t)}\left(\xi_{n,s},g\right)_{L^2(\Omega)}\mathcal{N}_s\xi_{n,s}\right)(t),\end{aligned}\quad(38)$$

then the equation (37) admits a unique solution $g\in G$ and moreover, $u^*(t)$ is the solution of the minimum energy control problem (27).

Proof. For any $y_T\in L^2(\omega)$, by Lemma 5, if system (1) is regionally exactly controllable in $L^2(\omega)$ at time T driven by above control input $u^*(t)$, we get that (31) defines a norm of space G. Let the completion of G with respect to the norm $\|\cdot\|_G$ again by G. By using the definition of operator Λ in (35), since $\chi_\omega^*\chi_\omega g=g$, one has

$$\begin{aligned}&\left(\Lambda g,\chi_\omega g\right)_{L^2(\omega)}=\left(-\psi(\cdot,T),\chi_\omega^*\chi_\omega g\right)_{L^2(\Omega)}\\&=\sum_{n=1}^{\infty}\int_0^T\left(BB^*\mathcal{N}_s v(\cdot,\tau),\mathcal{N}_s\xi_{n,s}\right)_{L^2(\mathbf{R}^n\backslash\Omega)}\frac{\left(\xi_{n,s},g\right)_{L^2(\Omega)}}{e^{\lambda_{n,s}(T-\tau)}}d\tau\\&=\int_0^T\left(BB^*\mathcal{N}_s v(\cdot,\tau),\sum_{n=1}^{\infty}\frac{\langle\xi_{n,s},g\rangle_{L^2(\Omega)}\mathcal{N}_s\xi_{n,s}}{e^{\lambda_{n,s}(T-\tau)}}\right)_{L^2(\mathbf{R}^n\backslash\Omega)}d\tau\\&=\int_0^T\|B^*\mathcal{N}_s v(\cdot,t)\|^2dt=\|g\|^2_G.\end{aligned}$$

This implies that $\Lambda:G\to L^2(\omega)$ is one to one. It then follows from the Theorem 2.1 of Lions (1971) that Λ is coercive. Therefore, we get that Eq.(37) has a unique solution $g\in G$.

In what follows, we show that $u^*(t)$ is the solution of the minimization problem (27).

Indeed, since $\chi_\omega y(x,T,u^*) = y_T$, for any $u \in L^2(0,T;\mathbf{R}^p)$, it leads to $\chi_\omega y(x,T,u^*) - \chi_\omega y(x,T,u) \equiv 0$ and besides,

$$(y(x,T,u^*) - y(x,T,u), g)_{L^2(\Omega)} = 0. \tag{39}$$

With this, we have

$$\left(\sum_{n=1}^{\infty} \int_0^T \frac{(B(u(\tau) - u^*(\tau)), \mathcal{N}_s \xi_{n,s})_{L^2(\mathbf{R}^n \backslash \Omega)} \xi_{n,s}}{e^{\lambda_{n,s}(T-\tau)}} d\tau, g \right)_{L^2(\Omega)}$$
$$= \int_0^T (u(\tau) - u^*(\tau)) B^* \mathcal{N}_s v(\cdot,\tau) d\tau = 0.$$

Further, since

$$\begin{aligned} J'(u^*)(u-u^*) &= \int_0^T u^*(t)(u(t) - u^*(t))dt \\ &= \int_0^T B^* \mathcal{N}_s v(x,t)(u(t) - u^*(t))dt = 0 \end{aligned} \tag{40}$$

holds true for all $u \in L^2(0,T;\mathbf{R}^p)$, by Chapter III of Lions (1971), it yields that u^* solves the minimum energy control problem (27) and the proof is finished.

4. AN EXAMPLE

Consider the following one dimensional nonlocal diffusion system with a zone actuator:

$$\begin{cases} y_t(x,t) + (-\triangle)^s y(x,t) = 0 \text{ in } (0,1) \times (0,5), \\ y(x,t) = \Upsilon_{[a_1,a_2]} u(t) \text{ in } (\mathbf{R} \backslash (0,1)) \times (0,5), \\ y(x,0) = 0 \text{ in } (0,1), \end{cases} \tag{41}$$

where $s = 0.3$, $Bu(t) = \Upsilon_{[a_1,a_2]} u(t)$ with $1 < a_1 \le a_2$ and $u \in L^2(0,5)$. By Kwaśnicki (2012); Kulczycki et al. (2010), one has $\lambda_{n,s} = (n\pi - 0.35\pi)^{0.6} + O\left(\frac{1}{n}\right)$,

$$\xi_{n,s}(x) \approx \sqrt{2} \sin((n - 0.35)\pi x + 0.35\pi) \tag{42}$$

and $\{\xi_{n,s}\}_{n \ge 1}$ forms a Riesz basic of $L^2(\Omega)$. Therefore,

$$D_5 u = \sum_{n=1}^{\infty} \int_0^5 e^{-\lambda_{n,s}(5-\tau)} u(\tau) d\tau \int_{a_1}^{a_2} \mathcal{N}_s \xi_{n,s}(x) dx \xi_{n,s}(x)$$

