Observer-based output feedback control for a boundary controlled fractional reaction diffusion system with spatially-varying diffusivity

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Abstract: This study is concerned with observer design and observer-based output feedback control for a fractional reaction diffusion (FRD) system with a spatially-varying (non-constant) diffusion coefficient by the backstepping method. The considered FRD system is endowed with only boundary measurable and actuation available. The contribution of this study is divided into three parts: first is the backstepping-based observer design for the FRD system with non-constant diffusivity, second is the output feedback controller generated by the integration of a separately backstepping-based feedback controller and the proposed observer to stabilise the FRD system with non-constant diffusivity, and the last is the Mittag–Leffler stability analysis of the observer error and the closed-loop FRD systems. Specifically, anti-collocated location of actuator and sensor is considered in the stabilisation problem of this system with Robin boundary condition at $x = 0$ and the boundary feedback controller for Dirichlet actuation at $x = 1$. By designing an invertible coordinate transformation to convert the observer error system into a Mittag–Leffler stable target system, the observer gains are obtained. They are used to design the output feedback control law for stabilising the closed-loop system. Finally, a numerical example is shown to validate the effectiveness of the authors' proposed method.

1 Introduction

1.1 Overview of prior literatures

In engineering application, the system state information is crucial to its control problem. Nevertheless, not all states are available in practical cases due to using restrictions of sensors or some unmeasurable variables. Recent years, several intensive research activities have been devoted to the observer design and the observer-based output feedback. Many of these activities have focused on integer-order differential systems, especially for distributed parameter systems (DPSs) governed by partial differential equations (PDEs). For these systems, the emphasis has been put on the problem of the Lunberger-like state observer design in the whole domain, see, e.g. [1, 2]. By the backstepping approach, the boundary observer design method has been announced in [3] for a class of PDEs. The main idea of the backstepping method in [4] is to transform a preset system into a selected stable target system. More basic knowledge and development of the backstepping approach can be found in [5, 6] and references therein.

Later on, a lot of related works have been arisen with the aid of this method, such as adaptive observer design for the ordinary differential equation-PDE (ODE–PDE) systems and parabolic PDEs with domain and boundary parameter uncertainties in [7, 8], and output feedback control of parabolic PDEs with moving boundary in [9]. Note that these works are related to the constant diffusivity. For non-constant diffusivity [10, 11], there are also many considerable research results [12, 13] on boundary observer design for the cascade PDE–ODE systems and semi-linear DPSs.

As we know, fractional differential equations have multiple applications in environmental science and biology, physics, and geophysics. Hence, great attention has been paid to this popular topic, see, e.g. [14, 15]. In particular, for the fractional reaction diffusion (FRD) system with constant diffusivity, the boundary stabilisation and the observer-based output feedback control problems have recently been addressed in [16–18]. Specifically, the work in [16] considered the boundary feedback control problem for the FRD system with Dirichlet or Neumann boundary conditions. After that the state-feedback and the output-feedback problems for the case of mixed or Robin boundary conditions were investigated in [17, 18].

1.2 Motivation

The observer design problem of fractional-order systems can be traced back to the work [19], which used the estimation of state in the whole domain to design observer and developed a stability criterion for these systems from the point of system matrix's argument. Unfortunately, solving the argument is too difficult to realise. It needs to find some other methods for the stability of fractional-order systems, e.g. the fractional Lyapunov method [20]. Additionally, despite the considerable contribution of the boundary stabilisation, observer design and output feedback control for DPSs (integer-order systems) in [3, 21, 22], very few results are available on the output feedback control for fractional-order systems with constant or non-constant diffusivity except for the one of fractional-order systems with constant diffusivity in [18].

Motivated by the advantage of the backstepping method for integer-order systems, observer design for DPSs and the fractional Lyapunov method for the stability of fractional-order systems, in this paper, we introduce the backstppeing approach into the FRD system with non-constant diffusivity for designing the observer and the output-feedback controller. Then the Mittag–Leffler stability of this controlled system is analysed by the fractional Lyapunov method. Here, the diffusivity is allowed to spatially-varying (non-constant), which can model pattern formation in inhomogeneous media [23, 24] and may mimic the diffusion effects of particles in the heterogeneous environment [25].

1.3 Main contribution

The contribution of this paper will be presented as the following aspects:

(1) Comparing with the integer-order case [3, 21, 22], the results in this paper can be viewed as a generalisation of the integer-order case. Specifically, compared to the case of integer-order system with constant diffusivity in [3, 22], the system in this paper was a fractional-order type and also subject to non-constant diffusivity so that the designed observer contained the diffusivity term and the stabilisation results of the observer error and closed-loop systems referred to the fractional Lyapunov stability (Mittag–Leffler stability) theory rather than the Lyapunov stability theory. Compared to the non-constant diffusivity case of the integer-order system in [21], our results are more difficult in some extent since the fractional-order type of our system makes the stabilisation problem complex and different from the integer-order one. Note that the case of an integer-order system with non-constant diffusivity in [21] only discussed the boundary stabilisation problem but here we investigate the observer design and observer-based output feedback control problems. They need to use the boundary state-feedback results, which has been put into the Appendix for reference.

(2) The boundary feedback control has recently been addressed in [16, 17] for the FRD system with constant diffusivity. In this paper, we do not focus on the constant diffusivity case. An important extension of the backstepping approach is the design of an observer and a stabilising output feedback controller for the FRD system with non-constant diffusivity, which makes the observer error and the closed-loop systems Mittag–Leffler stable. Here the output feedback control problem is solved for an FRD system with non-constant diffusivity, where the integration of the boundary feedback controller and the observer is based on separation principle, and only boundary measurement and actuation available.

(3) Compared to the output-feedback problems of fractional-order systems with constant diffusivity in [18, 19], the one discussed here give more freedom to choose the parameter value since the diffusion coefficient is spatially-varying (non-constant). In [19], the observer design is based on the fact measurement is available in the whole domain while this present problem needs only the boundary observation of the FRD system with non-constant diffusivity. Furthermore, the observer-based output feedback problem was considered for the constant-diffusivity FRD system in [18], which may be simpler than our present problem where the non-constraint diffusivity affects the FRD system and the stabilisation results need to be guaranteed by certain constraint conditions. From a theory point of view, this paper provides some insight into the observer-based output-feedback control for fractional-order systems with non-constant diffusivity.

1.4 Structure

The next section illustrates the output feedback problem for the FRD system with non-constant diffusivity. After it, Section 3 presents the observer design for this system. This result is combined with the boundary feedback controller (see the Appendix) to generate an output feedback controller in Section 4. Then, sufficient conditions for the Mittag–Leffler stability of the observer error and the closed-loop systems have been presented. The numerical simulation results are provided in Section 5 to demonstrate the validness of our proposed method. Finally, conclusions are contained in Section 6. To alleviate the presentation, the boundary stabilisation results are attached in the Appendix.

2 Problem settlement

In this paper, the dynamics equation and initial condition of the FRD system with non-constant diffusivity are described by

$$\begin{aligned} &{}_0^C D_t^\alpha u(x,t) = \vartheta(x) u_{xx}(x,t) + a(x) u(x,t), \quad x \in (0,1), \quad t > 0, \\ &u(x,0) = u_0(x), \ x \in [0,1] \end{aligned} \tag{1}$$

with the mixed boundary conditions

$$u_x(0,t) - p u(0,t) = 0, \quad t > 0, \tag{2}$$

$$u(1,t) = U(t), \quad t > 0, \tag{3}$$

$$\text{or} \quad u_x(1,t) = U(t), \quad t > 0, \tag{4}$$

$$\text{or} \quad u_x(1,t) + q u(1,t) = U(t), \quad t > 0, \tag{5}$$

where

$${}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{\partial u(x,\tau)}{\partial \tau} \mathrm{d}\tau, \quad 0 < \alpha < 1$$

denotes the Caputo time fractional-order derivative of α order [26], $\vartheta(x) > 0$ for $x \in [0,1]$ represents the non-constant diffusivity, $a(\cdot) \in C^1[0,1]$, $u_0(x)$ is the non-zero initial value, $p > 0$, $q > 0$, and $U(t)$ represents an input. In addition, this problem (i.e. FRD process with spatially-varying diffusivity) has another structure, which can be represented as ${}_0^C D_t^\alpha u(x,t) = (\partial/\partial x)(\vartheta(x) u_x(x,t)) + a(x) u(x,t)$. This system's dynamical equation is different from (1) and also got much attention [14]. It can be converted into ${}_0^C D_t^\alpha y(x,t) = \vartheta(x) y_{xx}(x,t) + (a(x) + (\vartheta'^2(x)/(4\vartheta(x)) - ((\vartheta''(x))/2)) y(x,t)$ with a change of variables $u = y/\sqrt{\vartheta(x)}$.

For the above system (1), the boundary conditions (2), (3) or (2), (4) can be taken as mixed boundary conditions, and (2), (5) are Robin boundary conditions. Here, we main focus on Robin boundary condition at $x = 0$ and Dirichlet actuation at $x = 1$ (i.e. (2), (3)), the cases of other boundary conditions are analogous and without much technical differences. For the FRD system (1), (2) without control, i.e. $u(1,t) = 0$, we can know that it is unstable if $a(\cdot)$ is large enough based on the stability results of fractional-order systems in [19, 27]. Specifically, the stability of FRD system is guaranteed by the fact that the roots of some polynomial lie outside the closed-angular sector

$$\left| \arg\left(\operatorname{spec}\left(\frac{\partial^2(\vartheta(x) u(x,t))}{\partial x^2} + a(x) \right) \right) \right| \le \frac{\alpha\pi}{2},$$

which has been proposed in [19, 27]. Although the eigenvalues of the operator

$$\frac{\partial^2(\vartheta(x) u(x,t))}{\partial x^2}$$

are negative (see, e.g. [28, Section 3]), the stability of the open-loop system (1)–(3) (with $U(t) = 0$) can also be lost if the value of $a(x)$ takes large enough. Thus, we need to design corresponding control to stabilise this system.

In this paper, we want to utilise an output feedback controller to make this system stable. First, design a Mittag–Leffler convergence observer for this system. Then, combine the proposed observer and the backstepping-based controller to design an output feedback controller to stabilise this controlled FRD system, which recurs to the backstepping-based boundary stabilisation results (see the Appendix for more details). The observer gains are determined by the type of actuation (i.e. Dirichlet actuation or someone else), and the setups of sensors and actuators, i.e. sensors and actuators are allocated at the same end or at the different end (collocated case or anti-collocated case). Actually, the two cases are analogous in technical aspects, including observer design and output feedback controller design. Thus, we only choose the anti-collocated case for discussion and give some relevant corollaries or remarks for the collocated case.

Remark 1: In the above system (1)–(3) ((1), (2), (4) or (1), (2), (5)), if $\alpha = 1$, this system will reduce to the integer-order case, whose observer design and observer-based output feedback problems have been investigated in [3].

