ISA Transactions 80 (2018) 203–211

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

Research article

Backstepping-based boundary control design for a fractional reaction diffusion system with a space-dependent diffusion coefficient

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ARTICLE INFO

Keywords:
Fractional reaction diffusion system with space-dependent diffusivity
Backstepping
Boundary feedback control
Mittag-Leffler stability

ABSTRACT

This paper presents a boundary feedback control design for a fractional reaction diffusion (FRD) system with a space-dependent (non-constant) diffusion coefficient via the backstepping method. The contribution of this paper is to generalize the results of backstepping-based boundary feedback control for a FRD system with a space-independent (constant) diffusion coefficient to the case of space-dependent diffusivity. For the boundary stabilization problem of this case, a designed integral transformation treats it as a problem of solving a hyperbolic partial differential equation (PDE) of transformation's kernel, then the well posedness of the kernel PDE is solved for the plant with non-constant diffusivity. Furthermore, by the fractional Lyapunov stability (Mittag-Leffler stability) theory and the backstepping-based boundary feedback controller, the Mittag-Leffler stability of the closed-loop FRD system with non-constant diffusivity is proved. Finally, an extensive numerical example for this closed-loop FRD system with non-constant diffusivity is presented to verify the effectiveness of our proposed controller.

1. Introduction

1.1. Summary of prior work

Recent years, the backstepping method has been widely used to solve a boundary stabilization problem of integer-order distributed parameter systems (DPSs) modeled by PDEs by designing an appropriate integral transformation. The pioneering work on stabilization problem of DPSs includes, the results on boundary control of linear partial integro-differential equations (P(I)DEs) via the backstepping approach [1], its dual backstepping-based output-feedback results [2], and other results on event-triggered observer-based output feedback control of spatially distributed processes [3]. Additionally, a predictor-based infinite-dimensional feedback for ordinary differential equation (ODE) systems with actuator delay was extended to a class of PDE-ODE cascades for its stabilization problem in Ref. [4]. The output feeback control problem of moving boundary parabolic PDEs was considered in Ref. [5], which formulated the observer design of a 1D unstable heat equation on the time-varying domain. It is worth to mention that a breakthrough has been made in extension of the ideas in Ref. [1] for the system with space-independent diffusivity to the case of space-dependent diffusion in Ref. [6].

1.2. Motivation

As we know, many realistic systems are modeled by fractional-order differential equations [7], such as fractional diffusion (FD) systems, FRD systems and etc. FRD systems [8] can exhibit some self-organization phenomena in biological and physical systems, and introduce the fractional derivative into these systems. They have a lot of applications in simulating process in physical [9], biology [10], and finance [11]. However, the results of the boundary feedback stabilization for the FRD systems are still relative few except for the work [12,13], which considered the backstepping-based boundary feedback control only for the FRD system with space-independent diffusivity. Motivated by the work on the boundary stabilization problem of PDEs with space-dependent diffusivity in Ref. [6], we introduce the boundary feedback controller into the FRD system with space-dependent diffusivity [14], which can be taken as one of the results to modelling pattern formation in inhomogeneous media [15].

1.3. Problem formulated

In this paper, we consider the Caputo time FRD system [12] with non-constant diffusivity, whose dynamics equation and initial condition

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https://doi.org/10.1016/j.isatra.2018.04.013
Received 6 August 2017; Received in revised form 5 April 2018; Accepted 24 April 2018
Available online 19 June 2018

are represented as

$$\begin{aligned} {}_0^C D_t^\alpha u(x,t) &= \vartheta(x)u_{xx}(x,t) + a(x)u(x,t), x\in(0,1), t>0, \\ u(x,0) &= u_0(x), \quad x\in[0,1], \end{aligned} \tag{1}$$

where the space-dependent diffusivity $\vartheta(x) > 0$ for $x \in [0, 1]$, $a(\cdot) \in C^1[0, 1]$, and $u_0(x)$ is the nonzero initial value, ${}_0^C D_t^\alpha(\cdot)$ represents the Caputo time fractional-order derivative [16]

$${}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-\tau)^\alpha}\frac{\partial u(x,\tau)}{\partial\tau}d\tau, \ 0<\alpha<1. \tag{2}$$

On the one hand, if the Caputo time fractional-order derivative of the state $u(x, t)$ is replaced by the integer-order derivative, the problem will reduce to the PDE with space-dependent diffusivity in Ref. [6]. On the other hand, if the space-dependent diffusion coefficient reduces to a constant, the problem will convert to the FRD systems with constant diffusivity in Refs. [12,13].

The model of a FRD equation (1) with the space-dependent diffusivity can describe inhomogeneous medium. The diffusion in the FRD process governed by the FRD equation (1) is subdiffusion, which arises in cases where there are temporal or spatial constraints such as occur in fractured and porous media [17], nonhomogeneous media [18], etc. For the spatially varying diffusivity, from the work [19], we know that a model with it may mimic the diffusion effect in a heterogeneous environment and describe the heterogeneous diffusion processes. In recent years, some intensive studies [20,21] on fractional differential equations with the space-dependent diffusivity have been emerged. Additionally, there are some potential applications of the FRD models with a space-dependent diffusion coefficient to many biological systems, for instance, chondrogenesis in the vertebrate limb [14,22]. This extension to the FRD system with non-constant diffusivity makes the results in Refs. [12,13] applicable to non-homogeneous media. Moreover, the introduction of FRD systems with space-dependent diffusivity enriches the family of fractional-order systems.

1.4. Main contribution and structure

The contribution of this paper can be divided into two aspects:

1) The boundary stabilization problem of PDEs with space-dependent diffusivity was studied in Ref. [6] utilizing the backstepping-based boundary feedback controller with the kernel under the constraint $k(0, 0) = 0$. In this paper, we analyze the kernel with this constraint replaced by the fact $k(0, 0)$ can be nonzero or zero constant. This can be seen more relax than the one in Ref. [6].
2) The most striking feature of the backstepping method is that the designed integral transformation can transform the integer-order system into the exponentially stable system by the Dirichlet/Neumann boundary feedback controller in Refs. [1,6]. For the FRD system with space-dependent diffusivity, in turn, it can Mittag-Leffler converge to the equilibrium point by the Robin boundary feedback controller, which implies the L^2 and H^1 Mittag-Leffler stability of this closed-loop plant. From theory point of the view, this paper provides some insights into the boundary feedback control of the fractional-order system with space-dependent diffusivity by the backstepping method.

This paper is organized as follows. We start in Section 2 with the problem statement. Section 3 mainly focuses on the well posedness of the gain kernel PDE. Then we illustrate the Mittag-Leffler stability arguments, and obtain our main theorem and its corollaries for cases of Dirichlet and Neumann boundary conditions in Section 4. The construction of boundary feedback controller is illustrated in Section 5. In Section 6, simulation studies are used to demonstrate the Mittag-Leffler stability of the closed-loop system with space-dependent diffusivity. Finally, conclusions and future work are contained in Section 7.

Notations: $L^2(0, 1)$ denotes the Hilbert space of a square integral function $u(x, t)$, $x \in (0, 1)$, $t \in [0, \infty)$ with the norm $\|u(x,t)\| = (\int_0^1 u^2(x,t)dx)^{1/2}$. $H^1(0, 1)$ represents the usual Sobolev space (see, e.g Ref. [23]) with the H^1 norm $\|u(x,t)\|_{H^1} = (u^2(0,t) + u^2(1,t) + \int_0^1 u_x^2(x,t)dx)^{1/2}$ in Section 4, $u(x, t) \in H^1(0, 1)$. In addition, $\vartheta_{\min}$ denotes the minimum value of $\vartheta(x)$, $x \in [0, 1]$. $\vartheta''_{\max}$ denotes the maximum value of $\vartheta''(x)$, $x \in [0, 1]$.

2. Mathematical modelling

In this section, we consider the FRD system (1) with the following boundary conditions

$$p_1 u_x(0,t) - p_2 u(0,t) = 0, \quad t>0 \tag{3}$$

$$q_1 u_x(1,t) + q_2 u(1,t) = U(t), \quad t>0, \tag{4}$$

or in another representation

$$u_x(0,t) - pu(0,t) = 0, \quad t>0 \tag{5}$$

$$u_x(1,t) + qu(1,t) = U(t), \quad t>0, \tag{6}$$

where $p = \frac{p_2}{p_1}$, $q = \frac{q_2}{q_1}$, $p_1, p_2, q_1, q_2, p, q \geq 0$ (p_1, p_2 can not be zero at the same time, likewise for q_1, q_2), and $U(t)$ is an input. If $p_1 = 0$, $q_1 = 0$ (i.e. $p = +\infty$, $q = +\infty$), $p_2 = 0$, $q_2 = 0$ (i.e. $p = 0$, $q = 0$), or p_1, p_2, q_1, $q_2 > 0$ (i.e. p, $q > 0$), above boundary conditions can be called Dirichlet boundary conditions, Neumann boundary conditions or Robin boundary conditions. Otherwise, it can be viewed as mixed boundary conditions. In this paper, we want to discuss the case of Robin boundary conditions, the other case are straightforward.