and

$$\begin{aligned} (D_5^* h)(t) &= \left(B^* \sum_{n=1}^{\infty} e^{-\lambda_{n,s}(5-t)} (\xi_{n,s}, h)_{L^2(0,1)} \mathcal{N}_s \xi_{n,s} \right)(t), \\ &= \sum_{n=1}^{\infty} e^{-\lambda_{n,s}(5-t)} (\xi_{n,s}, h)_{L^2(0,1)} \\ &\quad \times C_{n,s} \int_{a_1}^{a_2} \int_0^1 \frac{\xi_{n,s}(x) - \xi_{n,s}(y)}{(x-y)^{1.6}} dy dx. \end{aligned}$$

Taking into account that

$$\int_{a_1}^{a_2} \int_0^1 \frac{\xi_{n,s}(x) - \xi_{n,s}(y)}{(x-y)^{1.6}} dy dx = 0 \tag{43}$$

holds true for some $a_1, a_2 \in \mathbf{Q}$ with $a_2 \ge a_1 > 1$, we have $Ker(D_5^*) \ne \{0\}$. Then, condition $\langle 3 \rangle$ of Theorem 4 cannot be satisfied. This implies that system (41) is not exact controllability on whole domain $L^2(0,1)$.

However, let $a_1 = 2,\ a_2 = 2+\mu$. Since $\xi_{n,s}(x)$ is a periodic continuous function and $\xi_{n,s}(2) \ne 0$, there exists a small $\mu > 0$ such that

$$\int_2^{2+\mu} \int_0^1 \frac{\xi_{n,s}(x) - \xi_{n,s}(y)}{(x-y)^{1.6}} dy dx \ne 0. \tag{44}$$

Similarly, for a nonzero continuous function h, one can find for example, an interval $(\gamma, \gamma+\varepsilon)$ with $\varepsilon > 0$ satisfying

$$\int_\gamma^{\gamma+\varepsilon} \xi_{n,s}(x) h(x) dx \ne 0. \tag{45}$$

With these, let $\omega = (\gamma, \gamma+\varepsilon)$. Then $(D_5^* \chi_\omega^* h)(t) \ne 0$. This implies that system (41) is regionally exactly controllable in $L^2(\gamma, \gamma+\varepsilon)$ at time $T = 5$ under a zone actuator localized in $[2, 2+\mu]$.

Furthermore, given any target function $y_5 \in L^2(\omega)$, if system (41) is regionally exactly controllable in $L^2(\omega)$ at time $T = 5$, consider the following minimization problem:

$$\inf_u J(u) = \inf_u \left\{ \frac{1}{2} \int_0^5 u^2(t) dt : u \in U_T \right\}, \tag{46}$$

where $U_T = \{u \in L^2(0,5) : \chi_\omega y(x,5,u) = y_5\}$. By Theorem 6, if system (41) is regionally exactly controllable in $L^2(\omega)$ at time 5, the equation

$$\Lambda g = y_5 - \chi_\omega \widetilde{\psi}(x,5), \quad x \in \Omega \tag{47}$$

admits a unique solution $g \in G$, where $\Lambda g = -\chi_\omega \psi(\cdot,5)$, $\psi(x,t)$ is the solution of system

$$\begin{cases} \psi_t(x,t) + (-\triangle)^{0.3} \psi(x,t) = 0 \text{ in } (0,1) \times (0,5), \\ \psi(x,t) = \Upsilon_{[a_1,a_2]} u^*(t) \text{ in } (\mathbf{R} \backslash (0,1)) \times (0,5), \\ \psi(x,0) = 0 \text{ in } (0,1) \end{cases} \tag{48}$$

and

$$\begin{aligned} u^*(t) &= \sum_{n=1}^{\infty} e^{-\lambda_{n,s}(5-t)} (\xi_{n,s}, g)_{L^2(0,1)} \\ &\quad \times C_{n,s} \int_{a_1}^{a_2} \int_0^1 \frac{\xi_{n,s}(x) - \xi_{n,s}(y)}{(x-y)^{1.6}} dy dx. \end{aligned} \tag{49}$$

Besides, we get that $u^*(t)$ solves above minimum energy control problem (46).

5. CONCLUSION

This paper investigate the regional exact controllability problem of parabolic PDE systems with the fractional Laplacian, whose control input is localized on some subset of the system's exterior domain. Some equivalent conditions to achieve regional exact controllability of the considered system are presented. An approach on the minimum energy control problem is then explicitly derived by using HUM. Moreover, the presented results can be extended to more complex nonlocal distributed parameter systems. For instance, the problem of constrained regional control of time-space fractional diffusion systems with the fractional Laplacian under more complicated regional sensing and actuation configurations are of great interest. For more information on the potential topics related to space-fractional diffusion systems, we refer the reader to Ge et al. (2015) and the references therein.

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