

# FSS 2015 Debate Lecture

## Better Understanding Complexities via Fractional Calculus: from Extreme Events to Taoism

YangQuan Chen, Ph.D., Director,  
**MESA**(Mechatronics, Embedded Systems and Automation)**LAB**  
ME/EECS/SNRI/HSRI/CITRIS, School of Engineering,  
University of California, Merced

**E:** yqchen@ieee.org; *or*, yangquan.chen@ucmerced.edu  
**T:** (209)228-4672; **O:** SE1-254; **Lab:** Castle #22 (**T:** 228-4398)

October 1st, 2015. Wednesday 16:00-17:00PM  
Technical University of Cluj-Napoca, Romania

# Acknowledgements

- **Thank you, organizers of FSS15!**
- **Technical University of Cluj-Napoca, Romania**



Skip Ad in fractional minutes

## University of California, Merced



- The Research University of the Central Valley
- Centrally Located
  - Sacramento – 2 hrs
  - San Fran. – 2 hrs
  - Yosemite – 1.5 hrs
  - LA – 4 hrs
- Surrounded by farmlands and sparsely populated areas



# UC Merced



- Established 2005
- 1<sup>st</sup> research university in 21<sup>st</sup> century in USA.
- 6,685 Undergraduates
- 448 Grads (most Ph.Ds)
  
- Strong Undergraduate Research Presence (HSI, MSI)



## The MESA Lab

- **Mechatronics, Embedded Systems, Automation Lab**
- <http://mechatronics.ucmerced.edu>
- Lab Director: Prof. YangQuan Chen
  - Lab Manager: Brandon Stark
  - 5 Ph.D. Students
  - 1 MSc Student
  - 40+ Undergrads
  - 9 Visiting Ph.D. Students
  - 6 Visiting Professors/Scholars
- Unmanned Aerial Systems
- Cyber-Physical Systems
- Renewable Energy Systems
- Mechatronic Systems
- **Applied Fractional Calculus**





Data Drone Valley

Bruce J. West has been a research scientist and teacher for forty years. He is one of a handful of scientists in the world that understands complexity and who can explain its implications for modern society in everyday language.

In *Complex Worlds: Uncertain, Unequal and Unfair* he uses his understanding of complex networks to explain why the future cannot be made certain, why the same people are always at the center of controversy, and why only a select few get ahead. The emerging properties of complexity so prevalent in society stand in sharp contrast to how the greatest thinkers of the past and present believe the world ought to be.

West explores the question: Is the dissonance between what is true and what we believe ought to be true really that great? The answer is a resounding yes and he explains not only how but why.



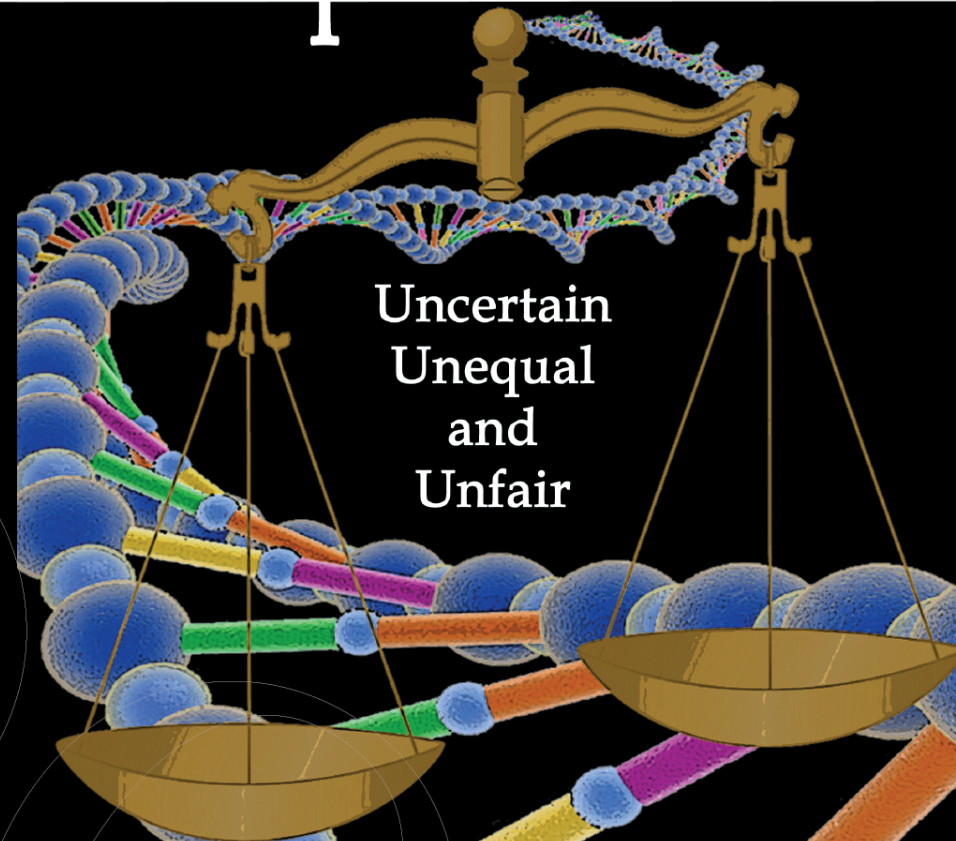
*Dr. Bruce J. West, Ph.D., FAPS, FARL has had three careers. The first was as an Industry Researcher in a small not-for-profit The La Jolla Institute, 1971-1989. The second was as a Full Professor and Physics Department Chair at the University of North Texas, 1989-1999. The third is as Chief Scientist of Mathematics for the U.S. Army Research Office, 1999-present.*



# Complex Worlds

Bruce J. West

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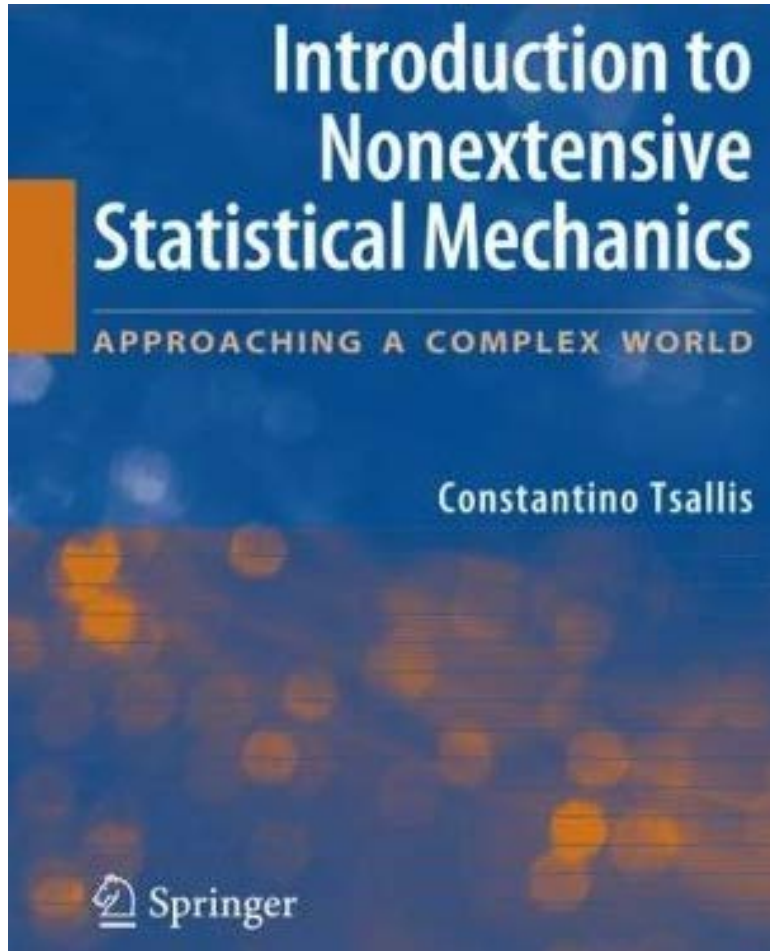
Uncertain  
Unequal  
and  
Unfair

Uncertain  
Unequal  
Unfair

# Bruce J. West



# Some other books, almost touched FC



"A deep and insightful book that is a joy to read. There are new ideas on every page, and none of them is obvious!"  
 —DANIEL GILBERT, Professor of Psychology at Harvard University and author of *Stumbling on Happiness*



Everything  
 Is Obvious\*

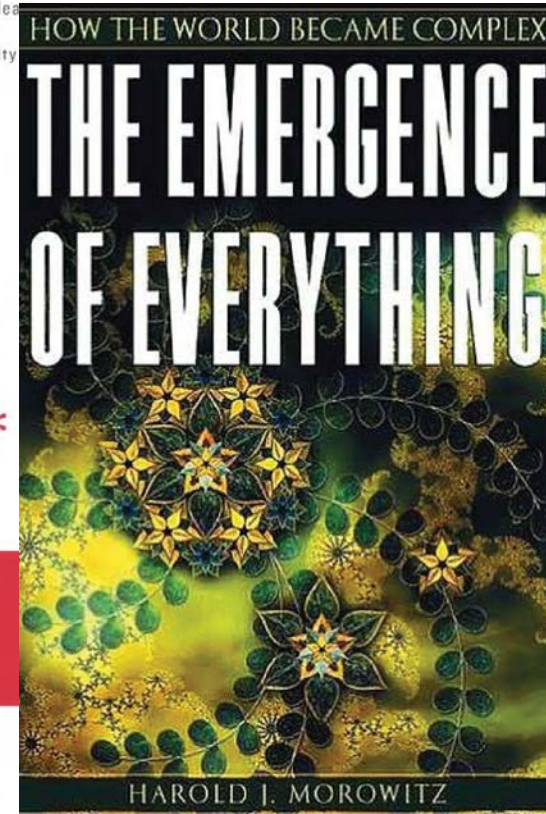
*How Common Sense Fails Us*

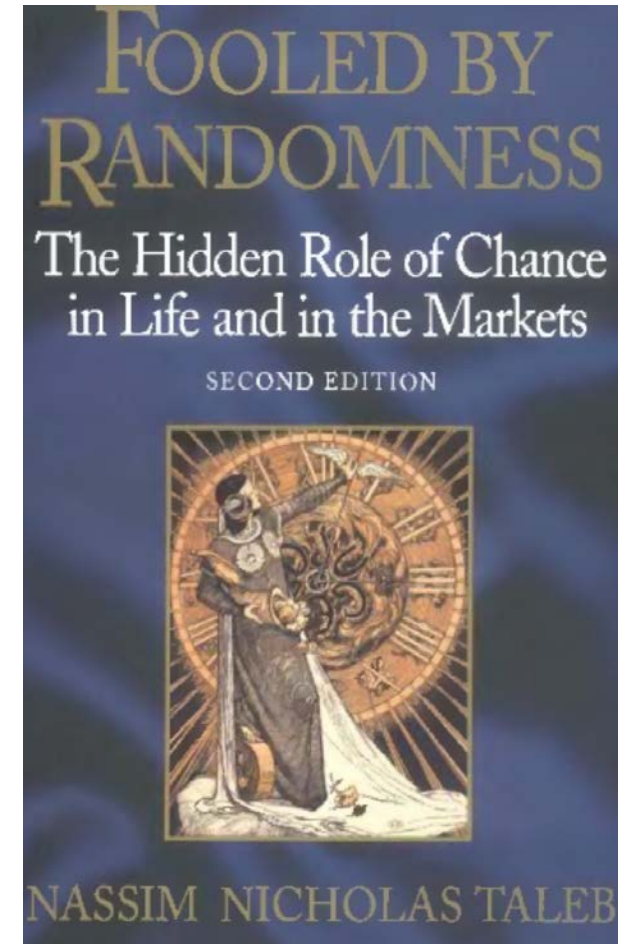
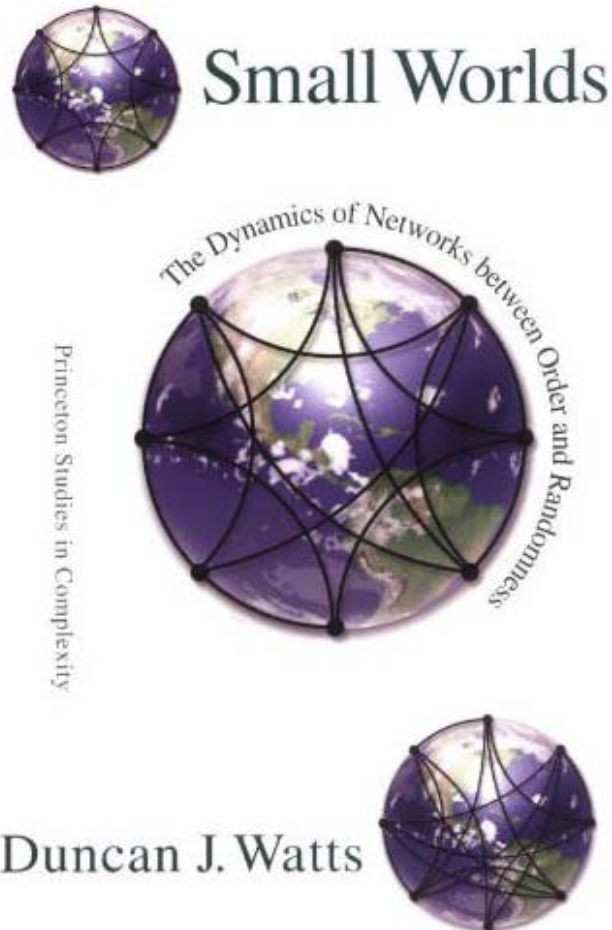
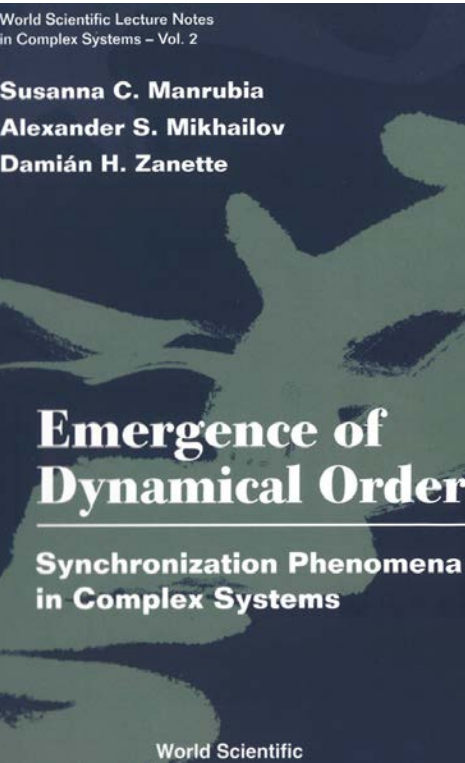
DUNCAN J. WATTS

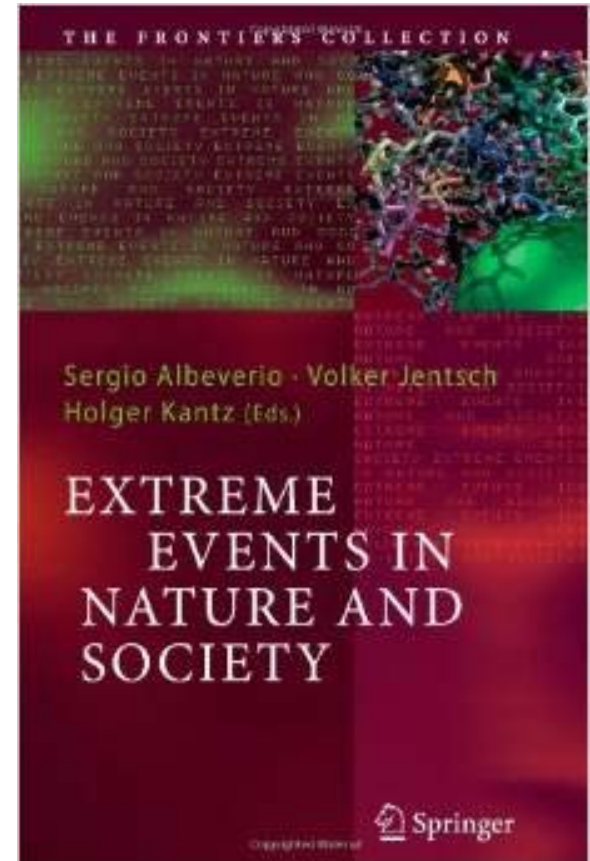
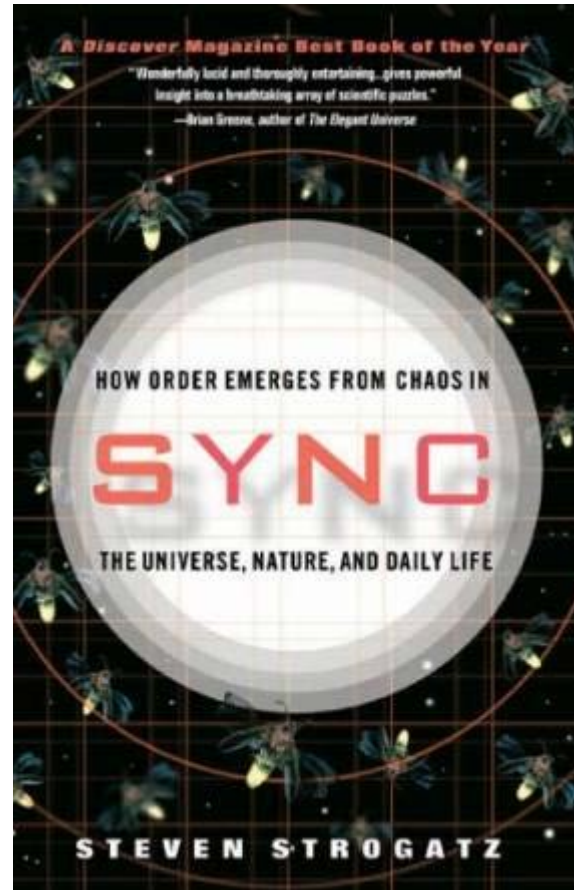
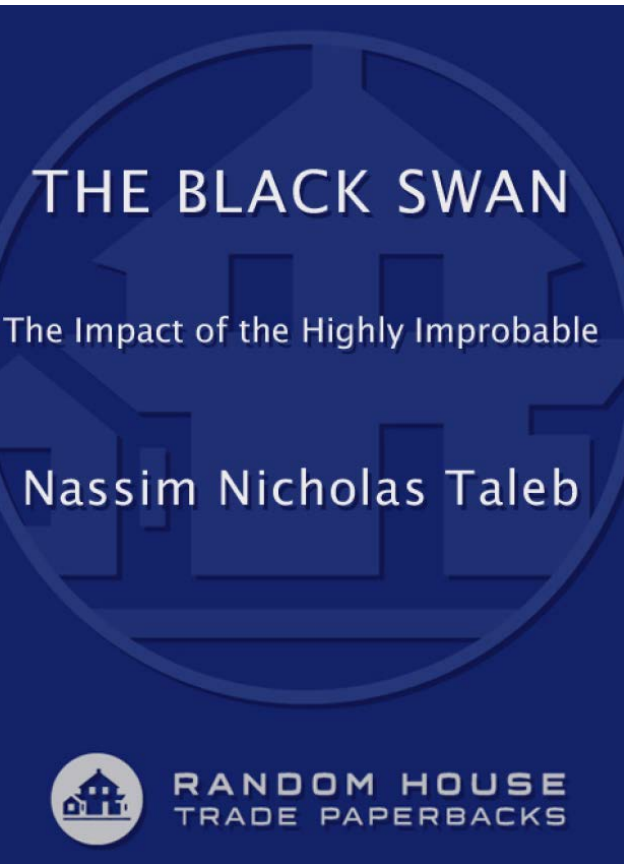


\*Once You Know the Answer

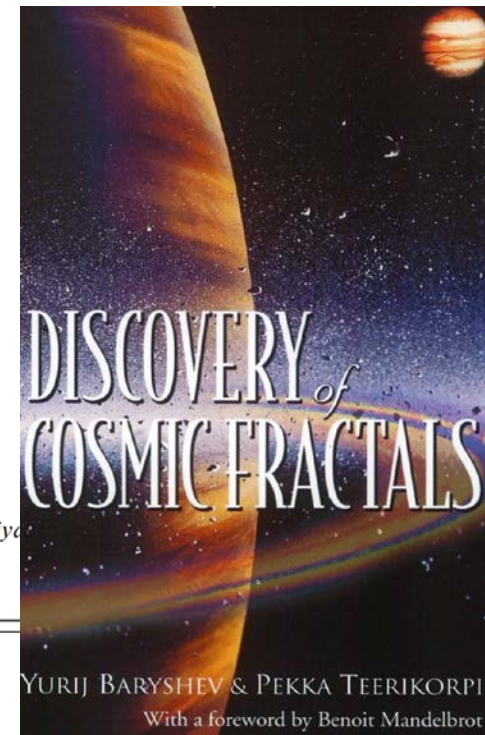
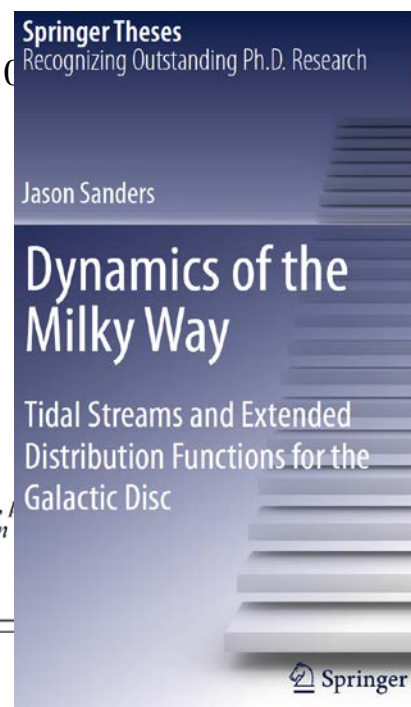
FSS15 Debate Lecture







ISSN 1062-8738, *Bulletin of the Russian Academy of Sciences. Physics*, 2015, Vol. 79, No. 5, pp. 646–649.  
Original Russian Text © V.V. Uchaikin, R.T. Sibatov, A.N. Byzykchi, 2015, published in



## Interpreting Data on Solar Cosmic Ray Fluxes via the Fractional Derivative Method

V. V. Uchaikin, R. T. Sibatov, and A. N. Byzykchi

*Ulyanovsk State University, Ulyanovsk, 432017 Russia*

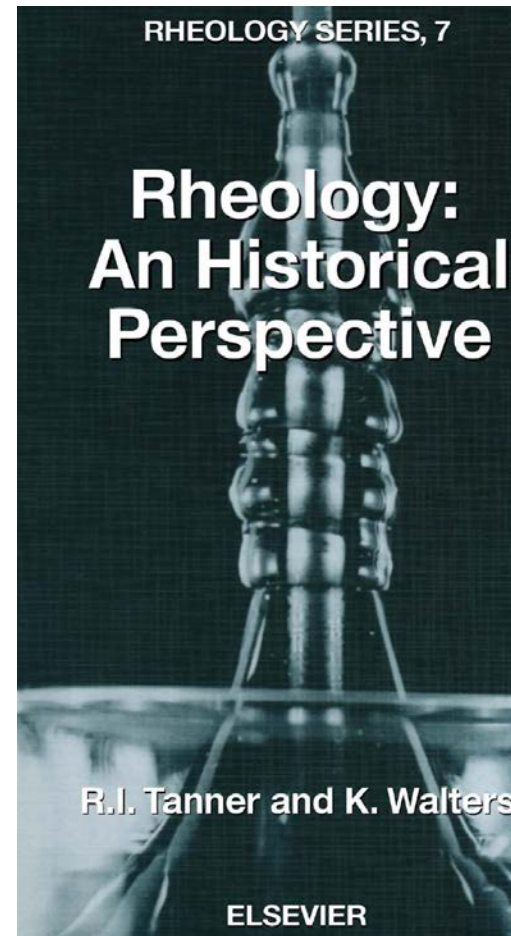
*e-mail: vuchaikin@gmail.com; ren\_sib@bk.ru; azy.baza@gmail.com*

**Abstract**—Solar cosmic ray propagation through the interplanetary magnetic field is considered as a random process of particles traveling along magnetic lines at a finite velocity of free motion and with a free path distributed according to an inverse power law. The propagator is presented as a sum of direct (nonscattered) flux (singular part of solution) and multiple scattered flux (regular part). In the long-time asymptotic, the regular part is described by an equation with a fractional-order derivative. Using analytical expressions for the propagator, we numerically calculate fluxes of energetic particles accelerated by shock waves generated by solar flares. The presented model is in better agreement with *Ulysses* and *Voyager 2* data than the Perri–Zimbardo model and may therefore be recommended for use in interpreting the results of further experiments.