By the results in Ref. [24], we know that the sufficient and necessary condition for stability of system (1), (3)–(4) (or (5)–(6) with $U(t) = 0$) is

$$|\arg(\mathrm{spec}(\widetilde{A} + a(x)))| \leq \frac{\alpha\pi}{2}, \tag{7}$$

where the operator $\tilde{A}$ is given by $(\widetilde{A}u)(x,t) = \frac{\partial^2(\vartheta(x)u(x,t))}{\partial x^2}$, i.e., the roots of some polynomial lie outside the closed angular sector. For this open-loop system, it will be unstable if $a(x)$ is positive and large enough even in the case that the eigenvalues of the operator $\tilde{A}$ are negative. Our purpose is to use the Robin boundary feedback controller to stabilize this system in terms of the backstepping method.

We utilize the following integral transformation [25]

$$w(x,t) = u(x,t) + \int_0^x k(x,y)u(y,t)dy \tag{8}$$

along with the Robin boundary feedback controller

$$\begin{aligned} q_1 u_x(1,t) + q_2 u(1,t) &= \left(q_2 - \frac{q_1 q_2^s}{q_1^s} - q_1 k(1,1)\right)u(1,t) \\ &\quad - \int_0^1 \left(q_1 k_x(1,y) + \frac{q_1 q_2^s}{q_1^s}k(1,y)\right)u(y,t)dy \end{aligned} \tag{9}$$

or simplified representation

$$\begin{aligned} u_x(1,t) + qu(1,t) &= (q - q^s - k(1,1))u(1,t) \\ &\quad - \int_0^1 (k_x(1,y) + q^s k(1,y))u(y,t)dy \end{aligned} \tag{10}$$

to transform system (1), (3) (or (5)) into a target system whose dynamic state equation and initial condition are described by

$$\begin{aligned} {}_0^C D_t^\alpha w(x,t) &= \vartheta(x)w_{xx}(x,t) - \lambda w(x,t), \ x\in(0,1), \ t>0 \\ w(x,0) &= w_0(x), \quad x\in[0,1] \end{aligned} \tag{11}$$

with boundary conditions

$$p_1^s w_x(0,t) - p_2^s w(0,t) = 0, \quad t>0 \tag{12}$$

$$q_1^s w_x(1,t) + q_2^s w(1,t) = 0, \quad t>0, \tag{13}$$

or in another representation

$$w_x(0, t) - p^s w(0, t) = 0, \quad t > 0 \tag{14}$$

$$w_x(1, t) + q^s w(1, t) = 0, \quad t > 0, \tag{15}$$

where $\lambda > 0$, $p^s = \frac{p_2^s}{p_1^s}$, $q^s = \frac{q_2^s}{q_1^s}$, $p_1^s, p_2^s, q_1^s, q_2^s, p^s, q^s \geq 0$ (p_1^s, p_2^s can not be zero at the same time, likewise for q_1^s, q_2^s), and $w_0(x) = u_0(x) + \int_0^x k(x, y)u_0(y)dy$. In order to further specify arguments on the stability of this target system, the notion of Mittag-Leffler is required.

Definition 1. [26] (Mittag-Leffler stability)

$$\text{If} \quad \|u(t)\| \leq (m[u(t_0)]E_\alpha(-M(t - t_0)^\alpha))^b, \tag{16}$$

where t_0 is the initial value of time, $\alpha \in (0, 1)$, $M \geq 0$, $b > 0$, $m(0) = 0$, $m(u)$ is nonnegative and satisfies locally Lipschitz condition on $u \in \mathbb{B} \in \mathbb{R}^n$ with the Lipschitz constant m_0, and $E_\alpha(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + 1)}$, $\forall\ \alpha > 0$, $t \in \mathbb{C}$ in Ref. [16], then the solution of the equation

$${}_{t_0}^{C}D_t^\alpha u(t) = f(t, u) \tag{17}$$

is said to be Mittag-Leffler stable. Here, in (17), $\alpha \in (0, 1)$, f is piecewise continuous in $t \in [t_0, \infty)$ and locally Lipschitz in u.

Remark 1. (Relationship of Mittag-Leffler stability and asymptotical stability) From above definition (1), we know that the system which meets the Mittag-Leffler stability is also asymptotically stable [26]. In addition, the Mittag-Leffler stability is also called the fractional Lyapunov stability since the role of its Mittag-Leffler function for the stability of fractional-order differential equations is similar to the one of exponential function for integer-order cases. More detail about the relationship between them can be found in [[13], Remark 1].

Based on above Definition 1, we see this target system can be L^2 and H^1 Mittag-Leffler stable under a certain stability condition (see the Proof of Theorem 1 for more details). Therefore, we need to establish the stability condition, then find out the kernel $k(x, y)$ in the integral transformation (8), which makes system (1), (3) (or (5)) with the controller (9) (or (10)) behave as the target system (11)–(13) (or (11), (14) and (15)).

Remark 2. If $q_2 = 0$ (i.e. $q = 0$) or $q_1 = 0$ (i.e. $q = +\infty$), the controller (9) or (10) reduces to the Dirichlet boundary feedback controller or the Neumann boundary feedback controller respectively. The discussion for boundary stabilization problems of Dirichlet and Neumann cases are similar to the above Robin case, so we omit them in this paper.

3. Analysis of kernel PDE

Next, we will find the kernel PDE. Based on the integral transformation (8) and its derivative on x, we easily find that $w(0, t) = u(0, t)$ and $w_x(0, t) = u_x(0, t) + k(0, 0)u(0, t)$. This, together with (5) and (14) implies $k(0, 0) = p^s - p$. Taking the second derivative of the integral transformation (8) on x, we get

$$\begin{aligned} w_{xx}(x, t) &= u_{xx}(x, t) + \tfrac{d}{dx}k(x, x)u(x, t) + k(x, x)u_x(x, t) \\ &\quad + k_x(x, x)u(x, t) + \int_0^x k_{xx}(x, y)u(y, t)dy. \end{aligned} \tag{18}$$

Aside, finding the Caputo time fractional-order derivative of (8), it leads that

$$\begin{aligned} {}_0^C D_t^\alpha w(x, t) &= {}_0^C D_t^\alpha u(x, t) + \int_0^x k(x, y){}_0^C D_t^\alpha u(y, t)dy \\ &= \vartheta(x)u_{xx}(x, t) + a(x)u(x, t) + k(x, x)\vartheta(x) \\ &\quad \times u_x(x, t) - k(x, 0)\vartheta(0)u_x(0, t) - [k_y(x, x) \\ &\quad \times \vartheta(x) + k(x, x)\vartheta'(x)]u(x, t) + [k_y(x, 0)\vartheta(0) \\ &\quad + k(x, 0)\vartheta'(0)]u(0, t) + \int_0^x [k_{yy}(x, y)\vartheta(y) \\ &\quad + 2k_y(x, y)\vartheta'(y) + k(x, y)\vartheta''(y) \\ &\quad + k(x, y)a(y)]u(y, t)dy. \end{aligned} \tag{19}$$

Substituting (18) and (19) into the first equation of (11), and combining the first equation of (1)/ce:cross-ref > and the boundary condition (5), we get

$$\begin{aligned} 0 &= \Big[\vartheta(x)\tfrac{d}{dx}k(x, x) + \vartheta(x)k_x(x, x) + \vartheta(x)k_y(x, x) - a(x) \\ &\quad - \lambda + \vartheta'(x)k(x, x)\Big]u(x, t) + [\vartheta(0)pk(x, 0) - \vartheta(0) \\ &\quad \times k_y(x, 0) - \vartheta'(0)k(x, 0)]u(0, t) + \int_0^x [\vartheta(x)k_{xx}(x, y) \\ &\quad - (\vartheta(y)k(x, y))_{yy} - a(y)k(x, y) - \lambda k(x, y)]u(y, t)dy. \end{aligned} \tag{20}$$

This, together with the notations $\frac{d}{dx}k(x, x) = k_x(x, x) + k_y(x, x)$ ($k_x(x, x) = k_x(x, y)|_{y=x}$, $k_y(x, x) = k_y(x, y)|_{y=x}$) and $k(0, 0) = p^s - p$, shows that $k(x, y)$ satisfies the below kernel PDE

$$\begin{cases} \vartheta(x)k_{xx}(x, y) - (\vartheta(y)k(x, y))_{yy} = (a(y) + \lambda)k(x, y) \\ k_y(x, 0) = (p - \vartheta'(0)/\vartheta(0))k(x, 0) \\ 2\vartheta(x)\tfrac{d}{dx}k(x, x) = -\vartheta'(x)k(x, x) + a(x) + \lambda \\ k(0, 0) = p^s - p \end{cases} \tag{21}$$

for $(x, y) \in \Xi = \{0 \leq y \leq x \leq 1\}$.