Definition 1 ([20, 29] Mittag–Leffler stability):

$$\text{If} \quad ||u(t)|| \le \left(h[u(t_0)]E_\beta(-\gamma(t-t_0)^\beta)\right)^b, \tag{6}$$

where t_0 is the initial value of time, $\beta \in (0,1)$, $\gamma \ge 0$, $b > 0$, $h(0) = 0$, $h(u)$ is non-negative and meets locally Lipschitz condition on $u \in \mathbb{B} \in \mathbb{R}^n$ with the Lipschitz constant h_0, and

$$E_\beta(t) := \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i\beta+1)}, \quad \forall \beta > 0, \ t \in \mathbb{C}$$

in [30], then the solution of the equation

$$ {}_{t_0}^{C}D_t^{\beta} u(t) = f(t,u) \tag{7}$$

is said to be Mittag–Leffler stable. In (7), $u(t_0)$ is the initial condition, $\beta \in (0,1)$, f is piecewise continuous in $t \in [t_0, \infty)$ and locally Lipschitz in u.

Definition 2 ([20] Equilibrium point): The constant u_0 is an equilibrium point of the Caputo fractional dynamic system (7), if and only if $f(t, u_0) = 0$, $t \in [t_0, \infty)$.

Based on the above Definition 2, we know $u(x,t) = 0$ is an equilibrium point of (1).

3 Observer design

Here, assume that the Dirichlet actuation is exerted at $x = 1$ and measurement is available at $x = 0$. Then the output is given by

$$y(t) = u(0,t). \tag{8}$$

In this case, the corresponding state observer for the plant (1)–(3) is designed as follows:

$$\begin{aligned} {}_0^C D_t^\alpha \hat{u}(x,t) &= \vartheta(x)\hat{u}_{xx}(x,t) + a(x)\hat{u}(x,t) + r_1(x) \\ &\quad \times (u(0,t) - \hat{u}(0,t)), \ x \in (0,1), t > 0, \\ \hat{u}(x,0) &= \hat{u}_0(x), \quad x \in [0,1], \end{aligned} \tag{9}$$

$$\hat{u}_x(0,t) = pu(0,t) + r_{10}(u(0,t) - \hat{u}(0,t)), \quad t > 0, \tag{10}$$

$$\hat{u}(1,t) = U(t), \quad t > 0, \tag{11}$$

where $r_1(x)$ and r_{10} represent observer gains to be designed, $\hat{u}_0(x)$ is the initial value.

Remark 2: Similar to the argument in [3, Section 3], the additional observer gain $p(u(0,t) - \hat{u}(0,t))$ in (10) is also used to eliminate the dependency on p in the error system. This makes (10) has the expression different from the one of extended Luenberger observer in [13, Section II].

The observer (9)–(11) for FRD systems mimics the fractional differential system in [19], whose observer is designed as the form of $d^\alpha \hat{x} = A\hat{x} + \hat{B}u - L(\hat{y} - y)$, $\hat{y} = C\hat{x}$ and used for the plant $d^\alpha x = Ax + Bu$, $Y = Cx$ ($d^\alpha x$ is the smooth derivative of α with respect to x). It is pointed out that in this paper the measurement is available at the end rather than the whole domain. Also, the way to find observer gains in the FRD system is based on the stabilisation solution problem in the Appendix.

With the observer error $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$, the following error system is obtained:

$$\begin{aligned} {}_0^C D_t^\alpha \tilde{u}(x,t) &= \vartheta(x)\tilde{u}_{xx}(x,t) + a(x)\tilde{u}(x,t), \\ &\quad -r_1(x)\tilde{u}(0,t) \ x \in (0,1), \quad t > 0, \\ \tilde{u}(x,0) &= \tilde{u}_0(x), \quad x \in [0,1], \\ \tilde{u}_x(0,t) &= -r_{10}\tilde{u}(0,t), \quad t > 0, \\ \tilde{u}(1,t) &= 0, \quad t > 0, \end{aligned} \tag{12}$$

where $\tilde{u}_0(x)$ is the initial value. Here, note that $\hat{u}(x,t) = 0$ and $\tilde{u}(x,t) = 0$ are the equilibrium points of (9) and (12) in terms of Definition 2.

The observer gains $r_1(x)$ and r_{10} are chosen to stabilise the system (12). To solve the Mittag–Leffler stabilisation problem of the system (12), we want to find an integral transformation

$$\tilde{u}(x,t) = \tilde{w}(x,t) + \int_0^x r(x,y)\tilde{w}(y,t)\,\mathrm{d}y \tag{13}$$

to map the system (12) into below target system:

$$\begin{aligned} {}_0^C D_t^\alpha \tilde{w}(x,t) &= \vartheta(x)\tilde{w}_{xx}(x,t) - \tilde{\lambda}\tilde{w}(x,t), \\ & \quad x \in (0,1), \quad t > 0, \\ \tilde{w}(x,0) &= \tilde{w}_0(x), \quad x \in [0,1] \\ \tilde{w}_x(0,t) &= 0, \quad t > 0, \\ \tilde{w}(1,t) &= 0, \quad t > 0, \end{aligned} \tag{14}$$

where $\tilde{\lambda} > 0$ can determine the observer convergence speed, which is usually different from λ of control design in Section 2. $\tilde{w}_0(x)$ is the initial value and $\tilde{w}_0(x) = \tilde{u}_0(x) - \int_0^x r(x,y)\tilde{w}_0(y)\,\mathrm{d}y$. It is noticeable that this target system is Mittag–Leffler stable under certain conditions (see the proof of Theorem 1 for more details).

The following is contributed to find out gain kernel $r(x,y)$ in (13) through a series computation and substitution. Based on integral transformation (13), we know that $\tilde{u}(0,t) = \tilde{w}(0,t)$. Then, taking the space derivative of (13), we have

$$\tilde{u}_x(x,t) = \tilde{w}_x(x,t) + r(x,x)\tilde{w}(x,t) + \int_0^x r_x(x,y)\tilde{w}(y,t)\,\mathrm{d}y \tag{15}$$

and

$$\begin{aligned} \tilde{u}_{xx}(x,t) &= \tilde{w}_{xx}(x,t) + \frac{\mathrm{d}}{\mathrm{d}x}r(x,x)\tilde{w}(x,t) + r(x,x)\tilde{w}_x(x,t) \\ &\quad + r_x(x,x)\tilde{w}(x,t) + \int_0^x r_{xx}(x,y)\tilde{w}(y,t)\,\mathrm{d}y. \end{aligned} \tag{16}$$

Next solving this integral transformation's Caputo time derivative with the α order, it is readily to obtain that

$${}_0^C D_t^\alpha \tilde{u}(x,t) = {}_0^C D_t^\alpha \tilde{w}(x,t) + \int_0^x r(x,y)\,{}_0^C D_t^\alpha \tilde{w}(y,t)\,\mathrm{d}y. \tag{17}$$

Substituting the first equations of (12), (14), and (16) into above equality (17), we get

$$\begin{aligned} 0 &= (\vartheta(x)\frac{\mathrm{d}}{\mathrm{d}x}r(x,x) + \vartheta(x)r_x(x,x) + a(x) + \tilde{\lambda} + \vartheta(x) \\ &\quad \times r_y(x,x) + \vartheta'(x)r(x,x))\tilde{w}(x,t) + \int_0^x (\vartheta(x)r_{xx}(x,y) \\ &\quad + a(x)r(x,y) - (\vartheta(y)r(x,y))_{yy} + \tilde{\lambda}r(x,y))\tilde{w}(y,t)\,\mathrm{d}y \\ &\quad + (-r_1(x) - r_y(x,0)\vartheta(0) - r(x,0)\vartheta'(0))\tilde{w}(0,t). \end{aligned} \tag{18}$$

By integral transformation (13), we have $\tilde{u}(1,t) = \tilde{w}(1,t) + \int_0^1 r(1,y)\tilde{w}(y,t)\,\mathrm{d}y$. This together with $\tilde{u}(1,t) = 0$, $\tilde{w}(1,t) = 0$, implies that $r(1,y) = 0$.

Furthermore, (18), together with $r(1,y) = 0$, and the notations $\frac{\mathrm{d}}{\mathrm{d}x}r(x,x) = r_x(x,x) + r_y(x,x)$ where $r_x(x,x) = r_x(x,y)|_{y=x}$, $r_y(x,x) = r_y(x,y)|_{y=x}$, yields the following kernel PDE of $r(x,y)$:

$$\begin{cases} \vartheta(x)r_{xx}(x,y) - (\vartheta(y)r(x,y))_{yy} = -\left(a(x) + \tilde{\lambda}\right)r(x,y) \\ 2\vartheta(x)\frac{\mathrm{d}}{\mathrm{d}x}r(x,x) = -\vartheta'(x)r(x,x) - a(x) - \tilde{\lambda} \\ r(1,y) = 0 \end{cases} \tag{19}$$

for $(x, y) \in \Theta = \{0 \le y \le x \le 1\}$. Aside, since $\tilde{u}_x(x, t) = \tilde{w}_x(x, t) + r(x, x)\tilde{w}(x, t) + \int_0^x r_x(x, y)\tilde{w}(y, t)\,\mathrm{d}y$, we can get $\tilde{u}_x(0, t) = \tilde{w}_x(0, t) + r(0, 0)\tilde{w}(0, t)$. It, together with $\tilde{u}_x(0, t) = -r_{10}\tilde{u}(0, t)$ and $\tilde{u}(0, t) = \tilde{w}(0, t)$, induces that $\tilde{w}_x(0, t) = -(r_{10} + r(0, 0))\tilde{w}(0, t)$. Comparing this with $\tilde{w}_x(0, t) = 0$ and considering (18) again, we see that observer gains need to be chosen as

$$r_1(x) = -\vartheta(0)r_y(x, 0) - \vartheta'(0)r(x, 0), \quad r_{10} = -r(0, 0). \quad (20)$$

Remark 3: In the above FRD system, if $\alpha = 1$ and $\vartheta(x) > 0, x \in [0, 1]$, then the corresponding observer gains of this integer-order system with non-constant diffusivity are same as (20) which can be derived from the results in [3, 21]. And when $\alpha = 1$ and $\vartheta(x) = \text{constant} = D$, the corresponding observer gains of the integer-order one with constant diffusivity are $r_1(x) = -Dr_y(x, 0), r_{10} = -r(0, 0)$ since $\vartheta(0) = D, \vartheta'(0) = 0$. It matches with the result in [21]. In addition, if $0 < \alpha < 1$ and $\vartheta(x) = \text{constant}$ (i.e. fractional systems with constant diffusivity), the observer gains are same as the integer-order case with constant diffusivity, which has been discussed in [18]. That said, the observer gains are associated with the diffusion coefficient (constant or non-constant).

For the above kernel PDE (20), if its solution could be found or the kernel exists, then observer gains also could be obtained. Therefore, we next prove that the solution of kernel PDE exists and is unique. Similar to the method in [3], we first use a change of variables like

$$\bar{r}(\bar{x}, \bar{y}) = r(x, y), \quad \bar{x} = 1 - y, \quad \bar{y} = 1 - x, \quad (21)$$

$$\bar{\vartheta}(\bar{y}) = \vartheta(x), \quad \bar{\vartheta}(\bar{x}) = \vartheta(y), \quad \bar{a}(\bar{y}) = a(x) \quad (22)$$

to make the kernel PDE (19) into

$$\begin{cases} (\bar{\vartheta}(\bar{x})\bar{r}(\bar{x}, \bar{y}))_{\bar{x}\bar{x}} - \bar{\vartheta}(\bar{y})\bar{r}_{\bar{y}\bar{y}}(\bar{x}, \bar{y}) = (\bar{a}(\bar{y}) + \tilde{\lambda})\,\bar{r}(\bar{x}, \bar{y}) \\ 2\bar{\vartheta}(\bar{y})\dfrac{\mathrm{d}}{\mathrm{d}\bar{y}}\bar{r}(\bar{y}, \bar{y}) = -\bar{\vartheta}'(\bar{y})\bar{r}(\bar{y}, \bar{y}) + \bar{a}(\bar{y}) + \tilde{\lambda} \\ \bar{r}(\bar{x}, 0) = 0 \end{cases} \quad (23)$$

for $(\bar{x}, \bar{y}) \in \Psi = \{0 \le \bar{y} \le \bar{x} \le 1\}$.

