The 21st IFAC World Congress, Germany,
July 12-17, 2020

External boundary regional controllability for nonlocal diffusion systems involving the fractional Laplacian

Fudong Ge

(Joint work with **YangQuan Chen, University of California, Merced**)

China University of Geosciences, Wuhan 430074, China

Email: gefd@cug.edu.cn

Contents

Motivation—Why regional controllability?

Controllability is one of the fundamental issues in control theory.

General dynamic systems

$$\begin{cases} y_t = f(t, y, u), \\ y(0) = y_0, \end{cases} \tag{1}$$

$t \to y(t)$ is the state (possibly infinite dimensional),
$t \to u(t)$ denotes the control.

Typical Controllability Question

For a given initial data y_0, can we find a control u such that the corresponding solution of equation (1) has a prescribed behavior?

Let $[0, T]$ be a finite time interval and y_T be a fixed target element.

- **Exact controllability:** Can I find a control u such that $y(T) = y_T$?
- **Null controllability:** If there exists a control u such that $y(T) = 0$?
- **Approximate controllability:** Can I find a control u such that $\|y(T) - y_T\|$ is as small as desired?

The finite-dimensional system case

$$y_t(t) = Ay(t) + Bu(t), \tag{2}$$

where $y \in \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$, $u \in \mathbf{R}^p$ and $B \in \mathbf{R}^{n\times p}$.

Theorem 1

Given $T > 0$, there is an equivalence among the following properties:

(1) System (2) is exactly controllable at time T;

(2) System (2) is approximately controllable at time T;

(3) The matrices A and B satisfy

$$\operatorname{rank}\left[B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B\right] = n.$$

REMARKS

✓ Approximate and exact controllability are equivalent.

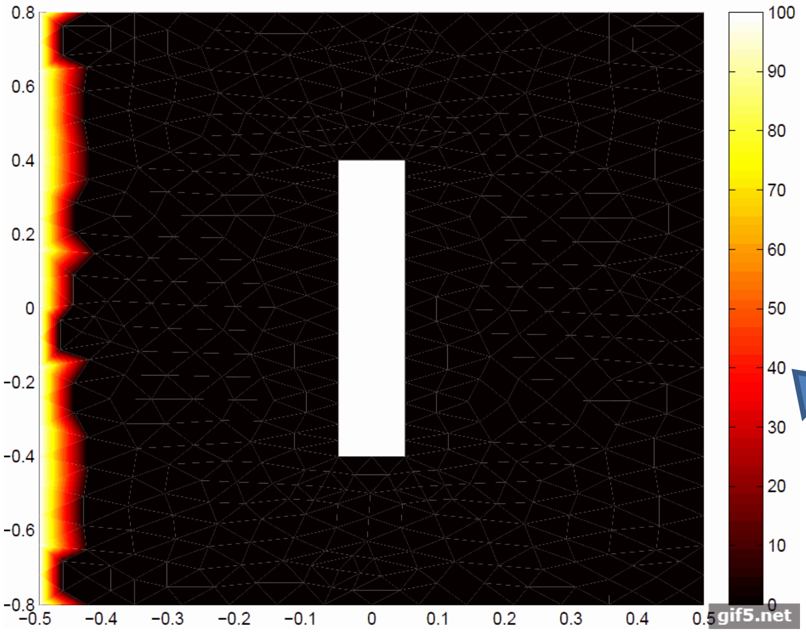
✓ The controllability of system (2) is independent of the time T.

Motivation—Why regional controllability?

The infinite dimensional diffusion system case

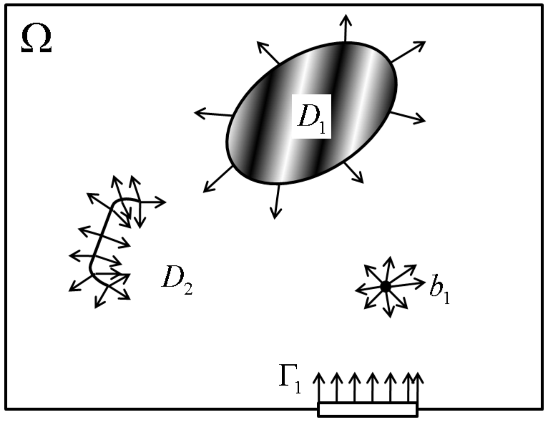
$$y_t(x,t) = f\left(y_{xx}(x,t), y_x(x,t), y(x,t)\right) + Bu(t) \qquad (3)$$

