Monotone iterative method for a class of fractional differential equations

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1 Abstract

Problem coonsidered:

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\begin{aligned}
&D^{\alpha}_{0+}u(t) = f(t, u(t)), \;\; t \in (0, h),\\
&t^{1-\alpha}u(t) \mid_{t=0} = u_0,
\end{aligned}
$$

where $0 < h < +\infty$, $f \in C([0, h] \times R, R)$, $D^{\alpha}_{0+}u(t)$ is the standard Riemann-Liouville fractional derivative, $0 < \alpha < 1$.

1. Some mistakes in the literatures are corrected;

2. A new condition on the nonlinear term is given to guarantee the equivalence;

3. The existence of maximal and minimal solutions for the problem is given.

2 Introduction

Some recent contributions to the theory of fractional differential equations initial value problems can be seen in [1].

In [7], the lower and upper solution method was used firstly to study the IVP

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \ \ t \in (0,1), \ \ (0 < \alpha < 1),$$
$$u(0) = 0,$$

where $f : [0,1]\times[0,+\infty) \to [0,+\infty)$ is continuous and $f(t,\cdot)$ is nondecreasing.

In [3, 6], the existence and uniqueness of solution of the IVP

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \ \ (0 < \alpha < 1; \ t > 0), \tag{2.1}$$
$$D_{0+}^{\alpha-1}u(0+) = u_0. \tag{2.2}$$

was obtained under the assumption that $f : [0,1] \times R \to R$ is Lipchitzian.

In [8], the existence and uniqueness of solution of the IVP

$$D^\alpha u(t) = f(t, u(t)), \quad t \in (0, T],$$
$$t^{1-\alpha} u(t) \mid_{t=0} = u_0$$

was discussed by using the method of lower and upper solutions and its associated monotone iterative method.

However, the maximum principle (Lemma 2.1 of [8]) is proved incorrectly. The author used the relation

$$I^1_{0+} D^\alpha_{0+} = I^{1-\alpha}_{0+}$$

although the conditions were not meet.

Motivated by the above references, we well focus our attention in this paper on the following problem

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \;\; t \in (0, h), \tag{2.3}$$

$$t^{1-\alpha}u(t) \mid_{t=0} = u_0, \tag{2.4}$$

where $f \in C([0, h] \times R, R)$, $D_{0+}^{\alpha}u(t)$ is the standard Riemann-Liouville fractional derivative, $0 < \alpha < 1$. The existence of the blow-up solution, that is to say $u \in C(0, h]$ and $\lim_{t \to 0+} u(t) = \infty$, is obtained by the use of the lower and upper solution method.

3 Preliminaries

Given $0 \le a < b < +\infty$ and $r > 0$, define a set

$$C_r[a,b] = \{u \mid u \in C(a,b], (t-a)^r u(t) \in C[a,b]\}.$$

Clearly, $C_r[a,b]$ is a linear space with the normal multiplication and addition.

Given $u \in C_r[a,b]$, define

$$\|u\| = \max_{t\in[a,b]} (t-a)^r |u(t)|,$$

then $(C_r[a,b], \|\cdot\|)$ is a Banach space.

Lemma 3.1. *[3] The linear initial value problem*

$$D_{0+}^{\alpha}u(t) + \lambda u(t) = q(t),$$
$$t^{1-\alpha}u(t)\,|_{t=0} = u_0,$$

where $\lambda \geq 0$ *is a constant and* $q \in L(0,h)$, *the solution*

$$u(t) = \Gamma(\alpha)u_0 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})q(s)ds.$$

Here, $E_{\alpha,\alpha}(t)$ *is a Mittag-Leffler function.*

Lemma 3.2. *For* $0 < \alpha \leq 1$, *the Mittag-Leffler type function* $E_{\alpha,\alpha}(-\lambda t^{\alpha})$ *satisfies*

$$0 \leq E_{\alpha,\alpha}(-\lambda t^{\alpha}) \leq \frac{1}{\Gamma(\alpha)}, \quad t \in [0, \infty), \ \ \lambda \geq 0.$$

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

The function $g(t) := E_{\alpha,\alpha}(-\lambda t^{\alpha})$, $t \in (0, +\infty)$ is completely monotonic, that is to say that $g(t)$ possesses of derivatives $g^{(n)}(t)$ for all $n = 0, 1, 2, \cdots$, and $(-1)^n f^{(n)}(t) \geq 0$ for all $t \in (0, \infty)$.