Then, by another change of variables

$$\begin{aligned} &\check{r}(\check{x}, \check{y}) = \bar{\vartheta}(\bar{x})^{3/4}\bar{\vartheta}(\bar{y})^{-1/4}\bar{r}(\bar{x}, \bar{y}), \\ &\check{\eta} = \psi(\bar{\eta}) = \sqrt{\bar{\vartheta}(0)}\int_0^{\bar{\eta}} \frac{\mathrm{d}\tau}{\sqrt{\bar{\vartheta}(\tau)}}, \quad \check{\eta} = \check{x}, \check{y}, \quad \bar{\eta} = \bar{x}, \bar{y}, \end{aligned} \quad (24)$$

the kernel PDE (23) becomes

$$\begin{cases} \check{r}_{\check{x}\check{x}}(\check{x}, \check{y}) - \check{r}_{\check{y}\check{y}}(\check{x}, \check{y}) = \dfrac{\check{\lambda}(\check{x}, \check{y})}{\bar{\vartheta}(0)}\check{r}(\check{x}, \check{y}) \\ \dfrac{\mathrm{d}\check{r}(\check{x}, \check{x})}{\mathrm{d}\check{x}} = \dfrac{1}{2\sqrt{\bar{\vartheta}(0)}}(\bar{a}(\bar{x}) + \tilde{\lambda}) \\ \check{r}(\check{x}, 0) = 0 \end{cases} \quad (25)$$

where

$$\begin{aligned} \check{\lambda}(\check{x}, \check{y}) = {} & \frac{3}{16}\Big(\frac{\bar{\vartheta}'^2(\bar{x})}{\bar{\vartheta}(\bar{x})} - \frac{\bar{\vartheta}'^2(\bar{y})}{\bar{\vartheta}(\bar{y})}\Big) - \frac{1}{4}(\bar{\vartheta}''(\bar{x}) - \bar{\vartheta}''(\bar{y})) \\ & + (\bar{a}(\bar{y}) + \tilde{\lambda}). \end{aligned}$$

Obviously, kernel PDE (25) can be taken as a specific form in class (73) from the Appendix (with $p - ((\vartheta'(0))/(4\vartheta(0))) = \infty$). Using Lemma 3, we can get the existence and uniqueness of the soultion of kernel PDE (19), which will be presented in below lemma.

Lemma 1: Suppose that $a(x) \in C^1[0, 1]$, the kernel PDE (19) with $r(x, y)$ given by (21), (24), (25) also has a unique solution which is bounded and twice continuously differentiable in $0 \le y \le x \le 1$.

Next, we will show a crucial lemma for the following proof of our main results.

Lemma 2 [31]: If $u(t) \in \mathbb{R}$ is a continuous and differentiable function. For any time $t \ge t_0 \ge 0$, one can readily show that

$$\frac{1}{2}{}_{t_0}^{C}D_t^{\alpha}u^2(t) \le u(t)\,{}_{t_0}^{C}D_t^{\alpha}u(t), \quad 0 < \alpha < 1.$$

Using the invertibility of integral transformation (68) in the Appendix, we can obtain that the integral transformation (13) is also invertible. This, together with the boundedness of kernel $r(x, y)$, implies that there exist constants M_1, M_2 to make the below inequalities hold:

$$\begin{aligned} \|\tilde{u}(x, t)\| &\le M_1\|\tilde{w}(x, t)\|, \\ \|\tilde{w}(x, 0)\| &\le M_1\|\tilde{u}(x, 0)\|, \end{aligned} \quad (26)$$

and

$$\begin{aligned} \|\tilde{u}(x, t)\|_{H^1} &\le M_2\|\tilde{w}(x, t)\|_{H^1}, \\ \|\tilde{w}(x, 0)\|_{H^1} &\le M_2\|\tilde{u}(x, 0)\|_{H^1}. \end{aligned} \quad (27)$$

Then, we will present the Mittag–Leffler stability analysis for the observer error system (12) in below the main theorem.

Theorem 1 (Mittag–Leffler stability of the observer error system for anti-collocated case): Assume that $a(x) \in C^1[0, 1]$, and the Laplace transformation of $\tilde{w}^2(x, t)$ exists, $x \in (0, 1), t \ge 0$. Let $r(x, y)$ be the solution of kernel PDE (19).

(1) For any $\tilde{u}_0(x) \in L^2(0, 1)$, the observer error system (12) with $r_1(x)$ and r_{10} provided in (19) and (20) is Mittag–Leffler stable at $\tilde{u}(x, t) = 0$ in the $L^2(0, 1)$ norm if the following constraint conditions hold:

$$\begin{cases} \vartheta'(0) < 0 \\ \tilde{\lambda} - \dfrac{\vartheta''_{\max}}{2} + \dfrac{\vartheta_{\min}}{4} > 0, \end{cases} \quad (28)$$

where $\vartheta''_{\max}$ and $\vartheta_{\min}$ denote the maximum value of the second-order derivative and the minimum value of $\vartheta(x)$, respectively.

(2) For any $\tilde{u}_0(x) \in H^1(0, 1)$, the observer error system (12) with $r_1(x)$ and r_{10} provided in (19) and (20) is Mittag–Leffler stable at $\tilde{u}(x, t) = 0$ in the $H^1(0, 1)$ norm.

Proof: Before beginning our proof, we first study the existence and uniqueness of the solution of the system (12) and a brief statement on its regularity properties. From the above analysis, we know that the integral transformation (13) is invertible. So if the solution of the target system (14) exist and is unique, the existence and uniqueness of the solution of the system (12) can also be obtained. Indeed, we can use separation of variables to prove the existence and uniqueness of the target system. Consider the target system (14), we solve the eigenvalues and the corresponding eigenfunctions of the operator

$$\mathscr{B} = \frac{\partial^2(\vartheta(x)\tilde{w}(x, t))}{\partial x^2} - \tilde{\lambda}.$$

Then, based on the conclusion in [28, 32], the solution of target system (14) exists and is unique, which can be given by $\tilde{w}(x, t) = \sum_{i=1}^{\infty}(\tilde{w}_0, X_i)E_{\alpha}(\mu_i t^{\alpha})X_i(x)$, where μ_i and X_i $(i = 0, 1, 2, \ldots)$ represent the eigenvalues and the corresponding eigenfunctions of the operator $\mathscr{B}$. Here note that $\mu_i < 0$, $i = 0, 1, 2, \ldots$ since the eigenvalues of the operator $\frac{\partial^2(\vartheta(x)\tilde{w}(x,t))}{\partial x^2}$ are negative. As illustrated in the above, the solution of the system (12) exists and is unique. The regularity of the solution of the fractional differential equations has been proved by the eigenfunction expansions in [32], which is also applicable to the solution's regularity here.

Now we continue our proof. The proof exploits the Mittag–Leffler stability of the target system (14) together with the invertibility of integral transformation (13) (i.e. inequalities (26), (27)). Specifically, for the L^1 Mittag–Leffler stability, we consider the Lyapunov functional as follows:

$$V(t,\tilde{w}(x,t)) = 1/2\int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x\,. \tag{29}$$

Then, taking the Caputo time fractional derivative of $V(t,\tilde{w}(x,t))$, we have ${}_0^CD_t^\alpha V(t,\tilde{w}(x,t)) = 1/2\int_0^1 {}_0^CD_t^\alpha \tilde{w}^2(x,t)\,\mathrm{d}x$. This, by Lemma 2, becomes

$${}_0^CD_t^\alpha V(t,\tilde{w}(x,t)) \le \int_0^1 \tilde{w}(x,t){}_0^CD_t^\alpha \tilde{w}(x,t)\,\mathrm{d}x\,. \tag{30}$$

Substituting the state equation of (14) into above inequality (30) and using the constraint conditions (28), we can get

$${}_0^CD_t^\alpha V(t,\tilde{w}(x,t)) \le -2SV(t,\tilde{w}(x,t)), \tag{31}$$

where $S = \tilde{\lambda} + (\vartheta_{\min}/4) - (\vartheta''_{\max}/2) > 0$.

Next is for the Mittag–Leffler stability statement. Based on the definition of the Caputo time fractional derivative in [30], we remark that $\tilde{w}(x,t)$ is continuously differentiable on $t \in [0,\infty)$ since $\tilde{w}(x,t)$ satisfies the state equation of the target system (14). Then we can conclude that $V(t,\tilde{w}(x,t))$ is also continuously differentiable. By the argument for the fractional-order Lyapunov direct method in [20, Proof of Theorem 5.1], we set a non-negative function $Y(t)$ for (31), which satisfies

$${}_0^CD_t^\alpha V(t,\tilde{w}(x,t)) + Y(t) = -2SV(t,\tilde{w}(x,t))\,. \tag{32}$$

The assumption of the Laplace transformation's existence of $\tilde{w}^2(\cdot,t)$ implies that the Laplace transforms of $V(t,\tilde{w}(x,t))$ and $Y(t)$ on t also exist. Thus we can take the Laplace transformation of (32), it is readily to get that

$$V(s) = \frac{s^{\alpha-1}V(0) - Y(s)}{s^\alpha + 2S}, \tag{33}$$

where $V(0) = V(0,\tilde{w}(x,0)) \ge 0$, $V(s) = \mathscr{L}\{V(t,\tilde{w}(x,t))\}$ and $Y(s) = \mathscr{L}\{Y(t)\}$ represent the Laplace transform of $V(t,w(x,t))$ and $N(t)$, respectively. $V(t,\tilde{w}(x,t))$ satisfying (29) is locally Lipschitz on $\tilde{w}(x,t)$. It obeys the fractional existence and uniqueness theorem [26]. Solving the inverse Laplace transform of (33), we have $V(t) = V(0)E_\alpha(-2St^\alpha) - Y(t) * [t^{\alpha-1}E_{\alpha,\alpha}(-2St^\alpha)]$, where $t \ge 0$, the symbol $*$ represents the convolution operator, $E_{\alpha,\beta}(t) := \sum_{k=0}^\infty \frac{t^k}{\Gamma(k\alpha+\beta)}$ and $E_\alpha(t) = E_{\alpha,1}(t)$. It is also the unique solution of (32) and can be deduce that $V(t) \le V(0)E_\alpha(-2St^\alpha)$ due to the fact $t^{\alpha-1} \ge 0$ and $E_{\alpha,\alpha}(-2\lambda t^\alpha) \ge 0$ (see [33]), $\alpha > 0, \lambda > 0$. This inequality, together with (29), induces that $||w(x,t)|| \le (2V(0)E_\alpha(-2St^\alpha))^{\frac{1}{2}}$, where $V(0) > 0$ for $\tilde{w}(x,0) \ne 0$ and $2V(0,\tilde{w}(x,0)) = 0$ if and only if $\tilde{w}(x,0) = 0$. Since $V(t,\tilde{w}(x,t))$ is locally Lipschitz on $\tilde{w}(x,t)$ ($V(0,\tilde{w}(x,0)) = 0$ if and only if $\tilde{w}(x,0) = 0$), we can conclude that $2V(0) = 2V(0,\tilde{w}(x,0))$ is Lipschitz on $\tilde{w}(x,0)$ and $2V(0,0) = 0$. Therefore, by Definition 1, the L^2 Mittag–Leffler stability of the target system (14) can be obtained. This, together with inequalities (26) and (27), can easily yield our result (1).