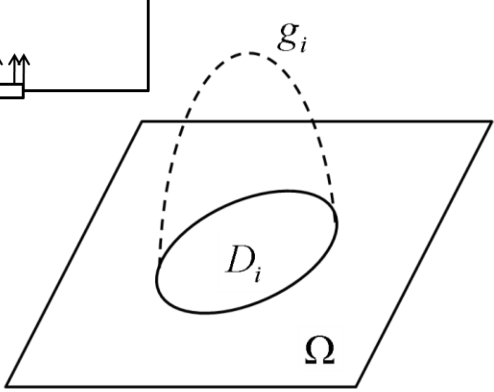
$x \in \Omega$, $t \in (0, \infty)$, $y(x,t)$ is the state, $Bu(t) = \sum_{i=1}^{p} P_{D_i} g_i(x) u_i(t)$ is the controller of Sakawa type [Y. Sakawa, SIAM Journal on Control, 1974].



Heating a metal block with a crack

A spatially homogeneous environment





Some Monographs

[1] J. L. Lions, Optimal control of systems governed by partial differential equations, Vol. 170, Springer Verlag, 1971.

[2] R. Glowinski, J. L. Lions, J. He, Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach[M], Cambridge University Press, 2008.

[3] I. Avdonin, S. A. Avdonin, S. A. Ivanov. Families of exponentials: the method of moments in controllability problems for distributed parameter systems[M]. Cambridge University Press, 1995.

[4] G. Fragnelli, D. Mugnai, Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations[M]. American Mathematical Society, 2016.

[5] Tucsnak M, Weiss G. Observation and control for operator semigroups[M]. Springer Science & Business Media, 2009.

[6] Khapalov A Y. Controllability of partial differential equations governed by multiplicative controls[M]. Springer, 2010.

[7] R. F. Curtain, H. Zwart, An introduction to infinite-dimensional linear systems theory, Vol. 21, Springer Science & Business Media, 2012.

[8] Khapalov A Y. Mobile Point Sensors and Actuators in the Controllability Theory of Partial Differential Equations[M]. Springer International Publishing, 2017.

[9] V. Barbu, Controllability and stabilization of parabolic equations[M]. Birkhäuser, 2018.

Motivation---Why regional controllability

Briefly speaking, regional controllability is to control the considered system to be controllability within some sub-region of its evolution whole domain. --- [El Jai et al.1995]

Proverb:

Money should be used more efficiently.

- More efficiently;
- Great help to discuss the systems which are not controllable on the whole domain;
- To improve the degree of controllability of the system only on a sub-region;
- Allow for a reduction in the number of actuators/sensors;
- Offer the potential to reduce the computational requirements.

Motivation--- Why regional controllability

Let $\omega \subseteq \Omega$ be a given region of positive Lebesgue measure and $y_T \in L^2(\omega)$ (the target function) be a given element. Consider now the restriction map

$$\chi_\omega : L^2(\Omega) \to L^2(\omega),$$

defined by $\chi_\omega z = z|_\omega$, is the projection operator on ω.

Definition 1

1) The considered system is said to be regionally exactly controllable in $L^2(\omega)$ at time T if for any $y_T \in L^2(\omega)$, a control $u \in U$ can be found such that

$$\chi_\omega y(\cdot, T, u) = y_T.$$

In particular, if $y_T \equiv 0$, the considered system is said to be regionally null controllable in $L^2(\omega)$ at time T.

2) The considered system is said to be regionally approximately controllable in $L^2(\omega)$ at time T if for any $y_T \in L^2(\omega)$, given $\varepsilon > 0$, there exists a control $u \in U$ satisfying

$$\|\chi_\omega y(\cdot, T, u) - y_T\|_{L^2(\omega)} \leq \varepsilon.$$

Regional idea makes sense.

Motivation—Why fractional Laplacian?

The past few decades have witnessed a significant development in the study of fractional Laplacian, which could efficiently describe the processes with interactions between two domains arising at a distance, i.e., long range interaction.

Different views of the fractional Laplacian in $\mathbf{R}^n$ would lead to various definitions, which, under some certain assumptions, are equivalent [10]. For example,

$$(-\triangle)^s\phi(x) = \frac{s4^s\Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)}\text{p.v.}\int_{\mathbf{R}^n}\frac{\phi(x)-\phi(y)}{|x-y|^{n+2s}}dy,\ \ s\in(0,1],$$

where $\phi\in\mathbf{R}^n$, p.v. denotes the Cauchy principle value integration.

However, when it comes down to a bounded domain Ω in $\mathbf{R}^n$, Warma in [11] cited that only knowing y at the boundary $\partial\Omega$ is not enough. The reason is that $(-\triangle)^s$ is nonlocal.

