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Mittag-Leffler convergent backstepping observers for coupled semilinear subdiffusion systems with spatially varying parameters

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ABSTRACT

The purpose of this paper is to investigate the observer-based boundary output feedback control for subdiffusion processes governed by coupled semilinear time fractional diffusion systems (TFDSs) with spatially varying parameters. For this, backstepping technique is used to Mittag-Leffler stabilize the coupled semilinear observer error dynamic systems. We then design an observer-based output feedback controller at the right boundary to realize the Mittag-Leffler stability of the closed-loop systems at hand. A numerical example is finally included to test our methods.

1. Introduction

Mittag-Leffler stability for systems with fractional derivatives is a fundamental issue. One pioneering work in this direction is [1] from the point of view of Lyapunov's direct method. A detailed overview of these and related results can be found in the survey [2] and the book [3]. However, from a practical point of view, researchers are more interested in the stabilization problem. There are many studies on the stabilization of finite dimensional fractional order systems such as in [4] by designing $PI^{\lambda}D^{\mu}$-controllers or in [5] based on linear matrix inequalities. However, few results are available for the stabilization of infinite dimensional fractional order systems. In [6], initial investigations on the boundary stabilization of fractional diffusion-wave equation were performed mainly by numerical simulations without giving a rigorous mathematical proof. In [7,8], the boundary feedback stabilization for a linear unstable time fractional reaction–diffusion system was studied by the backstepping method. Considering that semilinear models are suitable for a much wider range of physical phenomena, an extended Luenberger observer to solve the state observation problem for semilinear TFDSs with constant coefficients and Neumann boundary conditions (BCs) was presented in [9]. In the present paper, we extend these results and consider the output feedback control problem for a class of coupled semilinear TFDSs with spatially varying parameters and Robin BCs based on the fractional Lyapunov's method.

In general the full state information, that may be needed for feedback control or processing monitoring, is not available directly due to a lack of suitable sensors capturing the spatial and temporal dynamics. To overcome this limitation, similar to the methods introduced in [10], we here design a backstepping observer based on boundary measurements. With this, a dynamic boundary output feedback controller is obtained to achieve the desired performance of the closed-loop system at hand.

Motivated by these above considerations, in this paper, we deal with a class of coupled semilinear TFDSs of order $\alpha \in (0,1]$ as follows:

$$ {}_0^C D_t^{\alpha} y(x,t) = b(x) y_{xx}(x,t) + c(x) y_x(x,t) + f(x,t,y(x,t)) \tag{1}$$

for $(x,t) \in (0,l) \times (0,\infty)$, with $y = (y_1, y_2, \ldots, y_n)^T \in \left[L^2\left((0,l)\times(0,\infty)\right)\right]^n$, where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative with respect to t given by [11]

$$ {}_0^C D_t^{\alpha} y(\cdot,t) = \left({}_0 I_t^{1-\alpha} \frac{\partial y_1}{\partial t}(\cdot,t), \ldots, {}_0 I_t^{1-\alpha} \frac{\partial y_n}{\partial t}(\cdot,t) \right)^T $$

and ${}_0 I_t^{\alpha} y_i(\cdot,t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_i(\cdot,s) ds,\ i = 1,2,\ldots,n$ denotes the Riemann–Liouville fractional integral. Here $b(x)$, $c(x)$ and $f(x,t,y(x,t))$ are three functions to be specified later.

Note that the considered system (1) can be regarded as an extension of conventional non-fractional diffusion system, where

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the first order time derivative is generalized to a Caputo fractional derivative of order $\alpha \in (0, 1]$ and covers a great deal of applications. As cited in [12], it can be used to model subdiffusion processes in a spatially inhomogeneous environment. Typical examples include the flow through porous media with varying sources or sinks [13], or reheating processes of the heterogeneous metal slabs [14], etc.

It is worth mentioning that the past two decades have witnessed significant developments in the study of backstepping observer-based boundary output feedback control problem for non-fractional coupled diffusion systems. In [15], an output feedback boundary controller was proposed to stabilize a coupled PDE–ODE with interaction at the interface. The authors in [16,17] addressed the output feedback control problem for a set of coupled linear PDEs with different diffusions. Besides, in [16], the boundary stabilization for some classes of linear coupled reaction–diffusion processes with constant parameters was considered. The backstepping design of state feedback controller for non-fractional linear coupled diffusion systems with spatially-varying coefficients was studied in [18,19], where the spatially-varying coefficients were directly considered but they did not give the explicit expression of the kernel functions. Therefore, we claim that our results in this paper are still novel even for the special case when the order α in system (1) is equal to 1. Moreover, to the best of our knowledge, no results are available for the backstepping observer-based boundary output feedback control of coupled semilinear TFDSs.

Since the solutions of TFDSs are usually expressed with respect to Mittag-Leffler functions [2,9], the contribution of this paper is to consider the Mittag-Leffler stabilization of system (1) by exploiting an actuator and a sensor at opposite ends. This is called to be the anti-collocated setting. The results here can rather easily be extended to the collocated case when sensor and actuator are placed at the same end.

The structure of the paper is as follows. In Section 2, we formulate our problem and introduce some preliminary results to be used thereafter. The backstepping observer and the Mittag-Leffler stability of the observer error system are shown in Section 3. In Section 4, we consider the boundary feedback stabilization of the closed-loop system at hand. Simulation results in Section 5 illustrate the performance of controller and observer.

2. Problem formulation and preliminaries

As is illustrated in the Appendix, the considered system (1) is without loss of generality equivalent to the following simpler coupled semilinear system:

$$ {}_0^C D_t^{\alpha} y(x,t) = y_{xx}(x,t) + f(x,t,y(x,t)) \tag{2} $$

defined on $(x, t) \in (0, L) \times (0, \infty)$ with $L = \int_0^l b^{-\frac{1}{2}}(s)ds$, where $y = (y_1, y_2, \ldots, y_n)^T$ denotes the system variable and $f(x, t, y(x, t)) = (f_1(x, t, y(x, t)), f_2(x, t, y(x, t)), \ldots, f_n(x, t, y(x, t)))^T$. For the BCs, since only Dirichlet BCs are preserved under the proposed transformations and both Neumann BCs and Robin BCs are converted into Robin BCs, we consider the following general BCs

$$ \begin{cases} p_1 y_x(0,t) - p_2 y(0,t) = \theta, \\ q_1 y_x(L,t) + q_2 y(L,t) = u(t), \ \ t \in (0,\infty), \end{cases} \tag{3} $$

where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ is the actuation, $\theta = (0, \ldots, 0)^T$ and $p_i = \operatorname{diag}(p_{i1}, p_{i2}, \ldots, p_{in})$, $q_i = \operatorname{diag}(q_{i1}, q_{i2}, \ldots, q_{in})$, $i \in J_2 \triangleq \{1, 2\}$ are four constant matrices such that $p_{ij}, q_{ij} \geqslant 0, j \in J_n \triangleq \{1, 2, \ldots, n\}$ and $p_{1j}+p_{2j} > 0, q_{1j}+q_{2j} > 0$. The consistent initial condition reads

$$ y(x,0) = y_0(x) \triangleq (y_1^0(x), y_2^0(x), \ldots, y_n^0(x))^T. \tag{4} $$