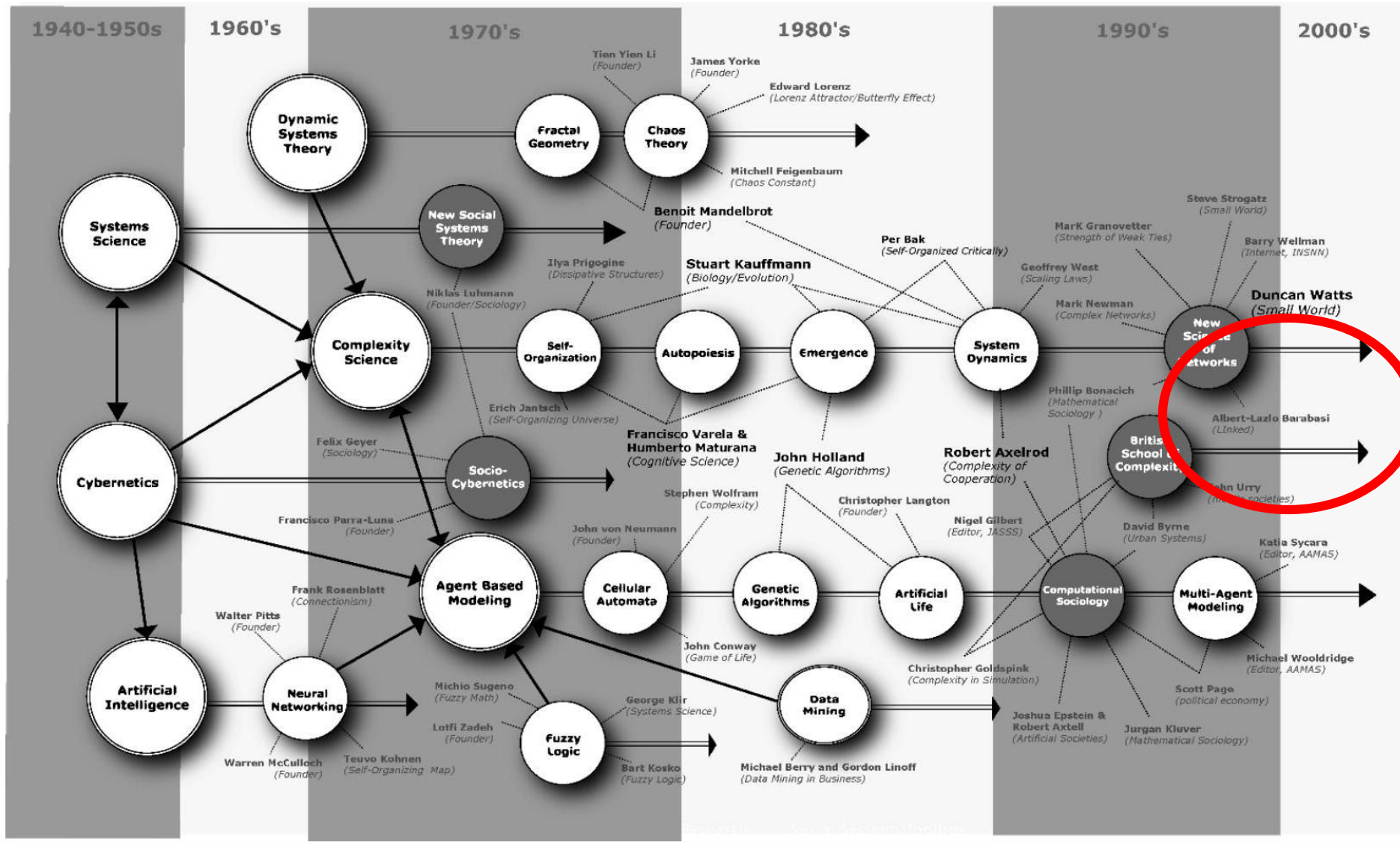
**DOI:** 10.3103/S1062873815050408

"We may express our concepts in Newtonian terms if we find this convenient but, if we do so, we must realize that we have made a translation into a language which is foreign to the system which we are studying."  
(1950)

G W Scott Blair



### Map 1: The New Science of Complexity



From Brian Castellani & Frederic William Hafferty (Eds.). "Sociology & Complexity Science -A New Field of Inquiry." Springer 2009.

My submission:

**Fractional dynamics  
point of view  
of  
complex systems for complexity  
characterization and regulation**

# Summary of the key messages

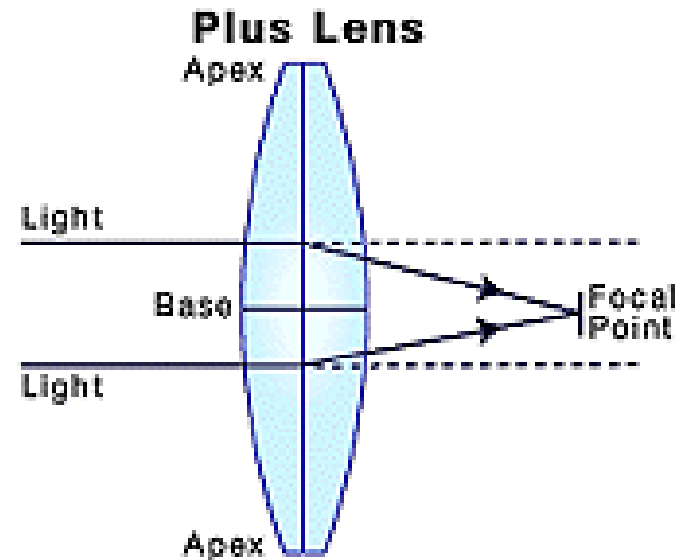
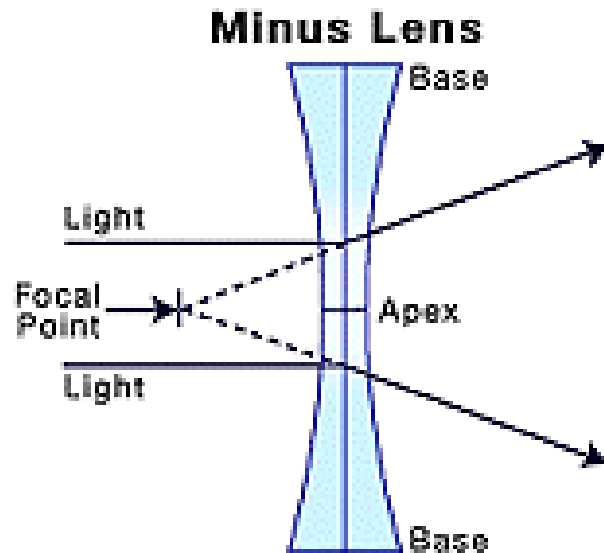
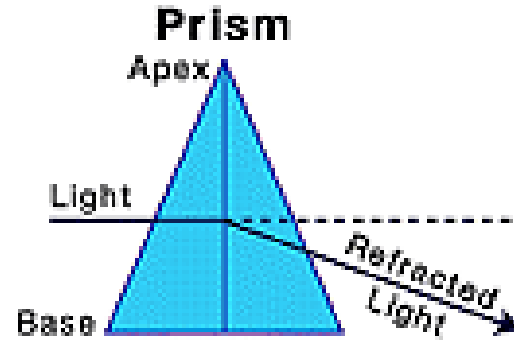
- Real worlds are complex
- We human beings perceive the complexity via our **mental prism** (lens)
- The prism (lens) is “IPL” (inverse power law) if we use integer order calculus, “Mittag-Leffer” if we use fractional calculus
- Tail matters (玄之又玄)

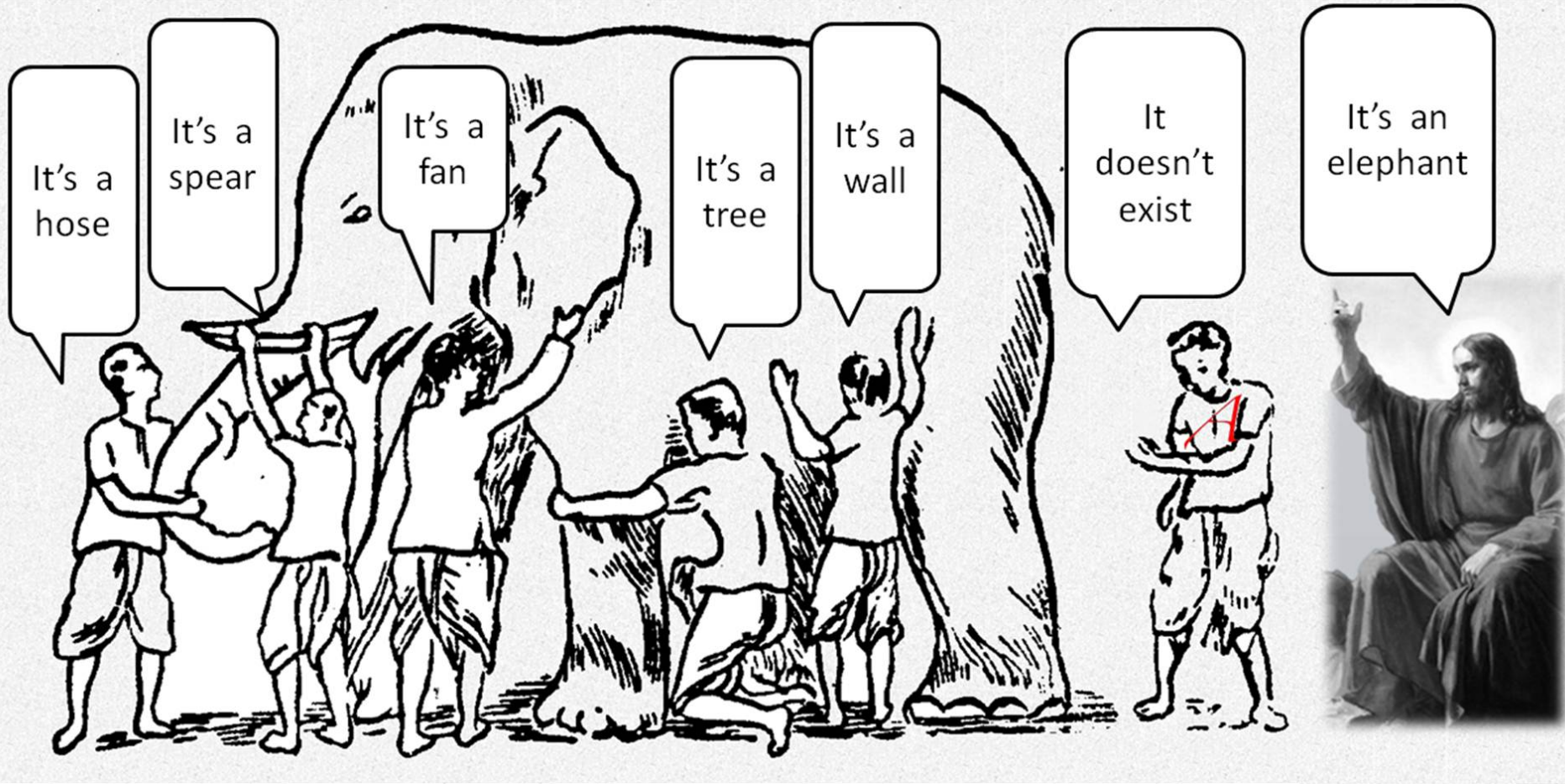




[http://en.wikipedia.org/wiki/The\\_Dark\\_Side\\_of\\_the\\_Moon#mediaviewer/File:Dsotm30.jpg](http://en.wikipedia.org/wiki/The_Dark_Side_of_the_Moon#mediaviewer/File:Dsotm30.jpg)

10/1/2015

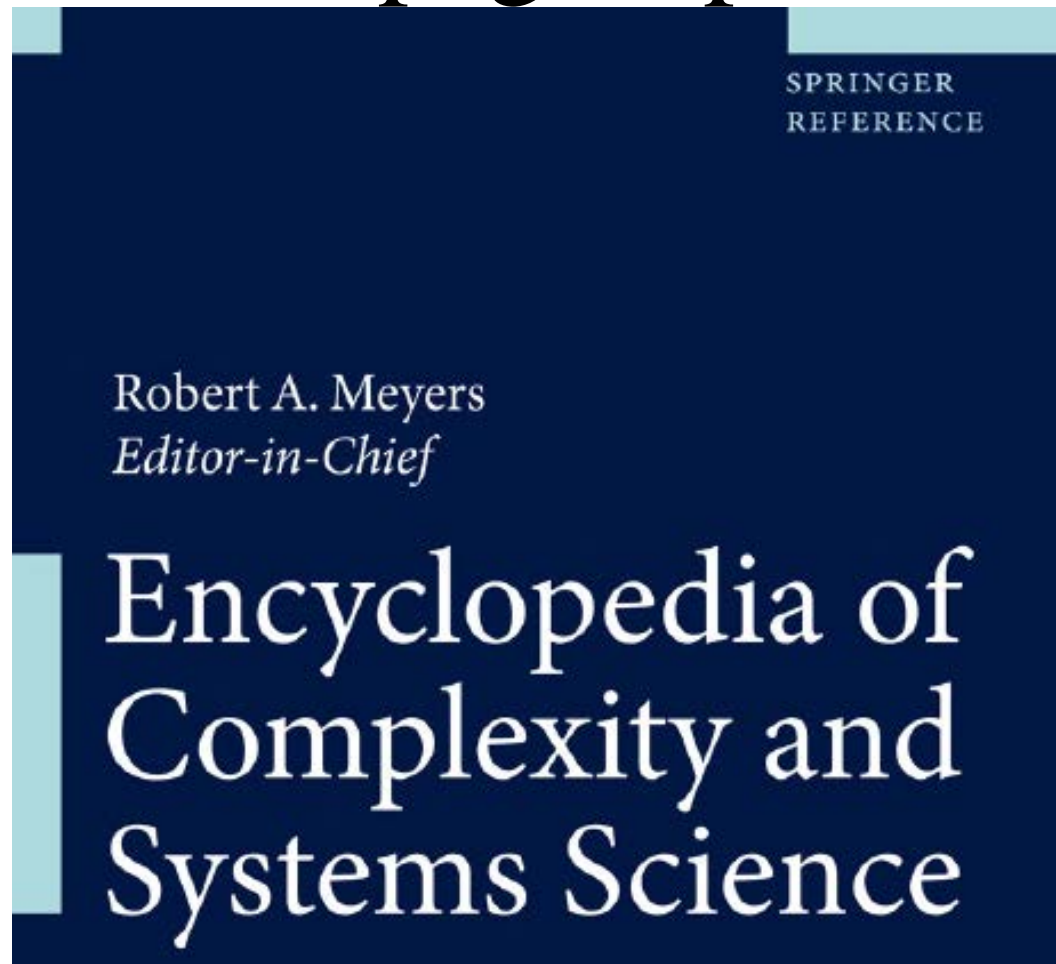




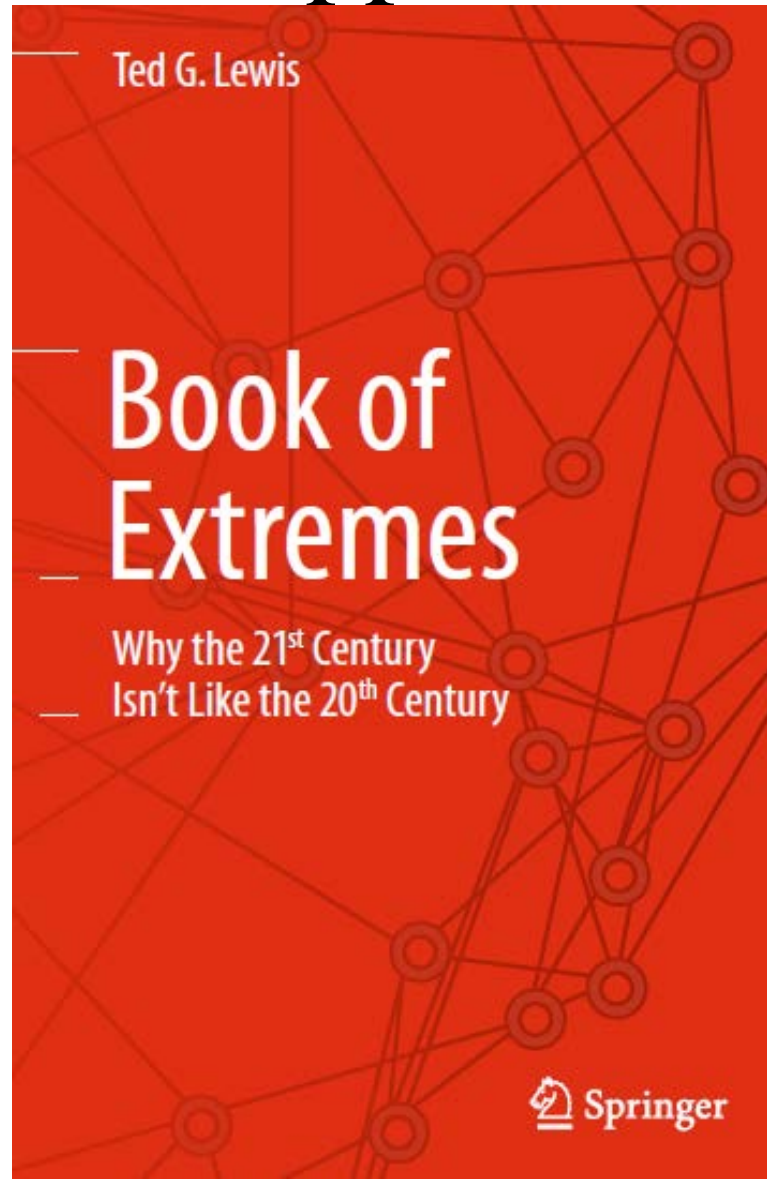
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“fractional calculus” appeared once  
10453 pages, p.1416



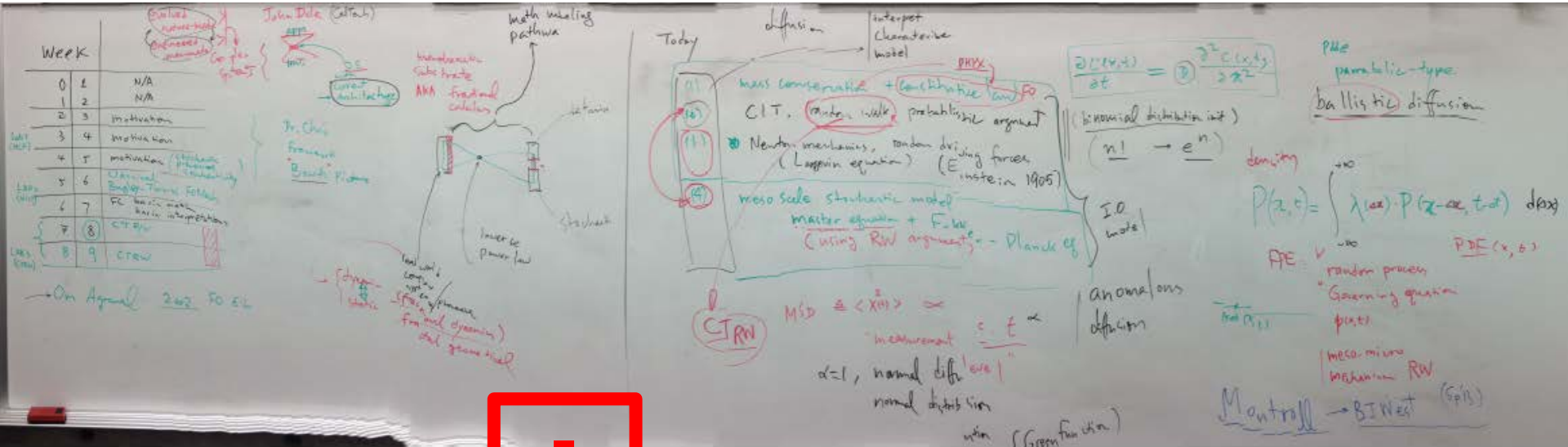
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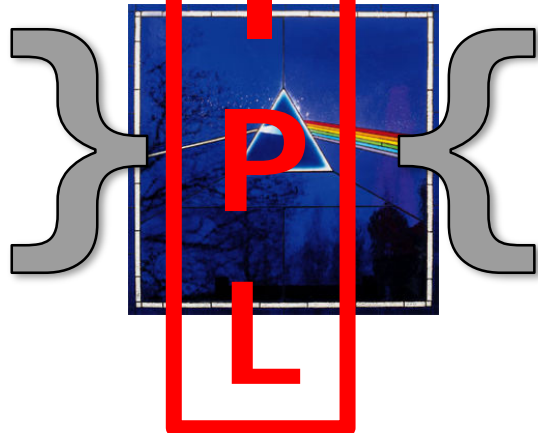
# Dr. Chen's theorem is not **axiom**

- “Power Law” should read “Mittag-Leffler Law”
- When you talk about “power law”, you are talking about fractional calculus.
- **Corollary: When you talk about “stretched exponential”, you are almost talking about fractional calculus!**

# From “bow tie” to “mental prism”



Complex systems. phenomena, behaviors, ...