Solving the third equation of (21) together with $k(0, 0) = p^s - p$, we obtain

$$k(x, x) = \frac{1}{2\sqrt{\vartheta(x)}}\int_0^x \frac{a(\tau) + \lambda}{\sqrt{\vartheta(\tau)}}d\tau + (p^s - p)\vartheta^{-1/2}(x)\vartheta^{1/2}(0). \tag{22}$$

Next, we want to convert (21) into one for applying the analysis derived from Ref. [13] to it. Similar to the argument in Ref. [6], we also convert this kernel PDE (21) into the canonical form utilizing the method of changes of variables. In order to simply the manipulation, we conclude the following changes of variables

$$\check{k}(\check{x}, \check{y}) = \vartheta^{-1/4}(x)\vartheta^{3/4}(y)k(x, y), \tag{23}$$

$$\check{x} = \phi(x), \check{y} = \phi(y), \phi(\xi) = \sqrt{\vartheta(0)}\int_0^\xi \frac{d\tau}{\sqrt{\vartheta(\tau)}}. \tag{24}$$

After a series of transformation and computation, the kernel PDE (21) becomes

$$\begin{cases} \check{k}_{\check{x}\check{x}}(\check{x}, \check{y}) - \check{k}_{\check{y}\check{y}}(\check{x}, \check{y}) = \frac{\check{a}(\check{x}, \check{y})}{\vartheta(0)}\check{k}(\check{x}, \check{y}) \\ \check{k}_{\check{y}}(\check{x}, 0) = \left(p - \frac{\vartheta'(0)}{4\vartheta(0)}\right)\check{k}(\check{x}, 0) \\ \frac{d}{d\check{x}}\check{k}(\check{x}, \check{x}) = \frac{a(\phi^{-1}(\check{x})) + \lambda}{2\sqrt{\vartheta(0)}} \\ \check{k}(0, 0) = \vartheta^{1/2}(0)(p^s - p), \end{cases} \tag{25}$$

where $\phi^{-1}(\cdot)$ is the inverse function of $\phi(\cdot)$, and

$$\check{a}(\check{x}, \check{y}) = \frac{3}{16}\left(\frac{\vartheta'^2(x)}{\vartheta(x)} - \frac{\vartheta'^2(y)}{\vartheta(y)}\right) + \frac{1}{4}(\vartheta''(y) - \vartheta''(x)) + a(y) + \lambda. \tag{26}$$

Note that, by a series of derivation and transformation, $k(x, x)$ (22) becomes

$$\check{k}(\check{x}, \check{x}) = \frac{1}{2\sqrt{\vartheta(0)}}\int_0^{\check{x}} (a(\phi^{-1}(\eta)) + \lambda)d\eta + \vartheta^{1/2}(0)(p^s - p), \tag{27}$$

which is matched with the third equation of above kernel PDE (25) and $\check{k}(0, 0) = \vartheta^{1/2}(0)(p^s - p)$.

We can see that the coefficient $\check{a}(\check{x}, \check{y})$ in (25)–(26) depends on $\check{x}$

and $\check{y}$, if the diffusion coefficient $\vartheta(x)$ takes a constant, $\check{a}(\check{x}, \check{y})$ depends only on $\check{y}$. Furthermore, by using the bound on this coefficient of the kernel PDE (25), the same Proof provided in [[13], Lemma 2] can apply to the well posedness of the PDE (25). As is illustrated above, we can obtain the following result.

Lemma 1. *Suppose that $a(y) \in C^1[0, 1]$, the kernel PDE (21) with $k(x, y)$ given by (23) also has a unique solution which is bounded and twice continuously differentiable in $0 \le y \le x \le 1$.*

Remark 3. It is noticeable that the difference between the kernel PDE provided in [[6], Section 2.3] and ours is the constraint $k(0, 0)$ for the kernel $k(x, y)$ can be nonzero or zero constant while the counterpart for the kernel is zero in Ref. [6]. It can be viewed as an extension of the one in Ref. [6].

4. Discussion on Mittag-Leffler stability

In this section, we will provide the Mittag-Leffler stability analysis for the controlled FRD system (1), (3) (or (5)), and (9) (or (10)). For the benefit of our theorem later, let us give an important definition and a crucial lemma first, which will be used for the Proof of our main theorem.

Definition 2. [26] (Equilibrium point) For the Caputo time fractional dynamic system ${}_0^C D_t^{\alpha} u(t) = f(t, u(t))$, the constant u_0 is an equilibrium point of it if and only if $f(t, u_0) = 0$.

From above Definition 2, it is easy to find the equilibrium point of the plant (1) is $u(x, t) = 0$.

Lemma 2. [27] *If $u(t) \in \mathbb{R}$ is a continuous and differentiable function, for any time $t \ge t_0 \ge 0$, it is easy to get*

$$\frac{1}{2}{}_{t_0}^{C}D_t^{\alpha}u^2(t) \le u(t){}_{C}^{t}{}_{t_0}{}_{\alpha}^{u}(t), \quad 0 < \alpha < 1.$$

As we know, the invertibility of the integral transformation (8) is needed to prove the Mittag-Leffler stability. Thanks to the Lemma 2.4 in Ref. [25], the existence of inverse transformation has been obtained. Then, we will illustrate our main theorem below.

Theorem 1. *Assume that $a(x) \in C^1[0, 1]$ and the Laplace transform of $w^2(x, t)$ exists for $(x, t) \in (0, 1) \times [0, \infty)$.*

(1) *For any initial value $u_0(x) \in L^2(0, 1)$, system (1), (3) (or (5)) under the Robin boundary feedback controller (9) (or (10)) with the gain kernel k (x, y) described by (23), (25) is Mittag-Leffler stable at u(x, t) = 0 (equilibrium point) in the $L^2(0, 1)$ norm if the non-constant diffusion coefficient $\vartheta(x)$ and parameter λ satisfy the following constraint condition*

$$\begin{cases} q^s\vartheta(1) + \frac{\vartheta'(1)}{2} > 0 \\ p^s\vartheta(0) - \frac{\vartheta'(0)}{2} - \frac{\vartheta_{\min}}{2} > 0 \\ \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0. \end{cases} \tag{28}$$

(2) *For any initial value $u_0(x) \in H^1(0, 1)$, system (1), (3) (or (5)) under the controller (9) (or (10)) with the gain kernel k(x, y) described by (23), (25) is Mittag-Leffler stable at u(x, t) = 0 (equilibrium point) in the $H^1(0, 1)$ norm.*

Proof. The proof can be viewed as a generalization of the proof in [[13], Theorem 3] and [[6], Theorem 2], since it is for the Mittag-Leffler stability of the FRD system with non-constant diffusivity.

(1). We will prove the L^2 Mittag-Leffler stability in the following steps.

Step 1. Considering the below Lyapunov function

$$V(t, w(x, t)) = \frac{1}{2}\int_0^1 w^2(x, t)\mathrm{d}x. \tag{29}$$

Then finding the Caputo time fractional-order derivative of the above function (29) by the integration by parts and using Lemma 2, we obtain

$$\begin{aligned} {}_0^C D_t^{\alpha} V(t, w(x, t)) &= \tfrac{1}{2}\int_0^1 {}_0^C D_t^{\alpha} w^2(x, t)\mathrm{d}x \\ &\le \int_0^1 w(x, t){}_0^C D_t^{\alpha} w(x, t)\mathrm{d}x \\ &= -q^s\vartheta(1)w^2(1, t) - p^s\vartheta(0)w^2(0, t) \\ &\quad - \int_0^1 \vartheta(x)w_x^2(x, t)\mathrm{d}x \\ &\quad - \int_0^1 \vartheta'(x)w(x, t)\mathrm{d}w(x, t) - \lambda\int_0^1 w^2(x, t)\mathrm{d}x. \end{aligned} \tag{30}$$