Let $Y = L^2(0, L)$, $E = L^2((0, L) \times (0, \infty))$ and $U = L^2(0, \infty)$ be three usual square integrable Hilbert spaces. One has $E = Y \times U$. To obtain our results, the following assumption on function $f : (0, L) \times [0, \infty) \times E^n \to E^n$ is assumed to hold:

Assumption 1. $f(x, t, \theta) = \theta$ and given a constant $R \geqslant 0, y, y^* \in E^n$ with $\|y\|_{E^n}, \|y^*\|_{E^n} \leqslant R$, there exists a positive constant $c = c(R)$ so that

$$ \begin{aligned} &\|f(\cdot,t,y(\cdot,t)) - f(\cdot,t,y^*(\cdot,t))\|_{Y^n} \\ &\leqslant c\,\|y(\cdot,t) - y^*(\cdot,t)\|_{Y^n} \end{aligned} \tag{5} $$

holds true for all $t \geqslant 0$ with the norm $\|y(\cdot,t)\|_{Y^n}^2 \triangleq \sum_{i=1}^n \|y_i(\cdot,t)\|_Y^2$, i.e.,$f$ is Lipschitz continuous in x and uniformly in t.

Introduce the operator $A \triangleq \frac{\partial}{\partial x^2}$ with the domain

$$ \mathcal{D}(A) = \left\{ \psi \in Y^n : \begin{array}{l} p_1\psi_x(0) - p_2\psi(0) = 0, \\ q_1\psi_x(L) + q_2\psi(L) = 0 \end{array} \right\}. \tag{6} $$

Then Lemma 2 of [20] yields that the eigenvalues $\{\lambda_n\}_{n\geqslant 1}$ of $(A, \mathcal{D}(A))$ satisfy $\lambda_n \leqslant 0$. However, we note that the open-loop system (2)–(4) (with $u(t) \equiv 0$) possesses arbitrarily many unstable eigenvalues if the semilinear vector f is large enough. Taking into account that not all needed spatially distributed state for feedback control could be measured by physical sensors, we design an observer-based boundary feedback controller. To this end, the output is assumed to be given as follows

$$ z(t) \triangleq y(0,t) = (y_1(0,t), y_2(0,t), \ldots, y_n(0,t))^T \tag{7} $$

with

$$ \det(p_1) \neq 0, \ \text{i.e., } p_{1j} \neq 0 \text{ for all } j \in J_n. \tag{8} $$

Let $E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}, \alpha > 0, t \geqslant 0$ denote the Mittag-Leffler function in two parameters. We write $E_{\alpha,1}(t) = E_\alpha(t)$ for short, when $\beta = 1$.

Definition 2.1 ([1]). The solution of system (2)–(4) is said to be Mittag-Leffler stable in the Y^n norm if there exist three constants $\sigma, \epsilon, b > 0$ such that

$$ \|y(\cdot,t)\|_{Y^n} \leqslant \sigma \|y_0\|_{Y^n} \{E_\alpha(-\epsilon t^\alpha)\}^b, \ t \geqslant 0. \tag{9} $$

3. Observer design and implementation

3.1. Observer design

Consider the following coupled semilinear observer

$$ \begin{cases} {}_0^C D_t^{\alpha} \hat{y}(x,t) = \hat{y}_{xx}(x,t) + f(x,t,\hat{y}(x,t)) \\ \quad + K_1(x)[z(t) - \hat{z}(t)] \text{ in } (0,L)\times(0,\infty), \\ p_1\hat{y}_x(0,t) - p_2\hat{y}(0,t) = K_2[z(t)-\hat{z}(t)] \text{ in } (0,\infty), \\ q_1\hat{y}_x(L,t) + q_2\hat{y}(L,t) = u(t) \text{ in } (0,\infty), \\ \hat{y}(x,0) = \hat{y}_0(x) \ \text{ in } (0,L), \end{cases} \tag{10} $$

where $\hat{y}(x, t) = (\hat{y}_1(x, t), \hat{y}_2(x, t), \ldots, \hat{y}_n(x, t))^T$ and $\hat{z}(t) = \hat{y}(0, t)$. Here the $n \times n$ matrices $K_1(x)$ and K_2 are the observer gains to be determined to ensure the stability of the observer error $e(x, t) \triangleq y(x, t) - \hat{y}(x, t)$. It then follows from (2)–(4) and (10) that $e(x, t)$ is governed by

$$ \begin{cases} {}_0^C D_t^{\alpha} e(x,t) = e_{xx}(x,t) - f(x,t,\hat{y}(x,t)) \\ \qquad + f(x,t,e(x,t)+\hat{y}(x,t)) \\ \qquad - K_1(x)e(0,t) \text{ in } (0,L)\times(0,\infty), \\ p_1 e_x(0,t) - (p_2 - K_2)\, e(0,t) = \theta \text{ in } (0,\infty), \\ q_1 e_x(L,t) + q_2 e(L,t) = \theta \text{ in } (0,\infty), \\ e(x,0) = e_0(x) \ \text{ in } (0,L). \end{cases} \tag{11} $$

To apply the backstepping technique, we consider the following target system

$$\begin{cases} {}_0^C D_t^\alpha \omega(x,t) = \omega_{xx}(x,t) - \Theta\omega(x,t) + \varphi \\ \qquad\qquad \text{in } (0,L)\times(0,\infty), \\ r_1\omega_x(0,t) - r_2\omega(0,t) = \theta \text{ in } (0,\infty), \\ s_1\omega_x(L,t) + s_2\omega(L,t) = \theta \text{ in } (0,\infty), \\ \omega(x,0) = \omega_0(x) \text{ in } (0,L), \end{cases} \tag{12}$$

where $\omega(x,t) = (\omega_1(x,t), \omega_2(x,t), \ldots, \omega_n(x,t))^T$, and r_i = diag $(r_{i1}, r_{i2}, \ldots, r_{in})$, $s_i = \text{diag}(s_{i1}, s_{i2}, \ldots, s_{in})$, $i \in J_2$ are four diagonal matrices satisfying Lemma 2 of [20], and $\varphi \triangleq \varphi(x,t,e(x,t),\hat{y}(x,t))$ is a function to be specified later.