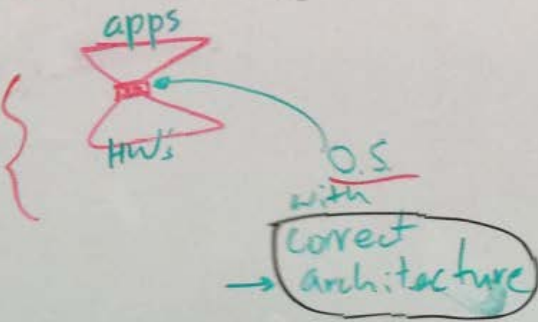
Scale-Free, Heavy-Tailedness, Long Range Dependence, Long Memory ...

John Dole (CalTech)

Slide-24/1024

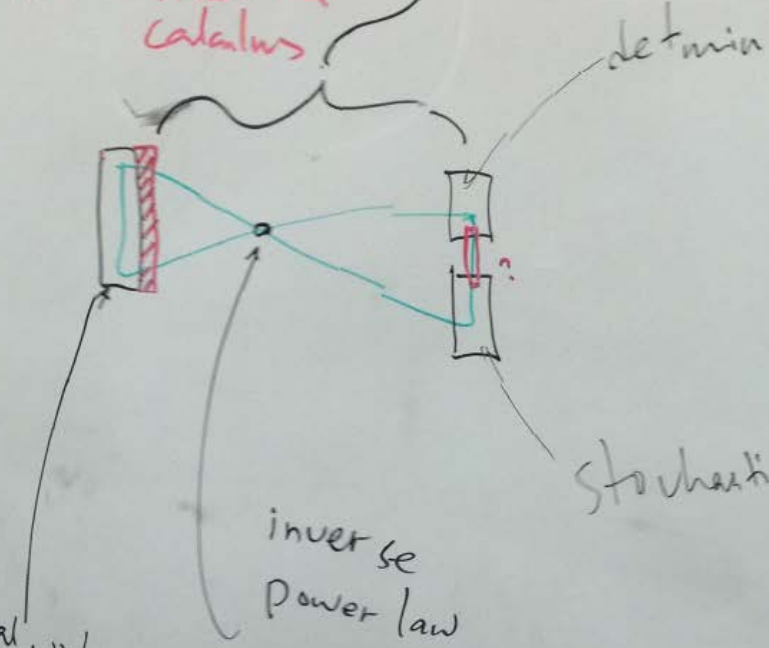
math wheeling pathwa

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mathematic Substrate  
AKA fractional calculus

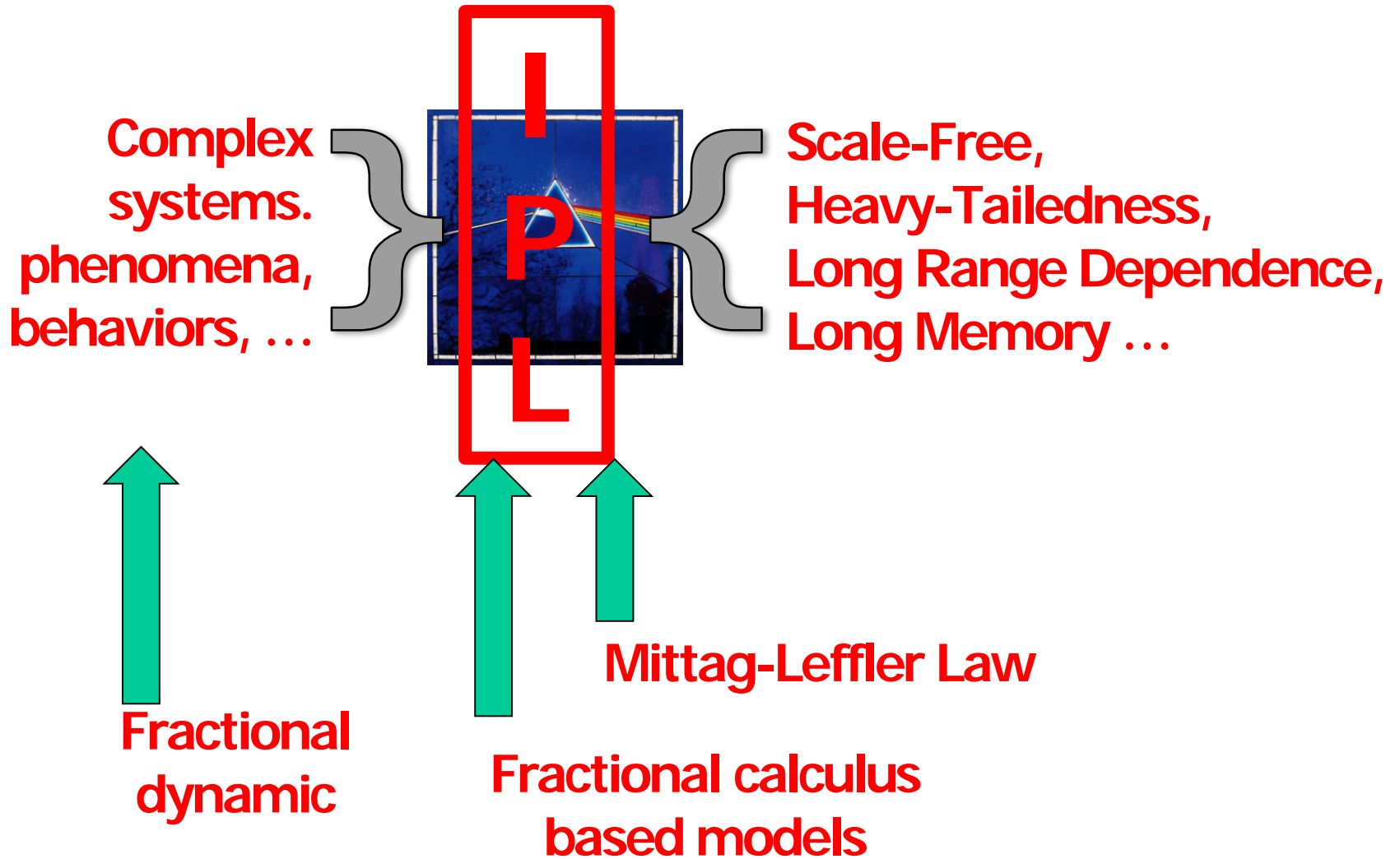
Dr. Chris  
framework  
"Bowtie" picture



Real world  
Complex  
system  
phenomena  
(fractional dynamics)  
fractal geometrical

{ dynamic ↔ static





# Integer Order Calculus

## Exponential Law

$$\dot{x}(t) = -bx(t), x(0) = x_0$$

$$x(t) = x_0 e^{-bt}$$

# Fractional Order Calculus

## Power Law

$$x^{(\alpha)}(t) = c, \quad \alpha \in R$$

$$x(t) = ct^\alpha / \Gamma(1 + \alpha)$$

$$(t^k)^{(\alpha)} = \frac{\Gamma(k + 1)}{\Gamma(k - \alpha + 1)} t^{k - \alpha}$$

*note*  $\Gamma(k + 1) = k!$

# Fractional Order Calculus

## Inverse Power Law

$$x^{(\alpha)}(t) = -bx(t), \quad x^{(\alpha-1)}(0) = x_0 \quad \alpha \in (0,1)$$

$$x(t) = x_0 t^{\alpha-1} E_{\alpha,\alpha}(-bt^\alpha)$$

**Mittag-Leffler function in two parameters:**

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \quad (\alpha > 0, \beta > 0)$$

$$E_{\alpha}(x) = E_{\alpha,1}(x)$$

$$E_{1,1}(x) = e^x$$

Professor Donald E. Knuth,  
creator of TEX:



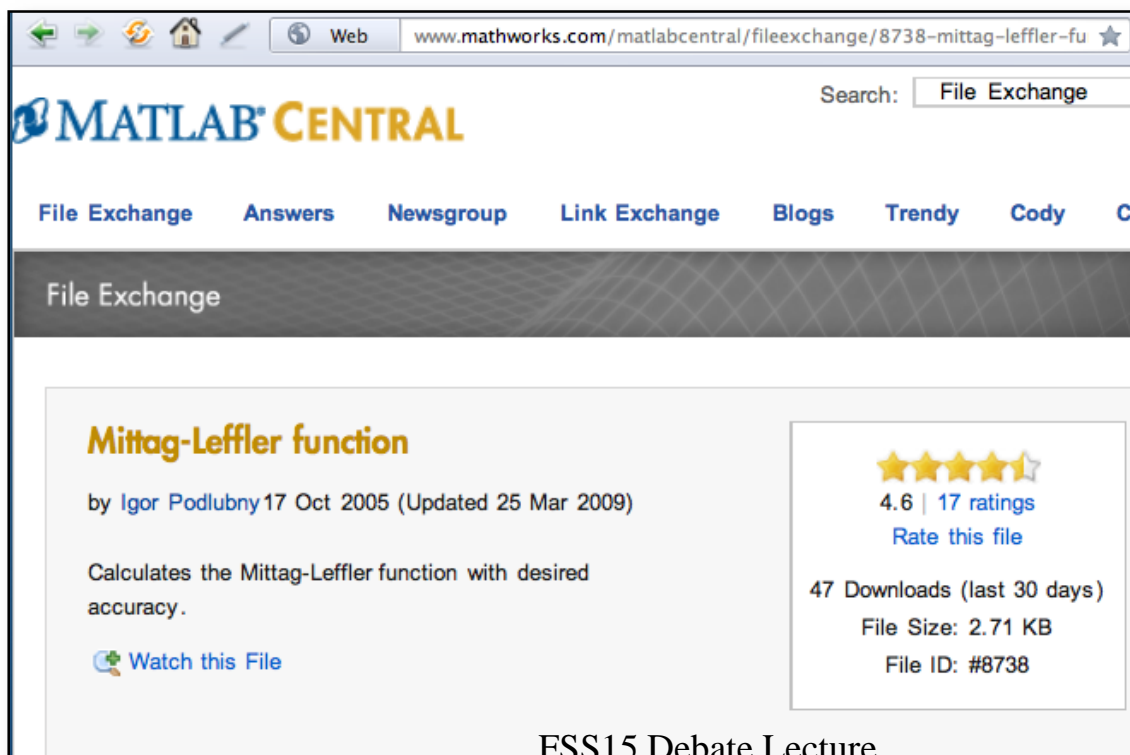
G. M. Mittag-Leffler  
(1846-1927)

"As far as the spacing in mathematics is concerned...I took Acta Mathematica, from 1910 approximately; this was a journal in Sweden ... Mittag-Leffler was the editor, and his wife was very rich, and they had the highest budget for making quality mathematics printing. So the typography was especially good in Acta Mathematica."

(Questions and Answers with  
Prof. Donald E. Knuth, Charles  
University, Prague, March 1996)

# The Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0)$$

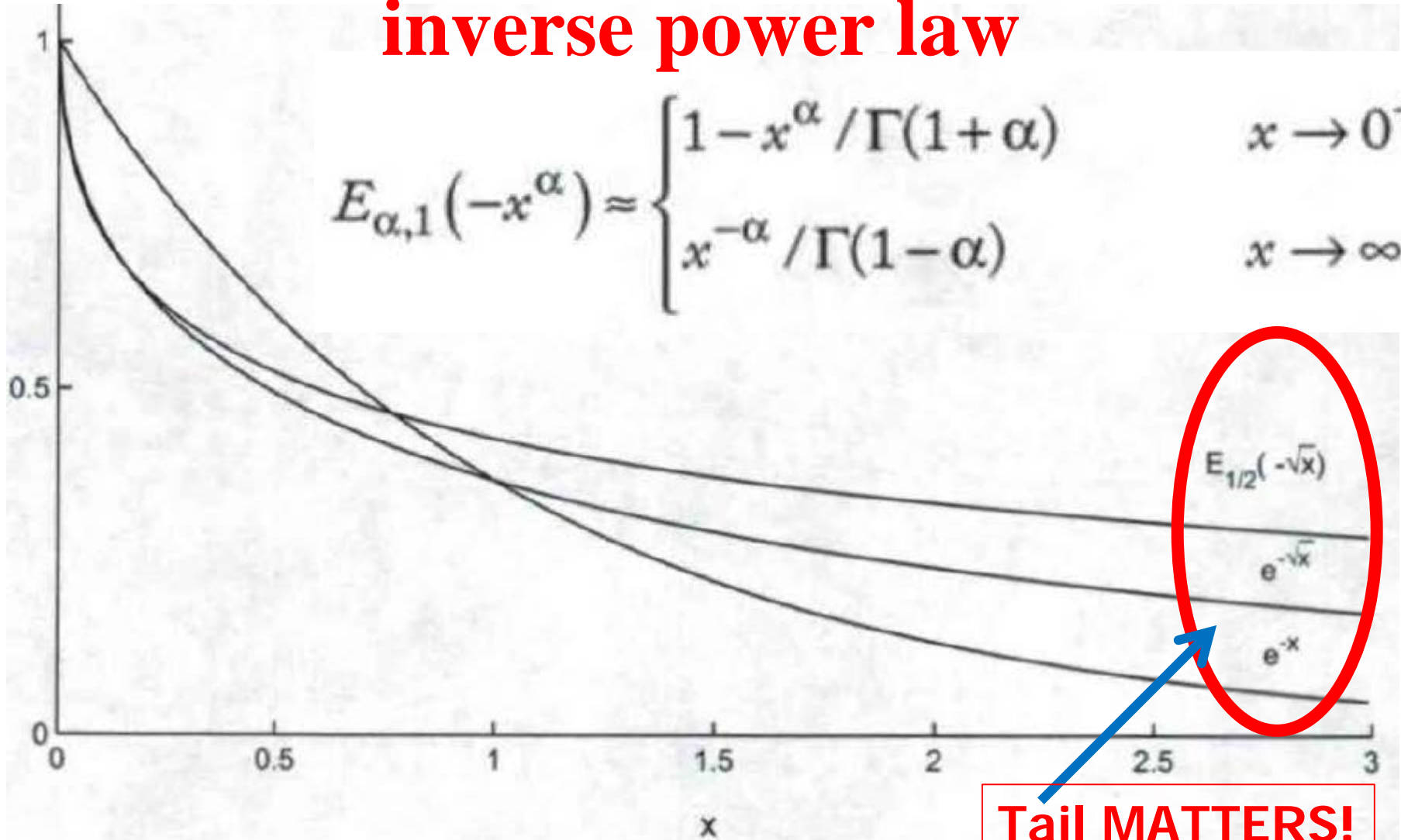


The screenshot shows a web browser window displaying a MATLAB Central File Exchange page. The browser's address bar shows the URL: [www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-fu](http://www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-fu). The page features the MATLAB Central logo and a search bar containing the text "File Exchange". Below the logo, there are navigation links for "File Exchange", "Answers", "Newsgroup", "Link Exchange", "Blogs", "Trendy", "Cody", and "Co". The main content area is titled "File Exchange" and displays the following information:

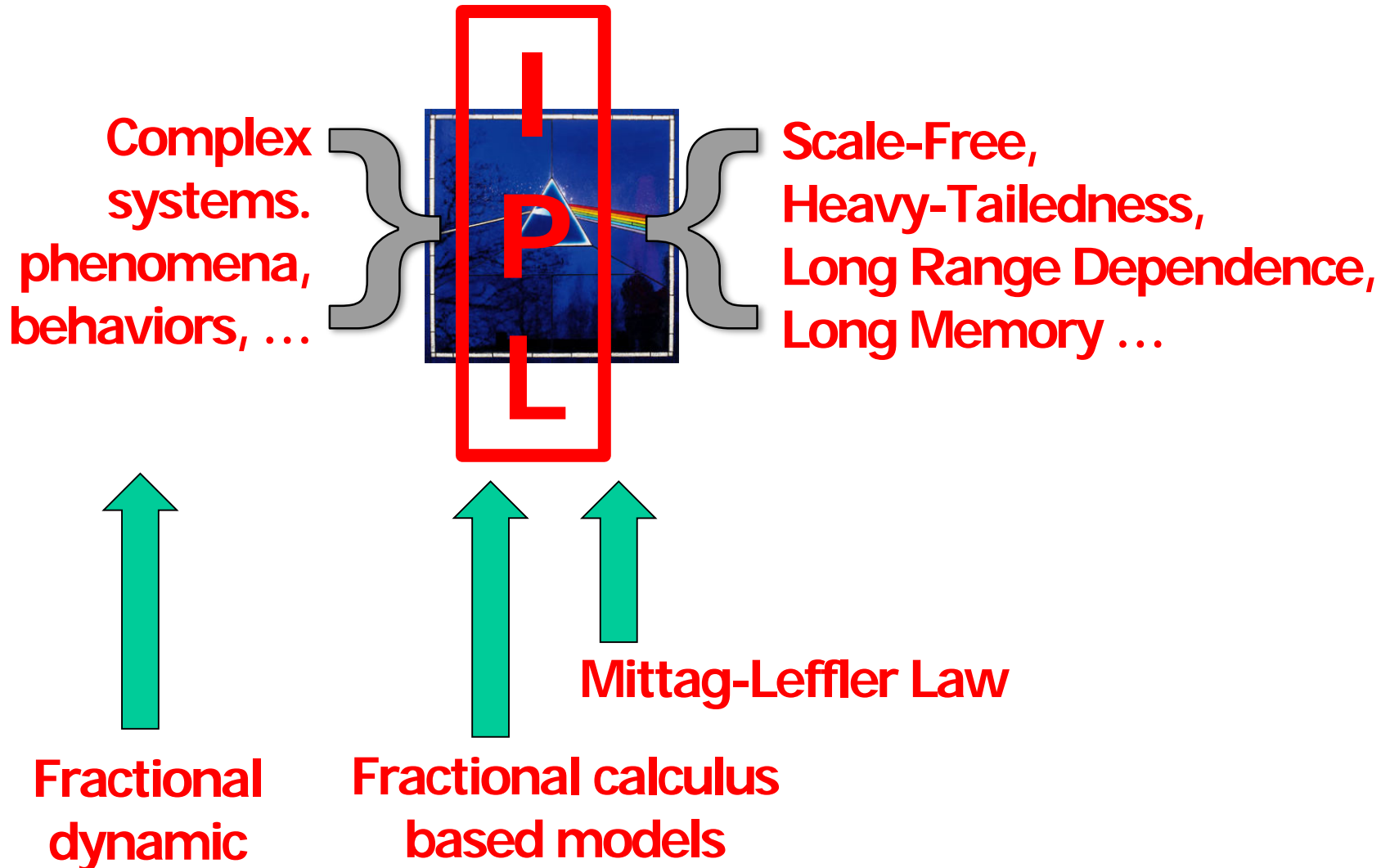
- Mittag-Leffler function**
- by Igor Podlubny 17 Oct 2005 (Updated 25 Mar 2009)
- Calculates the Mittag-Leffler function with desired accuracy.
- [Watch this File](#)
- 4.6 | 17 ratings
- [Rate this file](#)
- 47 Downloads (last 30 days)
- File Size: 2.71 KB
- File ID: #8738

# Root of long (algebraic) tail, or inverse power law

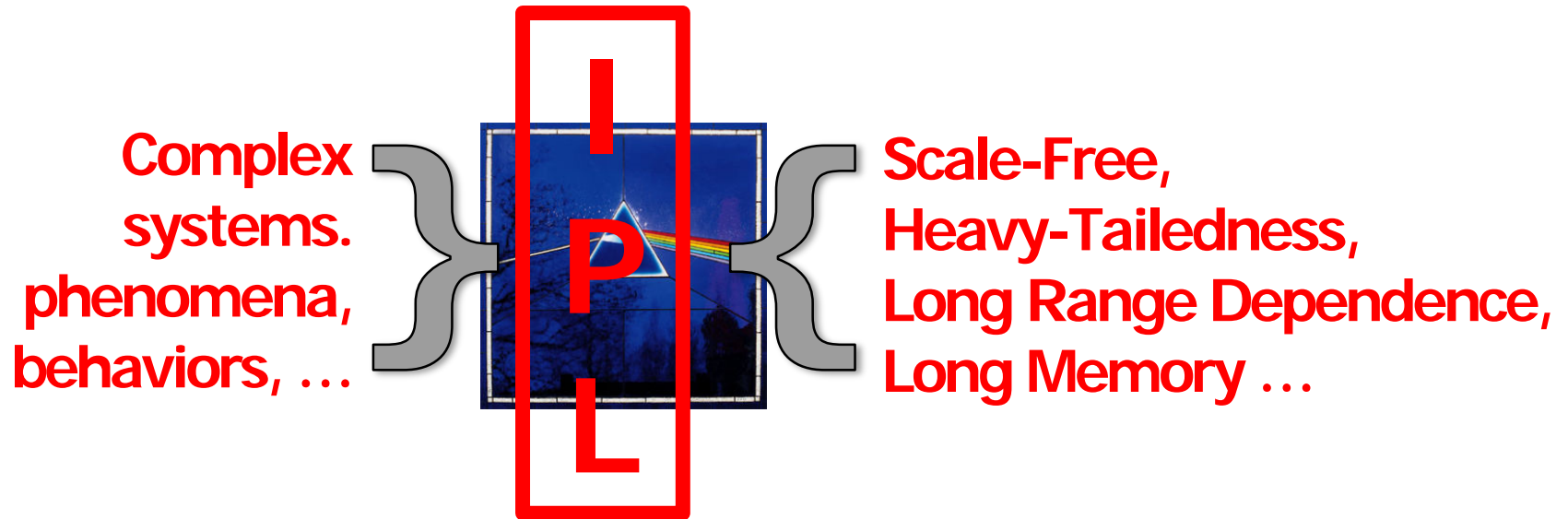
$$E_{\alpha,1}(-x^\alpha) \approx \begin{cases} 1 - x^\alpha / \Gamma(1 + \alpha) & x \rightarrow 0^+ \\ x^{-\alpha} / \Gamma(1 - \alpha) & x \rightarrow \infty \end{cases}$$