[10] M. Kwasnicki, Ten equivalent definitions of the fractional Laplace operator, Fractional Calculus and Applied Analysis 20 (1) (2017) 7–51.
[11] Warma M. Approximate Controllability from the Exterior of Space-Time Fractional Diffusive Equations[J]. SIAM Journal on Control and Optimization, 2019, 57(3): 2037-2063.

Motivation—Why fractional Laplacian?

The goal of this paper is to investigate regional exact controllability from the exterior of the following nonlocal diffusion system on a bounded domain $\Omega \subseteq \mathbf{R}^n$:

$$\begin{cases} y_t(x,t) + (-\triangle)^s y(x,t) = 0 \text{ in } \Omega \times (0,T), \\ y(x,t) = Bu(t) \text{ in } (\mathbf{R}^n \backslash \Omega) \times (0,T), \\ y(x,0) = y_0(x) \text{ in } \Omega, \end{cases} \tag{4}$$

- $Bu(t) = \sum_{i=1}^{p} \chi_{D_i} g_i(x) u_i(t)$ and $u = (u_1, u_2, \cdots, u_p) \in L^2(0,T;\mathbf{R}^p)$;
- Let $H_0^s(\Omega) = \{\phi \in H^s(\mathbf{R}^n) : \phi = 0 \text{ in } \mathbf{R}^n \backslash \Omega\}$ be the fractional Sobolev space endowed with the norm $\|\phi\|_{H_0^s(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dx dy \right)^{1/2}$;
- $(-\triangle)^s : H_0^s(\Omega) \to H^{-s}(\Omega)$ is a continuous operator as follows

$$(-\triangle)^s \phi(x) = (-\triangle)^s \widetilde{\phi}(x), \;\; s \in (0,1],$$

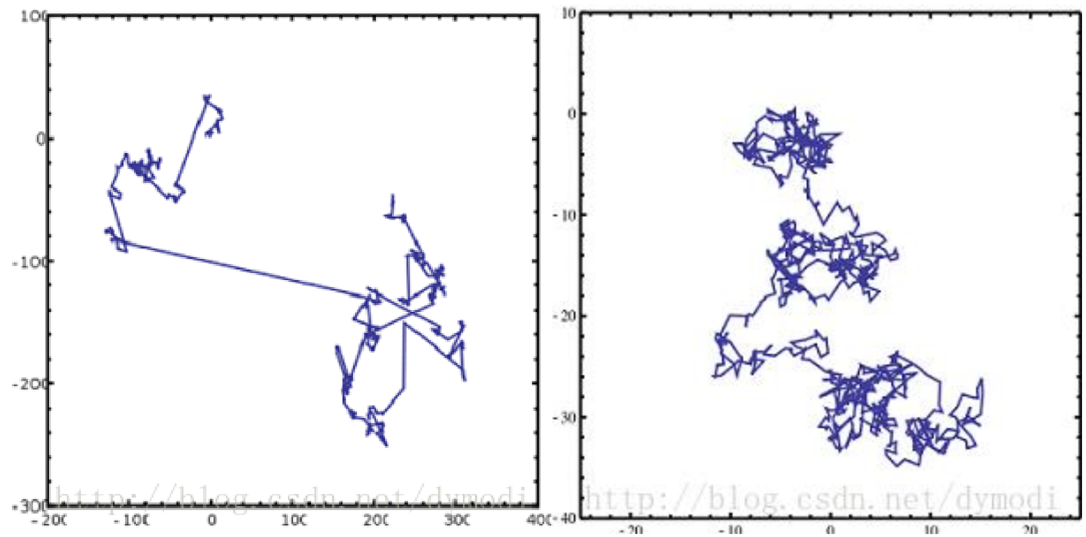
where $\widetilde{\phi}$ is an extension of ϕ defined as $\widetilde{\phi}(x) = \begin{cases} \phi(x), & x \in \Omega, \\ 0, & x \in \mathbf{R}^n \backslash \Omega. \end{cases}$

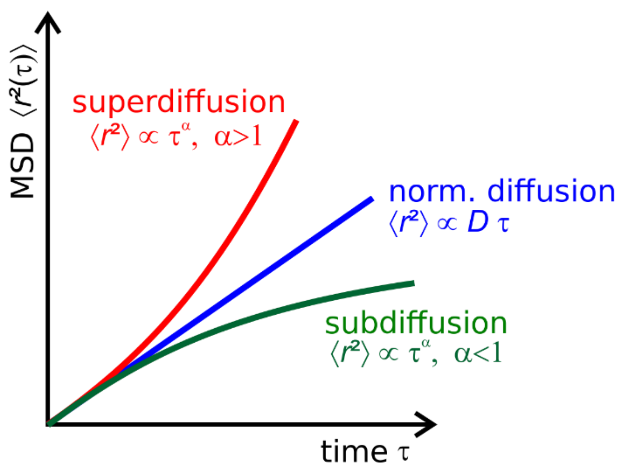
From a probabilistic point of view, it can be regarded as an infinitesimal generator of the stopped $\alpha-$stable Lévy motion [12].