Since

$$\begin{aligned} \int_0^1 \vartheta'(x)w(x, t)\mathrm{d}w(x, t) &= \vartheta'(1)w^2(1, t) - \vartheta'(0)w^2(0, t) \\ &\quad - \int_0^1 \vartheta''(x)w^2(x, t)\mathrm{d}x \\ &\quad - \int_0^1 \vartheta'(x)w(x, t)\mathrm{d}w(x, t), \end{aligned}$$

then we get

$$\begin{aligned} \int_0^1 \vartheta'(x)w(x, t)\mathrm{d}w(x, t) &= \tfrac{1}{2}\vartheta'(1)w^2(1, t) - \tfrac{1}{2}\vartheta'(0)w^2(0, t) \\ &\quad - \tfrac{1}{2}\int_0^1 \vartheta''(x)w^2(x, t)\mathrm{d}x. \end{aligned} \tag{31}$$

Substituting above equality (31) into (30), we further obtain

$$\begin{aligned} &{}_0^C D_t^{\alpha} V(t, w(x, t)) \\ &= -\left(q^s\vartheta(1) + \tfrac{1}{2}\vartheta'(1)\right)w^2(1, t) - \left(p^s\vartheta(0) - \tfrac{1}{2}\vartheta'(0)\right)w^2(0, t) \\ &\quad - \int_0^1 \vartheta(x)w_x^2(x, t)\mathrm{d}x - \int_0^1 \left(\lambda - \tfrac{1}{2}\vartheta''(x)\right)w^2(x, t)\mathrm{d}x. \end{aligned} \tag{32}$$

Applying Poincare's equality [[28], Lemma 2.1] to (32), it follows that

$$\begin{aligned} &{}_0^C D_t^{\alpha} V(t, w(x, t)) \\ &\le -\left(q^s\vartheta(1) + \tfrac{1}{2}\vartheta'(1)\right)w^2(1, t) \\ &\quad -\left(p^s\vartheta(0) - \tfrac{\vartheta'(0)}{2} - \tfrac{\vartheta_{\min}}{2}\right)w^2(0, t) \\ &\quad -\left(\lambda - \tfrac{\vartheta''_{\max}}{2} + \tfrac{\vartheta_{\min}}{4}\right)\int_0^1 w^2(x, t)\mathrm{d}x. \end{aligned} \tag{33}$$

Due to the assumption (28), we can further obtain

$$\begin{aligned} {}_0^C D_t^{\alpha} V(t, w(x, t)) &\le -\left(\lambda - \tfrac{\vartheta''_{\max}}{2} + \tfrac{\vartheta_{\min}}{4}\right)\int_0^1 w^2(x, t)\mathrm{d}x \\ &\le -2MV(t), \end{aligned} \tag{34}$$

where $M = \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0$.

Step 2. Note that $w(\cdot, t)$ is continuously differentiable on $t \in [0, \infty)$ as it satisfies the state equation of target system (11) and the definition of Caputo time fractional derivative [16]. Thus, $V(t, w(x, t))$ and ${}_0^C D_t^{\alpha} V(t, w(x, t))$ are continuously differentiable on $t \in [0, \infty)$. Since the Laplace transform of $w^2(x, t)$ exists, by the argument in [[29], Proof of Theorem 5], we can also get

$$V(t) \le V(0)E_{\alpha}(-2Mt^{\alpha}) \tag{35}$$

due to the fact of $t^{\alpha-1} \ge 0$ and $E_{\alpha,\alpha}(-2Mt^{\alpha}) \ge 0$ (see Ref. [30]), $\forall \alpha > 0, M > 0$. Note that $V(t) = V(t, w(x, t))$, $V(0) = V(0, w(x, 0))$.

By the equalities (29), (35) and $\|w(x, t)\| = (\int_0^1 w^2(x, t)\mathrm{d}x)^{1/2}$, it is readily to show

$$\|w(x, t)\| \le (2V(0)E_{\alpha}(-2Mt^{\alpha}))^{\frac{1}{2}}, \tag{36}$$

where $V(0) = V(0, w(x, 0)) > 0$ for $w(x, 0 \ne 0$ and $V(0, w(x, 0)) = 0$ if and only if $w(x, 0) = 0$. Based on Definition 1 and the facts $V(t, w(x, t))$ is locally Lipschitz with respect to $w(x, t)$ and $V(0, 0) = 0$ (i.e. $V(0, w(x, 0)) = 0$ when $w(x, 0) = 0$), it is easily obtain 2 $V(0, w(x,$

0)) is also Lipschitz on $w(x, 0)$ and $2V(0, 0) = 0$. Therefore, we can get the target system (11), (12) and (13) (or (14) and (15)) is Mittag-Leffler stable at $u(x, t) = 0$ in the $L^2(0, 1)$ norm.

Step 3. Based on Lemma 2.4 in Ref. [25], there exists constants γ, $\beta > 0$ to make the following inequalities true

$$\|u(x, t)\| \le \gamma\|w(x, t)\|, \quad \|w(x, 0)\| \le \gamma\|u(x, 0)\| \tag{37}$$

and

$$\begin{aligned}\|u(x, t)\|_{H^1} &\le \beta\|w(x, t)\|_{H^1},\\ \|w(x, 0)\|_{H^1} &\le \beta\|u(x, 0)\|_{H^1}.\end{aligned} \tag{38}$$

The inequality (37), together with (36), deduces that

$$\|u(x, t)\|^2 \le T_1\|u(x, 0)\|^2 E_\alpha(-2Mt^\alpha), \tag{39}$$

where $T_1 = \gamma^4$, $0 \le t < \infty$.

Last, using Definition 1 and the fact $V(t, w(x, t))$ is locally Lipschitz with respect to $w(x, t)$, we can get system (1), (3) and (9) (or (5) and (10)) is L^2 Mittag-Leffler stable at $u(x, t) = 0$.

(2). The Proof of the H^1 Mittag-Leffler stability will be presented. First, consider this below Lyapunov function

$$K(t, w_x(x, t)) = \int_0^1 w_x^2(x, t)\mathrm{d}x + p^s w^2(0, t) + q^s w^2(1, t). \tag{40}$$

Taking the Caputo time fractional-order derivative of above equality (40) and using Lemma 2, we get

$$\begin{aligned}{}_0^C D_t^\alpha K(t, w_x(x, t)) &= \int_0^1 {}_0^C D_t^\alpha w_x^2(x, t)\mathrm{d}x\\ &+ p^s\,{}_0^C D_t^\alpha w^2(0, t) + q^s\,{}_0^C D_t^\alpha w^2(1, t)\\ &\le 2\int_0^1 w_x(x, t)\,{}_0^C D_t^\alpha w_x(x, t)\mathrm{d}x\\ &+ 2p^s w(0, t)\,{}_0^C D_t^\alpha w(0, t)\\ &+ 2q^s w(1, t)\,{}_0^C D_t^\alpha w(1, t).\end{aligned} \tag{41}$$

For computing the term of $\int_0^1 w_x(x, t)\,{}_0^C D_t^\alpha w_x(x, t)\mathrm{d}x$, we multiply $w_{xx}(x, t)$ by the first equation of (11) and integrate the product from 0 to 1, then we obtain

$$\begin{aligned}&\int_0^1 w_{xx}(x, t)\,{}_0^C D_t^\alpha w(x, t)\mathrm{d}x\\ &= \int_0^1 \vartheta(x) w_{xx}^2(x, t)\mathrm{d}x + \lambda q^s w^2(1, t)\\ &+ \lambda p^s w^2(0, t) + \lambda\int_0^1 w_x^2(x, t)\mathrm{d}x\\ &= \int_0^1 \vartheta(x) w_{xx}^2(x, t)\mathrm{d}x + \lambda K(t, w_x(x, t)),\end{aligned} \tag{42}$$

since $w_x(0, t) = p^s w(0, t)$ and $w_x(1, t) = -q^s w(1, t)$.

By the aside of integration by parts, we compute the integration of above product again. Together with $w_x(0, t) = p^s w(0, t)$ and $w_x(1, t) = -q^s w(1, t)$, then one can easily show that

$$\begin{aligned}&\int_0^1 w_{xx}(x, t)\,{}_0^C D_t^\alpha w(x, t)\mathrm{d}x = -q^s w(1, t)\,{}_0^C D_t^\alpha w(1, t)\\ &- p^s w(0, t)\,{}_0^C D_t^\alpha w(0, t) - \int_0^1 w_x(x, t)\,{}_0^C D_t^\alpha w_x(x, t)\mathrm{d}x.\end{aligned} \tag{43}$$

Comparing (42) with (43), it is easy to obtain

$$\begin{aligned}&\int_0^1 w_x(x, t)\,{}_0^C D_t^\alpha w_x(x, t)\mathrm{d}x = -q^s w(1, t)\,{}_0^C D_t^\alpha w(1, t)\\ &- p^s w(0, t)\,{}_0^C D_t^\alpha w(0, t) - \int_0^1 \vartheta(x) w_{xx}^2(x, t)\mathrm{d}x - \lambda K(t, w_x(x, t)).\end{aligned} \tag{44}$$

Then substituting (44) into (41), we have

$$\begin{aligned}{}_0^C D_t^\alpha K(t, w_x(x, t)) &\le -2\int_0^1 \vartheta(x) w_{xx}^2(x, t)\mathrm{d}x - 2\lambda K(t, w_x(x, t))\\ &\le -2\lambda K(t, w_x(x, t)),\end{aligned} \tag{45}$$

since $\vartheta(x) > 0$ for $x \in [0, 1]$.