**Tail MATTERS!**







# Power Law

$$f(x) = ax^k$$

When  $k$  is negative: **Inverse power law**

**Scale-free**



**Scale invariance**

$$f(cx) = a(cx)^k = c^k f(x) \propto f(x).$$

- “**Scaling laws in cognitive sciences**” by CT Kello, GDA Brown, R Ferrer-i-Cancho, JG Holden, K Linkenkaer-Hansen, T. Trends in Cognitive Sciences 14 (5), 223-232, 2010

# In Different Contexts

- Scale-free networks (degree distributions)
- Pink noise (power spectrum)
- Probability density function (PDF)
- Autocorrelation function (ACF)
- Allometry ( $Y = a X^b$ )
- Anomalous relaxation (evolving over time)
- Anomalous diffusion (MSD versus time)
- Self-similar

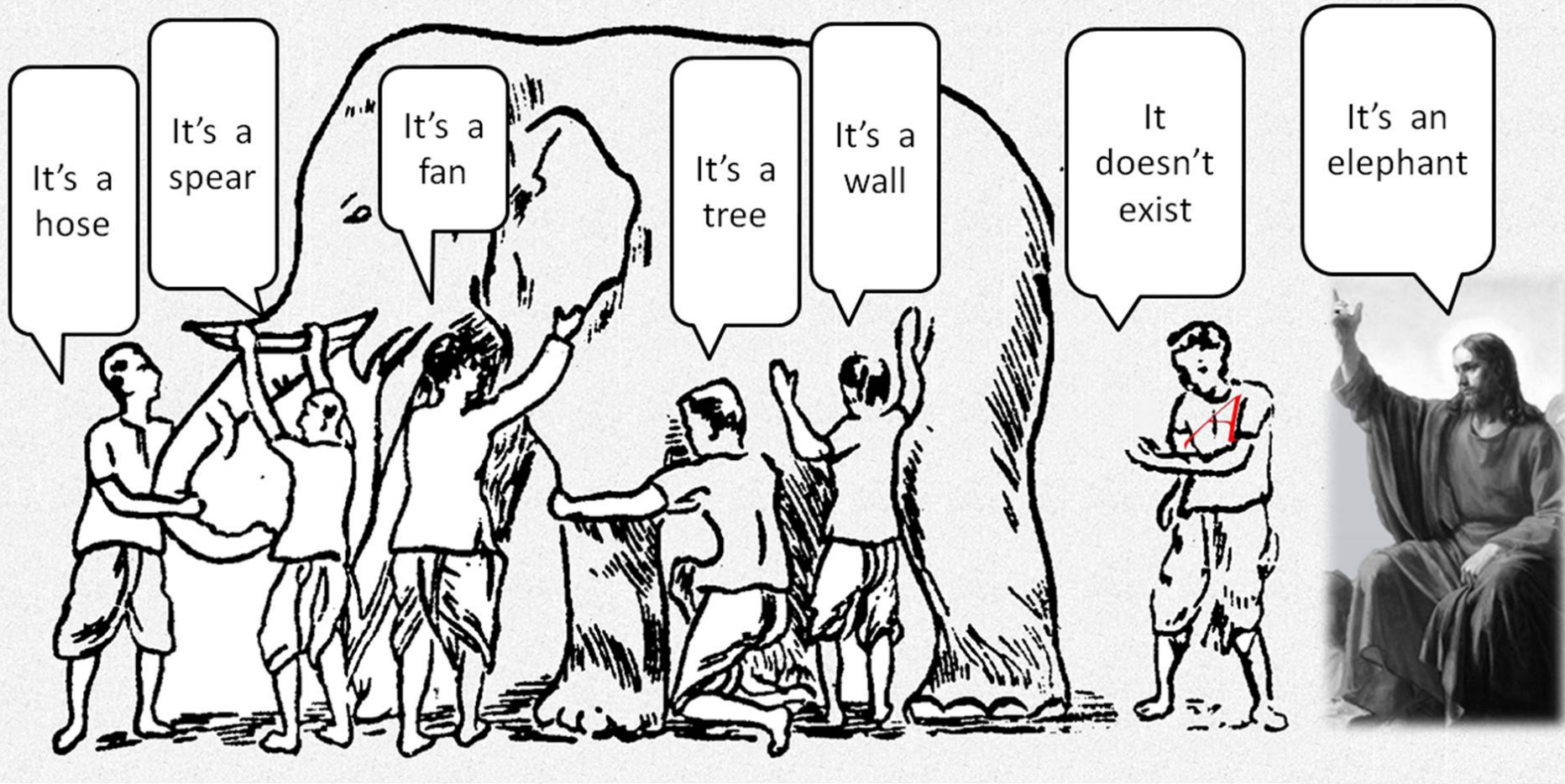
# Other connectedness to FC? (hidden)

- Fractal, irregular, anomalous, rough, Hurst
  - Multifractal, multi-scale, scale-rich
- Renormalization (?), Universality
- Extreme events– spikiness, bursty, intermittence
- Fluctuation in fluctuations; Variability,
- Emergence, Surprise, **Black swan**
- Nonlocality, Long term memory
- Complex (behavior, processes, network, fluid, dynamics, systems ...)
- When the forest is big, there are all types of birds ("It takes all kinds" 林子大了什么鸟都有), 20/80 rule(二八定律)

<http://www.cafepress.com/thepowerlawshop>

**I WENT TO A PHYSICS  
CONFERENCE AND ALL  
I GOT WAS A LOUSY  
POWER LAW.**



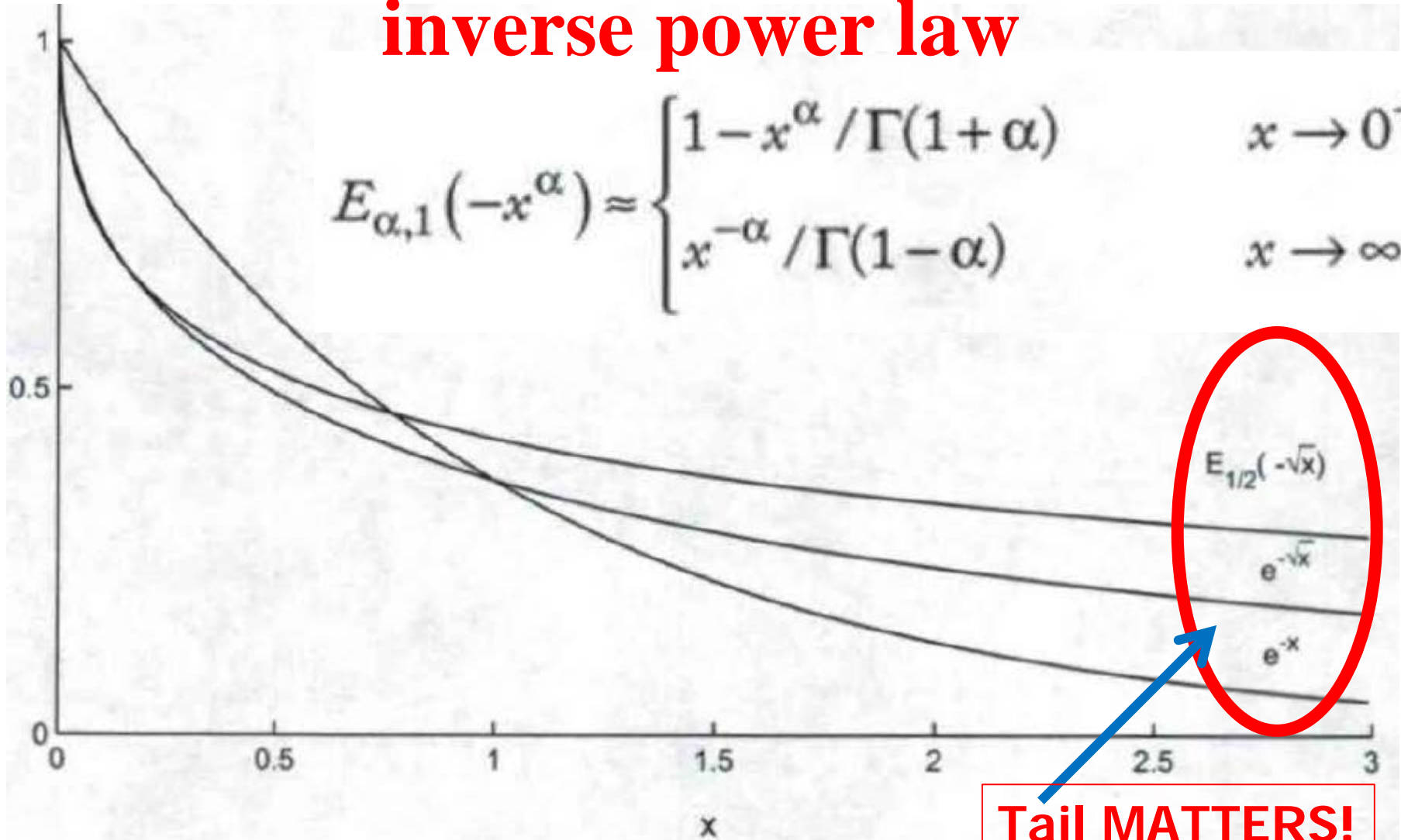


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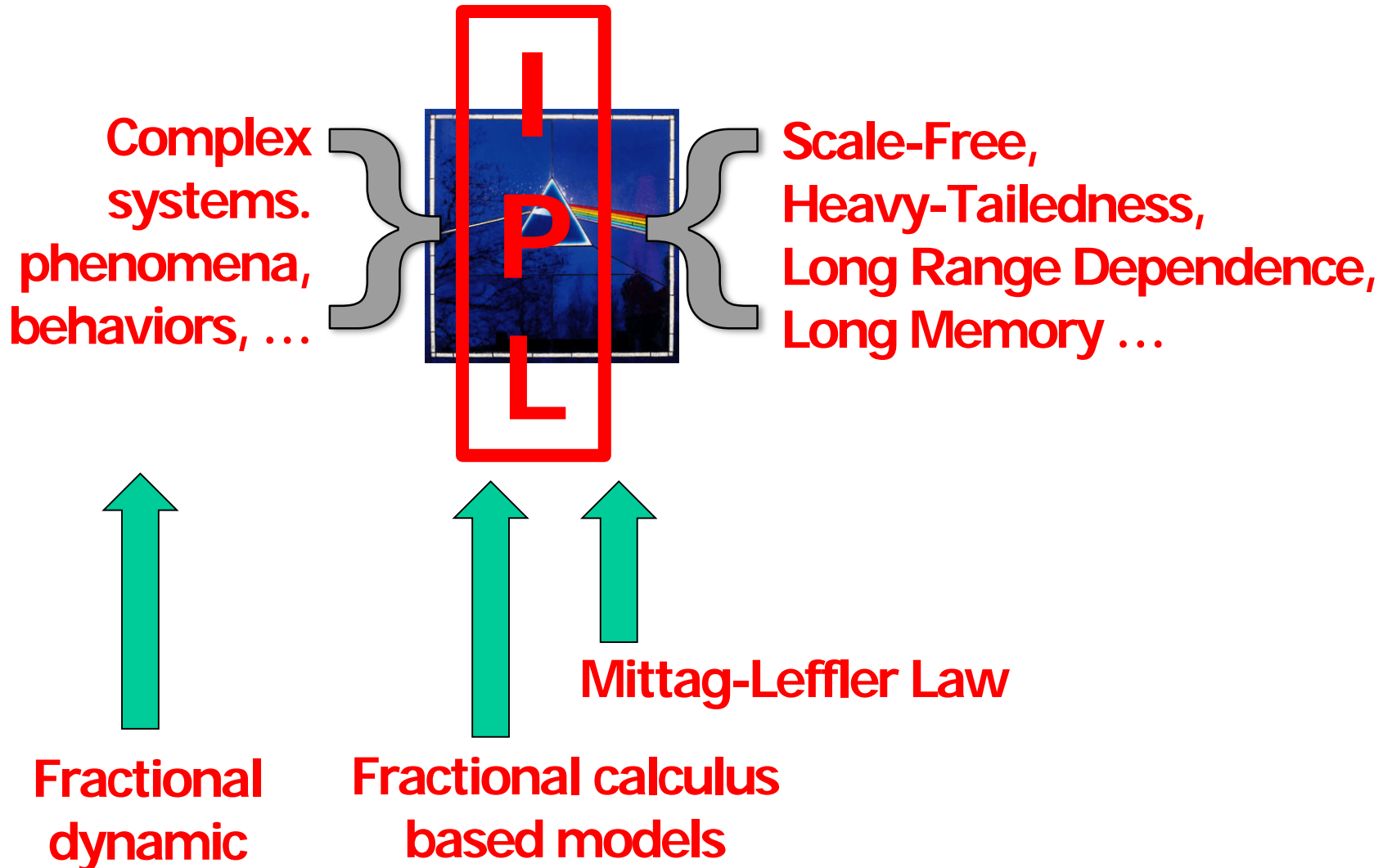
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# Root of long (algebraic) tail, or inverse power law

$$E_{\alpha,1}(-x^\alpha) \approx \begin{cases} 1 - x^\alpha / \Gamma(1 + \alpha) & x \rightarrow 0^+ \\ x^{-\alpha} / \Gamma(1 - \alpha) & x \rightarrow \infty \end{cases}$$



**Tail MATTERS!**





# “Fractional Order Thinking” or, “In Between Thinking”

- For example
  - Between integers there are non-integers;
  - Between logic 0 and logic 1, there is the “**fuzzy logic**”;
  - Between integer order splines, there are “**fractional order splines**”
  - Between integer high order moments, there are **noninteger order moments (e.g. FLOS)**
  - Between “integer dimensions”, there are **fractal dimensions**
  - **Fractional Fourier transform** (FrFT) – in-between time-n-freq.
  - Non-Integer order calculus (**fractional** order calculus – abuse of terminology.) (FOC)

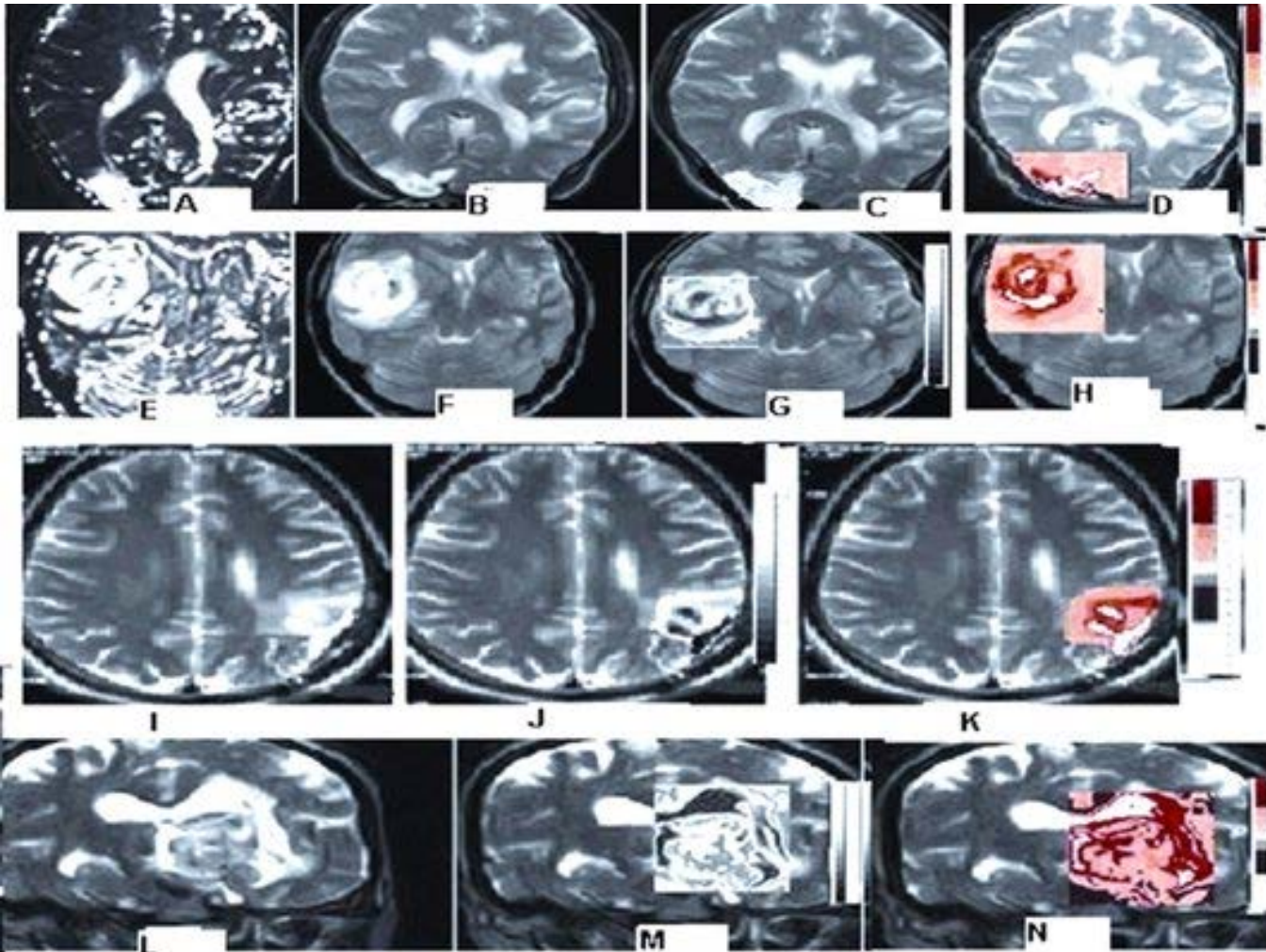


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# Complex relaxation in NMR



<http://en.wikipedia.org/wiki/Metastasis>

## Most common sites of Cancer metastasis and their symptoms

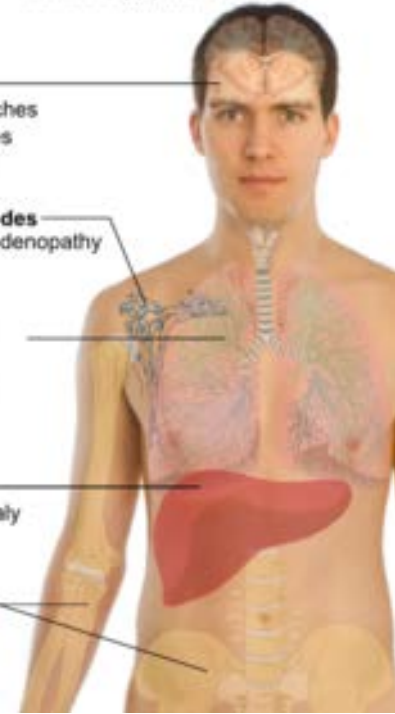
- Brain**
  - Headaches
  - Seizures
  - Vertigo