[12] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M. M. Meerschaert, M. Ainsworth, et al., What is the fractional Laplacian?, arXiv: 1801.09767, 2018.

Applications

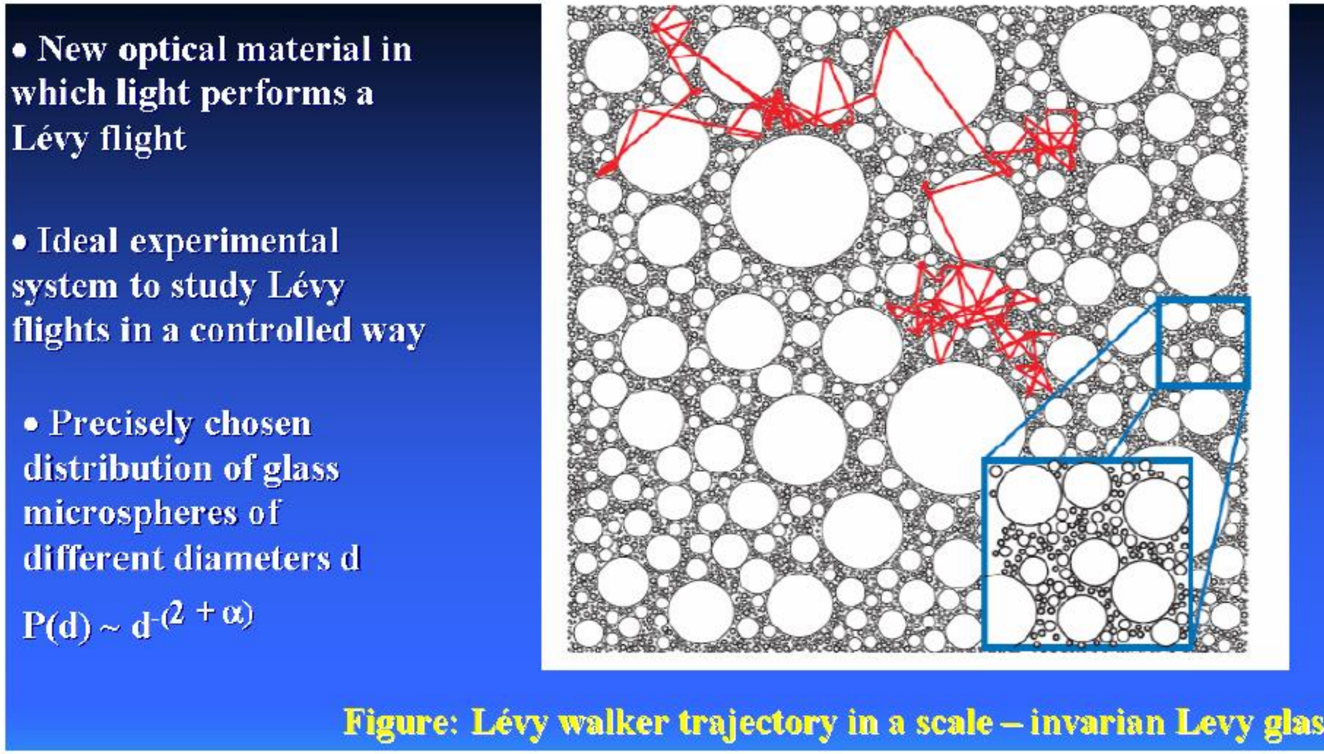
The considered parabolic systems with the fractional Laplacian could be used to well model a wide class of physical phenomena, including Levy flights and stochastic interfaces when traditional approaches appear to fail.