Similarly, we can further get

$$K(t) \le K(0) E_\alpha(-2\lambda t^\alpha),\ \forall\, t \in [0, \infty). \tag{46}$$

where $K(t) = K(t, w_x(x, t))$ and $K(0) = K(0, w_x(x, 0))$. The remainder Proof is the same as the one of (1) except $\|w(x, t)\| = (\int_0^1 w^2(x, t)\mathrm{d}x)^{1/2}$ and the inequality (39) replaced by $\|w(x, t)\|_{H^1} = (w^2(0, t) + w^2(1, t) + \int_0^1 w_x^2(x, t)\mathrm{d}x)^{1/2}$ and $\|u(x, t)\|_{H^1}^2 \le T_2\|u(x, 0)\|_{H^1}^2 E_\alpha(-2\lambda t^\alpha)$ ($T_2 = \frac{M}{m}\beta^4$, $M = \max\{1, p^s, q^s\} > 0$ and $m = \min\{1, p^s, q^s\} > 0$) respectively. Thus, we have proved system (1), (3) and (9) (or (5) and (10)) is Mittag-Leffler stable at $u(x, t) = 0$ in the $H^1(0, 1)$ norm.□

Remark 4. The hypothesis condition (28) in above theorem is relatively conservative. If we take a specific $\vartheta(x)$, the conservation can be relax or improved. And if the system parameter $\vartheta(x)$ is constant, the Mittag-Leffler stability results has been studied in [[13], Section 3.3, Section 4.2, Section 5.2].

Corollary 1. *(Mittag-Leffler Convergence for Dirichlet Boundary Conditions): Suppose that $a(x) \in C^1[0, 1]$ and the Laplace transform of $w^2(x, t)$ exists for $(x, t) \in (0, 1) \times [0, \infty)$.*

(1) *For any initial value $u_0(x) \in L^2(0, 1)$, system (1), (3) or (5) (with $p_1 = 0$ or $p = +\infty$) under the controller (9) or (10) (with $q_1 = 0$, $q_1^s = 0$ or $q = +\infty$, $q^s = +\infty$) with the gain kernel $k(x, y)$ provided by (23), (25) is L^2 Mittag-Leffler stable at $u(x, t) = 0$ (equilibrium point) if the non-constant diffusion coefficient $\vartheta(x)$ and parameter λ satisfy the following constraint condition*

$$\lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0. \tag{47}$$

(2) *For any initial value $u_0(x) \in H^1(0, 1)$, system (1), (3) or (5) (with $p_1 = 0$ or $p = +\infty$) under the controller (9) or (10) (with $q_1 = 0$, $q_1^s = 0$ or $q = +\infty$, $q^s = +\infty$) with the gain kernel $k(x, y)$ provided by (23), (25) is H^1 Mittag-Leffler stable at $u(x, t) = 0$ (equilibrium point).*

Corollary 2. *(Mittag-Leffler Convergence for Neumann Boundary Conditions): Suppose that $a(x) \in C^1[0, 1]$ and the Laplace transform of $w^2(x, t)$ exists for $(x, t) \in (0, 1) \times [0, \infty)$.*

(1) *For any initial value $u_0(x) \in L^2(0, 1)$, system (1), (3) or (5) (with $p_2 = 0$ or $p = 0$) under the controller (9) or (10) (with $q_2 = 0$, $q_2^s = 0$ or $q = 0$, $q^s = 0$) with the gain kernel $k(x, y)$ given by (23), (25) is L^2 Mittag-Leffler stable at $u(x, t) = 0$ (equilibrium point) if the non-constant diffusion coefficient $\vartheta(x)$ and parameter λ satisfy the following constraint condition*

$$\begin{cases}\frac{\vartheta'(1)}{2} > 0\\ \frac{\vartheta'(0)}{2} + \frac{\vartheta_{\min}}{2} < 0\\ \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0.\end{cases} \tag{48}$$

(2) *For any initial value $u_0(x) \in H^1(0, 1)$, system (1), (3) or (5) (with $p_2 = 0$ or $p = 0$) under the controller (9) or (10) (with $q_2 = 0$, $q_2^s = 0$ or $q = 0$, $q^s = 0$) with the gain kernel $k(x, y)$ given by (23), (25) is H^1 Mittag-Leffler stable at $u(x, t) = 0$ (equilibrium point).*

Remark 5. Some discussion on mixed boundary conditions.

i) If $p_2 = 0$ ($p = 0$), $q_1 = 0$ ($q = +\infty$), $p_1, q_2 > 0$ or $q_1 = 0$ ($q = +\infty$), $p_1, p_2, q_2 > 0$, i.e. $u_x(0, t) = 0$, $u(1, t) = U(t)$ or $p_1 u_x(0, t) - p_2 u(0, t) = 0$ ($u_x(0, t) - pu(0, t) = 0$), $u(1, t) = U(t)$. That is, the mixed boundary conditions are Neumann or Robin boundary condition at $x = 0$ and Dirichlet actuation at $x = 1$. Then, we can obtain the L^2 Mittag-Leffler stability of this closed-loop system under the following constraint condition

$$\begin{cases} \frac{\vartheta'(0)}{2} < 0 \\ \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0, \end{cases} \tag{49}$$

or

$$\begin{cases} p^s\vartheta(0) - \frac{\vartheta'(0)}{2} > 0 \\ \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0, \end{cases} \tag{50}$$

respectively.

ii) If $p_1 = 0$ ($p = +\infty$), $q_2 = 0$ ($q = 0$), $p_2, q_1 > 0$ or $q_2 = 0$ ($q = 0$), $p_1, p_2, q_1 > 0$, i.e. $u(0, t) = 0$, $u_x(1, t) = U(t)$ or $p_1u_x(0, t) - p_2u(0, t) = 0$ ($u_x(0, t) - pu(0, t) = 0$, $u_x(1, t) = U(t)$. That being said, the mixed boundary conditions are Dirichlet or Robin boundary condition at $x = 0$ and Neumann actuation at $x = 1$. Further, this closed-loop system is L^2 Mittag-Leffler stable under the following constraint condition

$$\begin{cases} \frac{\vartheta'(1)}{2} > 0 \\ \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0, \end{cases} \tag{51}$$

or

$$\begin{cases} \frac{\vartheta'(1)}{2} > 0 \\ p^s\vartheta(0) - \frac{\vartheta'(0)}{2} - \frac{\vartheta_{\min}}{2} > 0 \\ \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0, \end{cases} \tag{52}$$

respectively.

iii) If $p_1 = 0$ ($p = +\infty$), $p_2, q_1, q_2 > 0$ or $p_2 = 0$ ($p = 0$), $p_1, q_1, q_2 > 0$, i.e. $u(0, t) = 0$, $q_1u_x(1, t) + q_2u(1, t) = U(t)$ ($u_x(1, t) + qu(1, t) = U(t)$) or $u_x(0, t) = 0$, $q_1u_x(1, t) + q_2u(1, t) = U(t)$ ($u_x(1, t) + qu(1, t) = U(t)$). In other words, the mixed boundary conditions are Dirichlet or Neumann boundary condition at $x = 0$ and Robin actuation at $x = 1$. In this case, the L^2 Mittag-Leffler stability of the closed-loop system can be obtained under the following constraint condition

$$\begin{cases} q^s\vartheta(1) + \frac{\vartheta'(1)}{2} > 0 \\ \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0, \end{cases} \tag{53}$$

or

$$\begin{cases} q^s\vartheta(1) + \frac{\vartheta'(1)}{2} > 0 \\ \frac{\vartheta'(0)}{2} + \frac{\vartheta_{\min}}{2} < 0 \\ \lambda - \frac{\vartheta''_{\max}}{2} + \frac{\vartheta_{\min}}{4} > 0, \end{cases} \tag{54}$$

respectively.