In what follows, we transfer the observer error system (11) into (12) and design Θ, which is a $n \times n$ constant matrix with components μ_{ij} for $i,j \in J_n$, so that system (12) is Mittag-Leffler stable in the Y^n norm (see Theorem 3.1 for more details).

3.2. Stabilization of the observer error system (11)

With these preparation, consider the following integral transformation

$$\omega(x,t) = e(x,t) - \int_0^x G(x,\varsigma)e(\varsigma,t)d\varsigma. \tag{13}$$

By differentiating both sides of (13) with respect to x and t, together with $e(0,t) = \omega(0,t)$ and $\frac{d}{dx}G(x,x) = G_x(x,x) + G_\varsigma(x,x)$, we obtain

$$\begin{aligned} 0 &= {}_0^C D_t^\alpha \omega(x,t) - \omega_{xx}(x,t) + \Theta\omega(x,t) - \varphi \\ &= f(x,t,e(x,t)+\hat{y}(x,t)) - f(x,t,\hat{y}(x,t)) \\ &\quad - \int_0^x G(x,\varsigma) f(\varsigma,t,e(\varsigma,t)+\hat{y}(\varsigma,t))d\varsigma \\ &\quad + \int_0^x G(x,\varsigma) f(\varsigma,t,\hat{y}(\varsigma,t))d\varsigma - \varphi \\ &\quad + \left(2\frac{d}{dx}G(x,x) + \Theta\right) e(x,t) \\ &\quad + \int_0^x \left\{ \begin{array}{l} G_{xx}(x,\varsigma) - G_{\varsigma\varsigma}(x,\varsigma) \\ -\Theta G(x,\varsigma) \end{array} \right\} e(\varsigma,t)d\varsigma \\ &\quad + \left\{ \begin{array}{l} \int_0^x G(x,\varsigma)K_1(\varsigma)d\varsigma - G_\varsigma(x,0) \\ -K_1(x) + G(x,0)p_1^{-1}(p_2 - K_2) \end{array} \right\} e(0,t). \end{aligned}$$

Observing that this equation has to hold for all $(x,t) \in (0,L) \times (0,\infty)$, let

$$\begin{aligned} \varphi &\triangleq \varphi(x,t,e(x,t),\hat{y}(x,t)) \\ &= f(x,t,e(x,t)+\hat{y}(x,t)) - f(x,t,\hat{y}(x,t)) \\ &\quad - \int_0^x G(x,\varsigma) \left\{ \begin{array}{l} f(\varsigma,t,e(\varsigma,t)+\hat{y}(\varsigma,t)) \\ -f(\varsigma,t,\hat{y}(\varsigma,t)) \end{array} \right\} d\varsigma. \end{aligned} \tag{14}$$

The observer gain K_2 is obtained by evaluating the BCs at $x = 0$, i.e.,

$$K_2 = p_2 - p_1 G(0,0) - p_1 r_1^{-1} r_2, \tag{15}$$

where $\det(r_1) \neq 0$ is imposed, which is in consistent with (8). Choose $K_1(x)$ as

$$\begin{aligned} K_1(x) = &\int_0^x G(x,\varsigma)K_1(\varsigma)d\varsigma - G_\varsigma(x,0) \\ &+ G(x,0)\left(G(0,0) + r_1^{-1}r_2\right). \end{aligned} \tag{16}$$

Then we get that G has to fulfill

$$\begin{cases} G_{xx}(x,\varsigma) - G_{\varsigma\varsigma}(x,\varsigma) = \Theta G(x,\varsigma), 0 < \varsigma < x < L, \\ 2\frac{d}{dx}G(x,x) = -\Theta, \ 0 < x < L. \end{cases} \tag{17}$$

Since q_1 and s_1 are two diagonal matrices, in order to simplify the complexity of the transformation, we assume that

$$G(x,\varsigma) = \text{diag}(g_1(x,\varsigma), g_2(x,\varsigma), \ldots, g_n(x,\varsigma)). \tag{18}$$

With this, for the right side of the BCs, one has,

(1) if $q_1, s_1 = 0_{n\times n}$: $G(L,\varsigma) = 0_{n\times n}$;

(2) if $q_1, s_1 \neq 0_{n\times n}$ and $\exists i \in J_n$, $q_{1i} = s_{1i} = 0$:

$$\begin{cases} g_i(L,\varsigma) = 0, \ s_{1j}\dfrac{\partial g_j}{\partial x}(L,\varsigma) + s_{2j}g_j(L,\varsigma) = 0, \\ g_j(L,L) = \dfrac{q_{2j}}{q_{1j}} - \dfrac{s_{2j}}{s_{1j}}, \ j \in J_n, \ j \neq i; \end{cases} \tag{19}$$

(3) if $\det(q_1), \det(s_1) \neq 0$:

$$\begin{cases} s_1 G_x(L,\varsigma) + s_2 G(L,\varsigma) = 0, \\ G(L,L) = q_1^{-1}q_2 - s_1^{-1}s_2. \end{cases}$$

Remark 3.1. It is worth noting that the simplifying assumption (18) can be considered as an extension of that in [16,17], although it still imposes some constraints on the choice of the matrix Θ. It is supposed in [16,17] that $g_1(x,\varsigma) = g_2(x,\varsigma) = \cdots = g_n(x,\varsigma)$. This is due to the fact that in general, $G(x,\varsigma)\Lambda \neq \Lambda G(x,\varsigma)$ for $\Lambda \in \mathbb{R}^{n\times n}$ such as $\Lambda = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ when $n = 2$ and $g_1(x,\varsigma) \neq g_2(x,\varsigma)$.

With these preliminaries, design the matrix Θ as $\Theta \triangleq \text{diag}(\mu_{11}, \mu_{22}, \ldots, \mu_{nn})$ with $\mu_{ii} > 0$ and let $\text{diag}(c_1, c_2, \ldots, c_n) \triangleq s_1^{-1}s_2$ when $\det(q_1), \det(s_1) \neq 0$. Based on Section 5.1 of [21] and [22], we obtain the following result.