Levy flights

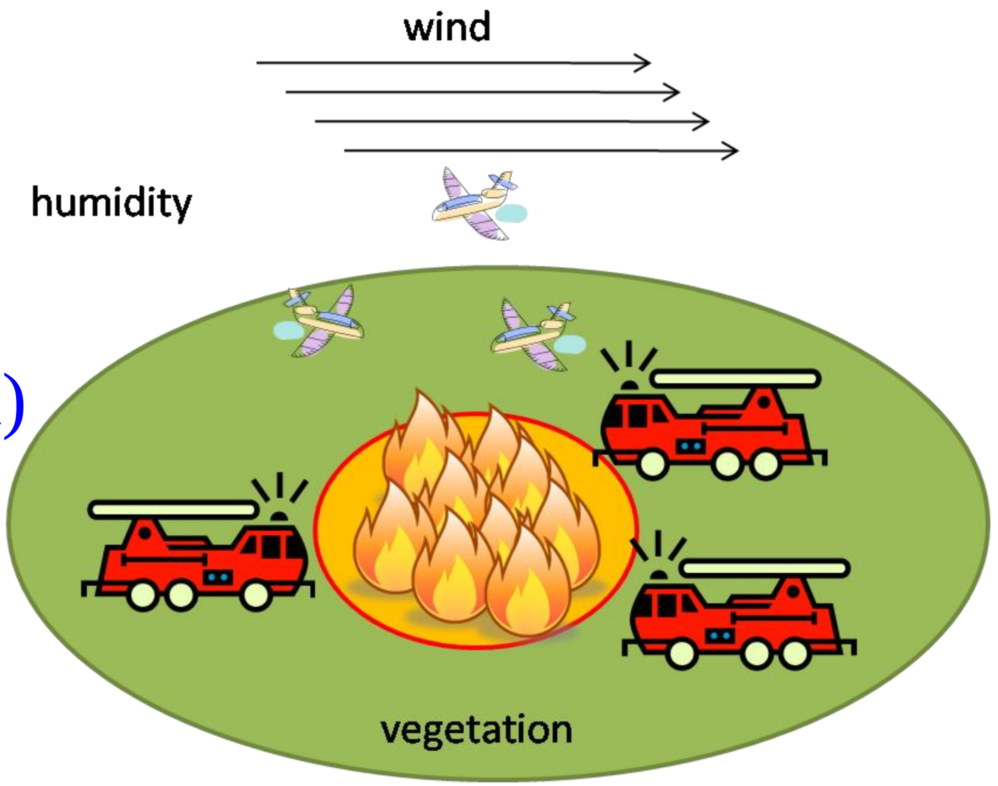
Levy flights--- It is faster than normal diffusions.



[1] Barthelemy P, Bertolotti J, Wiersma D S. A Lévy flight for light[J]. Nature, 2008, 453(7194): 495.

[13] Y. Chen, A. Fiorentino and L. D. Negro. A fractional diffusion random laser. Scientific Reports (2019) 9: 8686. https://doi.org/10.1038/s41598-019-44774-3 1

Wildfire (with wind) Control



Problems to be studied

1. Give some equivalent conditions to achieve regional exact controllability of the considered systems.
2. To approach the minimum energy control problem?

Suppose that $(\lambda_{n,s}, \xi_{n,s})$ is the eigenvalue-eigenfunction pair of operator $(-\triangle)^s$ under the boundary condition $\phi(x) = 0$ in $\mathbf{R}^n \backslash \Omega$, we have

1. If $s \in (0,1)$, $\lambda_{n,s}$ is strictly less than λ_n^s for any given n, where λ_n denotes the eigenvalue of $-\Delta$ under the Dirichlet boundary conditions.

   Further, we refer the reader to Theorem 1 of [14] for a particular case when $\Omega = (-1,1)$:

$$\lambda_{n,s} = \left(\frac{n\pi}{2} - \frac{(1-s)\pi}{4}\right)^{2s} + O\left(\frac{1}{n}\right) < \left(\frac{n^2\pi^2}{4}\right)^s = \lambda_n^s. \tag{5}$$

2. $\{\xi_{n,s}\}_{n\geq 1}$ forms a Riesz basic in $L^2(\Omega)$.

Solution:

$$y(x,t) = \sum_{n=1}^{\infty} \frac{(y_0, \xi_{n,s})_{L^2(\Omega)}}{e^{\lambda_{n,s} t}} \xi_{n,s}(x) - \sum_{n=1}^{\infty} \int_0^t \frac{(Bu(\tau), \mathcal{N}_s \xi_{n,s})_{L^2(\mathbf{R}^n \backslash \Omega)}}{e^{\lambda_{n,s}(t-\tau)}} d\tau \xi_{n,s}(x), \tag{6}$$

where

$$\mathcal{N}_s \phi(x) := C_{n,s} \int_\Omega \frac{\phi(x) - \phi(y)}{|x-y|^{n+2s}} dy, \; x \in (\mathbf{R}^n \backslash \Omega) \text{ with } C_{n,s} = \frac{s 4^s \Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)} \tag{7}$$

is a bounded operator for $\mathcal{D}\left((-\triangle)^s\right) \subseteq H^s(\mathbf{R}^n)$ to $L^2\left(\mathbf{R}^n \backslash \Omega\right)$ [14].

[14] Warma, M. (2019). Approximate controllability from the exterior of space-time fractional diffusive equations. SIAM Journal on Control and Optimization, 57(3), 2037–2063.