- Lymph nodes**
  - Lymphadenopathy

- Respiratory**
  - Cough
  - Hemoptysis
  - Dyspnea

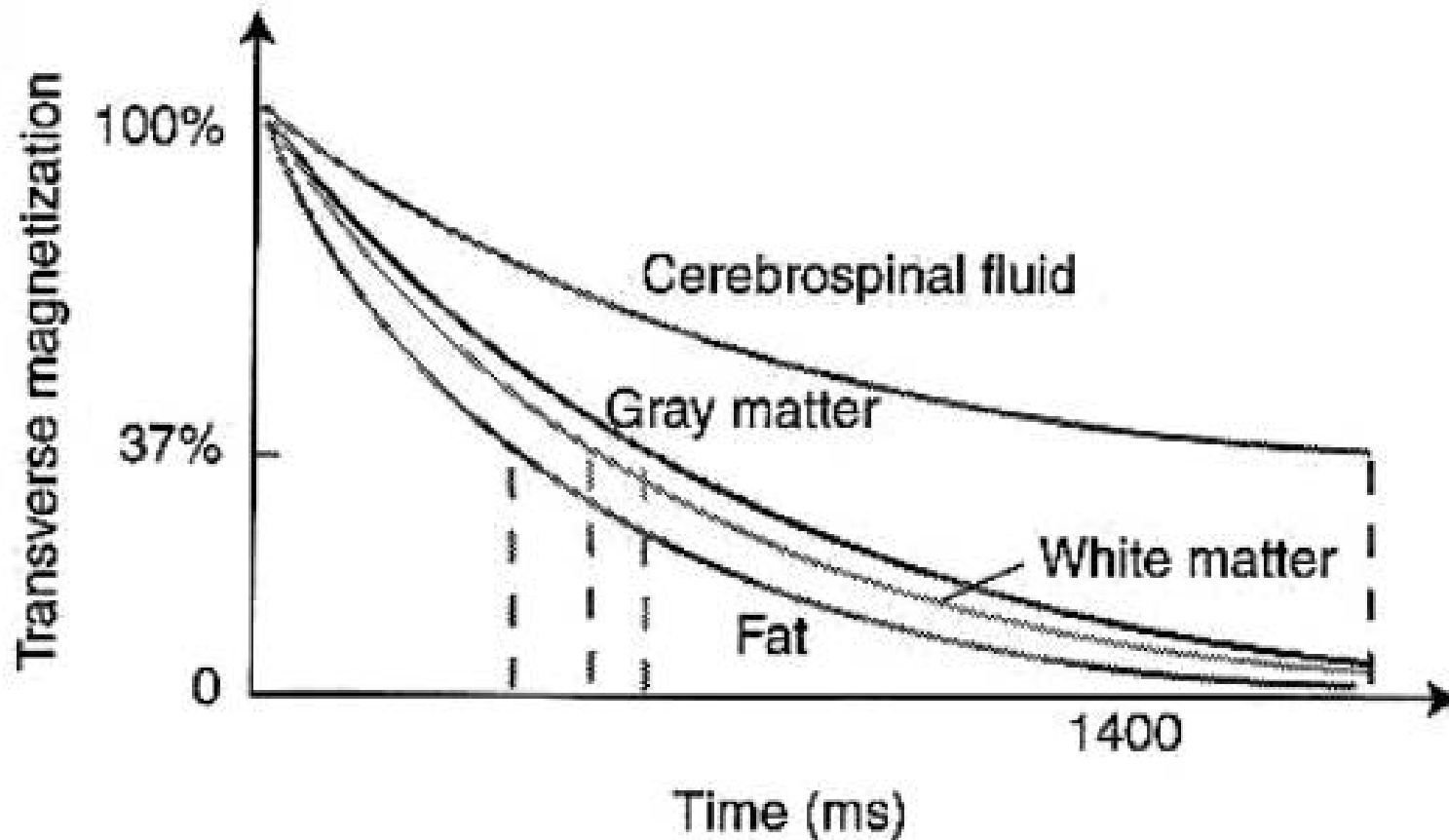
- Liver**
  - Hepatomegaly
  - Jaundice

- Skeletal**
  - Pain
  - Fractures



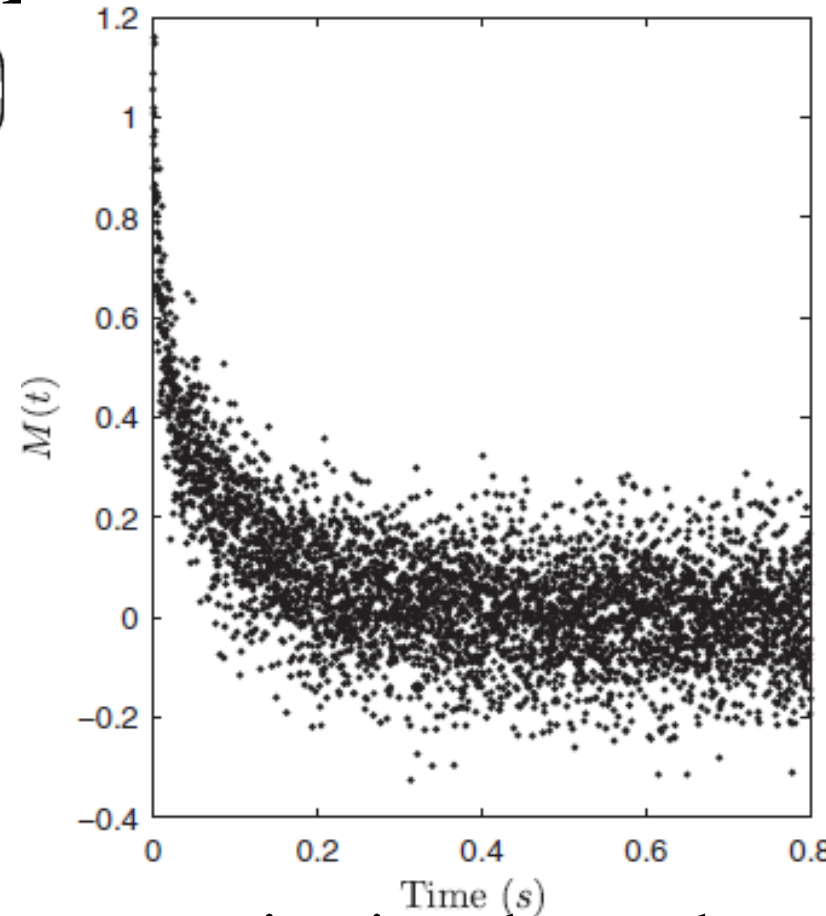
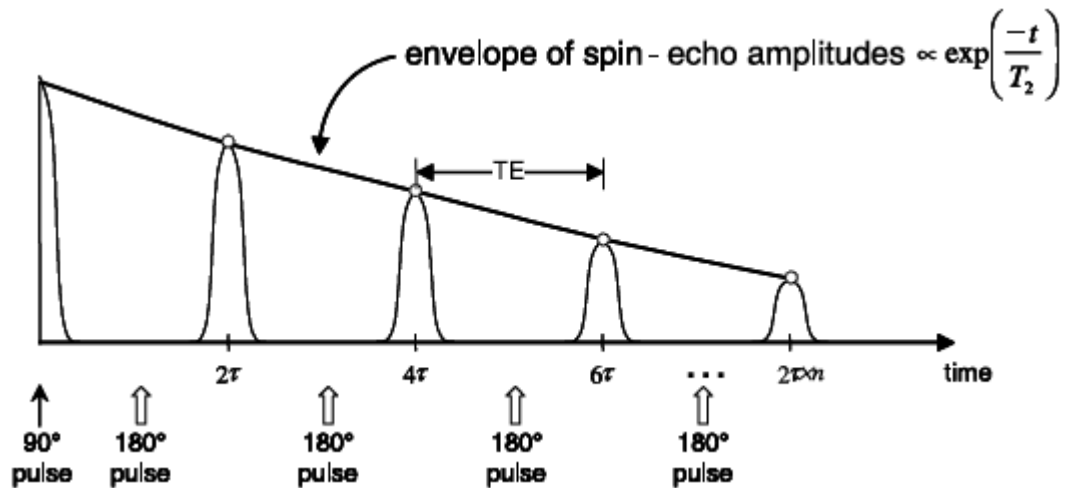
<http://www.ispub.com/journal/the-internet-journal-of-radiology/volume-13-number-1/in-vivo-mr-measurement-of-refractive-index-relative-water-content-and-t2-relaxation-time-of-various-brain-lesions-with-clinical-application-to-discriminate-brain-lesions.article-g08.fs.jpg>

# T2 relaxation in NMR



<http://hs.doversherborn.org/hs/bridgerj/DSHS/apphysics/NMR/T2.htm>

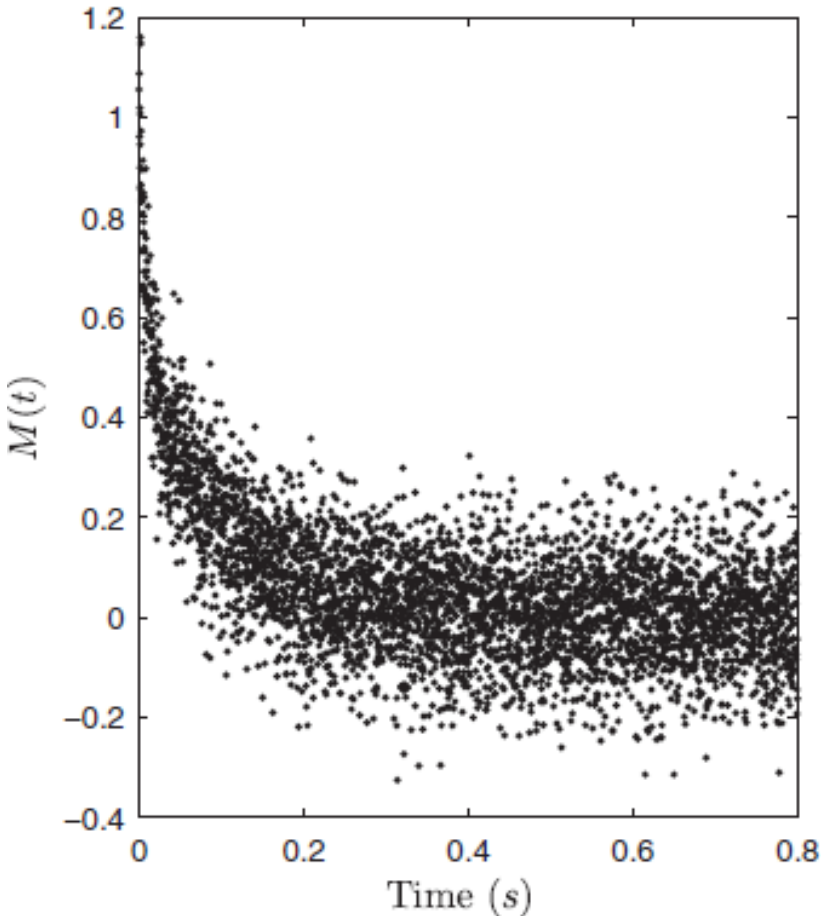
# T2 relaxation in NMR



magnetization decay data  
 $M(t)$  with poor SNR

Carr–Purcell–Meiboom–Gill  
(*CPMG*) pulse sequence, as  
shown in Fig. 1, is widely used to  
measure spin–spin *relaxation*  
time  $T_2$

# Complex relaxation: How to characterize or model it?

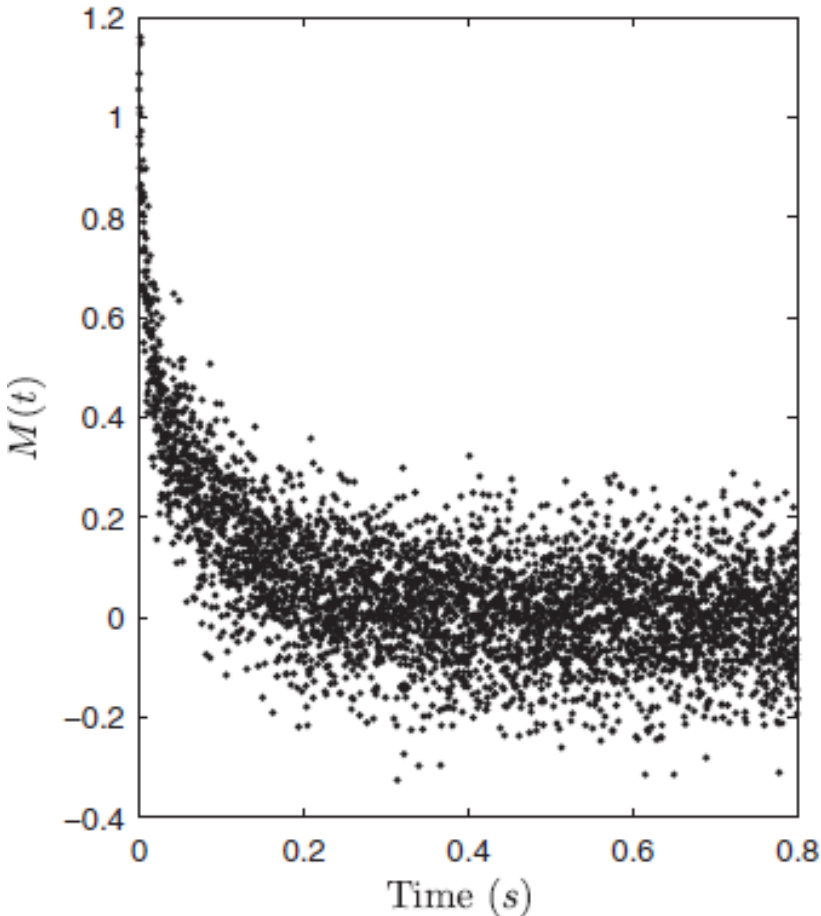


- Debye relaxation  $\exp(-t/\tau)$   
 $1/(1 + \tau s)$
- Distributed-parameter  
(infinite # of time constants)

$$\int_0^T \frac{f(\tau)}{\tau s + 1} d\tau$$

(H. Fröhlich, 1949)

# Complex relaxation: How to better characterize or model it?



- Cole-Cole (1941)

$$\frac{1}{(1 + \tau S^\alpha)}$$

- Distributed-parameter (infinite # time constants)

$$\int_0^T \frac{f(\tau)}{\tau S^\alpha + 1} d\tau$$

(Hu, Li and Chen, IEEE CDC 2010)

# More complex relaxation models

- Cole-Davidson

$$H_{\text{C-D}}(s) = \int_0^T \frac{f(\tau)}{(1 + \tau s)^\beta} d\tau$$

- Havriliak-Negami

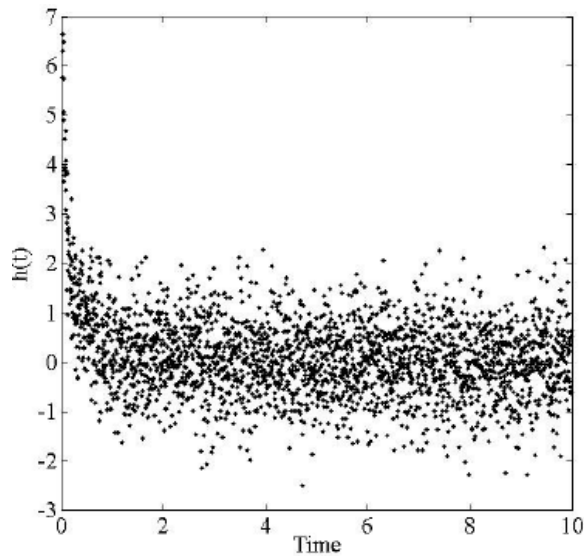
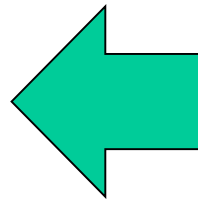
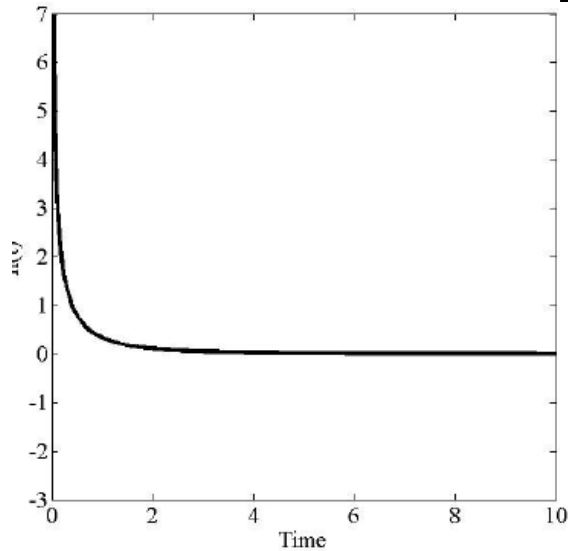
$$H_{\text{H-N}}(s) = \int_0^T \frac{f(\tau)}{(1 + \tau s^\alpha)^\beta} d\tau$$

- **Distributed-order case? Sure!**

$$H(s) = \int_0^1 \frac{f(\gamma)}{\tau s^\gamma + 1} d\gamma$$



# An illustration



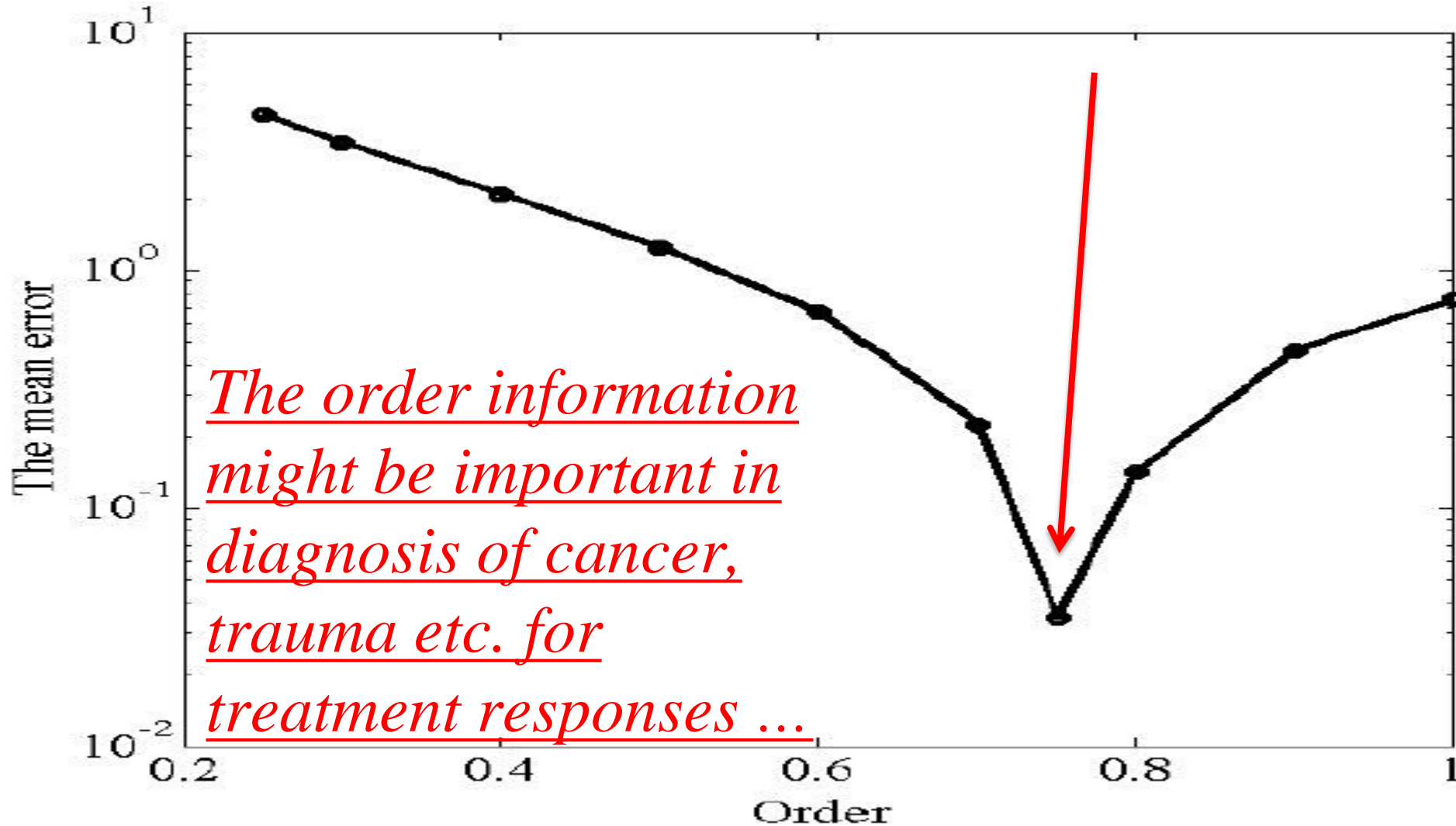
- Distributed-parameter (infinite time constants)

$$\int_0^T \frac{f(\tau)}{\tau s^\alpha + 1} d\tau$$

- $\alpha = 0.75, T=1$  sec.

$$f(\tau) = 1 + 2\tau$$

# Scanning the “order” and fitting



# Fractional noises / fluctuations

anti-persistent

Brownian motion, or  
Weiner process

persistent

fBm

$H = 0.25$

$H = 0.50$

$H = 0.75$

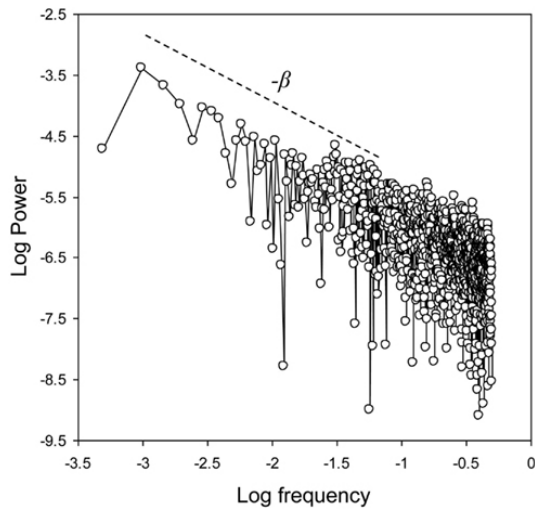
Normal Gaussian noise

fGn

$H = 0.25$

$H = 0.50$

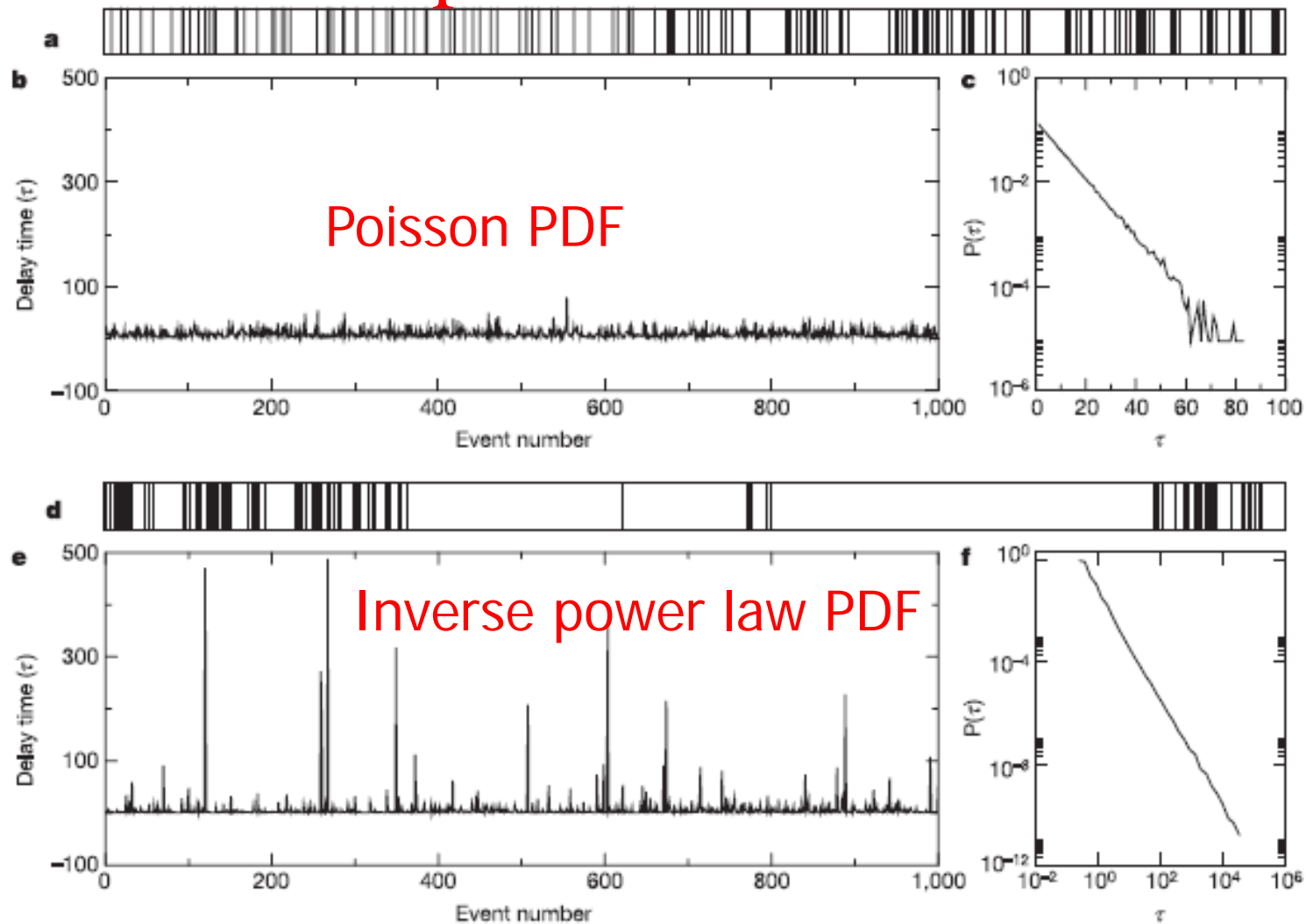
$H = 0.75$



[http://www.frontiersin.org/Fractal\\_Physiology/10.3389/fphys.2012.00208/full](http://www.frontiersin.org/Fractal_Physiology/10.3389/fphys.2012.00208/full)