5. Construction of boundary feedback controller

In this section, we want to show how to use our proposed method to obtain a gain kernel $k(x, y)$ for the corresponding boundary feedback controller through a specific case. We let $a(x) \equiv a \equiv const$, $p = p^s$, then system (1), (5) can turn into the following form

$$\begin{aligned} {}_0^CD_t^\alpha u(x, t) &= \vartheta(x)u_{xx}(x, t) + au(x, t),\ x \in (0, 1),\ t > 0 \\ u(x, 0) &= u_0(x), \quad x \in [0, 1] \\ u_x(0, t) - pu(0, t) &= 0, \quad t > 0. \end{aligned} \tag{55}$$

The open-loop system (55) with $u_x(1, t) + qu(1, t) = 0$, $q > 0$ is unstable when the constant a is positive and large enough (see the corresponding analysis in Section 2). Then, the corresponding target system is considered as follows

$$\begin{aligned} {}_0^CD_t^\alpha w(x, t) &= \vartheta(x)w_{xx}(x, t) - \lambda w(x, t),\ x \in (0, 1),\ t > 0 \\ w(x, 0) &= w_0(x), \quad x \in [0, 1] \\ w_x(0, t) - pw(0, t) &= 0, \quad t > 0. \end{aligned}$$

According to the results of Section 3, the kernel PDE (21) becomes

$$\begin{cases} \vartheta(x)k_{xx}(x, y) - (\vartheta(y)k(x, y))_{yy} = (a + \lambda)k(x, y) \\ k_y(x, 0) = (p - \vartheta'(0)/\vartheta(0))k(x, 0) \\ 2\vartheta(x)\frac{d}{dx}k(x, x) = -\vartheta'(x)k(x, x) + a + \lambda \\ k(0, 0) = 0 \end{cases} \tag{56}$$

for $(x, y) \in \Xi = \{0 \le y \le x \le 1\}$.

Using (23) and (24), we can also transform (56) into the following form

$$\begin{cases} \check{k}_{\check{x}\check{x}}(\check{x}, \check{y}) - \check{k}_{\check{y}\check{y}}(\check{x}, \check{y}) = \frac{\check{a}(\check{x},\check{y})}{\vartheta(0)}\check{k}(\check{x}, \check{y}) \\ \check{k}_{\check{y}}(\check{x}, 0) = \left(p - \frac{\vartheta'(0)}{4\vartheta(0)}\right)\check{k}(\check{x}, 0) \\ \frac{d}{d\check{x}}\check{k}(\check{x}, \check{x}) = \frac{a+\lambda}{2\sqrt{\vartheta(0)}} \\ \check{k}(0, 0) = 0, \end{cases} \tag{57}$$

where

$$\check{a}(\check{x}, \check{y}) = \frac{3}{16}\left(\frac{\vartheta'^2(x)}{\vartheta(x)} - \frac{\vartheta'^2(y)}{\vartheta(y)}\right) + \frac{1}{4}(\vartheta''(y) - \vartheta''(x)) + a + \lambda. \tag{58}$$

Similar to the arguments in [[6], Section 3.1], we also assume $\frac{3\vartheta'^2(x)}{16\vartheta(x)} - \frac{\vartheta''(x)}{4} = D = const$, likewise, for $\vartheta(y)$, $\frac{3\vartheta'^2(y)}{16\vartheta(y)} - \frac{\vartheta''(y)}{4} = D = const$. It has two solutions, and in this case we take one solution like

$$\vartheta(x) = \vartheta_0(1 + \rho_0(x - x_0)^2)^2, \tag{59}$$

where $\vartheta_0, \rho_0 > 0$, and x_0 is an arbitrary constant.

Then from (58) and above assumption, we can obtain $\check{a}(\check{x}, \check{y}) = a + \lambda$. The solution for the PDE (57) can be written as the following form based on [[1], Section VIII.B]

$$\begin{aligned} \check{k}(\check{x}, \check{y}) &= \sqrt{\vartheta(0)}\check{\lambda}\check{x}\frac{I_1(\sqrt{\check{\lambda}(\check{x}^2 - \check{y}^2)})}{\sqrt{\check{\lambda}(\check{x}^2 - \check{y}^2)}} - \frac{\check{p}\sqrt{\vartheta(0)}\check{\lambda}}{\sqrt{\check{\lambda} + \check{p}^2}} \times \int_0^{\check{x}-\check{y}} e^{-\check{p}\tau/2}I_0 \\ &\quad (\sqrt{\check{\lambda}(\check{x} + \check{y})(\check{x} - \check{y} - \tau)})\sinh\left(\frac{\sqrt{\check{\lambda} + \check{p}^2}}{2}\tau\right)d\tau, \end{aligned} \tag{60}$$

where $\check{\lambda} = \frac{a+\lambda}{\vartheta(0)}$, $\check{p} = p - \frac{\vartheta'(0)}{4\vartheta(0)}$, and I_i denotes a modified Bessel function of order i ($i = 0, 1$).

This, together with (23) and (59), also implies that

$$\begin{aligned} k(x, y) &= \frac{(1 + \rho_0(x - x_0)^2)^{1/2}}{\sqrt{\vartheta_0}(1 + \rho_0(y - x_0)^2)^{3/2}}\Bigg[\sqrt{\vartheta(0)}\check{\lambda}\check{x} \times \frac{I_1(\sqrt{\check{\lambda}(\check{x}^2 - \check{y}^2)})}{\sqrt{\check{\lambda}(\check{x}^2 - \check{y}^2)}} \\ &\quad - \frac{\check{p}\sqrt{\vartheta(0)}\check{\lambda}}{\sqrt{\check{\lambda} + \check{p}^2}} \times \int_0^{\check{x}-\check{y}} e^{-\check{p}\tau/2}I_0(\sqrt{\check{\lambda}(\check{x} + \check{y})(\check{x} - \check{y} - \tau)}) \\ &\quad \times \sinh\left(\frac{\sqrt{\check{\lambda} + \check{p}^2}}{2}\tau\right)d\tau\Bigg], \end{aligned} \tag{61}$$

where $\check{x} = \frac{1 + \rho_0x_0^2}{\sqrt{\rho_0}}(\text{atan}(\sqrt{\rho_0}(x - x_0)) + \text{atan}(\sqrt{\rho_0}x_0))$, and $\check{y} = \frac{1 + \rho_0x_0^2}{\sqrt{\rho_0}}(\text{atan}(\sqrt{\rho_0}(y - x_0)) + \text{atan}(\sqrt{\rho_0}x_0))$.

According to (61), we get the control kernels

$$\begin{aligned} k(1, y) &= \frac{(1 + \rho_0(1 - x_0)^2)^{1/2}}{\sqrt{\vartheta_0}(1 + \rho_0(y - x_0)^2)^{3/2}}\Bigg[\sqrt{\vartheta(0)}\check{\lambda}\check{x} \times \frac{I_1(\sqrt{\check{\lambda}(\check{x}^2 - \check{y}^2)})}{\sqrt{\check{\lambda}(\check{x}^2 - \check{y}^2)}} \\ &\quad - \frac{\check{p}\sqrt{\vartheta(0)}\check{\lambda}}{\sqrt{\check{\lambda} + \check{p}^2}} \times \int_0^{\check{x}-\check{y}} e^{-\check{p}\tau/2}I_0(\sqrt{\check{\lambda}(\check{x} + \check{y})(\check{x} - \check{y} - \tau)}) \\ &\quad \times \sinh\left(\frac{\sqrt{\check{\lambda} + \check{p}^2}}{2}\tau\right)d\tau\Bigg], \end{aligned} \tag{62}$$

and

$$\begin{aligned}k_x(1,y) &= \frac{\rho_0(1-x_0)(1+\rho_0(1-x_0)^2)^{-1/2}}{\sqrt{\vartheta_0}(1+\rho_0(y-x_0)^2)^{3/2}}[\sqrt{\vartheta(0)}\check{\lambda}\check{x} \\ &\times \frac{I_1(\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)})}{\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)}} - \frac{\check{p}\sqrt{\vartheta(0)}\check{\lambda}}{\sqrt{\check{\lambda}+\check{p}^2}}\int_0^{\check{x}-\check{y}} e^{-\check{p}\tau/2} \\ &\times I_0(\sqrt{\check{\lambda}(\check{x}+\check{y})(\check{x}-\check{y}-\tau)})\sinh\left(\frac{\sqrt{\check{\lambda}+\check{p}^2}}{2}\right. \\ &\left.\times \tau\right)\mathrm{d}\tau] + \frac{(1+\rho_0(1-x_0)^2)^{1/2}}{\sqrt{\vartheta_0}(1+\rho_0(y-x_0)^2)^{3/2}}\left[\frac{\sqrt{\vartheta(0)}\check{\lambda}\check{x}^2}{\check{x}^2-\check{y}^2}\right. \\ &\times I_2(\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)})\frac{1+\rho_0x_0^2}{1+\rho_0(1-x_0)^2} + \sqrt{\vartheta(0)}\check{\lambda} \\ &\times \frac{I_1(\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)})}{\sqrt{\check{\lambda}(\check{x}^2-\check{y}^2)}}\frac{1+\rho_0x_0^2}{1+\rho_0(1-x_0)^2} - I_0(0) \\ &\times \frac{\check{p}\sqrt{\vartheta(0)}\check{\lambda}}{\sqrt{\check{\lambda}+\check{p}^2}}e^{-\check{p}(\check{x}-\check{y})/2}\sinh\left(\frac{\sqrt{\check{\lambda}+\check{p}^2}}{2}(\check{x}-\check{y})\right) \\ &\times \frac{1+\rho_0x_0^2}{1+\rho_0(1-x_0)^2} - \frac{\check{p}\sqrt{\vartheta(0)}\check{\lambda}}{\sqrt{\check{\lambda}+\check{p}^2}}\int_0^{\check{x}-\check{y}} e^{-\check{p}\tau/2} \\ &\times \sinh\left(\frac{\sqrt{\check{\lambda}+\check{p}^2}}{2}\tau\right)I_1(\sqrt{\check{\lambda}(\check{x}+\check{y})(\check{x}-\check{y}-\tau)}) \\ &\left.\times \frac{\check{\lambda}\check{x}-\frac{1}{2}\check{\lambda}\tau}{\sqrt{\check{\lambda}(\check{x}+\check{y})(\check{x}-\check{y}-\tau)}}\frac{1+\rho_0x_0^2}{1+\rho_0(1-x_0)^2}\right],\end{aligned} \tag{63}$$