Lemma 3.1. *For the solution of the kernel function $G(x,\varsigma)$ governed by (17)–(19), let $\widehat{g}_i(x,\varsigma) = -\mu_{ii}(L-x)\frac{I_1\left(\sqrt{\mu_{ii}(2L-x-\varsigma)(x-\varsigma)}\right)}{\sqrt{\mu_{ii}(2L-x-\varsigma)(x-\varsigma)}}$, $i \in J_n$ and*

$$\begin{aligned} \widetilde{g}_j(x,\varsigma) = &-\mu_{jj}(L-\varsigma)\frac{I_1\left(\sqrt{\mu_{jj}(2L-x-\varsigma)(x-\varsigma)}\right)}{\sqrt{\mu_{jj}(2L-x-\varsigma)(x-\varsigma)}} \\ &+ \frac{c_j\mu_{jj}}{\sqrt{\mu_{jj}+c_j^2}}\int_0^{x-\varsigma} e^{\frac{-c_js}{2}} \sinh\left(\frac{\sqrt{\mu_{jj}+c_j^2}}{2}s\right) \times \\ &I_0\left(\sqrt{\mu_{jj}(2L-x-\varsigma)(x-\varsigma-s)}\right) ds, \ j \in J_n, \end{aligned}$$

where I_0 and I_1 are the modified zero order and first order Bessel function, respectively. The following conclusions hold true [21,22]:

(1) if $q_1, s_1 = 0_{n\times n}$:

$$G(x,\varsigma) = diag(\widehat{g}_1(x,\varsigma), \widehat{g}_2(x,\varsigma), \ldots, \widehat{g}_n(x,\varsigma));$$

(2) if $q_1, s_1 \neq 0_{n\times n}$ and $\exists i \in J_n$ such that $q_{1i} = s_{1i} = 0$: $g_i(x,\varsigma) = \widehat{g}_i(x,\varsigma)$ and $g_j(x,\varsigma) = \widetilde{g}_j(x,\varsigma)$ for $j \in J_n$, $j \neq i$;

(3) if $\det(q_1), \det(s_1) \neq 0$:

$$G(x,\varsigma) = diag(\widetilde{g}_1(x,\varsigma), \widetilde{g}_2(x,\varsigma), \ldots, \widetilde{g}_n(x,\varsigma)).$$

Moreover, for any given $t \geqslant 0$, the operator $\mathcal{E} : Y^n \to Y^n$ defined as

$$\omega(x,t) = (\mathcal{E}e)(x,t) = e(x,t) - \int_0^x G(x,\varsigma)e(\varsigma,t)d\varsigma$$

is invertible [22], i.e., $\mathcal{E}$ and its inverse $\mathcal{E}^{-1}$ given by $e(x,t) = (\mathcal{E}^{-1}\omega)(x,t) = \omega(x,t) + \int_0^x H(x,\varsigma)\omega(\varsigma,t)d\varsigma$ are all bounded.

Before verifying the stability of the observer error system (11), we give the following estimation result.

Lemma 3.2. *If Assumption 1 is satisfied, then for constant $R > 0$, there is a constant $M_1 = M_1(R) > 0$ such that the following* *inequality*

$$\begin{aligned}&\left(\omega(\cdot,t),\varphi(\cdot,t,e(\cdot,t),\hat{y}(\cdot,t))\right)_{Y^n}\\&\leqslant M_1\left\|\omega(\cdot,t)\right\|_{Y^n}\left\|e(\cdot,t)\right\|_{Y^n},\end{aligned}\tag{20}$$

is true for all $\|e(\cdot,t)+\hat{y}(\cdot,t)\|_{Y^n}, \|\hat{y}(\cdot,t)\|_{Y^n} \leqslant R$ *and* $t \geqslant 0$. *Here* $(\psi,\psi^*)_{Y^n} \triangleq \sum_{i=1}^{n}(\psi_i,\psi_i^*)_Y,\ \psi,\psi^*\in Y^n$ *denotes the inner product of the Hilbert space* Y^n.

Proof. By Lemma 3.1, G is bounded. Let $C^G \triangleq \max_{i\in J_n}\sup_{0\leqslant\xi\leqslant x\leqslant L}|g_i(x,\xi)|$. Subsequently, based on the definition (14) of φ, we get that

$$\left(\omega(\cdot,t),\varphi(\cdot,t,e(\cdot,t),\hat{y}(\cdot,t))\right)_{Y^n}=\sum_{i=1}^{n}\varphi_\omega^i(t),$$

where

$$\begin{aligned}&\varphi_\omega^i(t)=\textstyle\int_0^L\omega_i(x,t)\varphi_i(x,t,e(x,t),\hat{y}(x,t))dx\\&=\int_0^L\omega_i(x,t)\begin{pmatrix}f_i(x,t,e(x,t)+\hat{y}(x,t))\\-f_i(x,t,\hat{y}(x,t))\end{pmatrix}dx\\&\quad-\int_0^L\omega_i(x,t)\int_0^x g_i(x,\varsigma)\times\\&\qquad\begin{Bmatrix}f_i(\varsigma,t,e(\varsigma,t)+\hat{y}(\varsigma,t))\\-f_i(\varsigma,t,\hat{y}(\varsigma,t))\end{Bmatrix}d\varsigma dx.\end{aligned}\tag{21}$$

Then the Cauchy–Schwarz inequality yields that

$$\begin{aligned}|\varphi_\omega^i(t)|&\leqslant\left(1+\frac{L^{3/2}C^G}{\sqrt{3}}\right)\|\omega_i(\cdot,t)\|_Y\times\\&\quad\left\|f_i(x,t,e(x,t)+\hat{y}(x,t))-f_i(x,t,\hat{y}(x,t))\right\|_Y.\end{aligned}$$

Hence, taking into account Assumption 1,

$$\begin{aligned}&\left(\omega(\cdot,t),\varphi(\cdot,t,e(\cdot,t),\hat{y}(\cdot,t))\right)_{Y^n}\\&\leqslant c\left(1+\frac{C^GL^{3/2}}{\sqrt{3}}\right)\|\omega(\cdot,t)\|_{Y^n}\|e(\cdot,t)\|_{Y^n}.\end{aligned}\tag{22}$$

Taking $M_1=c\left(1+\frac{C^GL^{3/2}}{\sqrt{3}}\right)$ verifies the claim. □

For the inverse operator $\mathcal{E}^{-1}$ defined in Lemma 3.1, if $H(x,\varsigma)=\mathrm{diag}(h_1(x,\varsigma),h_2(x,\varsigma),\ldots,h_n(x,\varsigma))$, the following relationship holds true [10]

$$\begin{aligned}h_i(x,\varsigma)-g_i(x,\varsigma)&=\int_\varsigma^x h_i(x,s)g_i(s,\varsigma)ds\\&=\int_\varsigma^x g_i(x,s)h_i(s,\varsigma)ds.\end{aligned}\tag{23}$$

