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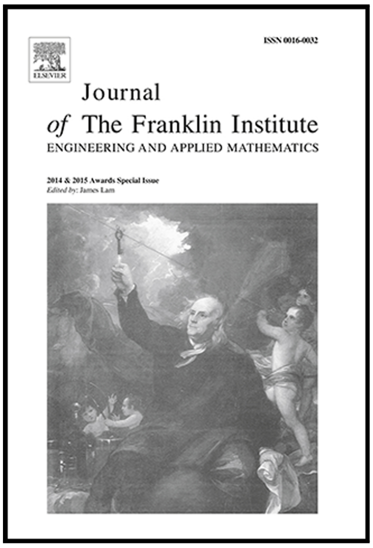

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Stability analysis of nonlinear Hadamard fractional differential system *

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Abstract

The stability of the zero solution of a class of nonlinear Hadamard type fractional differential system is investigated by utilizing a new fractional comparison principle. The novelty of this paper is based on some new fractional differential inequalities along the given nonlinear Hadamard fractional differential system. A comparison principle employing the new fractional differential inequality for scalar Hadamard fractional differential system is presented. Based on the new comparison principle, some sufficient conditions for the (generalized) stability and the (generalized) Mittag-Leffler stability are given.

Keywords: Stability; Hadamard fractional differential system; Fractional differential inequality; Fractional comparison principle.
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1 Introduction

In recent years, fractional calculus is a topic of growing interest based on the superiority of integrals and derivatives of complex order and the ability to model certain physical systems in a

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more adequate and precise fashion than integer order alternative. There are many applications in different fields such as electrical circuit, cosmology, control theory, biomedical engineering, economics, etc. In terms of applied mathematics to study many problems from several diverse disciplines of engineering and technical sciences, the fractional calculus is a powerful tool. For details , we refer the reader to the works in [1]-[7]. While the most common ones are the Riemann-Liouville and Caputo fractional operators, recently, there has been an increasing interest in the development of Hadamard fractional operators. Details and properties of the Hadamard fractional derivative and integral can be found in book [4] and papers [8]-[20].

Recently, fractional calculus in the control theory is widely seen. Fractional-order controller is playing a very vital role in almost every field of control subject. Stability is one of the important characteristics of the control problem. It is also an essential condition for any control problem. The initial work about stability of fractional order systems can be dated back to Matignon [21]. It has achieved great strides [22]-[29]. For its latest developments, readers of interest could refer to [30]-[38]. So far, there are several approaches to the study of the stability of fractional differential systems, one of which is the fractional comparison principle approach. The main difficulty is to establish a fractional comparison principle. To overcome this difficulty, we developed several fractional differential inequalities, which play a crucial role in this paper.

In this paper, the stability of the zero solution of nonlinear Caputo-type Hadamard fractional system is investigated. We establish a Hadamard type fractional differential inequality. Comparison principle using this new fractional differential inequality and scalar Hadamard fractional differential system is presented and sufficient conditions for the (generalized) stability and the (generalized) Mittag-Leffler stability are obtained.

2 Preliminaries

First of all, we summarize some important definitions and related lemmas.

Definition 2.1 *[4] The Hadamard fractional integral of order α for a function g is defined as*

$$^{H}I^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_{1}^{t}(\log\frac{t}{s})^{\alpha-1}\frac{g(s)}{s}ds,\qquad \alpha>0,$$

provided the integral exists.

Definition 2.2 *[4] The Hadamard fractional derivative of fractional order α for a function $g:[1,\infty)\to\mathbb{R}$ is defined as*

$$^{H}D^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^{n}\int_{1}^{t}(\log\frac{t}{s})^{n-\alpha-1}\frac{g(s)}{s}ds,\qquad n-1<\alpha<n, n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot)=\log_{e}(\cdot)$.