Lemma 3.3. *([2]) Suppose that* E *is an ordered Banach space,* $x_0, y_0 \in E$, $x_0 \leq y_0$, $D = [x_0, y_0]$, $T : D \to E$ *is an increasing completely continuous operator and* $x_0 \leq Tx_0$, $y_0 \geq Ty_0$*. Then the operator* T *has a minimal fixed point* x^* *and a maximal fixed point* y^**. If we let*

$$x_n = Tx_{n-1}, \;\; y_n = Ty_{n-1}, \quad n = 1, 2, 3, \cdots ,$$

then

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_2 \leq y_1 \leq y_0,$$

$$x_n \to x^*, \quad y_n \to y^*.$$

Zero-point theorem

$$x^5 - 20x^3 + 3x^2 - 1 \doteq f(x) = 0, \quad f(-1) > 0; \;\; f(0) < 0; \;\; f(10) > 0.$$

Definition 3.4. *A function* $v(t) \in C_{1-\alpha}[0, h]$ *is called as a lower solution of problem* (2.3), (2.4), *if it satisfies*

$$D^{\alpha}_{0+}v(t) \leq f(t, v(t)), \;\; t \in (0, h), \tag{3.1}$$

$$t^{1-\alpha}v(t) \mid_{t=0} \leq u_0. \tag{3.2}$$

Definition 3.5. *A function* $w(t) \in C_{1-\alpha}[0, h]$ *is called as an upper solution of problem* (2.3), (2.4), *if it satisfies*

$$D^{\alpha}_{0+}w(t) \geq f(t, w(t)), \;\; t \in (0, h), \tag{3.3}$$

$$t^{1-\alpha}w(t) \mid_{t=0} \geq u_0. \tag{3.4}$$

4 The existence of solutions

The following assumptions will be used in our main results:

$[S1]$ $f : [0,h] \times R \to R$ and there exist constants $A, B \geq 0$ and $0 < r_1 \leq 1 < r_2 < 1/(1-\alpha)$ such that for $t \in [0,h]$

$$|f(t,u) - f(t,v)| \leq A|u-v|^{r_1} + B|u-v|^{r_2}, \; u, v \in R. \tag{3.1}$$

$[S2]$ Assume that $f : [0,h] \times R \to R$ satisfies

$$f(t,u) - f(t,v) + \lambda(u-v) \geq 0, \quad \text{for} \quad \hat{u} \leq v \leq u \leq \tilde{u}, \tag{3.2}$$

where $\lambda \geq 0$ is a constant and $\hat{u}, \tilde{u}$ are lower and upper solutions of Problem (2.3), (2.4) respectively.

Theorem 3.1. *Suppose* $[S1]$ *holds. The function* u *solves the problem* (2.3), (2.4) *if and only if it is a fixed-point of the operator* $T : C_{1-\alpha}[0,h] \to C_{1-\alpha}[0,h]$ *defined by*

$$(Tu)(t) = \Gamma(\alpha)u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})[f(s,u(s)) + \lambda u(s)]ds.$$

Theorem 3.2. *Suppose [S1] holds. Then the operator* T *is a completely continuous operator.*

Theorem 3.3. *Assume* $[S1]$ *and* $[S2]$ *hold and* $v, w \in C_{1-\alpha}[0,h]$ *be lower and upper solutions of problem* (2.3), (2.4) *respectively such that*

$$\nu(t) < \omega(t), \ \ 0 \leq t \leq h. \tag{3.3}$$

Then, the fractional IVP (2.3), (2.4) *has a minimal solution* x^* *and a maximal solution* y^* *such that*

$$x^* = lim_{n\to\infty} T^n v, \qquad y^* = lim_{n\to\infty} T^n w.$$

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Thank you for your time and patience.