where $\check{x} = \frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\mathrm{atan}(\sqrt{\rho_0}(1-x_0)) + \mathrm{atan}(\sqrt{\rho_0}x_0))$, and $\check{y} = \frac{1+\rho_0x_0^2}{\sqrt{\rho_0}}(\mathrm{atan}(\sqrt{\rho_0}(y-x_0)) + \mathrm{atan}(\sqrt{\rho_0}x_0))$. They will be used in numerical simulations in Section 6.

As is illustrated in above Theorem 1, we obtain the controller (10) with gain kernels (62) and (63) can make system (55) Mittag-Leffler stable in the $L^2(0, 1)$ and $H^1(0, 1)$ norms.

6. Study on numerical simulation

We will present the boundary feedback controller for the Mittag-Leffler stability of the FRD system with space-dependent diffusivity. For this case, the numerical algorithm for Caputo-type advection-diffusion in Ref. [31], together with finite-difference approximation method and the approach of using difference to estimate differential, is used to solve the FRD system. We take the spatial stepsize $h = \frac{S}{X}$ and temporal stepsize $\mu = \frac{R}{T}$, i.e., the space domain $0 < x < S$, grid points $X + 1$, and the time domain $0 < t < R$, grid points $T + 1$.

In this non-constant diffusivity case, we let discretization parameters $S = 1$, $R = 0.6$, $X = 20$, $T = 300$, and $\vartheta_0 = 1$, $\rho_0 = 1$, $x_0 = 0$ for the space-dependent diffusivity, i.e. $\vartheta(x) = (1 + x^2)^2$ which meets the constraint condition (28). The system parameters are chosen as $\alpha = 0.7$, $a(x) \equiv 10$, $p^s = p = 1$, $\lambda = 10$, $q^s = q = 2$ together with the initial condition $u_0(x) = 10x(1 - x)$. For better understanding our simulation implementation, we present the relevant algorithm below.

Algorithm 1
Implementation of the controlled FRD system with space-dependent diffusivity.

Step 1: Solving kernel $k(1, y)$, $k_x(1, y)$ based on (62), (63) respectively, then obtain the controller (10).
Step 2: Inputting parameters $u_0(x)$, $\vartheta(x)$ and t_0.
Step 3: Computing function $u(x, t)$ with $u_0(x) = u(x, t_0)$ according to (1), (5), (10).
Step 4: Initializing $n = 0$, $n = 0, 1, \cdots, T - 1$;
  while $n \geq 0$ do $t_{n+1} = t_n + R/T$, $t_n = t_0 + nR/T$;
    then update $u_{n+1}(x) = u(x, t_{n+1})$, $n = n + 1$;
  end.

With this algorithm and above given parameters, we show our simulation results. Fig. 1 shows the space-dependent diffusivity $\vartheta(x)$. The gain kernels (62) and (63) for the controller (10) are presented in Fig. 2.

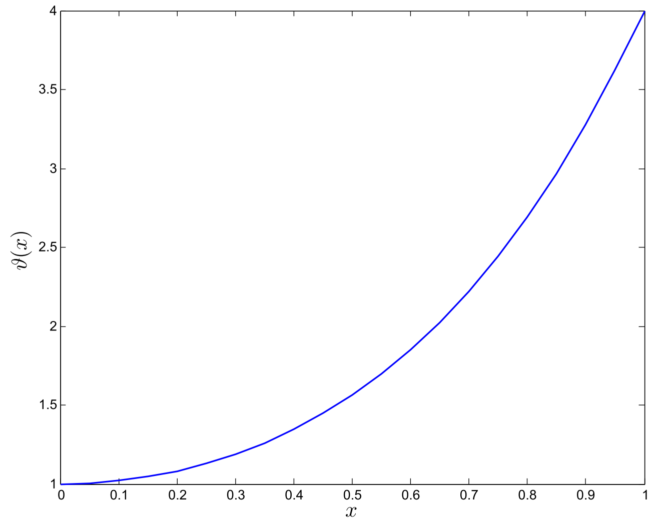


Fig. 1. The space-dependent diffusion efficient (parameter function $\vartheta(x) = (1 + x^2)^2$) of system (1), (3).

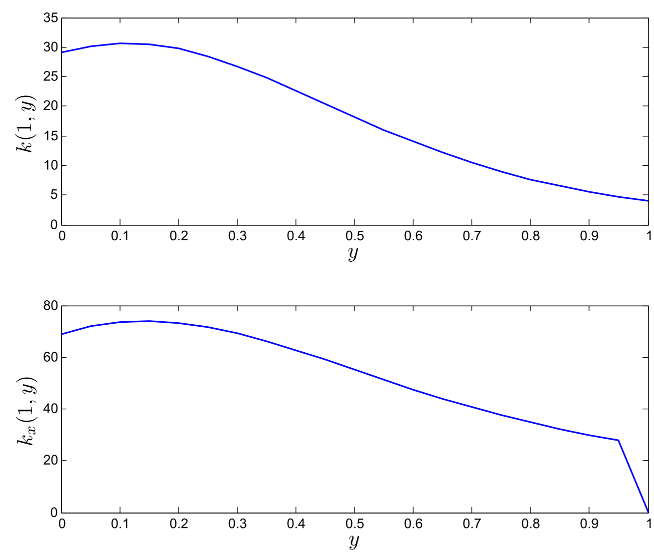


Fig. 2. The gain kernels $k(1, y)$ and $k_x(1, y)$ from (62) and (63).

It is noticeable that we removed $k_x(1, y)$ at end $y = 1$ (i.e. $k_x(1, 1)$) to avoid its denominator being zero (in other words, we set $k_x(1, 1) = 0$ see Fig. 2). The simulation results on state L^2 norm in logarithmic scale of the open-loop system (1), (5) (with $u_x(1, t) + 2u(1, t) = 0$) and the closed-loop system (1), (5), (10) are shown in Fig. 3, which illustrates the Robin boundary feedback controller (10) can make this closed-loop system L^2 Mittag-Leffler stable (state norm converge to zero). In Fig. 4(b), we can see this closed-loop system can be H^1 Mittag-Leffler stabilized (state converges to zero for all x) under the controller (10). The state's space curve's projection on XOZ plane further verify the Mittag-Leffler convergence of the closed-loop system, as shown in Fig. 5 (b). Obviously, the proposed Robin boundary feedback controller has good control effects for the FRD system with space-dependent diffusivity.