Definition 2.3 *[4] The Caputo-type Hadamard fractional derivative of fractional order α for a function $g:[1,\infty)\to\mathbb{R}$ is defined as*

$$^{H}_{C}D^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_{1}^{t}(\log\frac{t}{s})^{n-\alpha-1}(t\frac{d}{dt})^{n}\frac{g(s)}{s}ds,\qquad n-1<\alpha<n, n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.1 *[4] If g is a function such that ${}_{C}^{H}D^{\alpha}g(t)$ and ${}^{H}D^{\alpha}g(t)$ exist, then*

$$ {}_{C}^{H}D_{t_0}^{\alpha}g(t) = {}^{H}D_{t_0}^{\alpha}g(t) - \sum_{k=0}^{n-1}\frac{(t\frac{d}{dt})^k g(t_0)}{\Gamma(k-\alpha+1)}(\log\frac{t}{t_o})^{k-\alpha}, $$

and when $0 < [\alpha] < 1$, then

$$ {}_{C}^{H}D_{t_0}^{\alpha}g(t) = {}^{H}D_{t_0}^{\alpha}g(t) - \frac{g(t_0)}{\Gamma(1-\alpha)}(\log\frac{t}{t_0})^{-\alpha}. $$

Definition 2.4 *[4] The one and two parameter Mittag-Leffler functions are defined as*

$$ E_{q_1}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(q_1k+1)} $$

$$ E_{q_1,q_2}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(q_1k+q_2)}. $$

Definition 2.5 *Let $x = 0$ be the zero solution of ${}_{C}^{H}D_{t_0}^{\alpha}x(t) = f(t,s)$ with $\alpha \in (0,1)$ and if $f \in C([1,+\infty)\times\mathbb{R}^n,\mathbb{R}^n)$. The zero solution $x = 0$ is said to be*
$\triangleright$ stable if for $\forall\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, such that $\|x(t_0)\| < \delta(\varepsilon)$, implies $\|x(t)\| \le \varepsilon$ for $t \ge t_0$. The zero solution $x = 0$ is said to be unstable, if $\exists\varepsilon_0 > 0$, $\forall\delta > 0$, $\exists x(t_0)$, $\|x(t_0)\| < \delta$, but $\exists t_1 \ge t_0$ such that $\|x(t_1)\| \ge \varepsilon$.
$\triangleright$ asymptotically stable if it is stable and $\lim_{t\to+\infty} x(t) = 0$.

Definition 2.6 *(Mittag-Leffler Stability) The solution of ${}_{C}^{H}D_{t_0}^{\alpha}x(t) = f(t,x)$ is said to be Mittag-Leffler stable if*

$$ \|x(t)\| \le \{m[x(t_0)]E_{\alpha}(-\lambda(\log\frac{t}{t_0})^{\alpha})\}^j, $$

where t_0 is the initial time, $\alpha \in (0,1)$, $\lambda \ge 0$, $j > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{R}^n$ with Lipschitz constant m_0.

Definition 2.7 *(Generalized Mittag-Leffler Stability) The solution of ${}_{C}^{H}D_{t_0}^{\alpha}x(t) = f(t,x)$ is said to be Generalized Mittag-Leffler stable if*

$$ \|x(t)\| \le \{m[x(t_0)](\log\frac{t}{t_0})^{-\rho}E_{\alpha,1-\rho}(-\lambda(\log\frac{t}{t_0})^{\alpha})\}^j, $$

where t_0 is the initial time, $\alpha \in (0,1), -\alpha < \rho < 1-\alpha$, $\lambda \ge 0$, $j > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{R}^n$ with Lipschitz constant m_0.

Remark 2.1 *Mittag-Leffler Stability and Generalized Mittag-Leffler Stability imply asymptotic stability.*

Definition 2.8 *If a continuous function of $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing, and $\varphi(1) = 0$, we call φ a K-class function, denoted by $\varphi \in K$. Here $\mathbb{R}^+ = [0,\infty)$.*

3 Stability of Caputo-type Hadamard fractional system

Consider the stability of the following Caputo-type Hadamard fractional differential system

$$ {}_{C}^{H}D_{t_0}^{\gamma}x(t) = f(t,x), \tag{3.1}$$

with the initial condition $x(t_0) = x_0$, where $t_0 \geq 1,\ 0 < \gamma < 1, f \in C([1, +\infty) \times \mathbb{D}, \mathbb{R}^n),\ f(t, 0) \equiv 0,\ \mathbb{D} \in \mathbb{R}^n$ be a domain containing the origin.