FSS15 Debate Lecture

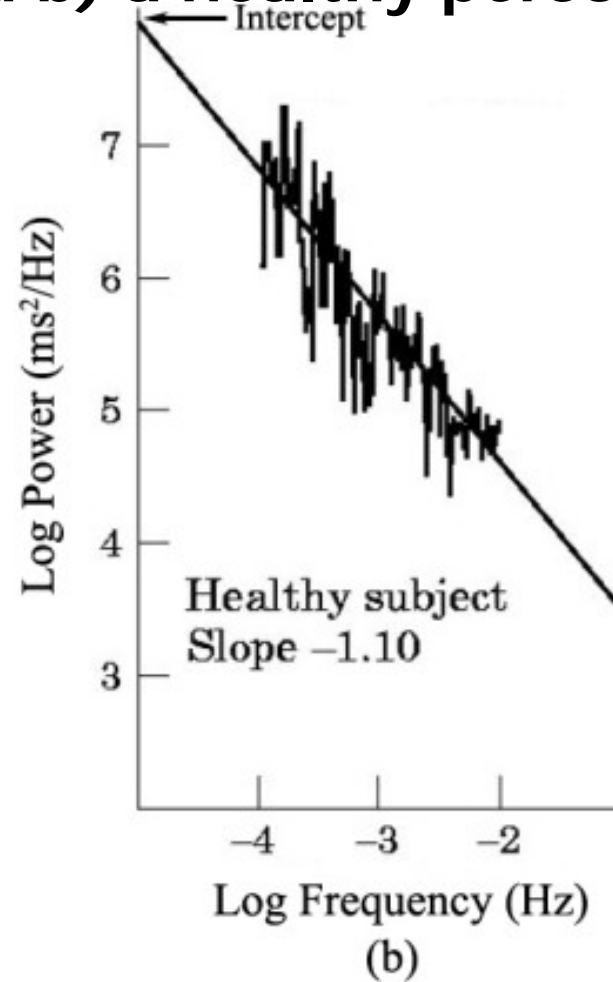
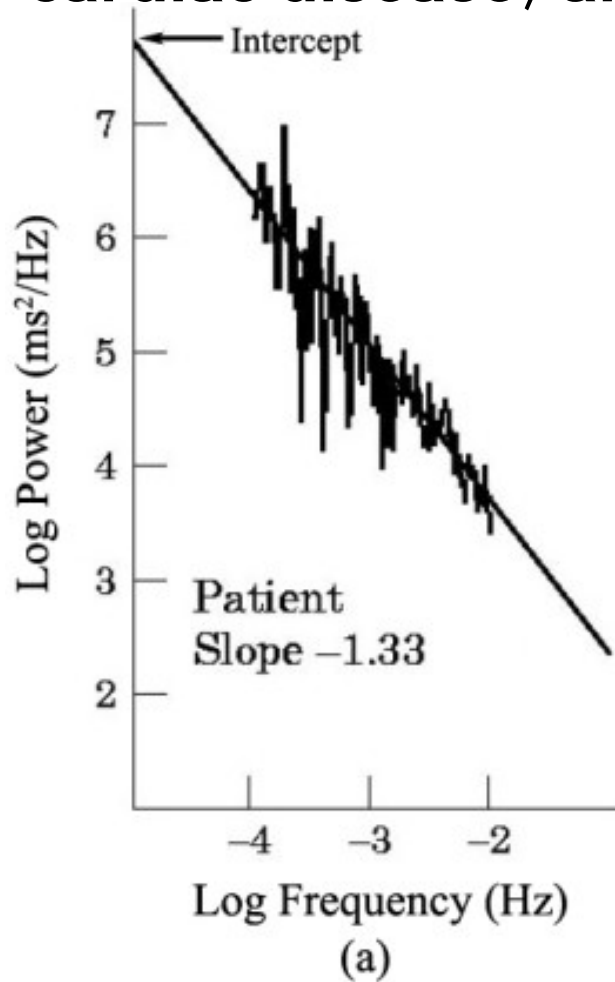
# Spikiness/Burstiness



A.-L. Barabási. The origin of bursts and heavy tails in human dynamics. *Nature* 435 207–211 (2005).

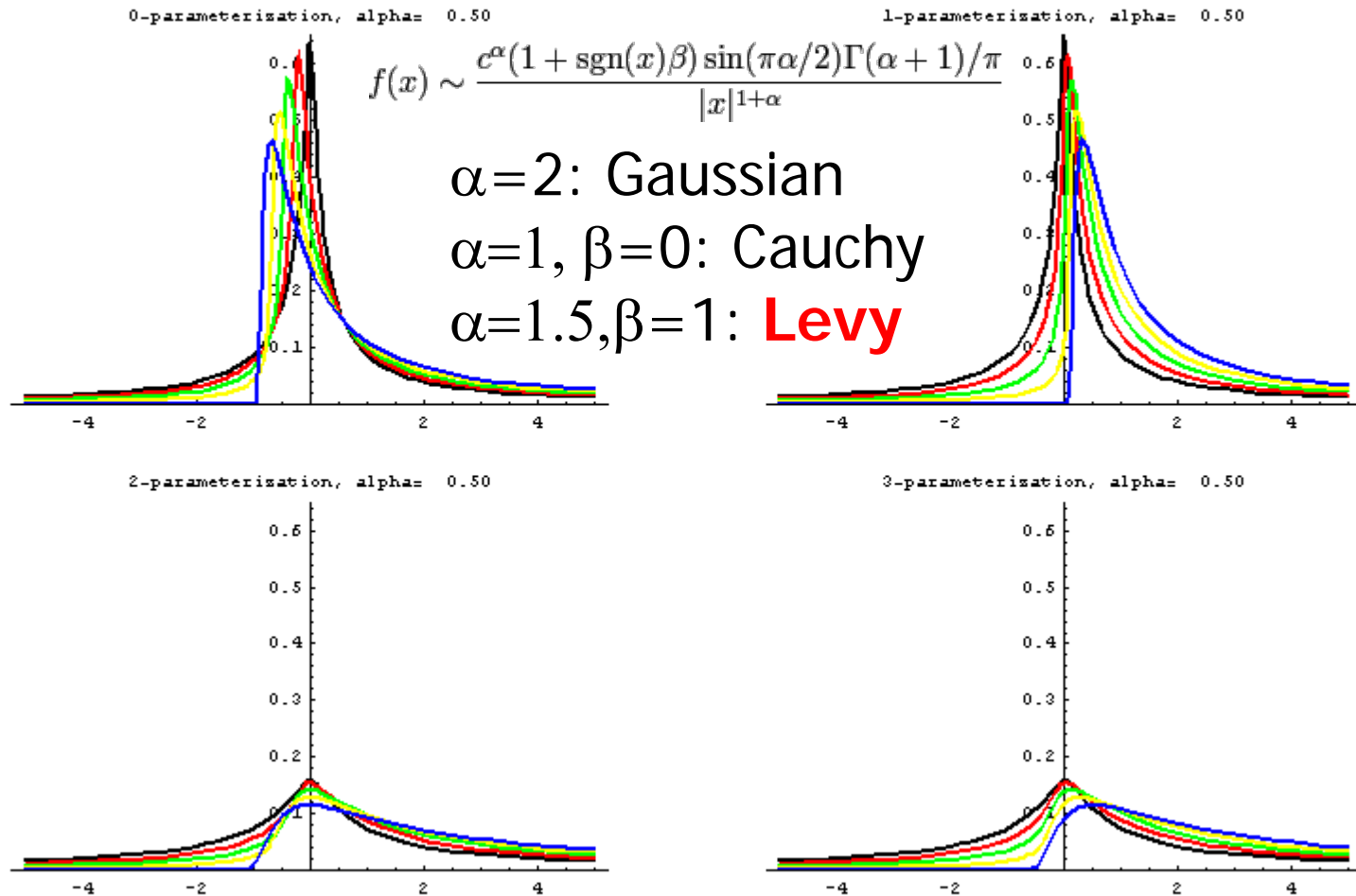
<http://seeingcomplexity.wordpress.com/2011/03/07/global-android-activations-and-the-power-law/>  
10/1/2015

Examples of the power law slope in a) a patient with cardiac disease; and b) a healthy person.



Phyllis K. Stein, Ph.D., Anand Reddy, M.D. "Non-Linear **Heart Rate Variability** and Risk Stratification in Cardiovascular Disease"  
Indian Pacing and Electrophysiology Journal (ISSN 0972-6292), 5(3): 210-220 (2005)

# Stable Distributions



<http://academic2.american.edu/~jpnolan/stable/stable.html>

# Connection to FC via PDF

- “*Fractional Calculus and Stable Probability Distributions*” (1998) by Rudolf Gorenflo , Francesco Mainardi <http://arxiv.org/pdf/0704.0320.pdf>

$$\frac{\partial u}{\partial t} = D(\alpha) \frac{\partial^\alpha u}{\partial |x|^\alpha}, \quad -\infty < x < +\infty, \quad t \geq 0,$$

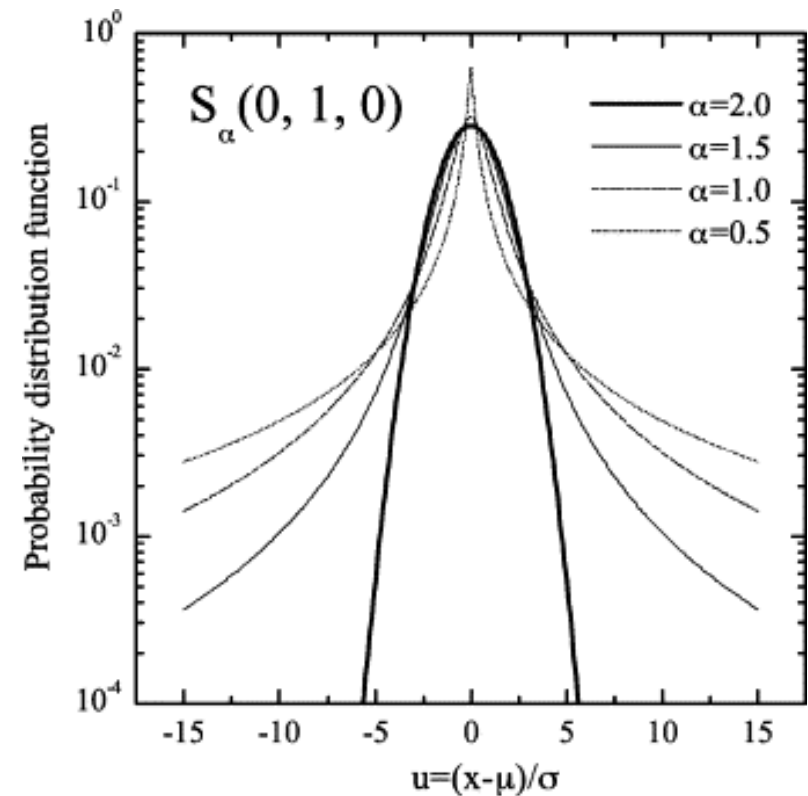
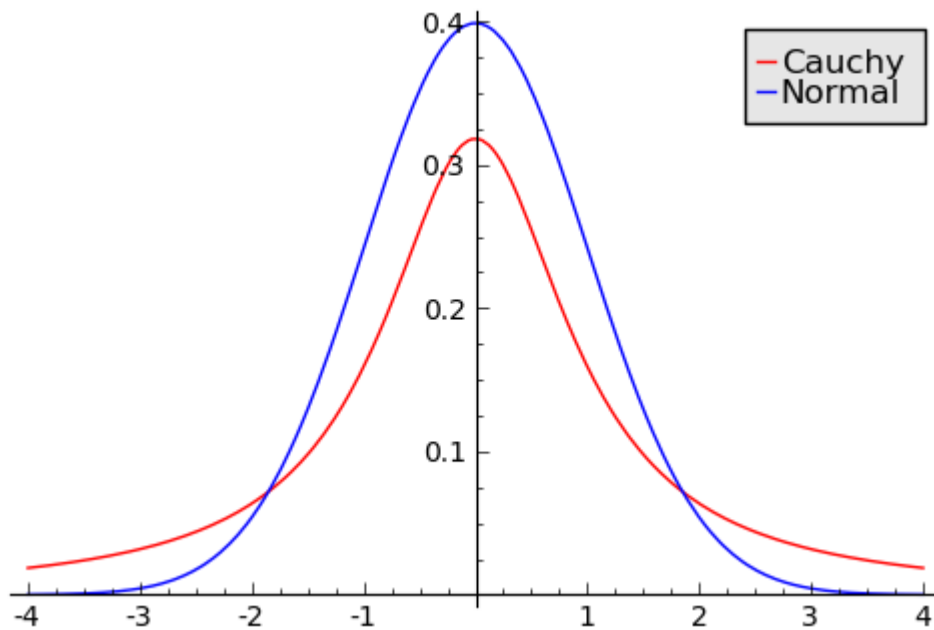
with  $u(x, 0) = \delta(x) \quad 0 < \alpha \leq 2$

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = D(\beta) \frac{\partial^2 u}{\partial x^2}, \quad x \geq 0, \quad t \geq 0,$$

with  $u(0, t) = \delta(t) \quad 0 < \beta < 1$

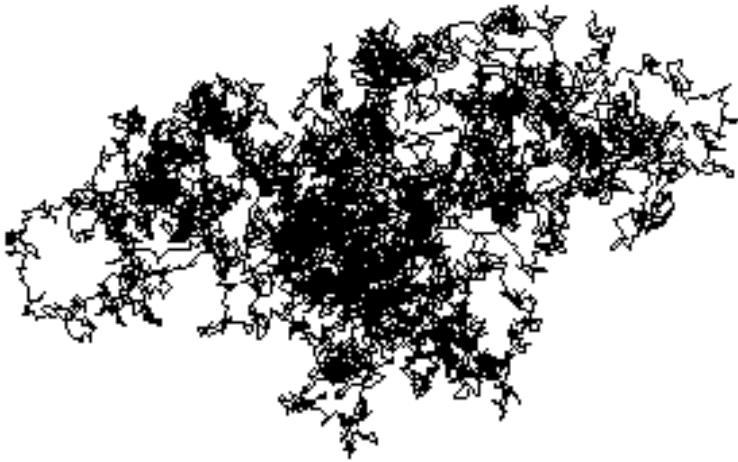
# Heavy tail, fat tail

$$P[X > x] \sim x^{-\alpha}$$

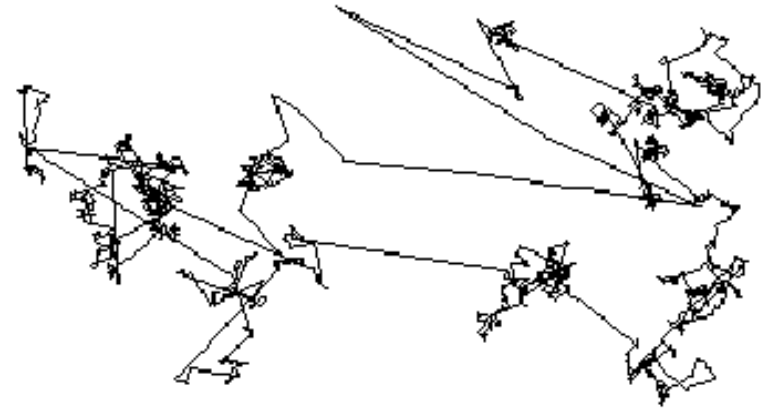




# Long jumps, intermittence



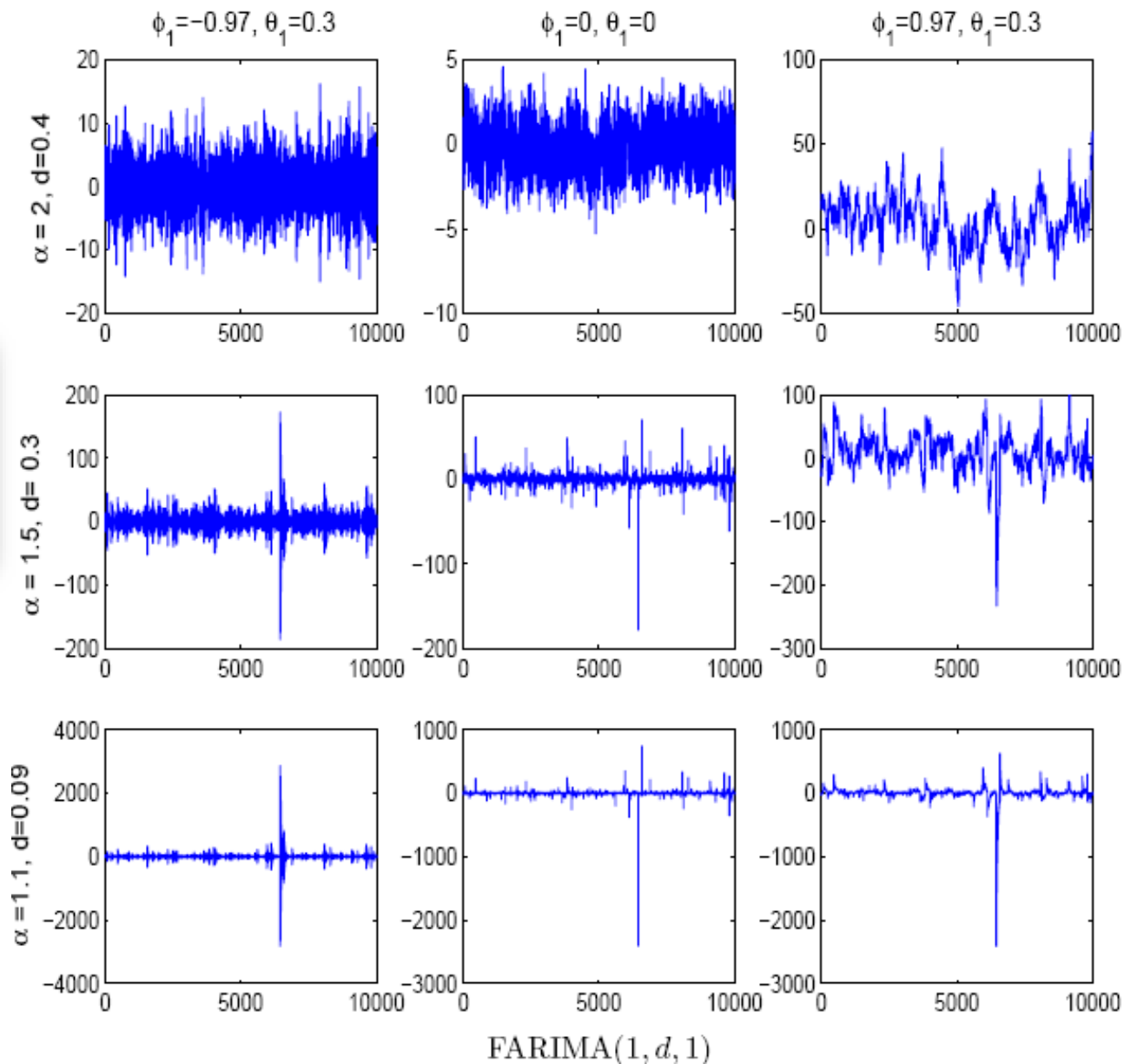
Brownian motion

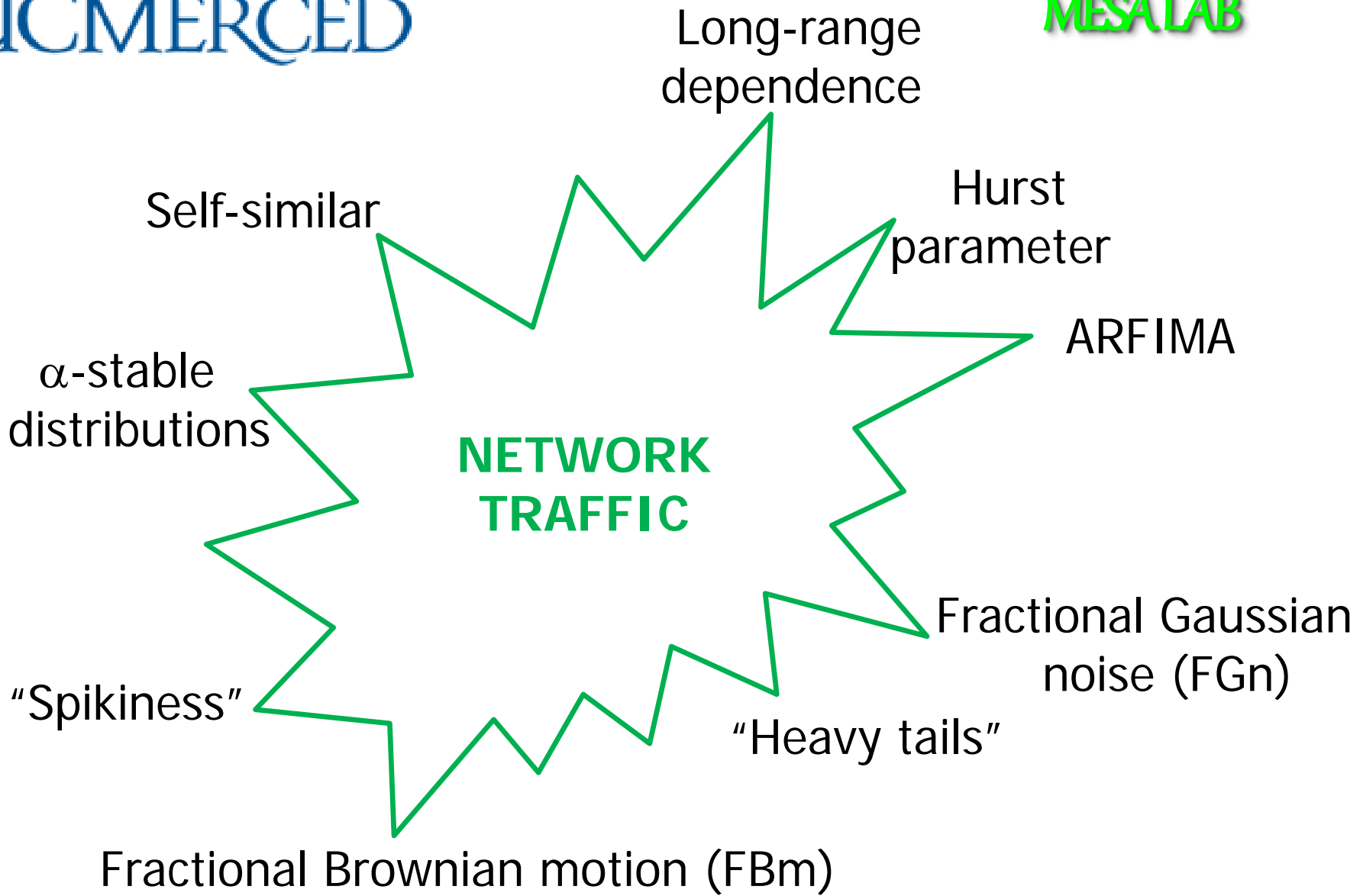


Levy flights

# Spikiness/Burstiness

[7] S. Stoev, and M.S. Taqqu, "Simulation Methods for Linear Fractional Stable Motion and FARIMA Using the Fast Fourier Transform". *Fractals*, 2004.