Main Results

Consider the operator $D_T : L^2(0, T; \mathbf{R}^p) \to L^2(\Omega)$ given by

$$(D_T u)(x) = \sum_{n=1}^{\infty} \int_0^T \frac{(Bu(\tau), \mathcal{N}_s \xi_{n,s})_{L^2(\mathbf{R}^n \setminus \Omega)}}{e^{\lambda_{n,s}(T-\tau)}} d\tau \xi_{n,s}(x) \ \forall u \in L^2(0, T; \mathbf{R}^p).$$

Theorem 1.

Given $T > 0$, there is an equivalence among the following three properties:

$\langle 1 \rangle$ System (4) is regionally exactly controllable in $L^2(\omega)$ at time T;

$\langle 2 \rangle$ $Im(\chi_\omega D_T) = L^2(\omega)$;

$\langle 3 \rangle$ For any $\varphi \in L^2(\omega)$, a constant $\gamma > 0$ can be found satisfying

$$\|\varphi\|_{L^2(\omega)} \leq \gamma \|D_T^* \chi_\omega^* \varphi\|_{L^2(0,T;\mathbf{R}^p)}. \tag{8}$$

Here D_T^* denotes the adjoint operator of D_T and χ_ω^* is the adjoint operator of χ_ω given by

$$\chi_\omega^* y(x) = \begin{cases} y(x), & x \in \omega, \\ 0, & x \in \Omega \setminus \omega. \end{cases} \tag{9}$$

Proof : Omitted.

Suppose that the following two assumptions hold:

(A_1) B is a densely defined operator and B^* exists;

(A_2) $(BK_\alpha(t))^*$ exists and $(BK_\alpha(t))^* = K_\alpha^*(t)B^*$.

Obviously, (A_1) and (A_2) hold when $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$.

Given any target function $y_T \in L^2(\omega)$, in what follows, we focus on discussing the following minimum energy control problem for regional exact controllability of system (4):

$$\inf_u J(u) = \inf_u \left\{ \frac{1}{2} \int_0^T \|u(t)\|_{\mathbf{R}^p}^2 dt : u \in U_T \right\}, \tag{10}$$

where $U_T = \{u \in L^2\left(0, T; \mathbf{R}^p\right) : \chi_\omega y(x, T, u) = y_T\}$ is a nonempty closed convex set.

Actually, this minimization problem admits a unique solution, denoted by u_k.

Lemma 1. If system (4) is regionally exactly controllable in $L^2(\omega)$ at time T, then

$$g \in G \to \|g\|_G^2 = \int_0^T \|B^* \mathcal{N}_s v(\cdot, t)\|_{\mathbf{R}^p}^2 dt$$

defines a norm on $G := \left\{g \in L^2(\Omega) : g = 0 \text{ in } \Omega \backslash \omega\right\}$.

Proof. Omitted.

Define operator Λ: $G \to L^2(\Omega)$ as follows

$$\Lambda g = -\chi_\omega \psi(\cdot, T), \tag{11}$$

where $\psi(x,t)$ is the solution of system

$$\begin{cases} \psi_t(x,t) + (-\triangle)^s \psi(x,t) = 0 \text{ in } \Omega \times (0,T), \\ \psi(x,t) = BB^* \mathcal{N}_s v(x,t) \text{ in } (\mathbf{R}^n \backslash \Omega) \times (0,T), \\ \psi(x,0) = 0 \text{ in } \Omega \end{cases} \tag{12}$$

Suppose that $\widetilde{\psi}(t)$ satisfies

$$\begin{cases} \widetilde{\psi}_t(x,t) + (-\triangle)^s \widetilde{\psi}(x,t) = 0 \text{ in } \Omega \times (0,T), \\ \widetilde{\psi}(x,t) = 0 \text{ in } (\mathbf{R}^n \backslash \Omega) \times (0,T), \\ \widetilde{\psi}(x,0) = y_0(x) \text{ in } \Omega. \end{cases} \tag{13}$$

Main Results

For any target function $y_T \in L^2(\omega)$, the minimum energy control to achieve regional exact controllability of y_T at time T is equal to solving the equation

$$\Lambda g = y_T - \chi_\omega \widetilde{\psi}(x,T), \quad x \in \Omega. \tag{14}$$

Now we are in a position to give our main result.