The H^1 Mittag–Leffler stability of the system (12) can also be proved by combining the fact the target system (14) is the H^1 Mittag–Leffler stable (consider the Lyapunov functional $G(t,\tilde{w}(x,t)) = \int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x$) and the invertibility of integral transformation (13). □

Remark 4: If $\alpha = 1$, the Mittag–Leffler stability of system (12) will turn into the exponential stability of the integer-order system. The Mittag–Leffler stability of fractional systems is analogous to the exponentially stability of integer-order systems and can infer the asymptotical stability. Since the fractional derivative has memory and hereditary effects, the fractional Lyapunov method and classical Lyapunov method is slightly different for the stabilities of fractional-order and integer-order systems. It is worth to point out that Lemma 1 in [31] makes the fractional Lyapunov method developed in [29] applicable for the stability of the fractional system. In the above proof, by Lemma 2 (Lemma 1 in [31], the Caputo time fractional derivative of $V(t,\tilde{w}(x,t))$ turns into (30) and the corresponding computation result becomes (31), which are similar to the ones in integer-order case like $\dot{V}(t,\tilde{w}(x,t)) = \int_0^1 w(x,t)w_t(x,t)\,\mathrm{d}x$ and $\dot{V}(t,\tilde{w}(x,t)) \le -RV(t,\tilde{w}(x,t))$ $(R > 0)$, respectively. The argument on the Mittag–Leffler stability in the above proof is for the fractional-order system, which replaces the discussion on the exponentially stability of the integer-order system. All of them show the presentence of the fractional derivative and the differences on the proof between fractional-order and integer-order cases.

This result can be extended to the collocated case, i.e. measurement is available at the same end (i.e. $x = 1$) as actuation, which will be concluded in below corollary. For the collocated case, the state observer of (1)–(3) is given by

$$\begin{aligned} {}_0^CD_t^\alpha \hat{u}(x,t) &= \vartheta(x)\hat{u}_{xx}(x,t) + a(x)\hat{u}(x,t) + r_1(x) \\ &\quad \times (u_x(1,t) - \hat{u}_x(1,t)),\ x \in (0,1),\ t > 0, \\ \hat{u}(x,0) &= \hat{u}_0(x),\ x \in [0,1], \\ \hat{u}_x(0,t) &= p\hat{u}(0,t),\ t > 0, \\ \hat{u}(1,t) &= U(t) + r_{10}(u_x(1,t) - \hat{u}_x(1,t)),\ t > 0, \end{aligned} \tag{34}$$

where $r_1(x)$ and r_{10} denote observer gains. Then the observer error system can be obtained

$$\begin{aligned} {}_0^CD_t^\alpha \tilde{u}(x,t) &= \vartheta(x)\tilde{u}_{xx}(x,t) + a(x)\tilde{u}(x,t) \\ &\quad - r_1(x)\tilde{u}_x(1,t), \quad x \in (0,1), \quad t > 0, \\ \tilde{u}(x,0) &= \tilde{u}_0(x), \quad x \in [0,1], \\ \tilde{u}_x(0,t) &= p\tilde{u}(0,t), \quad t > 0, \\ \tilde{u}(1,t) &= -r_{10}\tilde{u}_x(1,t), \quad t > 0. \end{aligned} \tag{35}$$

With the integral transformation

$$\tilde{u}(x,t) = \tilde{w}(x,t) + \int_x^1 r(x,y)\tilde{w}(y,t)\,\mathrm{d}y, \tag{36}$$

we can transform the system (35) into

$$\begin{aligned} {}_0^CD_t^\alpha \tilde{w}(x,t) &= \vartheta(x)\tilde{w}_{xx}(x,t) - \tilde{\lambda}\tilde{w}(x,t), \\ &\quad x \in (0,1), \quad t > 0, \\ \tilde{w}(x,0) &= \tilde{w}_0(x), \quad x \in [0,1], \\ \tilde{w}_x(0,t) &= p^s\tilde{w}(0,t), \quad t > 0, \\ \tilde{w}(1,t) &= 0, \quad t > 0, \end{aligned} \tag{37}$$

where $p^s > 0$, $\tilde{w}_0(x) = \tilde{u}_0(x) - \int_x^1 r(x,y)\tilde{w}_0(y)\,\mathrm{d}y$. Based on Definition 1, we can find that the target system (37) is the Mittag–Leffler stable under certain conditions which is similar to the counterpart of anti-collocated case (see the proof of Theorem 1 for more details).

Similar to the arguments in anti-collocated, we can get

$$r_1(x) = -\vartheta(1)r(x,1), \quad r_{10} = 0, \tag{38}$$

where $r(x, y)$ satisfies the below PDE:

$$\begin{cases} \vartheta(x) r_{xx}(x, y) - (\vartheta(y) r(x, y))_{yy} = -\left(a(x) + \tilde{\lambda}\right) r(x, y) \\ 2\vartheta(x) \dfrac{\mathrm{d}}{\mathrm{d}x} r(x, x) = a(x) + \tilde{\lambda} - \vartheta'(x) r(x, x) \\ r_x(0, y) = p r(0, y) \\ r(0, 0) = p^s - p \end{cases} \tag{39}$$

for $(x, y) \in \Phi = \{0 \le y \le x \le 1\}$.

This equation also can be converted into the one in class of (73) in the Appendix with a change of variables

$$\begin{aligned} \check{r}(\check{x}, \check{y}) &= \vartheta(x)^{-1/4} \vartheta(y)^{3/4} r(x, y), \\ \check{x} &= \psi(y) = \sqrt{\vartheta(0)} \int_0^y \frac{\mathrm{d}\tau}{\sqrt{\vartheta(\tau)}}, \\ \check{y} &= \psi(x) = \sqrt{\vartheta(0)} \int_0^x \frac{\mathrm{d}\tau}{\sqrt{\vartheta(\tau)}}. \end{aligned} \tag{40}$$

It follows that the existence and uniqueness of the solution of kernel PDE (39) and the invertibility of the transformation (36) can also be obtained.

Corollary 1 (Mittag–Leffler stability of the observer error system for collocated case): Suppose that $a(x) \in C^1[0, 1]$, there exists Laplace transformation of $\tilde{w}^2(x, t)$ for $x \in (0, 1), t \ge 0$, and $r(x, y)$ be the solution of (39).

(i) For any initial value $\tilde{u}_0(x) \in L^2(0, 1)$, the observer error system (35) with observer gains $r_1(x)$ and r_{10} described by (38) and (39) is L^2 Mittag–Leffler stable under the following constraint conditions:

$$\begin{cases} p^s \vartheta(0) - \dfrac{\vartheta'(0)}{2} > 0 \\ \tilde{\lambda} - \dfrac{\vartheta''_{\max}}{2} + \dfrac{\vartheta_{\min}}{4} > 0. \end{cases} \tag{41}$$

(ii) For any initial value $\tilde{u}_0(x) \in H^1(0, 1)$, the observer error system (35) with observer gains $r_1(x)$ and r_{10} governed by (38) and (39) is H^1 Mittag–Leffler stable.

4 Output feedback boundary control

Theorem 2 (Mittag–Leffler stability of the closed-loop system for anti-collocated case): Let $k(1, y)$ be derived from (72), (73), observer gains $r_1(x)$, r_{10} be given by (19), (20) in the Appendix, and $\tilde{\lambda} \ge \lambda > 0$. Assume that the Laplace transformations of $\hat{w}^2(x, t)$ and $\tilde{w}^2(x, t)$ exist for $x \in (0, 1), t \ge 0$, and $a(x) \in C^1[0, 1]$.

(a) For any initial $u_0, \hat{u}_0 \in L^2(0, 1)$, the system (1)–(3) with the Dirichlet boundary feedback controller

$$U(t) = -\int_0^1 k(1, y) \hat{u}(y, t)\,\mathrm{d}y \tag{42}$$

and the observer (9)–(11), (42) is L^2 Mittag–Leffler stable at $u(x, t) = 0, \hat{u}(x, t) = 0$ if the following constraint conditions hold:

$$\begin{cases} \vartheta'(0) < 0 \\ \vartheta_{\min} - \dfrac{1}{2} > 0 \\ \vartheta_{\min} - \dfrac{1}{2} + 4\lambda - 2\vartheta''_{\max} > 0. \end{cases} \tag{43}$$

(b) For any initial $u_0, \hat{u}_0 \in H^1(0, 1)$, the system (1)–(3) with the Dirichlet boundary feedback controller (42) and the observer (9)–(11), (42) is H^1 Mittag–Leffler stable at $u(x, t) = 0, \hat{u}(x, t) = 0$.

Proof:

(a) This proof is a generalisation of the FRD system with constant diffusivity in [18, Theorem 4]. The transformations (13) and

$$\hat{w}(x, t) = \hat{u}(x, t) + \int_0^x k(x, y) \hat{u}(y, t)\,\mathrm{d}y \tag{44}$$

can convert the system including the observer (9)–(11), (42) and the observer error system (12) into the integrated system of $(\hat{w}(x, t), \tilde{w}(x, t))$ as follows:

$$\begin{cases} {}_0^C D_t^\alpha \hat{w}(x, t) = \vartheta(x) \hat{w}_{xx}(x, t) - \lambda \hat{w}(x, t) \\ \qquad + \{r_1(x) + \int_0^x k(x, y) r_1(y)\,\mathrm{d}y\} \tilde{w}(0, t) \\ \hat{w}(x, 0) = \hat{w}_0(x) \\ \hat{w}_x(0, t) = p^s \hat{w}(0, t) + (p + r_{10}) \tilde{w}(0, t) \\ \hat{w}(1, t) = 0 \\ {}_0^C D_t^\alpha \tilde{w}(x, t) = \vartheta(x) \tilde{w}_{xx}(x, t) - \tilde{\lambda} \tilde{w}(x, t) \\ \tilde{w}(x, 0) = \tilde{w}_0(x) \\ \tilde{w}_x(0, t) = 0 \\ \tilde{w}(1, t) = 0, \end{cases} \tag{45}$$

where $p, p^s > 0$, $\hat{w}_0(x)$ and $\tilde{w}_0(x)$ are initial values that satisfy (44) and (13), respectively. We first remark that the existence and uniqueness of the solution of the closed-loop system (1)–(3), controller (42) and observer (9)–(11), (42) hold since the integral transformations of (13) and (44) are invertible and the existence and uniqueness of the solutions of the system of $\hat{w}(x, t)$ and the system of $\tilde{w}(x, t)$ in (45) both hold, and the regularity of its solution can also been proved (the proofs are similar to the ones in Theorem 1). It is noticeable that the integrated system of $(\hat{w}(x, t), \tilde{w}(x, t))$ is driven by $\tilde{w}(0, t)$, and the system of $\tilde{w}(x, t)$ and the system of $\hat{w}(x, t)$ without the term of $w(0, t)$ are L^2 and H^1 Mittag–Leffler stabilities. Consider the following Lyapunov functional:

$$V(t, (\hat{w}(x, t), \tilde{w}(x, t))) = \frac{Q}{2} \int_0^1 \tilde{w}^2(x, t)\,\mathrm{d}x + \frac{1}{2} \int_0^1 \hat{w}^2(x, t)\,\mathrm{d}x, \tag{46}$$

where Q is a positive constant that needs to be designed later.