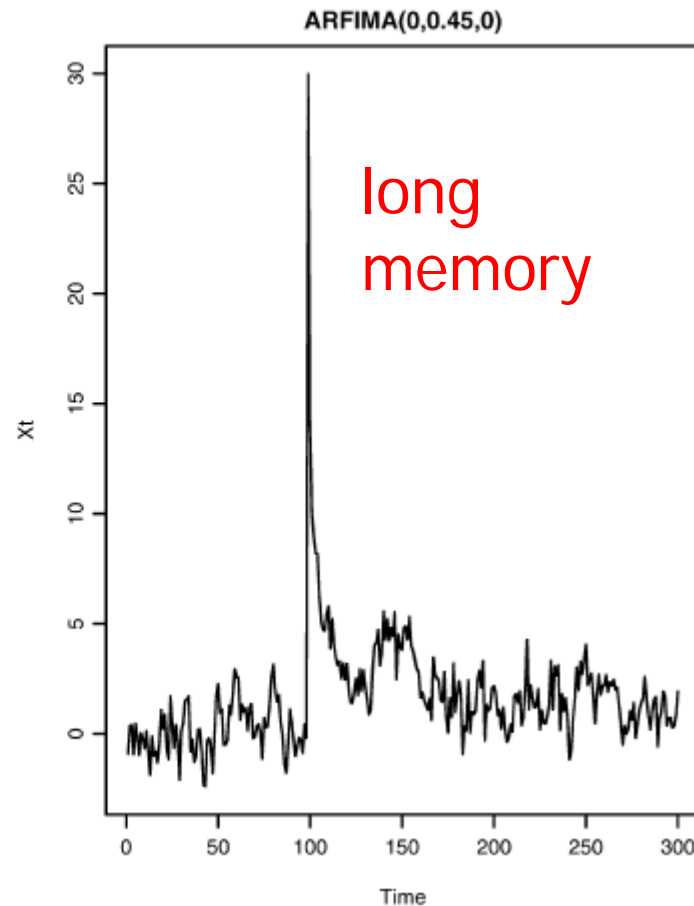
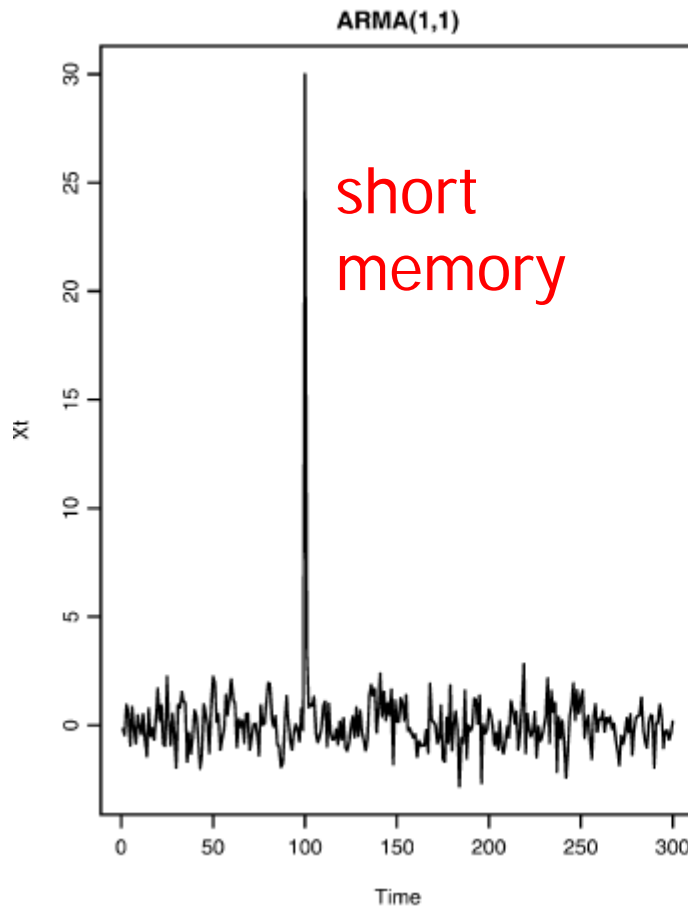


# Long-range dependence

- History: The first model for long range dependence was introduced by Mandelbrot and Van Ness (1968)
- Value: financial data
  - communications networks data
  - video traffic
  - biocorrosion data

# Motivations: Long Memory or LRD

Concept of persistency – shot responses



# SRD vs LRD

- **Short-range dependent (SRD)** processes are characterized by an autocorrelation function which decays exponentially fast;
- **Long-range dependence (LRD)** processes exhibit a much slower decay of the correlations - their autocorrelation functions typically obey some inverse power law.

[http://en.wikipedia.org/wiki/Long-range\\_dependency](http://en.wikipedia.org/wiki/Long-range_dependency)

Mathematically,

- A stationary process is said to have long-range correlations if its **covariance function**  $C(n)$  (assume that the process has finite second-order statistics) decays slowly as  $n \rightarrow \infty$ , i.e. for  $0 < \alpha < 1$ ,

$$\lim_{n \rightarrow \infty} \frac{C(n)}{n^{-\alpha}} = c,$$

where  $c$  is a finite, positive constant.

- The weakly-stationary time-series  $X(t)$  is said to be long range dependent if its **spectral density** obeys  $f(\lambda) \sim C_f |\lambda|^{-\beta}$  as  $\lambda \rightarrow 0$ , for some  $C_f > 0$  and some real parameter  $\beta \in (0, 1)$ .

# Heavy tailedness and Long tailedness

- The distribution of a random variable  $X$  with distribution function  $F$  is said to have a **heavy right tail** if  $\lim_{x \rightarrow \infty} e^{\lambda x} \Pr[X > x] = \infty$  for all  $\lambda > 0$ .
- The distribution of a random variable  $X$  with distribution function  $F$  is said to have a long right tail if for all  $t > 0$   $\lim_{x \rightarrow \infty} \Pr[X > x + t | X > x] = 1$ ,

NOTE: long-tailed distributions are heavy-tailed, but the converse is false

if you know the situation is bad, it is probably worse than you think.

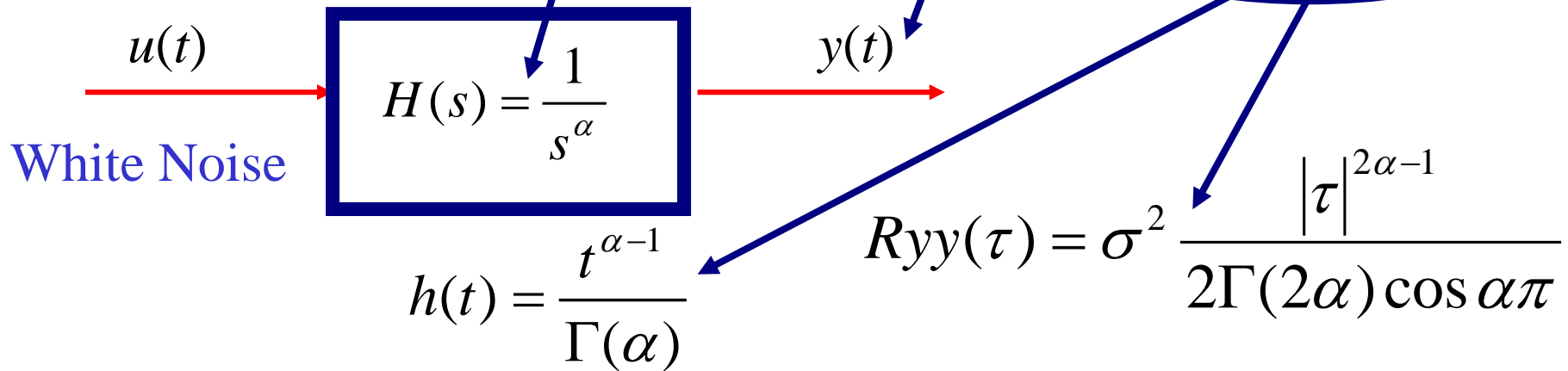


# 100+ years of LRD/HT research

- Distribution of wealth, income of individuals (20-80 rule 二八定律)
- City sizes vs. ranks - given the population, what is the city rank?
- Graphs of gene regulatory & protein-protein networks are scale free
- Long neuron inter-spike intervals in depressed mice
- Internet and WWW - scale free network (graph): fault tolerant, hubs are both the strength and Achilles' heels
- Scene lengths in VBR and MPEG video are heavy-tailed
- Computer files, Web documents, frequency of access are heavy-tailed
- Stock price fluctuations and company sizes
- Inter occurrence of catastrophic events, earthquakes - applications to reinsurance
- Frequency of words in natural languages (often called Zipf's law)

Ubiquity of Power Laws, Jankovic, 2007

# Fractional Calculus, LRD, Power Law,



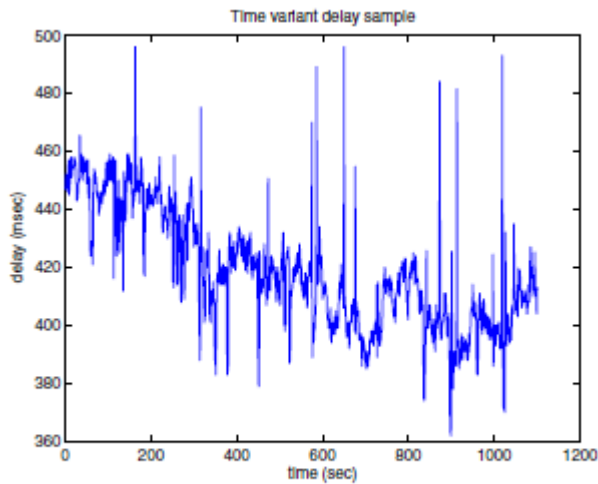
$y(t)$  is a Brownian motion when  $\alpha=1$ , i.e.,  $1/f^2$  process.

$1/f^{2\alpha}$  noise (signal) generation via fractional dynamic system

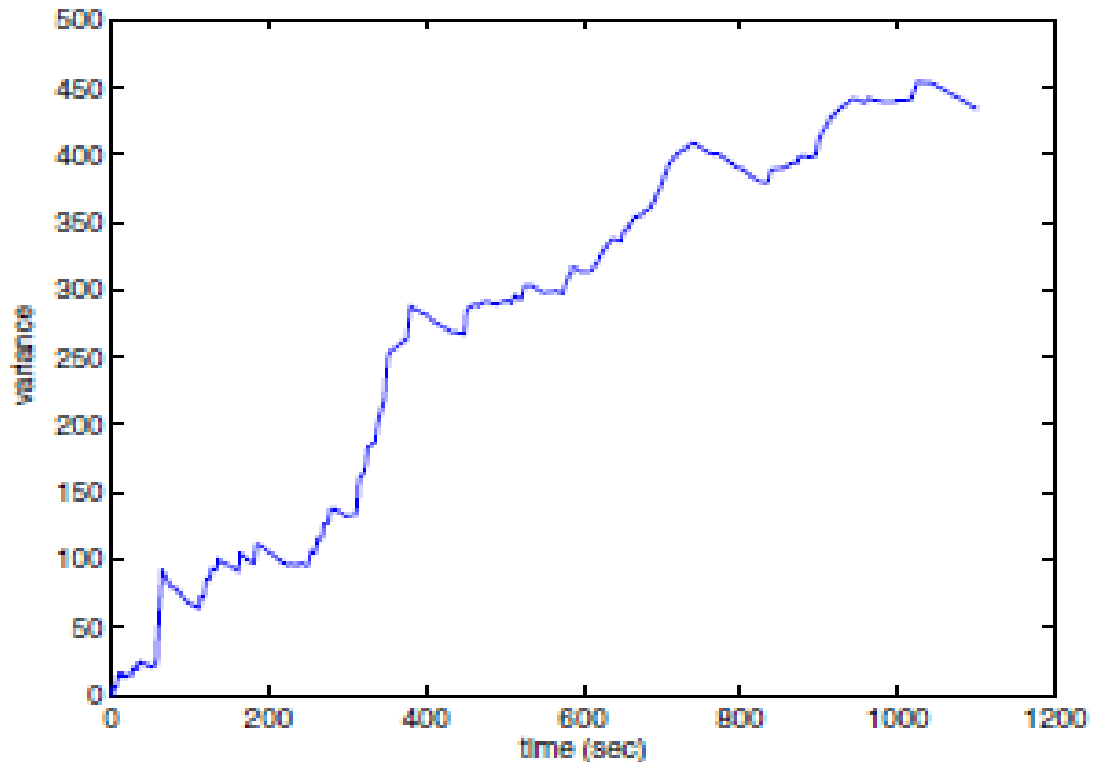
Power laws in

- Signal/Systems
- Probability distribution
- Random processes (correlation functions)

# Running variance estimate is not convergent!

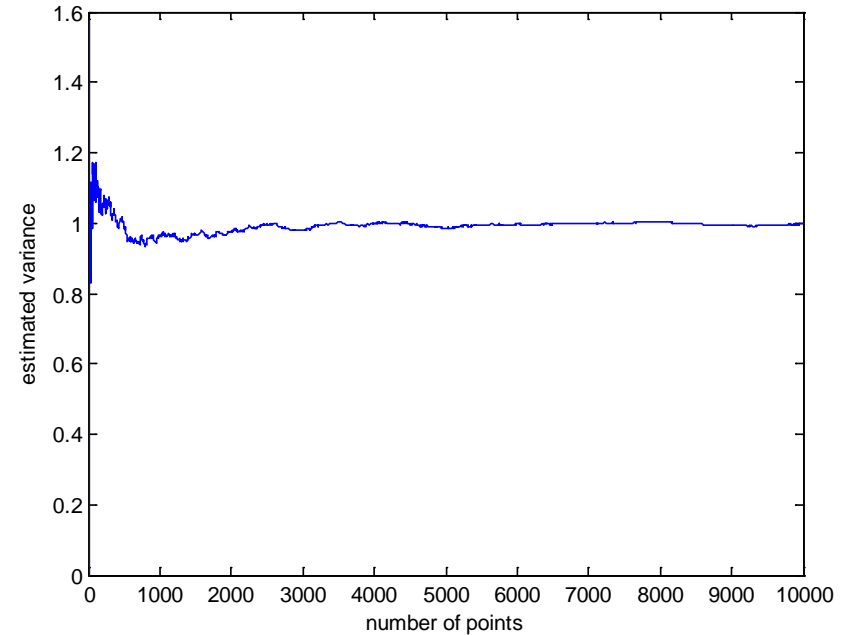
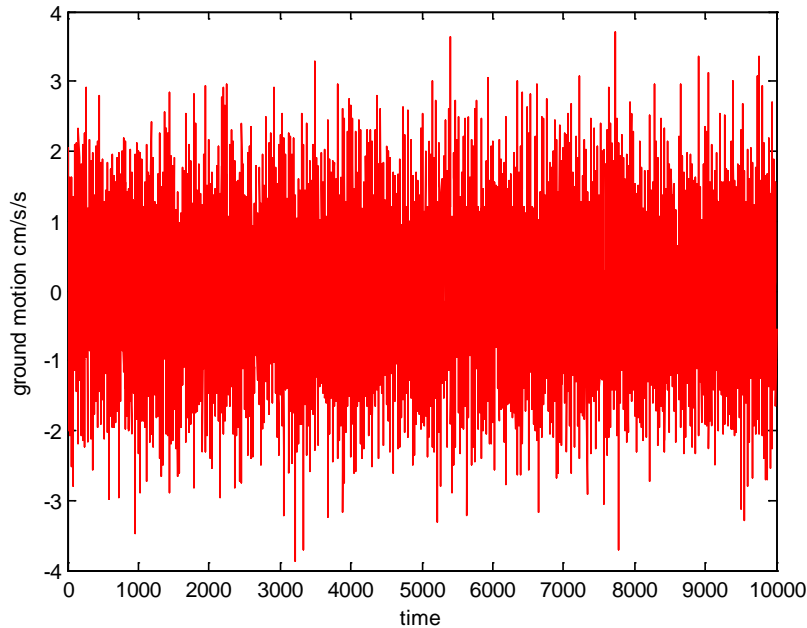


(a) Network delay samples



(b) infinite or divergent variance

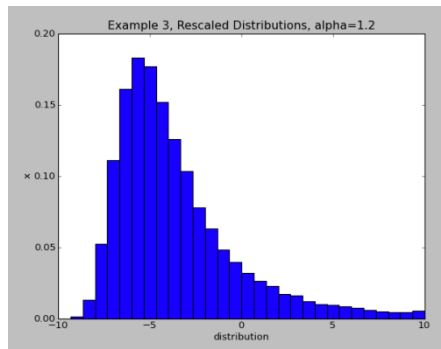
# Noise - 1



Normal distribution  $N(0,1)$     Sample Variance

## Statistics, Sociology, Natural Phenomenon, and Economics

### Statistics



[http://cours-physique.lps.ens.fr/index.php/TD4\\_Errors\\_2012\\_Fluctuations](http://cours-physique.lps.ens.fr/index.php/TD4_Errors_2012_Fluctuations)

### Sociology



<http://www.dailymail.co.uk/news/article-2164536/BBCs-coverage-Arab-Spring-sporadic-ignoring-uprisings-failed-favour-big-stories-Libya-Egypt.html>

### Nature



<http://www.shutterstock.com/s/global+warming/search.html>

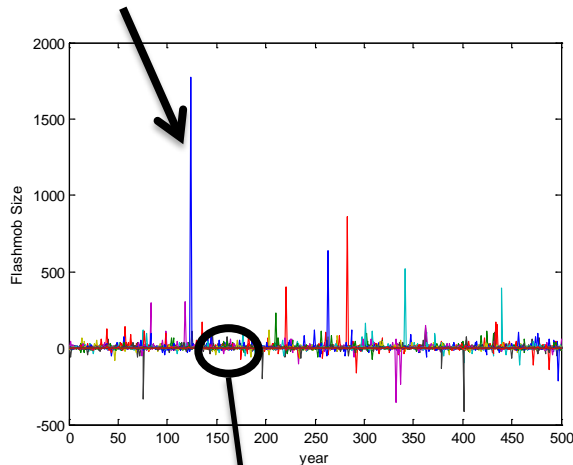
### Economics



<http://blogs.swa-jkt.com/swa/10321/tag/economics/>

# Reality is More of a Levy Walk

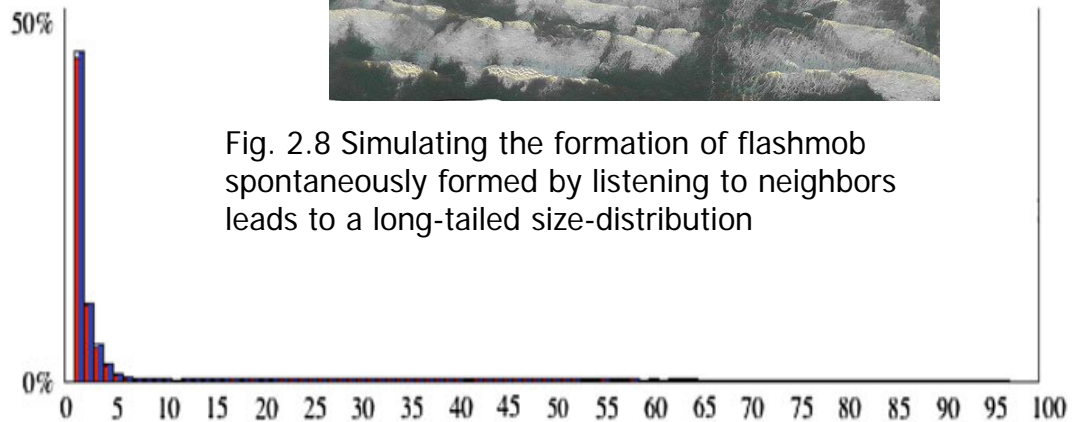
*EXTREME* events are *NATURAL*. What are some causes??