Theorem 2. Given any target function $y_T \in L^2(\omega)$, if system (4) is regionally exactly controllable in $L^2(\omega)$ at time T under the control input $u^*(t)$ given by

$$\begin{aligned} u^*(t) &= B^* \mathcal{N}_s v(x,t) \\ &= \left(B^* \sum_{n=1}^{\infty} e^{-\lambda_{n,s}(T-t)} \left(\xi_{n,s}, g\right)_{L^2(\Omega)} \mathcal{N}_s \xi_{n,s} \right)(t), \end{aligned} \tag{15}$$

then the equation (14) admits a unique solution $g \in G$.

Moreover, $u^*(t)$ is the solution of the minimum energy control problem (10).

Proof. Omitted.

An Example

Consider the following parabolic system with a zone actuator

$$\begin{cases} y_t(x,t) + (-\triangle)^s y(x,t) = 0 \text{ in } (0,1)\times(0,5), \\ y(x,t) = \chi_{[a_1,a_2]} u(t) \text{ in } (\mathbf{R}\backslash(0,1))\times(0,5), \\ y(x,0) = 0 \text{ in } (0,1), \end{cases} \tag{16}$$

By [14], one has $\lambda_{n,s} = (n\pi - 0.35\pi)^{0.6} + O\left(\frac{1}{n}\right)$, $\xi_{n,s}(x) \approx \sqrt{2}\sin\left((n-0.35)\pi x + 0.35\pi\right)$ and

$$(D_5^* h)(t) = \sum_{n=1}^{\infty} e^{-\lambda_{n,s}(5-t)} \left(\xi_{n,s}, h\right)_{L^2(0,1)} C_{n,s} \int_{a_1}^{a_2} \int_0^1 \frac{\xi_{n,s}(x)-\xi_{n,s}(y)}{(x-y)^{1.6}} dydx.$$

Taking into account that

$$\int_{a_1}^{a_2} \int_0^1 \frac{\xi_{n,s}(x) - \xi_{n,s}(y)}{(x-y)^{1.6}} dydx = 0$$

holds true for some $a_1, a_2 \in \mathbf{Q}$ with $a_2 \geq a_1 > 1$, we have $Ker(D_5^*) \neq \{0\}$. Then, condition $\langle 3 \rangle$ of Theorem 1 cannot be satisfied. This implies that system (15) is not exact controllability on whole domain $L^2(0,1)$.

An Example

However, let $a_1 = 2,\ a_2 = 2 + \mu$. Since $\xi_{n,s}(x)$ is a periodic continuous function and $\xi_{n,s}(2) \neq 0$, there exists a small $\mu > 0$ such that

$$\int_2^{2+\mu} \int_0^1 \frac{\xi_{n,s}(x) - \xi_{n,s}(y)}{(x-y)^{1.6}} dy dx \neq 0. \tag{17}$$

Similarly, for a nonzero continuous function h, one can find for example, an interval $(\gamma, \gamma + \varepsilon)$ with $\varepsilon > 0$ satisfying

$$\int_\gamma^{\gamma+\varepsilon} \xi_{n,s}(x) h(x) dx \neq 0. \tag{18}$$

With these, let $\omega = (\gamma, \gamma + \varepsilon)$. Then $(D_5^* \chi_\omega^* h)(t) \neq 0$.

This implies that system (15) is regionally exactly controllable in $L^2(\gamma, \gamma + \varepsilon)$ at time $T = 5$ under a zone actuator localized in $[2, 2 + \mu]$.

An Example

In addition, consider the following minimization problem:

$$\inf_u J(u) = \inf_u \left\{ \frac{1}{2} \int_0^5 u^2(t)dt : u \in U_T \right\}, \tag{19}$$

where $U_T = \{u \in L^2(0,5) : \chi_\omega y(x,5,u) = y_5\}$. By Theorem 2, if system (15) is regionally exactly controllable in $L^2(\omega)$ at time 5, the equation

$$\Lambda g = y_5 - \chi_\omega \widetilde{\psi}(x,5), \quad x \in \Omega \tag{20}$$

admits a unique solution $g \in G$, where $\Lambda g = -\chi_\omega \psi(\cdot,5)$ with $\psi(x,t)$ solving

$$\begin{cases} \psi_t(x,t) + (-\triangle)^{0.3}\psi(x,t) = 0 \ \text{ in } (0,1)\times(0,5), \\ \psi(x,t) = \chi_{[a_1,a_2]}u^*(t) \text{ in } (\mathbf{R}\backslash(0,1))\times(0,5), \\ \psi(x,0) = 0 \text{ in } (0,1) \end{cases} \tag{21}$$