Using Lemma 2 and the method of integration of parts for the Caputo time fractional derivative of (46) of α order along with the trajectory of (45), we can get

$$\begin{aligned} &{}_0^C D_t^\alpha V(t, (\hat{w}(x, t), \tilde{w}(x, t))) \\ &= \frac{Q}{2} \int_0^1 {}_0^C D_t^\alpha \tilde{w}^2(x, t)\,\mathrm{d}x + \frac{1}{2} \int_0^1 {}_0^C D_t^\alpha \hat{w}^2(x, t)\,\mathrm{d}x \\ &\le Q \int_0^1 \tilde{w}(x, t) {}_0^C D_t^\alpha \tilde{w}(x, t)\,\mathrm{d}x + \int_0^1 \hat{w}(x, t) {}_0^C D_t^\alpha \hat{w}(x, t)\,\mathrm{d}x \\ &= Q \int_0^1 \tilde{w}(x, t) \big(\vartheta(x) \tilde{w}_{xx}(x, t) - \lambda \tilde{w}(x, t)\big)\,\mathrm{d}x + \int_0^1 \hat{w}(x, t) \\ &\quad \times \Big(\vartheta(x) \hat{w}_{xx}(x, t) - \lambda \hat{w}(x, t) + \Big\{r_1(x) + \int_0^x k(x, y) r_1(y)\,\mathrm{d}y\Big\} \\ &\quad \times \tilde{w}(0, t)\Big)\,\mathrm{d}x \\ &= Q\Big(-\int_0^1 \vartheta'(x) \tilde{w}(x, t) \tilde{w}_x(x, t)\,\mathrm{d}x - \int_0^1 \vartheta(x) \tilde{w}_x^2(x, t)\,\mathrm{d}x\Big) \\ &\quad - Q\tilde{\lambda} \int_0^1 \tilde{w}^2(x, t)\,\mathrm{d}x - \vartheta(0) \hat{w}(0, t) \big(p^s \hat{w}(0, t) + (p + r_{10}) \end{aligned} \tag{47a}$$

$$\times \tilde{w}(0,t)) - \int_0^1 \vartheta'(x)\hat{w}(x,t)\hat{w}_x(x,t)\,\mathrm{d}x - \int_0^1 \vartheta(x)\hat{w}_x^2(x,t)\,\mathrm{d}x$$
$$-\lambda\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x + \int_0^1 \hat{w}(x,t)\Big\{r_1(x) + \int_0^x k(x,y)r_1(y)\,\mathrm{d}y\Big\}$$
$$\times \tilde{w}(0,t)\,\mathrm{d}x. \tag{47b}$$

Since $\tilde{w}(1,t) = 0, \hat{w}(1,t) = 0$, by integration of parts, we can easily get

$$\begin{aligned}
&\int_0^1 \vartheta'(x)\tilde{w}(x,t)\tilde{w}_x(x,t)\,\mathrm{d}x \\
&\quad = -\frac{1}{2}\Big(\vartheta'(0)\tilde{w}^2(0,t) + \int_0^1 \vartheta''(x)\tilde{w}^2(x,t)\,\mathrm{d}x\Big) \\
&\int_0^1 \vartheta'(x)\hat{w}(x,t)\hat{w}_x(x,t)\,\mathrm{d}x \\
&\quad = -\frac{1}{2}\Big(\vartheta'(0)\hat{w}^2(0,t) + \int_0^1 \vartheta''(x)\hat{w}^2(x,t)\,\mathrm{d}x\Big).
\end{aligned} \tag{48}$$

Substituting (48) into (47), we have

$$\begin{aligned}
&{}_0^C D_t^\alpha V(t,(\hat{w}(x,t),\tilde{w}(x,t))) \\
&\le Q\Big(\frac{1}{2}\vartheta'(0)\tilde{w}^2(0,t) + \frac{1}{2}\int_0^1 \vartheta''(x)\tilde{w}^2(x,t)\,\mathrm{d}x - \int_0^1 \vartheta(x) \\
&\quad\times \tilde{w}_x^2(x,t)\,\mathrm{d}x\Big) - Q\tilde{\lambda}\int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x - p^s\vartheta(0)\hat{w}^2(0,t) \\
&\quad -(p + r_{10})\vartheta(0)\hat{w}(0,t)\tilde{w}(0,t) + \frac{1}{2}\vartheta'(0)\hat{w}^2(0,t) + \frac{1}{2}\int_0^1 \vartheta''(x) \\
&\quad\times \hat{w}^2(x,t)\,\mathrm{d}x - \int_0^1 \vartheta(x)\hat{w}_x^2(x,t)\,\mathrm{d}x - \lambda\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x \\
&\quad + \int_0^1 \hat{w}(x,t)\{r_1(x) + \int_0^x k(x,y)r_1(y)\,\mathrm{d}y\}\tilde{w}(0,t)\,\mathrm{d}x.
\end{aligned} \tag{49}$$

Based on the first constraint condition in (43), i.e. $\vartheta'(0) < 0$, we can further get

$$\begin{aligned}
&{}_0^C D_t^\alpha V(t,(\hat{w}(x,t),\tilde{w}(x,t))) \\
&\le Q\Big(\frac{1}{2}\int_0^1 \vartheta''(x)\tilde{w}^2(x,t)\,\mathrm{d}x - \int_0^1 \vartheta(x)\tilde{w}_x^2(x,t)\,\mathrm{d}x\Big) - Q\tilde{\lambda} \\
&\quad\times \int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x - (p + r_{10})\vartheta(0)\hat{w}(0,t)\tilde{w}(0,t) + \frac{1}{2}\int_0^1 \vartheta''(x)
\end{aligned} \tag{50a}$$

$$\begin{aligned}
&\times \hat{w}^2(x,t)\,\mathrm{d}x - \int_0^1 \vartheta(x)\hat{w}_x^2(x,t)\,\mathrm{d}x - \lambda\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x \\
&+ \int_0^1 \hat{w}(x,t)\{r_1(x) + \int_0^x k(x,y)r_1(y)\,\mathrm{d}y\}\tilde{w}(0,t)\,\mathrm{d}x.
\end{aligned} \tag{50b}$$

Aside by Lemma 2.1 in [4], it is easy to obtain

$$\begin{aligned}
\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x &\le 4\int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x, \\
\int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x &\le 4\int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x,
\end{aligned} \tag{51}$$

since $\hat{w}(1,t) = 0$ and $\tilde{w}(1,t) = 0$. Similar to the arguments in [4, pp. 56–57], the following estimates holds by Poincaré and Young inequalities

$$\begin{aligned}
&-(p + r_{10})\vartheta(0)\hat{w}(0,t)\tilde{w}(0,t) \\
&\quad \le \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x + (p + r_{10})^2\vartheta^2(0)\int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x,
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&\int_0^1 \hat{w}(x,t)\{r_1(x) + \int_0^x k(x,y)r_1(y)\,\mathrm{d}y\}\tilde{w}(0,t)\,\mathrm{d}x \\
&\quad \le R\tilde{w}(0,t)\int_0^1 \hat{w}(x,t)\,\mathrm{d}x \\
&\quad \le \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x + 4R^2\int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x,
\end{aligned} \tag{53}$$

where $R = \max_{0<x<1}\{r_1(x) + \int_0^x k(x,y)r_1(y)\,\mathrm{d}y\}$.

With these estimates (51), (52), the equality (50) turns into

$$\begin{aligned}
&{}_0^C D_t^\alpha V(t,(\hat{w}(x,t),\tilde{w}(x,t))) \\
&\le \frac{Q}{2}\vartheta''_{\max}\int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x - Q\vartheta_{\min}\int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x - Q\tilde{\lambda} \\
&\quad\times \int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x + \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x + \vartheta^2(0)(p + r_{10})^2 \\
&\quad\times \int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x + \frac{1}{2}\vartheta''_{\max}\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x - \vartheta_{\min} \\
&\quad\times \int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x - \lambda\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x \\
&\quad + \int_0^1 \hat{w}(x,t)\{r_1(x) + \int_0^x k(x,y)r_1(y)\,\mathrm{d}y\}\tilde{w}(0,t)\,\mathrm{d}x
\end{aligned} \tag{54}$$

Then, this, together with the equality (53) further implies that

$$\begin{aligned}
&{}_0^C D_t^\alpha V(t,(\hat{w}(x,t),\tilde{w}(x,t))) \\
&\le \frac{Q}{2}\vartheta''_{\max}\int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x - Q\vartheta_{\min}\int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x - Q\tilde{\lambda} \\
&\quad\times \int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x + \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x + \vartheta^2(0)(p + r_{10})^2 \\
&\quad\times \int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x + \frac{1}{2}\vartheta''_{\max}\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x - \vartheta_{\min} \\
&\quad\times \int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x - \lambda\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x + \frac{1}{4}\int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x \\
&\quad + 4R^2\int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x
\end{aligned} \tag{55a}$$

$$\begin{aligned}
&= -\Big(Q\tilde{\lambda} - \frac{Q}{2}\vartheta''_{\max}\Big)\int_0^1 \tilde{w}^2(x,t)\,\mathrm{d}x - (Q\vartheta_{\min} - (p + r_{10})^2 \\
&\quad\times \vartheta^2(0) - 4R^2)\int_0^1 \tilde{w}_x^2(x,t)\,\mathrm{d}x - \Big(\vartheta_{\min} - \frac{1}{2}\Big)\int_0^1 \hat{w}_x^2(x,t)\,\mathrm{d}x \\
&\quad -\Big(\lambda - \frac{1}{2}\vartheta''_{\max}\Big)\int_0^1 \hat{w}^2(x,t)\,\mathrm{d}x.
\end{aligned} \tag{55b}$$

Consider the second constraint condition in (43), the second equality in (51) and the assumption $\tilde{\lambda} > \lambda$ for the above estimate (55), then we can get

$$
\begin{aligned}
&{}_0^C D_t^\alpha V(t,(\hat{w}(x,t),\tilde{w}(x,t)))\\
&\le -\frac{1}{4}(Q\vartheta_{\min}-(p+r_{10})^2\vartheta^2(0)-4R^2+4Q\lambda-2Q\vartheta''_{\max})\\
&\quad\times\int_0^1\tilde{w}^2(x,t)\,\mathrm{d}x-\frac{1}{4}\Big(\vartheta_{\min}-\frac{1}{2}+4\lambda-2\vartheta''_{\max}\Big)\\
&\quad\times\int_0^1\hat{w}^2(x,t)\,\mathrm{d}x.
\end{aligned}\tag{56}
$$

Next, to make the right part of (56) to meet a multiple of the right part of (46), Q is set to satisfy the following equality:

$$
\begin{aligned}
&Q\vartheta_{\min}-(p+r_{10})^2\vartheta^2(0)-4R^2+4Q\lambda-2Q\vartheta''_{\max}\\
&\quad=Q\Big(\vartheta_{\min}-\frac{1}{2}+4\lambda-2\vartheta''_{\max}\Big),
\end{aligned}\tag{57}
$$

which implies that $Q=2(p+r_{10})^2\vartheta^2(0)+8R^2$.