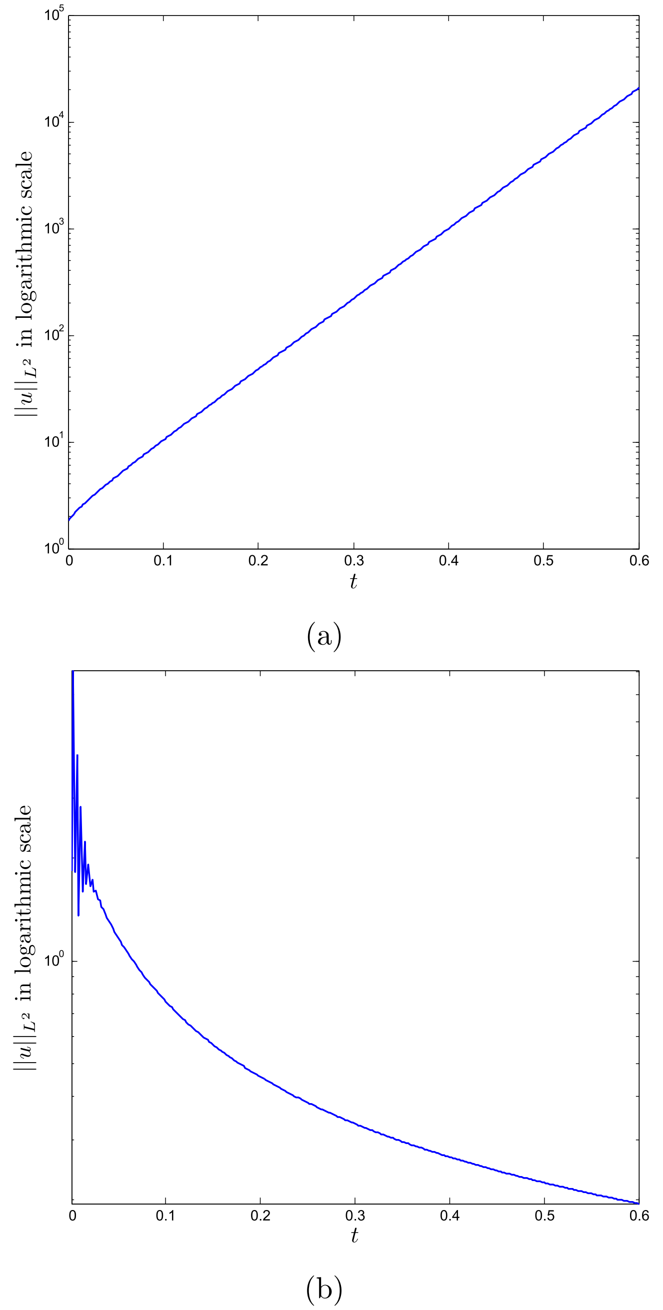


Fig. 3. State L^2 norms in logarithmic scale of the open-loop system (1), (5)($u_x(1, t) + 2u(1, t) = 0$) and the closed-loop system (1), (5) under the Robin boundary feedback controller (10).

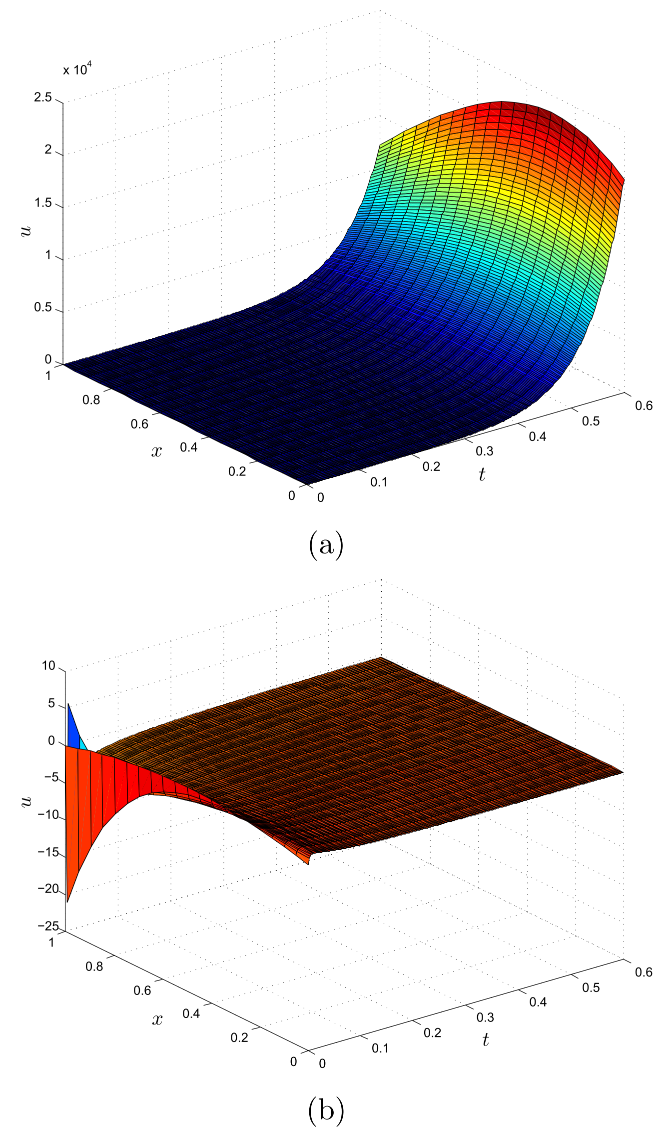


Fig. 4. Evolution of state of the open-loop system (1), (5) ($u_x(1, t) + 2u(1, t) = 0$) and the closed-loop system (1), (5) under the Robin boundary feedback controller (10).

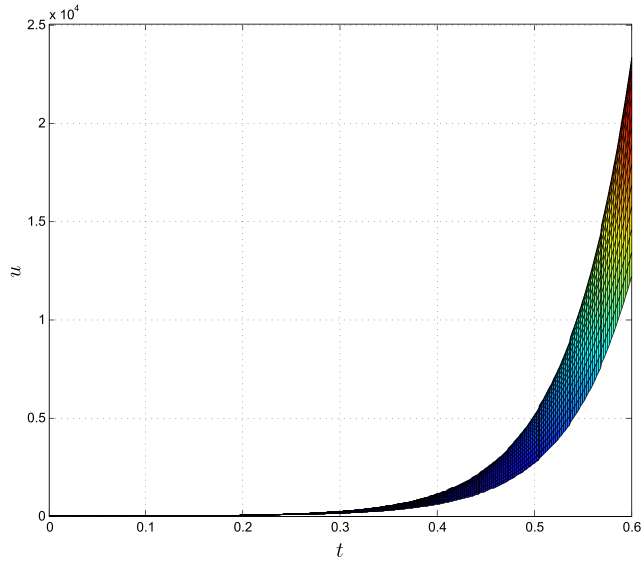
7. Conclusions and future work

This paper discussed the boundary feedback control problem of the FRD system with a space-dependent diffusion coefficient via the backstepping method, which can be taken as the extension of the boundary feedback stabilization problem of the FRD system with a space-independent diffusion coefficient. With the help of integral transformation, the closed-loop system under the Robin boundary feedback controller was mapped into a Mittag-Leffler stable target system. It is worth pointing out that the well posedness of the gain kernel PDE with the nonzero-value or zero-value $k(0, 0)$ was solved with the help of the results in [[13], Lemma 2]. Extensive simulation studies on the closed-loop system with space-dependent diffusivity revealed that the Robin boundary feedback controller can be used to stabilize this system, i.e. this system can approach to Mittag-Leffler stable in the $L^2(0, 1)$ and $H^1(0, 1)$ norms.

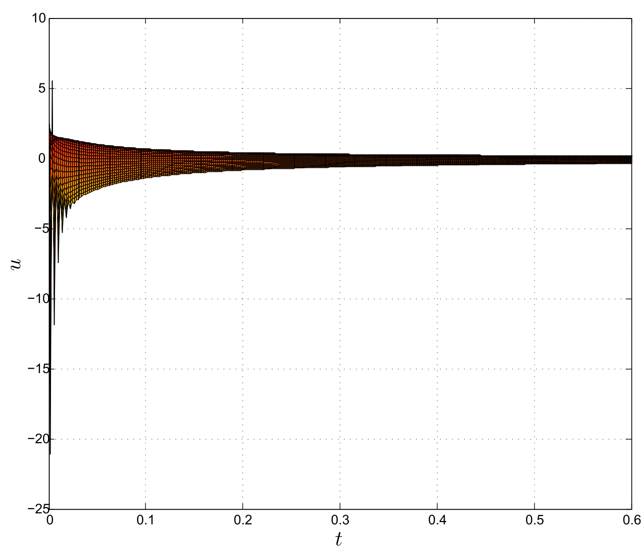
Future work is devoted to observer-based output feedback control for the FRD system with a non-constant diffusion coefficient by the backstepping approach, and boundary control for the FRD system with a time-varying reaction coefficient.

Author contributions

The first author conceived the central idea and wrote the initial draft of the paper, the second author conducted the analysis, and the third author contributed to refining the ideas and the revision.



(a)



(b)

Fig. 5. Projection of the system state's space curve on the XOZ plane without control ($u_x(1, t) + 2u(1, t) = 0$) and under the Robin boundary feedback controller (10).

Acknowledgements

This work is partially supported by National Natural Science Foundation of China (61174021,61473136), the Fundamental Research Funds for the Central Universities (JUSRP51322B), the 111 Project (B12018) and Jiangsu Innovation Program for Graduates (Project No. KYLX15_1170).

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