Let $C^H \triangleq \max_{i\in J_n}\sup_{0\leqslant\xi\leqslant x\leqslant L}|h_i(x,\xi)|$. One has

$$\|e(\cdot,t)\|_{Y^n}\leqslant\left(1+C^H\sqrt{L}\right)\|\omega(\cdot,t)\|_{Y^n}.\tag{24}$$

Theorem 3.1. *Suppose that Assumption 1 holds and let* $\|e_0\|_{Y^n}\leqslant R_0$ *for some constant* $R_0>0$. *If there exist constants* $R>R_0$, $\varepsilon>0$ *and a matrix* Θ *such that*

$$\begin{aligned}\lambda_\Theta&\triangleq\min\{\mu_{11},\mu_{22},\ldots,\mu_{nn}\}\\&\geqslant c\left(1+\frac{C_GL^{3/2}}{\sqrt{3}}\right)\left(1+C^H\sqrt{L}\right)+\varepsilon.\end{aligned}\tag{25}$$

Then system (11) *with* $K_1(x)$ *and* K_2 *satisfying* (15) and (16) *is Mittag-Leffler stable in the* Y^n *norm and* $\|e(\cdot,t)\|_{Y^n}\leqslant R$.

The following lemma plays a central role in the proof of Theorem 3.1.

Lemma 3.3 (*[23]*). *Let* $\rho:[0,\infty)\to\mathbb{R}$ *be a differentiable function. Then, for any given* $t\geqslant 0$,

$$\frac{1}{2}{}_0^CD_t^\alpha\rho^2(t)\leqslant\rho(t){}_0^CD_t^\alpha\rho(t),\ \alpha\in(0,1].\tag{26}$$

Proof of Theorem 3.1.

Note that system (11) can be equivalently converted into (12) via the integral transform (13) if $K_1(x)$ and K_2 are chosen satisfying (16) and (15). In what follows, based on (24), we focus on studying the stability of target system (12).

Since $\omega_i(\cdot,t)$ is a differentiable function, let

$$W(t)=\frac{1}{2}\|\omega(\cdot,t)\|_{Y^n}^2=\frac{1}{2}\int_0^L\omega^T(x,t)\omega(x,t)dx.\tag{27}$$

Lemma 3.3 yields that

$$\begin{aligned}&{}_0^CD_t^\alpha W(t)=\frac{1}{2}\int_0^L{}_0^CD_t^\alpha\left[\omega^T(x,t)\omega(x,t)\right]dx\\&\leqslant\int_0^L\sum_{i=1}^n\omega_i(x,t){}_0^CD_t^\alpha\omega_i(x,t)dx\\&=\sum_{i=1}^n\left\{\frac{\partial\omega_i(L,t)}{\partial x}\omega_i(L,t)-\frac{\partial\omega_i(0,t)}{\partial x}\omega_i(0,t)\right\}\\&\quad-\sum_{i=1}^n\left\{\int_0^L\left\{\frac{\partial\omega_i}{\partial x}(x,t)\right\}^2dx+\mu_{ii}\int_0^L\{\omega_i(x,t)\}^2dx\right\}\\&\quad+\left(\omega(\cdot,t),\varphi(\cdot,t,e(\cdot,t),\hat{y}(\cdot,t))\right)_{Y^n}.\end{aligned}$$

Thanks to the boundary conditions of system (12), one has $\frac{\partial\omega_i(0,t)}{\partial x}\omega_i(0,t)=\frac{r_{2i}}{r_{1i}}\{\omega_i(0,t)\}^2$ and

$$\frac{\partial\omega_i(L,t)}{\partial x}\omega_i(L,t)=\begin{cases}-\frac{s_{2i}}{s_{1i}}\{\omega_i(L,t)\}^2 & \text{if } s_{1i}>0\\0, & \text{if } s_{1i}=0.\end{cases}$$

This, together with Lemma 4.1 and (24), leads to

$$\begin{aligned}&{}_0^CD_t^\alpha W(t)\leqslant-\sum_{i=1}^n\mu_{ii}\int_0^L\{\omega_i(x,t)\}^2dx\\&\qquad\qquad+\left(\omega(\cdot,t),\varphi(\cdot,t,e(\cdot,t),\hat{y}(\cdot,t))\right)_{Y^n}\\&\leqslant-2\left(\lambda_\Theta-c\left(1+\frac{C^GL^{3/2}}{\sqrt{3}}\right)\left(1+C^H\sqrt{L}\right)\right)W(t)\\&\leqslant-2\varepsilon W(t),\end{aligned}$$

where the latter inequality follows from (25).

In addition, define $N(t)\triangleq{}_0^CD_t^\alpha W(t)+2\varepsilon W(t)$. Then $N(t)\leqslant 0$ for all $t\geqslant 0$. By Section 4.1.3 of [11], one has

$$\begin{aligned}W(t)&=E_\alpha\left(-2\varepsilon t^\alpha\right)W(0)\\&\quad+\int_0^t\frac{E_{\alpha,\alpha}(-2\varepsilon(t-s)^\alpha)}{(t-s)^{1-\alpha}}N(s)ds.\end{aligned}\tag{28}$$

Observing that $E_{\alpha,\alpha}(-2\varepsilon t^{\alpha}) > 0$ when $t \geqslant 0$, $\varepsilon > 0$ and $\alpha \in (0, 1]$ [24], it follows that

$$W(t) \leqslant E_{\alpha}\left(-2\varepsilon t^{\alpha}\right) W(0) = \frac{\|\omega_0\|^2_{Y^n}}{2} E_{\alpha}\left(-2\varepsilon t^{\alpha}\right).$$

Recall that $\|\omega_0\|_{Y^n} \leqslant \left(1 + C^G\sqrt{L}\right)\|e_0\|_{Y^n}$. This, together with (24), results in

$$\begin{aligned}\|e(\cdot,t)\|_{Y^n} &\leqslant \left(1 + C^H\sqrt{L}\right)\sqrt{2W(t)}\\ &\leqslant \left(1 + C^H\sqrt{L}\right)\left(1 + C^G\sqrt{L}\right)\|e_0\|_{Y^n} E_{\alpha}^{\frac{1}{2}}\left(-2\varepsilon t^{\alpha}\right).\end{aligned}$$

Then system (11) is Mittag-Leffler stable in the Y^n norm if there exists a strictly positive ε fulfilling (25).