This, together with (46) and (56), induces that

$$
{}_0^C D_t^\alpha V(t,(\hat{w}(x,t),\tilde{w}(x,t)))\le-\frac{1}{2}MV(t,\hat{w}(x,t),\tilde{w}(x,t)),\tag{58}
$$

where $M=\vartheta_{\min}-\frac{1}{2}+4\lambda-2\vartheta''_{\max}>0$, which is based on the last constraint condition in (43).

It is noticeable that $\hat{w}(\cdot,t)$ and $\tilde{w}(\cdot,t)$ are continuously differentiable on $t\in[0,\infty)$ since they both satisfy the state equations of the integrated target system (45). Thus, $V(t,(\hat{w}(x,t),\tilde{w}(x,t)))$ is also continuously differentiable on $t\in[0,\infty)$ and locally Lipschitz on $(\hat{w}(x,t),\tilde{w}(x,t))$. By the same deduction as the argument in Theorem 1, we can obtain L^2 Mittag–Leffler stability of the integrated target system (45). Combining it with invertibility of the integral transformation (13) and (44), it follows that the system of $(\hat{u}(x,t),\tilde{u}(x,t))$ is L^2 Mittag–Leffler stable at $\hat{u}(x,t)=0$, $\tilde{u}(x,t)=0$, which induces L^2 Mittag–Leffler stability of the closed-loop system of $(u(x,t),\hat{u}(x,t))$.

(b) It follows that the system of $\tilde{w}(x,t)$ and the system of $\hat{w}(x,t)$ without the term of $w(0,t)$ are H^1 Mittag–Leffler stabilities by considering Lyapunov functional $G(t,\tilde{w}(x,t))=\int_0^1\tilde{w}_x^2(x,t)\,\mathrm{d}x$ and $K(t,\hat{w}(x,t))=\int_0^1\hat{w}_x^2(x,t)\,\mathrm{d}x+p^s\hat{w}^2(0,t)$, respectively. The integrated system of $(\hat{w}(x,t),\tilde{w}(x,t))$ is driven by $\tilde{w}(0,t)$, and the relationship between them is cascade. Thus, the integrated system of $(\hat{w}(x,t),\tilde{w}(x,t))$ is Mittag–Leffler stable in the $H^1(0,1)$ norm. Furthermore, we can obtain the system of $(\hat{u}(x,t),\tilde{u}(x,t))$ is the H^1 Mittag–Leffler stable, which is related to the system of $(\hat{w}(x,t),\tilde{w}(x,t))$ and the invertibility of integral transformation (13) and (44). So we have proved H^1 Mittag–Leffler stability of the system of $(u(x,t),\hat{u}(x,t))$. □

Remark 5: The constraint conditions (43) include the constraint conditions (28) and the constraint conditions (74) (see the Appendix) due to the fact that the conditions $\vartheta'(0)<0$ and $\vartheta_{\min}-\frac{1}{2}+4\lambda-2\vartheta''_{\max}>0$ in (43) can lead to

$$
p^s\vartheta(0)-\frac{\vartheta'(0)}{2}>0,\quad\lambda-\frac{\vartheta''_{\max}}{2}+\frac{\vartheta_{\min}}{4}>0
$$

and

$$
\tilde{\lambda}-\frac{\vartheta''_{\max}}{2}+\frac{\vartheta_{\min}}{4}>0
$$

(since $\vartheta(x)>0,\forall x\in[0,1]$ and $\tilde{\lambda}>\lambda$).

Remark 6: If $\alpha=1$, the closed-loop system (1)–(3), controller (42) and observer (9)–(11), (42) reduces the integer-order one, i.e. the corresponding Mittag–Leffler stability replaces by the corresponding exponentially stability. The difference in proof of Theorem 2 between the fractional-order and the integer-order cases is same as the one in Theorem 1, we refer to Remark 4 for more details. For the stability of this closed-loop system, it can also solve the solution of this system's equation without applying the fractional Lyapunov method in [20] (see [34]. The method and proof in this paper seems to be easier than solving its solution directly with the help of the fractional Lyapunov method and Lemma 1 in [31].

For better of understanding the output feedback control design for an anti-collocated case, we are ready to give a specific example to illustrate it here.

Example 1: Consider $a(x)\equiv a=\text{const}$ in the system (1)–(3). We know that this open-loop system with $U(t)=0$ is unstable if a is a large enough positive value (see the arguments on stability of fractional-order systems provided in [19, 27]). In this case, it needs to design a controller to make this system Mittag–Leffler stable. Here, we first design an observer, then combine it with the Dirichlet boundary feedback controller (69) developed in the Appendix to form an output feedback controller as follows:

$$
U(t)=-\int_0^1k(1,y)\hat{u}(y,t)\,\mathrm{d}y,\quad t>0.\tag{59}
$$

The observer copied the system (1)–(3), (59) (with $a(x)\equiv a=const$) is given by

$$
\begin{aligned}
&{}_0^C D_t^\alpha\hat{u}(x,t)=\vartheta(x)\hat{u}_{xx}(x,t)+a\hat{u}(x,t)+r_1(x)(u(0,t)\\
&\qquad-\hat{u}(0,t)),\quad x\in(0,1),\quad t>0,\\
&\hat{u}(x,0)=u_0(x),\quad x\in[0,1],\\
&\hat{u}_x(0,t)=pu(0,t)+r_{10}(u(0,t)-\hat{u}(0,t)),\quad t>0,\\
&\hat{u}(1,t)=-\int_0^1k(1,y)\hat{u}(y,t)\,\mathrm{d}y,\quad t>0.
\end{aligned}\tag{60}
$$

Then we can obtain the observer error system (12) $(a(x)=a)$. Using the integral transformation (13) to map this system into the corresponding target system (14). Based on the analysis on PDE of kernel $r(x,y)$ in Section 3, we can get the kernel PDEs (19) and (25) (with $a(x)=a$). There are two solutions of $(3\bar{\vartheta}'^2(\bar{\eta})/16\bar{\vartheta}(\bar{\eta}))-((\bar{\vartheta}''(\bar{\eta}))/4)=\text{const}$, $\eta=x,y$ (see [21, Section 3] for more details). Here, we take one solution $\bar{\vartheta}(\bar{\eta})=\bar{\vartheta}_0(1+\rho_0(\bar{\eta}-x_0)^2)^2$, $\eta=\bar{x},\bar{y}$, where $\bar{\vartheta}_0>0$, ρ_0 and x_0 are arbitrary constants. According to it and (22), we have

$$
\vartheta(\eta)=\bar{\vartheta}_0(1+\rho_0(1-\eta-x_0)^2)^2,\quad\eta=x,y.\tag{61}
$$

As is illustrated above, we find that $\check{\lambda}(\check{x},\check{y})=a+\tilde{\lambda}$. Thus, with the help of the results in [4, Page 35], and the change of variables (21), (22), (24), we can get $r(x,y)$ as follows:

$$
r(x,y)=\vartheta(x)^{1/4}\vartheta(y)^{-3/4}\sqrt{\vartheta(1)}b\check{y}\frac{I_1(\sqrt{b(\check{x}^2-\check{y}^2)})}{\sqrt{b(\check{x}^2-\check{y}^2)}},\tag{62}
$$

where

$$
b=(a+\tilde{\lambda})/\bar{\vartheta}(0),\quad\check{x}=\frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(1-y-x_0))+\operatorname{atan}(\sqrt{\rho_0}x_0)),
$$

and

$$
\check{y}=\frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(1-x-x_0))+\operatorname{atan}(\sqrt{\rho_0}x_0)).
$$

Moreover, we have

$$
\begin{aligned}
r_1(x) = & -\frac{1}{4}\vartheta(x)^{1/4}\vartheta(0)^{-3/4}\vartheta'(0)\sqrt{\vartheta(1)}b\check{y}\frac{I_1(\sqrt{b(\check{x}^2-\check{y}^2)})}{\sqrt{b(\check{x}^2-\check{y}^2)}} \\
& +\vartheta(x)^{1/4}\vartheta(0)^{-1/4}\vartheta(1)\frac{b\check{x}\check{y}}{\check{x}^2-\check{y}^2}I_2(\sqrt{b(\check{x}^2-\check{y}^2)})
\end{aligned} \tag{63}
$$

and

$$
\begin{aligned}
r_{10} = & -\vartheta(0)^{-1/2}\frac{a+\tilde{\lambda}}{2\sqrt{\vartheta(1)}}\frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(1-x_0)) \\
& +\operatorname{atan}(\sqrt{\rho_0}x_0)),
\end{aligned} \tag{64}
$$

where

$$
\check{x} = \frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(1-x_0)) + \operatorname{atan}(\sqrt{\rho_0}x_0)).
$$

According to Theorem 1, the observer error system (12) ($a(x) = a$) with the observer gains (63), (64) is Mittag–Leffler stable at $\tilde{u}(x,t) = 0$ in the $L^2(0,1)$ and $H^1(0,1)$ norms.
Next, we want to solve the kernel $k(1,y)$, which is given by (71)–(73) in the Appendix. Here assume that $p^s = p$. Combining this assumption with $a(x) = a = const$, then the kernel PDEs of $k(x,y)$ and $\check{k}(\check{x},\check{y})$ becomes (71) and (73) (with $a(x) = a$, $k(0,0) = \check{k}(0,0) = 0$) in the Appendix. The results in [22, Section VIII.B] together with (72) and (61), induce that the control kernel $k(1,y)$ as follows:

$$
\begin{aligned}
k(1,y) = & \frac{(1+\rho_0x_0^2)^{1/2}}{\sqrt{\bar{\vartheta}_0}(1+\rho_0(1-y-x_0)^2)^{3/2}}\Big[\sqrt{\vartheta(0)}\check{\lambda}\check{x} \\
& \times\frac{I_1(\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)})}{\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)}} - \frac{\check{p}\sqrt{\vartheta(0)}\check{\lambda}}{\sqrt{\check{\lambda}+\check{p}^2}} \\
& \times\int_0^{\check{x}-\check{y}} e^{-\check{p}\tau/2}I_0\Big(\sqrt{\check{\lambda}(\check{x}+\check{y})(\check{x}-\check{y}-\tau)}\Big) \\
& \times\sinh\Big(\frac{\sqrt{\check{\lambda}+\check{p}^2}}{2}\tau\Big)\mathrm{d}\tau\Big],
\end{aligned} \tag{65}
$$

where

$$
\check{x} = \frac{1+\rho_0(1-x_0)^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(1-x_0)) + \operatorname{atan}(\sqrt{\rho_0}x_0)),
$$

and

$$
\check{y} = \frac{1+\rho_0(1-x_0)^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(1-x_0)) - \operatorname{atan}(\sqrt{\rho_0}(1-y-x_0))), \quad \check{\lambda} = \frac{a+\lambda}{\vartheta(0)}.
$$

Obviously, consider Theorem 2, it is easy to obtain the L^2 and H^1 Mittag–Leffler stabilities of the system (1)–(3) ($a(x) = a$) with the observer (60) and the controller (59).