Self-Similar Statistics

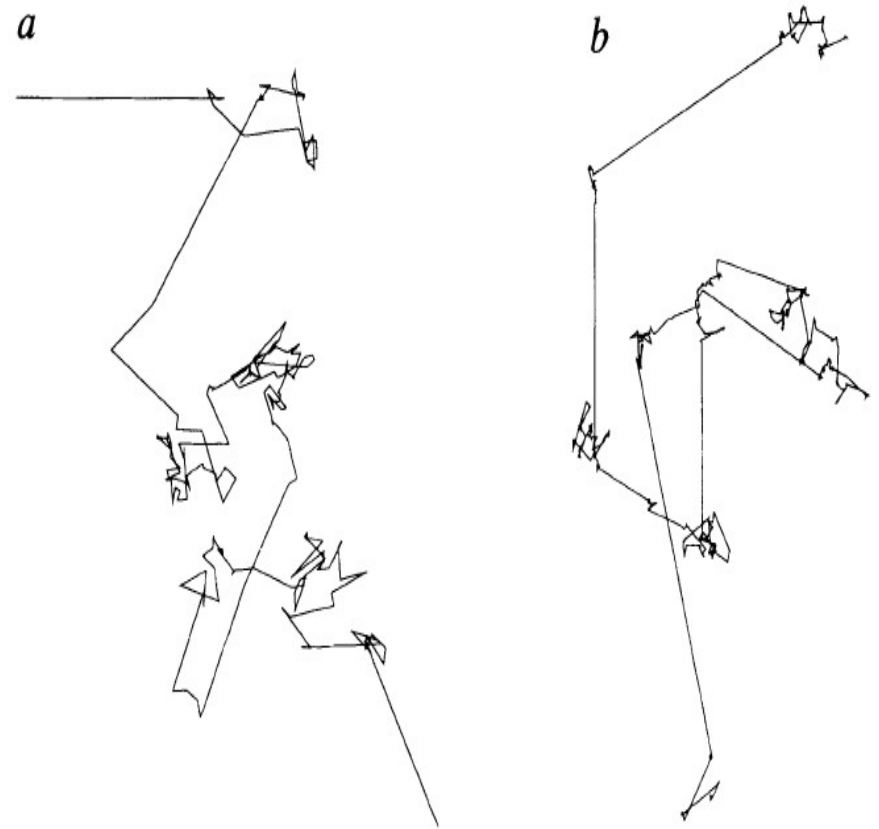


Fig. 2.8 Simulating the formation of flashmob spontaneously formed by listening to neighbors leads to a long-tailed size-distribution



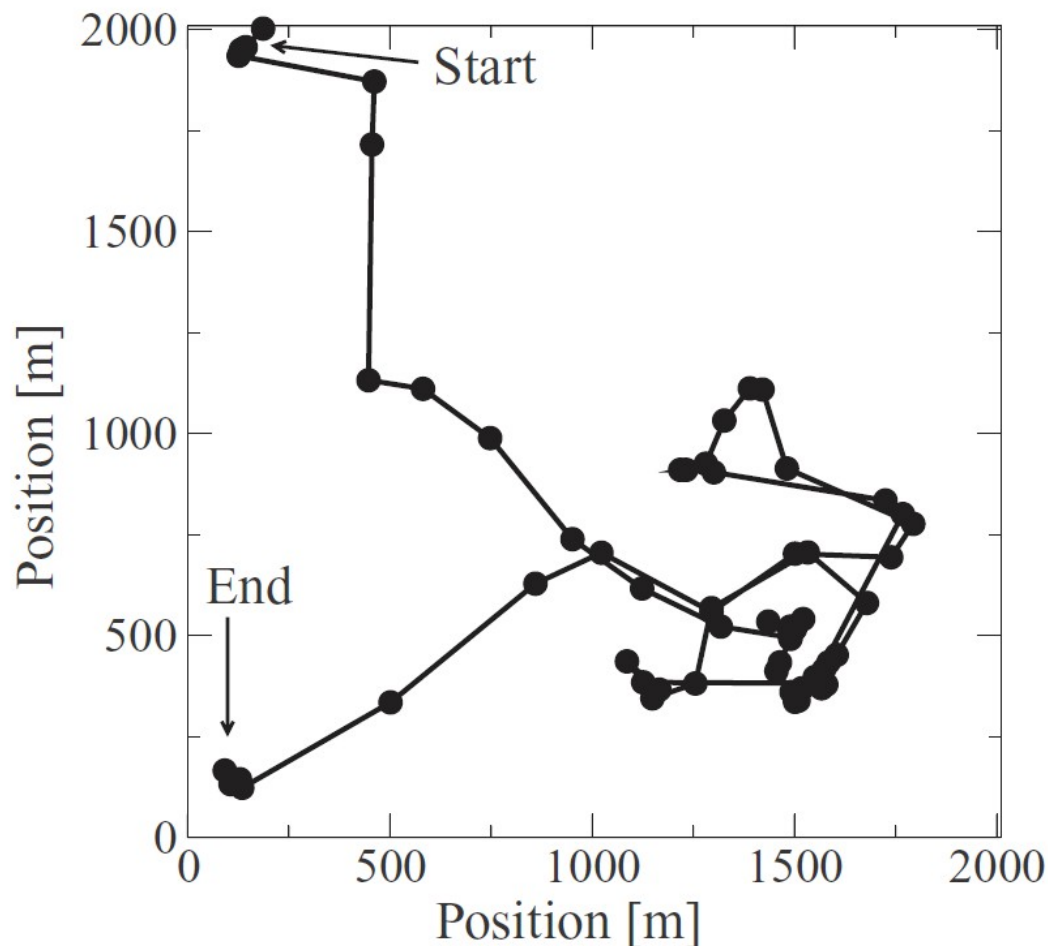


Wandering albatrosses



flight search patterns

G.M. Viswanathan, et al. *Nature* 381 (1996) 413–415.



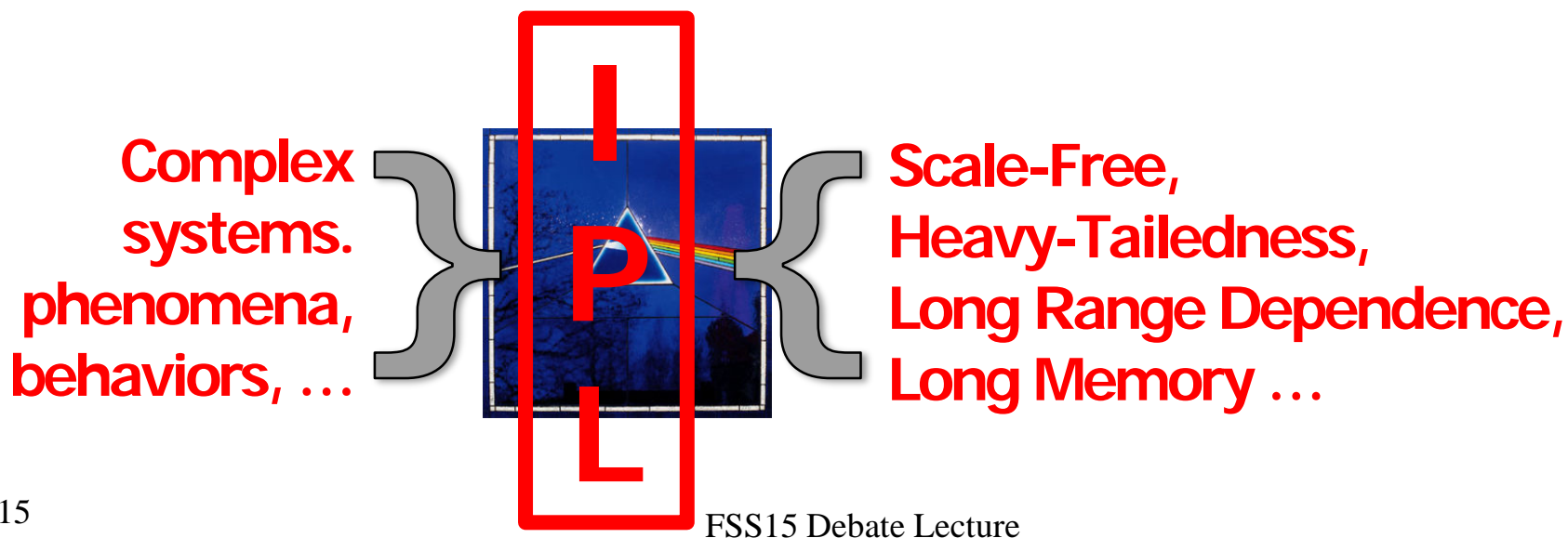
GPS traces of  
human move  
in NCSU campus

I. Rhee, et al. *IEEE INFOCOM 2008 Proceedings*, Phoenix, Arizona: Curran Associates.



# Recap – key msg

- Real worlds are complex
- We human beings perceive the complexity via our **mental prism** (lens)
- The prism (lens) is “IPL” (inverse power law) if we use integer order calculus, “Mittag-Leffer” if we use fractional calculus



# In Different Contexts

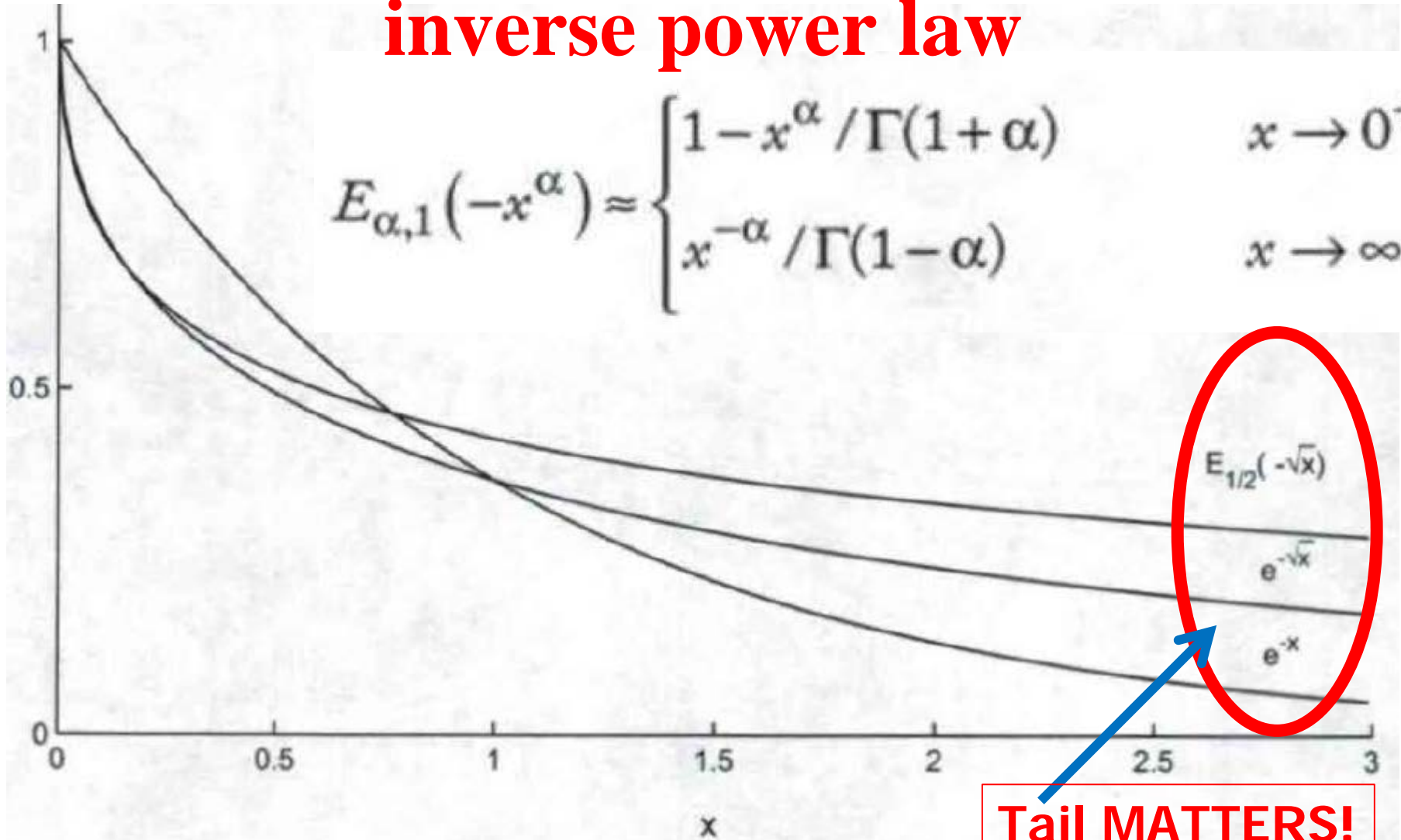
- Scale-free networks (degree distributions)
- Pink noise (power spectrum)
- Probability density function (PDF)
- Autocorrelation function (ACF)
- Allometry ( $Y = a X^b$ )
- Anomalous relaxation (evolving over time)
- Anomalous diffusion (MSD versus time)
- Self-similar

# Other connectedness to FC? (hidden)

- Fractal, irregular, anomalous, rough, Hurst
  - Multifractal, multi-scale, scale-rich
- Renormalization (?), Universality
- Extreme events– spikiness, bursty, intermittence
- Fluctuation in fluctuations; Variability,
- Emergence, Surprise, **Black swan**
- Nonlocality, Long term memory
- Complex (behavior, processes, network, fluid, dynamics, systems ...)
- When the forest is big, there are all types of birds ("It takes all kinds" 林子大了什么鸟都有), 20/80 rule(二八定律)

# Root of long (algebraic) tail, or inverse power law

$$E_{\alpha,1}(-x^\alpha) \approx \begin{cases} 1 - x^\alpha / \Gamma(1 + \alpha) & x \rightarrow 0^+ \\ x^{-\alpha} / \Gamma(1 - \alpha) & x \rightarrow \infty \end{cases}$$

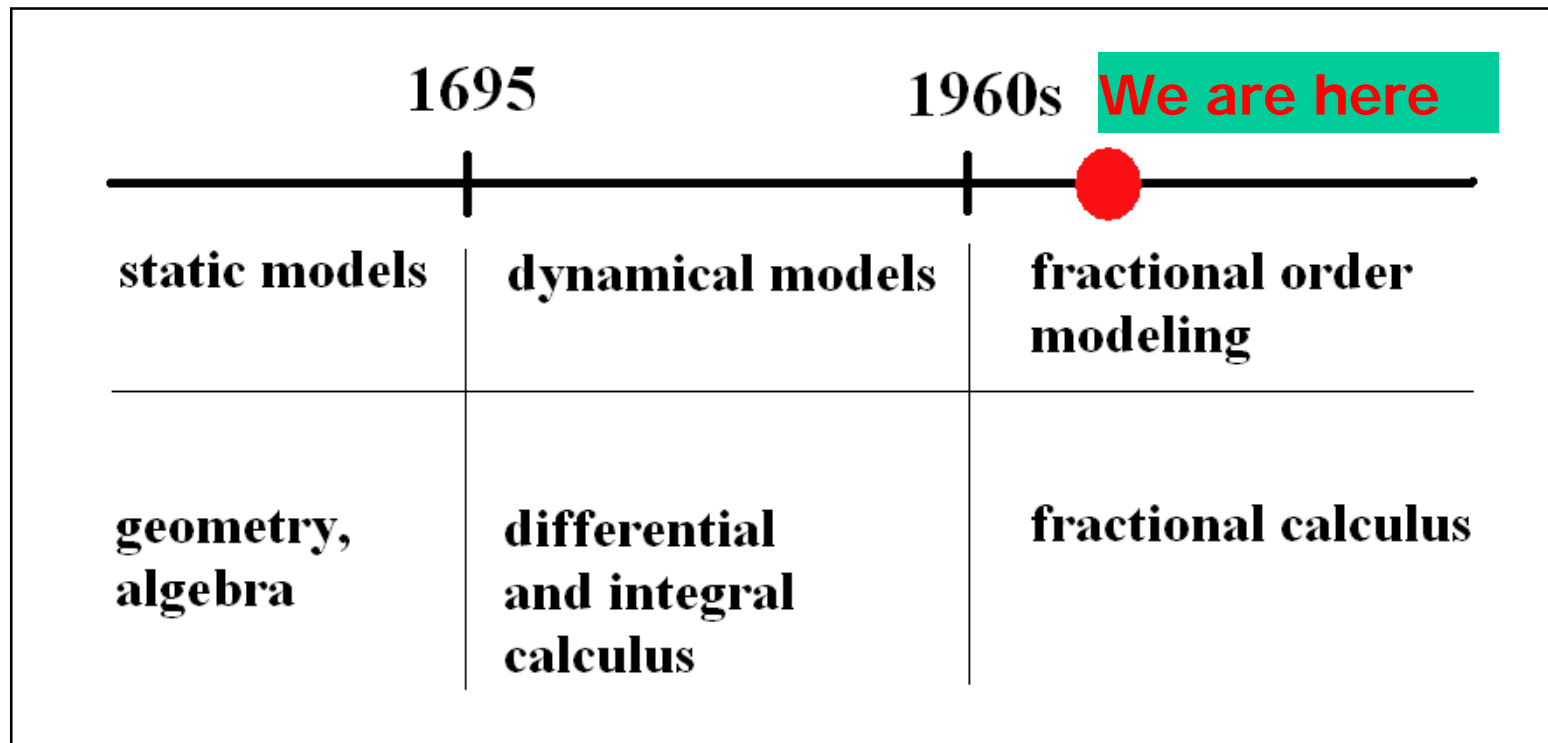


**Tail MATTERS!**

## G.W. Scott Blair (1950)

- “We may express our concepts in Newtonian terms if we find this convenient but, if we do so, we must realize that we have made a translation into a language which is foreign to the system which we are studying.”

**Fractional Calculus: a response to more advanced characterization of our more complex world at smaller scale**





Integer-Order Calculus



Fractional-Order Calculus

Slide credit: Richard L. Magin, ICC12

## Rule of thumb for “Fractional Order Thinking”

- Self-similar
- Scale-free/Scale-invariant
- Power law
- Long range dependence (LRD)
- $1/f^a$  noise
- Porous media
- Particulate
- Granular
- Lossy
- Anomaly
- Disorder
- Soil, tissue, electrodes, bio, nano, network, transport, diffusion, soft matters (**bio**x) ...



**Signal/system**  
 I D.V.M. (1)  
 II a R.V. (2) probability  
 III B R.P.V. (3) random process

**System**  
 (Frank-free) other Factors  
 U ⊗ System → y  
 F → F<sub>in</sub> + Signal  
 C System (S+β)<sup>α</sup>

**granular dynamic system**  
 John Doyle '88  
 Cal Tech  
 IIFAC FDA'06 .pt  
 L&R 11/5/05  
 L&R U=K<sup>α</sup>  
 metallurgy  
 powders  
 Controllable porosity plug

**Mathematical & Physical Concepts:**  
 z = X + Y  
 f<sub>z</sub> = f<sub>X</sub> \* f<sub>Y</sub> (convolution / superposition)  
 PWL + Convolution (superposition)  
 Long tail process R<sub>xx</sub> ∝ e<sup>-αt</sup>  
 Power law exp.  
 E(x<sub>in</sub>) ∝ (1+z)  
 = R<sub>xx</sub>  
 f(t) ∝ 1/t  
 Γ(α) α=1  
 P(X|Y) = P(X) / P(Y)  
 FRET  
 Cause → effect  
 "evolution" / "dynamic" axis  
 FOC  
 multi-scale  
 CDF

**Diagrams:**  
 Block diagram: U ⊗ System → y  
 Block diagram: f<sub>in</sub> + Signal → C System → (S+β)<sup>α</sup>  
 Block diagram: f(t) → L [1/s<sup>α</sup>] → y  
 Graph: Long tail process R<sub>xx</sub> ∝ e<sup>-αt</sup> vs t  
 Graph: f(t) ∝ 1/t vs t  
 Graph: CDF vs x

A snap shot of discussion board of Igor Podlubny and YangQuan Chen in Sept. 2005

<http://220.178.124.24:8080/wbbbs/archiver/?tid-16226.html>

# New wisdom equipped with FC

- **道可道，非常道。** ----- 世间万物的运行规律是  
可以被描述的，但它们并非一成不变的。

**Non-normal way:**  
**Fractional Calculus!**  
**Heavytailedness**

way  
by non-normal

be explained

could

The nature's rule (of complexity)

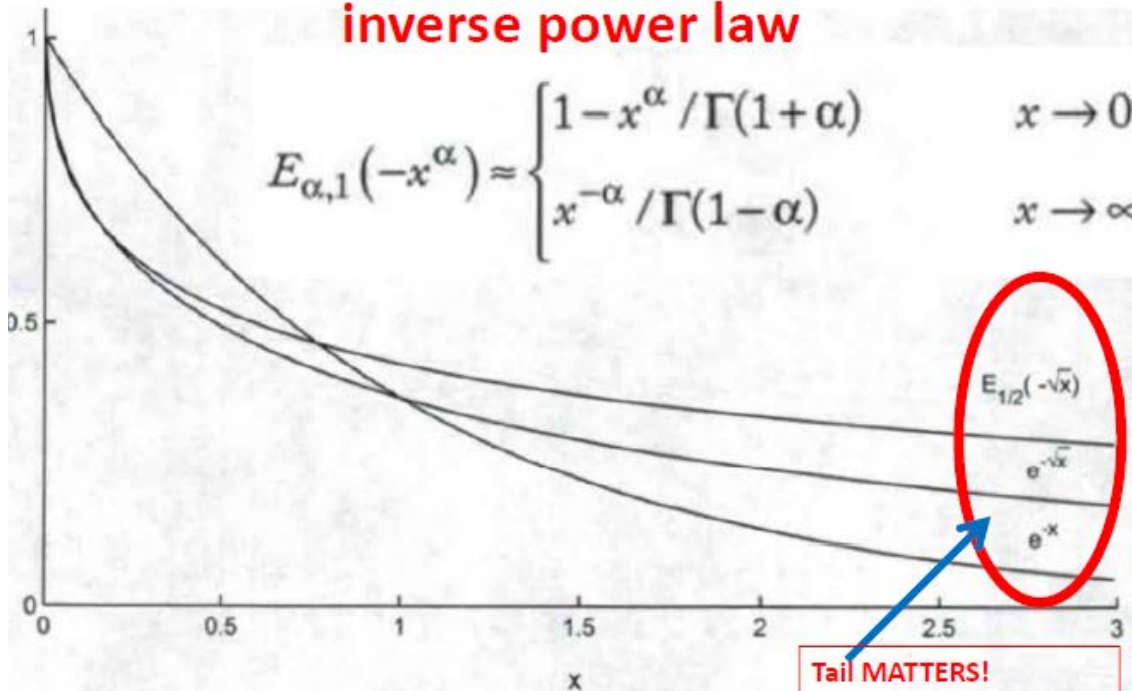
<http://220.178.124.24:8080/wbbbs/archiver/?tid-16226.html>

## New wisdom equipped with FC

- **玄之又玄，众妙之门。** ----- 了解这类对立统一体相互转变的规律，就是通向对世间万物理解的大门。

Root of long (algebraic) tail, or  
inverse power law

$$E_{\alpha,1}(-x^\alpha) \approx \begin{cases} 1 - x^\alpha / \Gamma(1+\alpha) & x \rightarrow 0^+ \\ x^{-\alpha} / \Gamma(1-\alpha) & x \rightarrow \infty \end{cases}$$



Non-normal way:

**Fractional**  
**Calculus!**  
**Heavytailedness**

# New wisdom equipped with FC

- "God is in the detail"
- "The Devil is in the detail"
  - [http://en.wikipedia.org/wiki/The\\_Devil\\_is\\_in\\_the\\_detail](http://en.wikipedia.org/wiki/The_Devil_is_in_the_detail)
- "God is in **the tail**"
- "The Devil is in **the tail**"

# Conclusions

- 7/13/1865 - “Go west, young man. Go West and grow up with the country.” – Horace Greeley (1811-1872)



[http://upload.wikimedia.org/wikipedia/commons/1/12/American\\_progress.JPG](http://upload.wikimedia.org/wikipedia/commons/1/12/American_progress.JPG)

- **“Go Fractional. It’s urgent!”** – YangQuan Chen

# Thank you for your attention!

- Q/A

# Fractional Calculus View of Complexity

Tomorrow's Science

Bruce J. West



 **CRC Press**  
Taylor & Francis Group  
A SCIENCE PUBLISHERS BOOK