In addition, if $\|e_0\|_{Y^n} \leqslant R_0$, since C^G and C^H are bounded, there always exists a constant $R > R_0$ such that $\left(1 + C^H\sqrt{L}\right)\left(1 + C^G\sqrt{L}\right) R_0 < R$. The proof is finished. □

4. Observer-based output feedback controller

Let us consider the following backstepping transformation

$$\omega^*(x,t) = y(x,t) - \int_0^x D(x,\varsigma) y(\varsigma,t) d\varsigma, \tag{29}$$

where $\omega_0^*(x) = y_0(x) - \int_0^x D(x,\varsigma) y_0(\varsigma) d\varsigma$ and

$$D(x,\varsigma) = \operatorname{diag}(d_1(x,\varsigma), d_2(x,\varsigma), \ldots, d_n(x,\varsigma)). \tag{30}$$

Proceeding as before for system (2)–(4), we get that

$$\begin{aligned}&{}_0^C D_t^{\alpha}\omega^*(x,t) - \omega^*_{xx}(x,t) + \widehat{\Theta}\omega^*(x,t) - \widehat{\varphi}\\ &= f(x,t,y(x,t)) - \int_0^x D(x,\varsigma) f(\varsigma,t,y(\varsigma,t)) d\varsigma - \widehat{\varphi}\\ &\quad + \left(\widehat{\Theta} + 2\frac{d}{dx}D(x,x)\right) y(x,t)\\ &\quad + \int_0^x \left\{\begin{array}{l} D_{xx}(x,\varsigma) - D_{\varsigma\varsigma}(x,\varsigma)\\ -\widehat{\Theta} D(x,\varsigma)\end{array}\right\} y(\varsigma,t) d\varsigma\\ &\quad + \left(D(x,0) p_1^{-1} p_2 - D_{\varsigma}(x,0)\right) y(0,t).\end{aligned}$$

Introduce

$$\begin{aligned}\widehat{\varphi} &\triangleq \varphi(x,t,y(x,t))\\ &= f(x,t,y(x,t)) - \textstyle\int_0^x D(x,\varsigma) f(\varsigma,t,y(\varsigma,t)) d\varsigma\end{aligned}$$

and choose D to satisfy

$$\begin{cases} D_{xx}(x,\varsigma) - D_{\varsigma\varsigma}(x,\varsigma) = \widehat{\Theta} D(x,\varsigma),\ 0 < \varsigma < x < L,\\ D(x,x) = \dfrac{-\widehat{\Theta}}{2} x,\ 0 < x < L,\\ D_{\varsigma}(x,0) = D(x,0) p_1^{-1} p_2,\ \ 0 < x < L. \end{cases} \tag{31}$$

Then system (2)–(4) can be converted into

$$\begin{cases} {}_0^C D_t^{\alpha}\omega^*(x,t) = \omega^*_{xx}(x,t) - \widehat{\Theta}\omega^*(x,t) + \widehat{\varphi}\\ \qquad\qquad\qquad \text{in } (0,L)\times(0,\infty),\\ r_1\omega^*_x(0,t) - r_2\omega^*(0,t) = \theta \text{ in } (0,\infty),\\ s_1\omega^*_x(L,t) + s_2\omega^*(L,t) = \theta \text{ in } (0,\infty),\\ \omega^*(x,0) = \omega_0^*(x) \ \text{ in } (0,L) \end{cases} \tag{32}$$

with $r_1^{-1} r_2 = p_1^{-1} p_2$ if the controller is defined by

(1) when $q_1 = s_1 = 0_{n\times n}$:

$$u(t) = q_2 \int_0^L D(L,\varsigma)\hat{y}(\varsigma,t) d\varsigma;$$

(2) when $q_1, s_1 \neq 0_{n\times n}$ and $\exists i \in J_n$ satisfying $q_{1i} = s_{1i} = 0$:

$$\begin{cases} u_i(t) = q_{2i}\displaystyle\int_0^L d_i(L,\varsigma)\hat{y}_i(\varsigma,t) d\varsigma,\\ u_j(t) = \left(q_{2j} + q_{1j} d_j(L,L)\right)\hat{y}_j(L,t)\\ \qquad - q_{1j} s_{1j}^{-1} s_{2j}\hat{y}_j(L,t)\\ \qquad - q_{1j}\displaystyle\int_0^L \frac{\partial d_j}{\partial x}(L,\varsigma)\hat{y}_j(\varsigma,t) d\varsigma\\ \qquad - q_{1j} s_{1j}^{-1} s_{2j}\displaystyle\int_0^L d_j(L,\varsigma)\hat{y}_j(\varsigma,t) d\varsigma,\\ \qquad j \in J_n,\ j \neq i; \end{cases} \tag{33}$$

(3) when $\det(q_1), \det(s_1) \neq 0$:

$$\begin{aligned}&u(t) = \left(q_2 + q_1 D(L,L) - q_1 s_1^{-1} s_2\right)\hat{y}(L,t)\\ &-q_1\int_0^L \left(D_x(L,\varsigma) + s_1^{-1} s_2 D(L,\varsigma)\right)\hat{y}(\varsigma,t) d\varsigma.\end{aligned}$$

For the solution of the kernel system (31), similarly, design the matrix $\widehat{\Theta}$ as

$$\widehat{\Theta} \triangleq \operatorname{diag}(\widehat{\mu}_{11}, \widehat{\mu}_{22}, \ldots, \widehat{\mu}_{nn}) \text{ with } \widehat{\mu}_{jj} > 0,\ j \in J_n$$

and let $\operatorname{diag}(\widehat{c}_1, \widehat{c}_2, \ldots, \widehat{c}_n) \triangleq p_1^{-1} p_2$. By [21] and Theorem 10 of [22], we give the following result as done for Lemma 3.1.

Lemma 4.1. *The solution to the system (31) is*

$$\begin{aligned}&d_j(x,\varsigma)\\ &= \frac{\widehat{c}_j\widehat{\mu}_{jj}}{\sqrt{\widehat{\mu}_{jj} + \widehat{c}_j^2}}\int_0^{x-y} e^{-\frac{\widehat{c}_j\tau}{2}} I_0\left(\sqrt{\widehat{\mu}_{jj}(x+y)(x-y-\tau)}\right)\\ &\quad + \sinh\left(\frac{\sqrt{\widehat{\mu}_{jj} + \widehat{c}_j^2}}{2}\tau\right) d\tau - \mu_{jj} x \frac{I_1\left(\sqrt{\mu_{jj}\left(x^2 - \varsigma^2\right)}\right)}{\sqrt{\mu_{jj}\left(x^2 - \varsigma^2\right)}},\end{aligned}$$

$j \in J_n$. Besides, the operator $\widehat{\mathcal{E}} : Y^n \to Y^n$ given by

$$\omega^*(x,t) = (\widehat{\mathcal{E}} y)(x,t) = y(x,t) - \int_0^x D(x,\varsigma) y(\varsigma,t) d\varsigma$$

and its inverse $\widehat{\mathcal{E}}^{-1}$ are all bounded.