Remark 7: The result of output feedback boundary stabilisation can also be generalised to the collocated case with certain constraints conditions for system parameters. Similar to the anti-collocated case, the output feedback controller is also the combination of the observer and the backstepping controller. We are ready to present a brief example as below.

Example 2: Consider the plant (1)–(3) with $a(x) = a$. Suppose the measurement and actuation are both available at $x = 1$. The corresponding observer is the one (34) (with $a(x) = a$). Here assume that $p^s = p$. Based on the argument on the observer design of the collocated case in Section 3, we can get the PDE of $r(x,y)$ in (36) and the observer gains $r_1(x) = -\vartheta(1)r(x,1)$, $r_{10} = 0$. This PDE is same as (39) (a instead of $a(x)$, $r(0,0) = 0$). By the change of variables (40), the PDE of $r(x,y)$ becomes the one (73) ($\check{r}(\check{x},\check{y})$ instead of $\check{k}(\check{x},\check{y})$, $\check{a}(\check{x},\check{y}) = a+\lambda$, $\check{k}(0,0) = 0$). Similar to Example 1, here $\vartheta(x) = \bar{\vartheta}_0(1+\rho_0(x-x_0)^2)^2$. According to [22, Section VIII-B], we can get $\check{r}(\check{x},\check{y})$ and $r(x,y)$ (see (40)). This, together with the observer gains, implies that

$$
\begin{aligned}
r_1(x) = & -\frac{\vartheta(1)(1+\rho_0(x-x_0)^2)^{1/2}}{\sqrt{\bar{\vartheta}_0}(1+\rho_0(1-x_0)^2)^{3/2}}\Big[\sqrt{\vartheta(0)}\bar{\lambda}\check{x} \\
& \times\frac{I_1(\sqrt{\bar{\lambda}(\check{x}^2-\check{y}^2)})}{\sqrt{\bar{\lambda}(\check{x}^2-\check{y}^2)}} - \frac{\bar{p}\sqrt{\vartheta(0)}\bar{\lambda}}{\sqrt{\bar{\lambda}+\bar{p}^2}}\int_0^{\check{x}-\check{y}} e^{-\bar{p}\tau/2} \\
& \times I_0\Big(\sqrt{\bar{\lambda}(\check{x}+\check{y})(\check{x}-\check{y}-\tau)}\Big)\sinh\Big(\frac{\sqrt{\bar{\lambda}+\bar{p}^2}}{2}\tau\Big)\mathrm{d}\tau\Big]
\end{aligned}
$$

where

$$
\check{x} = \frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(1-x_0) + \operatorname{atan}(\sqrt{\rho_0}x_0))),
$$

$$
\begin{aligned}
\check{y} = & \frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(x-x_0) + \operatorname{atan}(\sqrt{\rho_0}x_0))), \quad \bar{\lambda} = \frac{a+\tilde{\lambda}}{\vartheta(0)}, \\
\bar{p} = & p - \frac{\vartheta'(0)}{4\vartheta(0)}.
\end{aligned}
$$

Combining the observer (34) (with $a(x) = a$) and the controller (69) in the Appendix, it infers the output controller $u(1,t) = U(t) = -\int_0^1 k(1,y)\hat{u}(y,t)\,\mathrm{d}y$ with

$$
\begin{aligned}
k(1,y) = & \frac{(1+\rho_0(1-x_0)^2)^{1/2}}{\sqrt{\bar{\vartheta}_0}(1+\rho_0(y-x_0)^2)^{3/2}}\Big[\sqrt{\vartheta(0)}\check{\lambda}\check{x}\frac{I_1(\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)})}{\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)}} \\
& -\frac{\bar{p}\sqrt{\vartheta(0)}\check{\lambda}}{\sqrt{\check{\lambda}+\bar{p}^2}}\int_0^{\check{x}-\check{y}} e^{-\bar{p}\tau/2}I_0\Big(\sqrt{\check{\lambda}(\check{x}+\check{y})(\check{x}-\check{y}-\tau)}\Big) \\
& \times\sinh\Big(\frac{\sqrt{\check{\lambda}+\bar{p}^2}}{2}\tau\Big)\mathrm{d}\tau\Big],
\end{aligned}
$$

where

$$
\check{x} = \frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(1-x_0)) + \operatorname{atan}(\sqrt{\rho_0}x_0)),
$$

$$
\check{y} = \frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\operatorname{atan}(\sqrt{\rho_0}(y-x_0)) + \operatorname{atan}(\sqrt{\rho_0}x_0)), \quad \check{\lambda} = \frac{a+\lambda}{\vartheta(0)}.
$$

5 Simulation study

In this part, we use a same prototypical case of the FRD system in [17] for numerical simulation except the non-constant diffusivity. By the Caputo derivative numerical algorithm in [35, 36] together with the finite difference method and the approach of differential evaluated by difference, we carry out simulation computations for the anti-collocated case of this plant. The system parameters have the values as $\alpha = 0.7$, $a(x) = 10$, $p^s = p = 1$, $\lambda = 10$, $\tilde{\lambda} = 20$, $u_0(x) = 10x(1-x)$, $\hat{u}_0(x) = 7x(1-x)$. The non-constant diffusivity is chosen as $\vartheta(x) = (1+(1-x)^2)^2$, i.e. $\bar{\vartheta}_0 = 1$, $\rho_0 = 1$, $x_0 = 0$. These parameters and the diffusivity are set to match with the constraint conditions (28), (43), and the assumption $\tilde{\lambda} \geq \lambda$. According to these given parameter values, we will present our numerical simulation results.

In Fig. 1, we show the shape of spatially-varying diffusivity, kernel gain (65) for control law, and observer gain (63) for the closed-loop system. Note that here we removed the end value of observer gain (i.e. $r_1(0)$) and take $r_1(0) = 0$ in our simulation due to the fact that the denominator of $r_1(0)$ is zero. With this observer gain, the observer error system can approach to L^2 and H^1 Mittag–Leffler stablities, as shown in Fig. 2. Based on the simulation

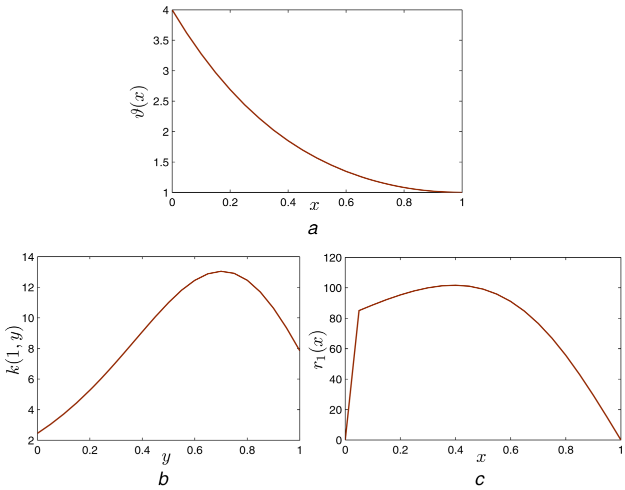

Fig. 1 *Diffusivity* $\vartheta(x) = (1 + (1 - x)^2)^2$, *control kernel* $k(1, y)$ *and observer gain* $r_1(x)$

(a) Diffusivity, ***(b)*** Control kernel, ***(c)*** Observer gain

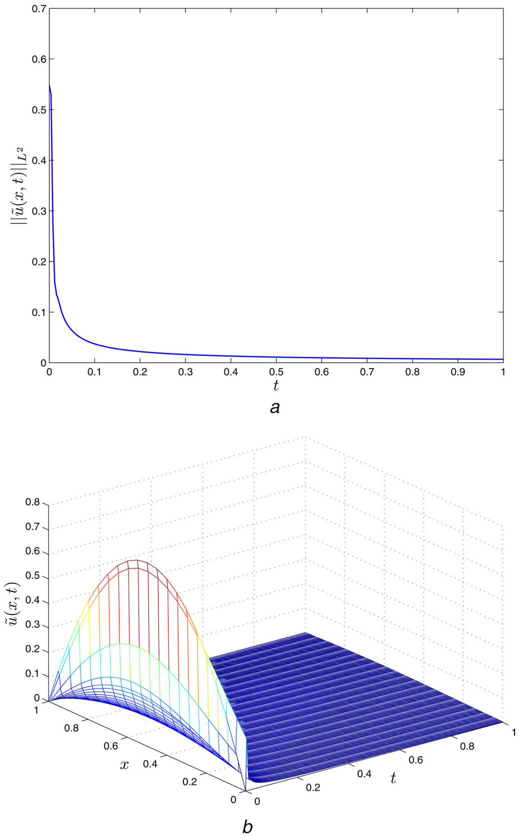

Fig. 2 *Simulation results of observer error system (12) with observer gains (20)*

(a) Evolution of state L^2 norm, ***(b)*** Evolution of state

results in Fig. 3, we can see that the closed-loop system (1) and (2) with the observer (9)–(11), (42) and the controller (3), (42) is the Mittag–Leffler stable at $u(x, t) = 0$ in the L^2 and H^1 norms, i.e. state norm and state both converge to zero. In addition, the control effort is also presented here. In Fig. 4, we present the output in cases of normal and with white noise (20 dB of signal-to-noise ratio). Fig. 5 illustrates that this closed-loop system with measurement noise still robustly converges to zero, which implies the robustness of the proposed observer on measure noise in some extent.

6 Conclusions

In this contribution, a combination of observer design and backstepping feedback controller has been successfully employed

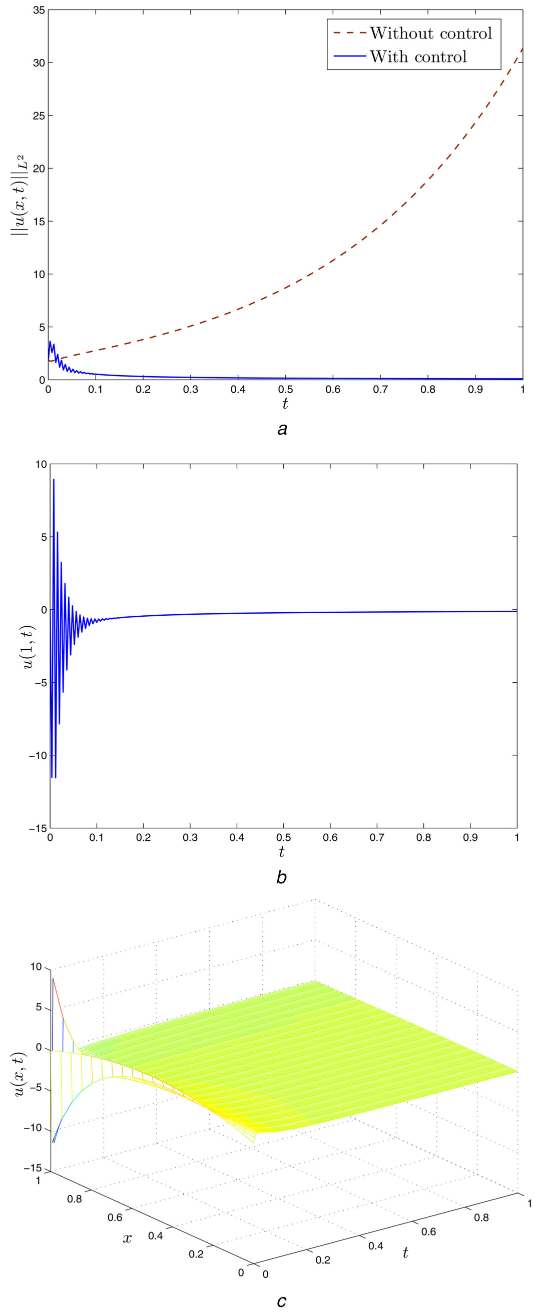

Fig. 3 *Simulation of open-loop and closed-loop responses of system (1), (2)*

(a) Comparison of state L^2 norms of open-loop (with $u(1, t) = 0$) and closed-loop (under controller (3), (42)), ***(b)*** Control effort of stabilising controller (3), (42), ***(c)*** State evolution of the closed-loop system with observer (9)–(11), (42) and controller (3), (42)

to form the output feedback controller, which is used to stabilise the FRD system with non-constant diffusivity. Then sufficient conditions for the output feedback stabilisation problem has been obtained based on the fractional Lyapunov stability (Mittag–Leffler stability) theory and the backstepping method. To some extent, these conditions are relatively conservative since the diffusivity of the system is a general form of spatially-varying function rather than a specific one. Namely, for a specific diffusivity, the estimates can be improved or more relaxed.