and

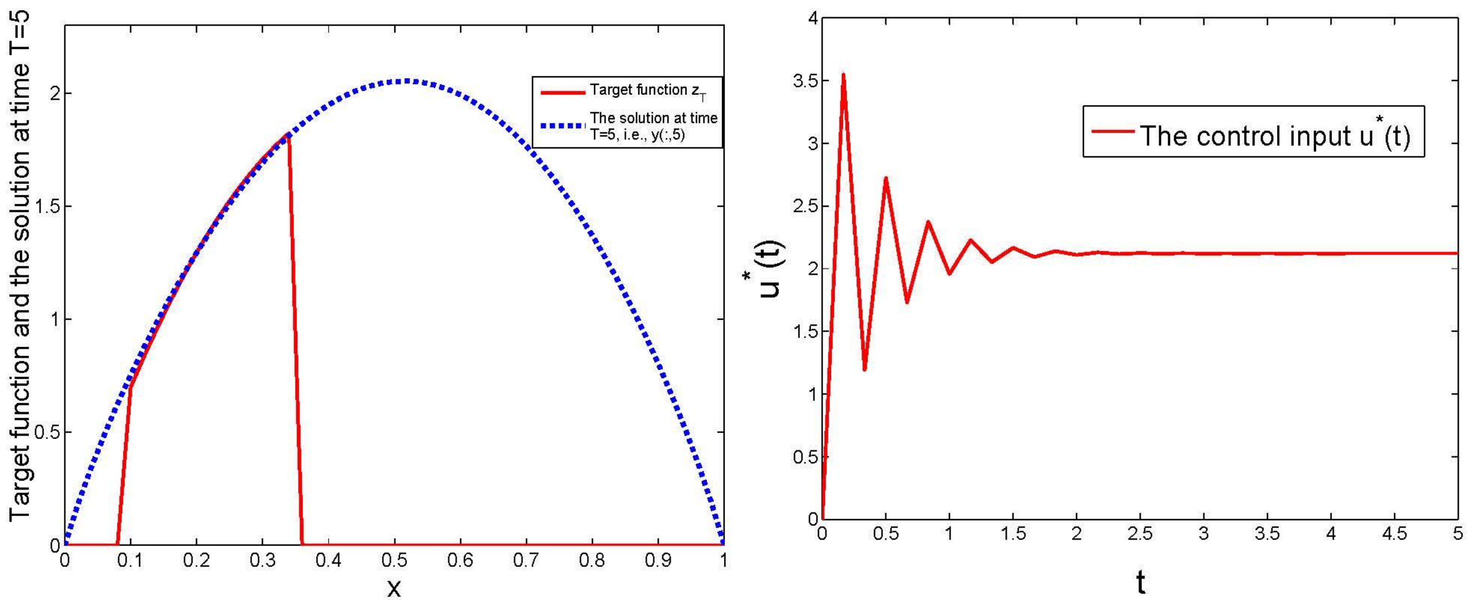
$$u^*(t) = \sum_{n=1}^{\infty} e^{-\lambda_{n,s}(5-t)} \left(\xi_{n,s}, g\right)_{L^2(0,1)} C_{n,s} \int_{a_1}^{a_2}\int_0^1 \frac{\xi_{n,s}(x)-\xi_{n,s}(y)}{(x-y)^{1.6}} dydx. \tag{22}$$

Moreover, we get that this $u^*(t)$ solves above minimum control problem (10).

An Example

Continue to consider system (15) with $a_2 = a_1 = \sigma = 0.47$. In this case, we say that it is excited by a pointwise actuator located at $\sigma = 0.47$. Let $\omega = \left(\frac{\sqrt{2}}{15}, \frac{\sqrt{2}}{15} + 0.25\right) \subseteq [0, 1]$. Assume that the target function is given by

$$y_T(x) = \begin{cases} 0, \quad 0 \le x < \frac{\sqrt{2}}{15}; \\ -9x^2 + 9x - 0.25, \ \frac{\sqrt{2}}{15} \le x \le \frac{\sqrt{2}}{15} + 0.25; \\ 0, \quad \frac{\sqrt{2}}{15} + 0.25 < x \le 1. \end{cases}$$



(1) The final states at time T =5 and the target function.

(2) The control input $u^*(t)$.

Figure 1: Evolution of target function $y(x, 5)$ at $T = 5$ and the corresponding control input.

Summary and Extensions

This paper investigate the regional exact controllability problem of parabolic PDE systems with the fractional Laplacian, whose control input is localized on some subset of the system's exterior domain. Some equivalent conditions to achieve regional exact controllability of the considered system are presented. An approach on the minimum energy control problem is then explicitly derived by using HUM.

Moreover, the presented results can be extended to more complex nonlocal distributed parameter systems. For instance, the problem of constrained regional control of time-space fractional diffusion systems with the fractional Laplacian under more complicated regional sensing and actuation configurations are of great interest.

Thank you for your attention!

Questions or comments?