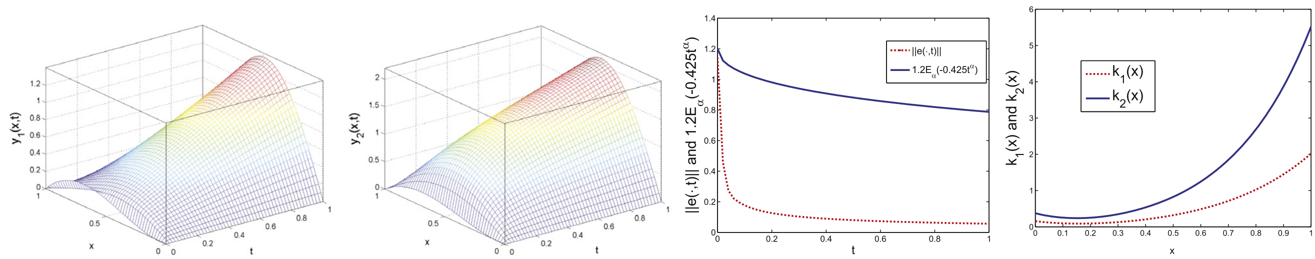
Since $f(x,t,\theta) = \theta$, if Assumption 1 holds, a constant $M_2 = c\left(1 + \frac{C^D L^{3/2}}{\sqrt{3}}\right) > 0$ with $C^D = \max_{i\in J_n}\sup_{0\leqslant\xi\leqslant x\leqslant L}|d_i(x,\xi)|$ can be found such that

$$\begin{aligned}&\left(\omega^*(\cdot,t), \widehat{\varphi}(\cdot,t,y(\cdot,t))\right)_{Y^n}\\ &\leqslant M_2\left\|\omega^*(\cdot,t)\right\|_{Y^n}\|y(\cdot,t)\|_{Y^n},\ \ t > 0\end{aligned} \tag{34}$$

holds for all $\|y(\cdot,t)\|_{Y^n} \leqslant R$ as a consequence of Lemma 4.1. Besides, the inverse transformation of (29) given by $y(x,t) = \omega^*(x,t) + \int_0^x V(x,\varsigma)\omega^*(\varsigma,t) d\varsigma$ is bounded. If $V(x,\varsigma) = \operatorname{diag}(v_1(x,\varsigma), v_2(x,\varsigma), \ldots, v_n(x,\varsigma))$, let $C^V \triangleq \max_{i\in J_n}\sup_{0\leqslant\xi\leqslant x\leqslant L}|v_i(x,\xi)|$. We have

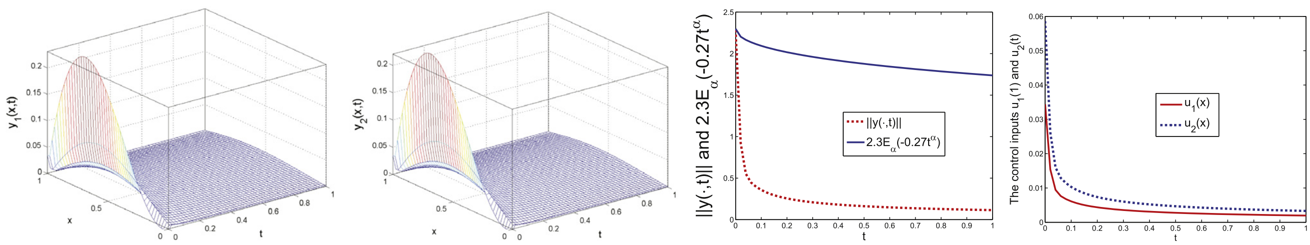
$$\|y(\cdot,t)\|_{Y^n} \leqslant \left(1 + C^V\sqrt{L}\right)\|\omega^*(\cdot,t)\|_{Y^n} \tag{35}$$

and then obtain the following result. Since its proof is very analogous to the one for Theorem 3.1, we omit the details.



(a) Solution $y_1(x,t)$ with $u_i \equiv 0$. (b) Solution $y_2(x,t)$ with $u_i \equiv 0$. (c) Y^2-norm of $\|e(\cdot,t)\|_{Y^2}$. (d) Weight $K(x) = \operatorname{diag}(k_1(x), k_2(x))$.

Fig. 1. Simulation results for the system (2) with $u_i \equiv 0$ and the observer error system.



(a) Solution $y_1(x,t)$. (b) Solution $y_2(x,t)$. (c) Y^2-norm of $\|y(\cdot,t)\|_{Y^2}$. (d) The control inputs.

Fig. 2. The dynamic behavior of closed-loop system and the corresponding control inputs.

Theorem 4.1. *Suppose Assumption 1 holds and let $\|y_0\|_{Y^n} \leqslant R_0$ for some $R_0 > 0$. If there exist constants $\widehat{R} > R_0$, $\widehat{\varepsilon} \in (0, \varepsilon)$ and a matrix $\widehat{\Theta}$ such that*

$$\lambda_{\widehat{\Theta}} \triangleq \min\left\{\hat{\mu}_{11}, \hat{\mu}_{22}, \ldots, \hat{\mu}_{nn}\right\} \geqslant c\left(1 + \frac{C^D L^{3/2}}{\sqrt{3}}\right)\left(1 + C^V\sqrt{L}\right) + \widehat{\varepsilon}, \tag{36}$$

then the system consisting of the plant (2)–(4), observer (10) and controller (33) is closed-loop Mittag-Leffler stable in the Y^n norm and $\|y(\cdot,t)\|_{Y^n} \leqslant \widehat{R}$.

Remark 4.1. The condition $\widehat{\varepsilon} \in (0, \varepsilon)$ is given to guarantee that the convergence speed of the observer is higher than that of the controller, which is used to stabilize the closed-loop system at hand.

5. A numerical example

Consider system (2) with $n = 2$, $\alpha = 0.5$, $L = 1$ and

$$f_1(x, t, y_1, y_2) = \frac{1}{100} y_1\left(1 + \sin(2\pi x) - y_2\right), \quad f_2(x, t, y_1, y_2) = \frac{1}{100(1+t)} y_2 y_1. \tag{37}$$

Given $R > 0$ such that $\|y\|_{E^2} \leqslant R$, clearly, Assumption 1 holds true for $c = (2 + R)/100$. Let $y_0(x) = 2x(1-x)(x-0.1)I_{2\times2}$, $p_1 = q_1 = I_{2\times2}$, $q_2 = 3I_{2\times2}$ and $p_2 = 0_{2\times2}$. According to (a) and (b) of Fig. 1, we get that the solution of the considered TFDSs with $u \equiv 0$ is unstable.