Future research work could consider the extension of the backstepping transformation method to the boundary stabilisation problem of coupled FRD systems with different constant diffusivity or different non-constant (spatially-varying) diffusivity, whose dynamics is governed by fractional PDEs

$$ {}_0D_t^{\alpha} U(x, t) = \Theta U_{xx}(x, t) + \Psi U(x, t), \qquad (66) $$

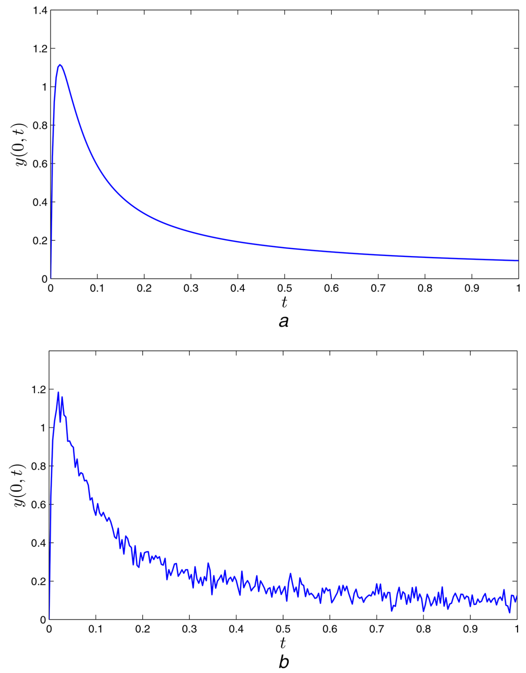

Fig. 4 *Output* $y(0, t)$
(a) Normal case, ***(b)*** With noise case

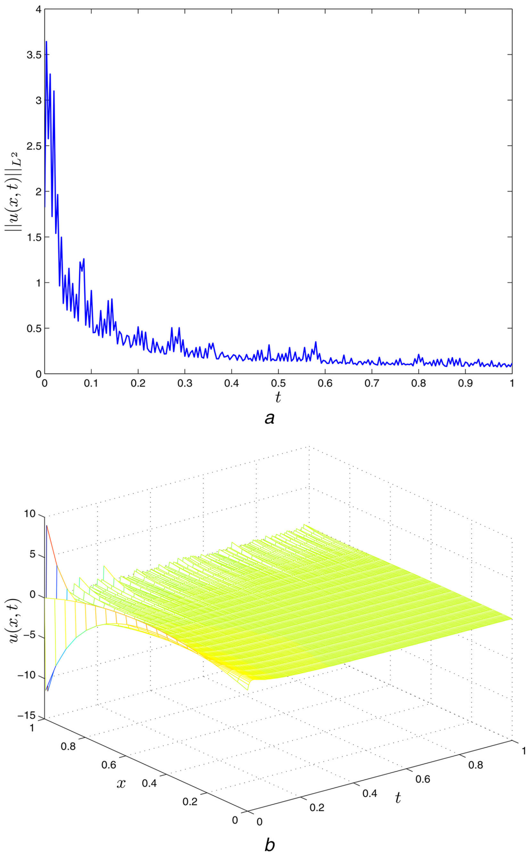

Fig. 5 *Simulation results of closed-loop system with noise*
(a) Evolution of state L^2 norm, ***(b)*** Evolution of state

or

$$ {}_0D_t^{\alpha}U(x,t) = \Theta(x)U_{xx}(x,t) + \Psi U(x,t), \tag{67} $$

with the boundary conditions $U_x(0, t) = \mathbf{0}$ and $U(1, t) = U_c(t)$, where the state vector $U(x, t)$ and the input vector $U_c(1, t)$ are $1 \times n$ matrices, diffusion coefficients Θ, $\Theta(x)$ and reaction coefficient Ψ denote $n \times n$ matrices. Moreover, the backstepping-based output feedback boundary control problem for these coupled systems, would be a challenging and interesting research topic.

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9 Appendix

9.1 Outline of backstepping boundary feedback control design

By designing a backstepping-like boundary feedback controller, the stabilisation problem of FRD system (1)–(3) has been solved with the help of the below integral transformation:

$$w(x,t) = u(x,t) + \int_0^x k(x,y)u(y,t)\,\mathrm{d}y\,. \tag{68}$$

Using this integral transformation, we convert the FRD system (1) and (2) with the Dirichlet boundary feedback controller

$$u(1,t) = -\int_0^1 k(1,y)u(y,t)\,\mathrm{d}y \tag{69}$$

into a target system

$$\begin{aligned} {}_0^C D^{\alpha} w(x,t) &= \vartheta(x) w_{xx} - \lambda w(x,t), x \in (0,1), t > 0, \\ w_0(x) &= w(x,0), x \in [0,1], \\ w_x(0,t) &= p^s w(0,t), t > 0 \\ w(1,t) &= 0, t > 0, \end{aligned} \tag{70}$$

where $\lambda > 0$, $p^s > 0$, and $w_0(x) = u_0(x) + \int_0^x k(x,y)u_0(y)\,\mathrm{d}y$. Here the above target system (70) can also be Mittag–Leffler stable if its parameters meet certain conditions. Our objective is establish the kernel $k(x,y)$ of the integral transformation (68). Similar to the arguments in [17, Section 3.1], we can show that kernel $k(x,y)$ satisfies the below PDE:

$$\begin{cases} \vartheta(x) k_{xx}(x,y) - (\vartheta(y)k(x,y))_{yy} = (a(y) + \lambda)k(x,y) \\ k_y(x,0) = (p - \vartheta'(0)/\vartheta(0))k(x,0) \\ 2\vartheta(x)\dfrac{\mathrm{d}}{\mathrm{d}x} = -\vartheta'(x)k(x,x) + a(x) + \lambda \\ k(0,0) = p^s - p \end{cases} \tag{71}$$

for $(x,y) \in \Pi = 0 \le y \le x \le 1$.

Based on the analysis in [21, Section 2], we also use a change of variables

$$\begin{aligned} &\breve{k}(\breve{x},\breve{y}) = \vartheta^{-1/4}(x)\vartheta^{3/4}(y)k(x,y), \\ &\breve{x} = \psi(x), \breve{y} = \psi(y), \psi(\eta) = \sqrt{\vartheta(0)}\int_0^{\eta} \frac{\mathrm{d}\tau}{\sqrt{\vartheta(\tau)}} \end{aligned} \tag{72}$$

to convert above kernel PDE (71) into a canonical one as follows:

$$\begin{cases} \breve{k}_{\breve{x}\breve{x}}(\breve{x},\breve{y}) - \breve{k}_{\breve{y}\breve{y}}(\breve{x},\breve{y}) = \dfrac{\breve{a}(\breve{x},\breve{y})}{\vartheta(0)}\breve{k}(\breve{x},\breve{y}) \\ \breve{k}_{\breve{y}}(\breve{x},0) = \left(p - \dfrac{\vartheta'(0)}{4\vartheta(0)}\right)\breve{k}(\breve{x},0) \\ \dfrac{\mathrm{d}}{\mathrm{d}\breve{x}}\breve{k}(\breve{x},\breve{x}) = \dfrac{a(\psi^{-1}(\breve{x})) + \lambda}{2\sqrt{\vartheta(0)}} \\ \breve{k}(0,0) = \vartheta^{1/2}(0)(p^s - p), \end{cases} \tag{73}$$

where

$$\breve{a}(\breve{x},\breve{y}) = \frac{3}{16}\left(\frac{\bar{\vartheta}'^2(\bar{x})}{\bar{\vartheta}(\bar{x})} - \frac{\bar{\vartheta}'^2(\bar{y})}{\bar{\vartheta}(\bar{y})}\right) - \frac{1}{4}(\bar{\vartheta}''(\bar{x}) - \bar{\vartheta}''(\bar{y})) + (a(y) + \lambda).$$

and $\psi^{-1}(\cdot)$ represents the inverse function of $\psi(\cdot)$. In this PDE, the parameter $\breve{a}(\breve{x},\breve{y})$ depends only on $\breve{y}$ if $\vartheta(x)$ is equal to a constant. To prove the existence and uniqueness of the solution of kernel PDE (73), the proof given in [17, 37] can apply here by using the bound of this parameter. Based on the above analysis, we will give the below main results.

Lemma 3: Suppose that $a(y) \in C^1[0,1]$, the kernel PDE (71) with $k(x,y)$ given by (72), (73) also has a unique solution which is bounded and twice continuously differentiable in $0 \le y \le x \le 1$.

Moreover, based on Lemma 2.4 in [37], the integral transformation (68) is invertible. Thus, it is easy to prove the controller (69) can stabilise the system (1) and (2).

Theorem 3: Assume that $a(x) \in C^1[0,1]$ and the Laplace transform of $w^2(x,t)$ exists for $(x,t) \in (0,1) \times [0,\infty)$.

(i) For any initial value $u_0 \in L^2(0,1)$, the system (1), (2) under the controller (69) (Dirichlet boundary feedback controller) with the gain kernel $k(x,y)$ described by (72), (73) is Mittag–Leffler stable at $u(x,t) = 0$ in the $L^2(0,1)$ norm if the following constraint condition holds:

$$\begin{cases} p^s\vartheta(0) - \dfrac{\vartheta'(0)}{2} > 0 \\ \lambda - \dfrac{\vartheta''_{\max}}{2} + \dfrac{\vartheta_{\min}}{4} > 0. \end{cases} \tag{74}$$

(ii) For any initial value $u_0 \in H^1(0,1)$, the system (1), (2) under the controller (69) with the gain kernel $k(x,y)$ described by (72), (73) is Mittag–Leffler stable at $u(x,t) = 0$ in the $H^1(0,1)$ norm.

Proof: The proof is very similar to the one of Theorem 1, so we omit it here. □