As stated in Section 3, the considered system can be estimated by designing an observer. Define $\hat{y}_0(x) = x(1-x)(x-0.1)I_{2\times2}$, $r_1 = s_1 = I_{2\times2}$, $s_2 = 3I_{2\times2}$ and $r_2 = 0_{2\times2}$. Then $e_0(x) = x(1-x)(x-0.1)I_{2\times2}$. Based on the backstepping approach, set $\Theta = \operatorname{diag}(1, 0.5)$, one has $\max\{|g_i(x,\varsigma)|, |h_i(x,\varsigma)|\} \leqslant N_i e^{2N_i x}$, $i \in J$, where $N_i = \mu_{ii}(1 + e^{-3})$ [22]. Then (25) holds true for $\varepsilon = 0.425$. As a result, (c) of Fig. 1 implies that the observer error system is Mittag-Leffler stable in the Y^2 norm. By (17),(19), one has $\frac{d}{dx}G(x,x) = -\frac{\Theta}{2}$ and $G(L,L) = q_1^{-1}q_2 - s_1^{-1}s_2 = 0_{2\times2}$. These imply $G(0,0) = \frac{\Theta}{2}$, $K_2 = \operatorname{diag}(2.5, 2.75)$ and $K_1(x) = \operatorname{diag}(k_1(x), k_2(x))$ to be plotted in (d) of Fig. 1.

Based the statements in Section 4, since $p_2 = r_2 = 0_{2\times2}$, let $\widehat{\Theta} = \operatorname{diag}(0.5, 0.3)$, we get that (36) holds true for $\widehat{\varepsilon} = 0.27$. The evolution of the solution obtained by the observer-based output feedback controller and its corresponding control inputs are shown in Fig. 2. These confirm our theoretical results.

6. Conclusion

In this paper, backstepping based observer design and output feedback control are addressed for n coupled semilinear TFDSs. For this, the dynamic equivalence of the observer error system and a Mittag-Leffler stable target system is deduced by an approximate choice of the output injection weights. These results are combined with a state observer to obtain a dynamic output feedback controller at the right boundary that ensure closed-loop Mittag-Leffler stability of the considered system.

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Appendix. Transformation into (2)–(4)

Rewrite the considered system (1) with appropriate initial conditions and BCs as follows

$$\begin{cases} {}_0^C D_\tau^\alpha y(\zeta,\tau) = b(\zeta) y_{\zeta\zeta}(\zeta,\tau) + c(\zeta) y_\zeta(\zeta,\tau) \\ \quad + \hat{f}(\zeta,\tau,y(\zeta,\tau)) \text{ in } (0,l)\times(0,\infty), \\ m_1 y_\zeta(0,t) - m_2 y(0,t) = 0, \ \text{ in } (0,\infty), \\ n_1 y_\zeta(l,t) + n_2 y(l,t) = 0 \text{ in } (0,\infty), \\ y(\zeta,0) = y^0(\zeta) \ \text{ in } (0,l), \end{cases} \tag{38}$$

We next show the transformation based on the results in [25,26] with certain modifications due to the Caputo fractional derivative nature of our problem.

Assume that $b(\zeta) \triangleq b_0(\zeta)I_{n\times n}$, $b_0 \in C^2(0,l)$ with $b_0(\zeta) \in [b^l, b^u] \subseteq (0,\infty)$ and $c(\zeta) \triangleq \operatorname{diag}(c_1(\zeta), c_2(\zeta), \ldots, c_n(\zeta))$, $c_i \in C^1(0,l)$ with $|c_i(\zeta)| \leqslant c^u < \infty$. By introducing

$$\tau \mapsto t = \bar{\tau}(\tau) \triangleq \tau,\ \zeta \mapsto x = \bar{\zeta}(\zeta) := \int_0^\zeta b_0^{-\frac{1}{2}}(s)ds,$$
$$y_i(\zeta,\tau) \mapsto y_i(x,t)e^{-\vartheta_i(x)},\ i \in J_n$$

with $\vartheta_i(x) = \int_0^x \frac{\left(b_0(\zeta)\frac{d^2\bar{\zeta}(\zeta)}{d\zeta^2} + c_i(\zeta)\frac{d\bar{\zeta}(\zeta)}{d\zeta}\right)\Big|_{\zeta=\bar{\zeta}^{-1}(s)}}{2}ds$, and setting $L = \int_0^l b_0^{-\frac{1}{2}}(s)ds$, it follows that the considered system is equivalent to

$$ {}_0^C D_t^\alpha y(x,t) = y_{xx}(x,t) + f(x,t,y(x,t)) \tag{39}$$

in $(x,t) \in (0,L) \times (0,\infty)$, where $f(x,t,y(x,t)) = (f_1(x,t,y(x,t)),\ldots,f_n(x,t,y(x,t)))$ with

$$f_i(x,t,y(x,t)) = e^{-\vartheta_i(x)}\hat{f}_i(x,t,y(x,t)e^{-\vartheta(x)}) + \left[\left(\frac{d\vartheta_i(x)}{dx}\right)^2 - \frac{d^2\vartheta_i(x)}{dx^2} - \theta_i(x)\frac{d\vartheta_i(x)}{dx}\right]y_i(x,t). \tag{40}$$

The corresponding initial condition follows from $y_i^0(x) \mapsto y_i^0(\bar{\zeta}^{-1}(x))e^{-\vartheta_i(x)}$ and the BCs satisfy

$$\begin{aligned} &m_{1i} \mapsto p_{1i} \triangleq m_{1i}, \quad n_{1i} \mapsto q_{1i} \triangleq n_{1i}\frac{e^{-\vartheta_i(L)}}{\sqrt{b_0(L)}},\\ &m_{2i} \mapsto p_{2i} \triangleq m_{1i}\frac{d\vartheta_i(0)}{dx} + m_{2i}\sqrt{b_0(0)},\\ &n_{2i} \mapsto q_{2i} \triangleq n_{2i}e^{-\vartheta_i(L)} - n_{1i}\frac{e^{-\vartheta_i(L)}\frac{d\vartheta_i(L)}{dx}}{\sqrt{b_0(L)}}. \end{aligned} \tag{41}$$

Then we obtain the standard form (2)–(4).

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