First, the general Caputo-type Hadamard fractional comparison principle will be presented. Here, we always assumes in the paper that there exists a unique continuously differentiable solution $x(t)$ to (3.1) with the initial condition x_0.
Comparison results will be used for scalar fractional differential system of the type

$$ {}_{C}^{H}D_{t_0}^{\gamma}u(t) = y(t,u), \tag{3.2}$$

with the initial condition

$$u(t_0) = u_0, \tag{3.3}$$

where $y \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$ is Lipschitz in u, $y(t, 0) \equiv 0$.

Lemma 3.1 *Let $h : [t_0, T) \to \mathbb{R}$ be a locally Hölder continuous such that for any $t_1 \in [t_0, T)$, we have $h(t_1) = 0$ and $h(t) \leq 0$ for $t_0 \leq t \leq t_1$. Then it follows that ${}^{H}D_{t_0}^{\gamma}h(t_1) \geq 0,\ 0 < \gamma < 1$.*
Proof. *We know that*

$$ {}^{H}D_{t_0}^{\gamma}h(t) = \frac{1}{\Gamma(1-\gamma)}(t\frac{d}{dt})\int_1^t (\log\frac{t}{s})^{-\gamma}\frac{h(s)}{s}ds, \qquad \gamma \in (0,1),$$

Let $H(t) = \int_1^t (\log\frac{t}{s})^{-\gamma}\frac{h(s)}{s}ds$. Consider for $a > 0$,

$$\begin{aligned} H(t_1) - H(t_1 - a) &= \int_1^{t_1}(\log\frac{t_1}{s})^{-\gamma}\frac{h(s)}{s}ds - \int_1^{t_1-a}(\log\frac{t_1-a}{s})^{-\gamma}\frac{h(s)}{s}ds \\ &= \int_1^{t_1-a}[(\log\frac{t_1}{s})^{-\gamma} - (\log\frac{t_1-a}{s})^{-\gamma}]\frac{h(s)}{s}ds + \int_{t_1-a}^{t_1}(\log\frac{t_1}{s})^{-\gamma}\frac{h(s)}{s}ds \\ &= I_1 + I_2. \end{aligned}$$

Since $(\log\frac{t_1}{s})^{-\gamma} - (\log\frac{t_1-a}{s})^{-\gamma} < 0$ for $1 \leq s \leq t_1 - a$ and $h(s) \leq 0$, we have $I_1 \geq 0$. Hence,

$$H(t_1) - H(t_1 - a) \geq \int_{t_1-a}^{t_1}(\log\frac{t_1}{s})^{-\gamma}\frac{h(s)}{s}ds = I_2.$$

Since $h(t)$ is locally Hölder continuous and $h(t_1) = 0$, there exists a constant $K(t_1) > 0$, such that, for $t_1 - a \leq s \leq t_1 + a$,

$$-K(t_1)(t_1 - s)^{\lambda} \leq h(s) \leq K(t_1)(t_1 - s)^{\lambda}$$

where $\lambda > 0$ is such that $\lambda - \gamma > 0$ and $0 < \lambda < 1$. Then we have

$$I_2 \geq -K(t_1) \int_{t_1-a}^{t_1} (\log \frac{t_1}{s})^{-\gamma} (t_1 - s)^{\lambda} \frac{ds}{s}.$$

Applying the differential mean value theorem, we get

$$\log t_1 - \log s = \frac{1}{\xi}(t_1 - s), \qquad \xi \in (s, t_1).$$

Then

$$\begin{aligned} I_2 &\geq -K(t_1) \int_{t_1-a}^{t_1} (\log \frac{t_1}{s})^{-\gamma} (t_1 - s)^{\lambda} \frac{ds}{s} \geq -\xi^{\gamma} K(t_1) \int_{t_1-a}^{t_1} (t_1 - s)^{\lambda-\gamma} \frac{ds}{s} \\ &\geq -\frac{\xi^{\gamma} K(t_1)}{t_1 - s} \int_{t_1-a}^{t_1} (t_1 - s)^{\lambda-\gamma} ds = \frac{-\xi^{\gamma} K(t_1) a^{1+\lambda-\gamma}}{(t_1 - a)(1 + \lambda - \gamma)}. \end{aligned}$$

Hence

$$H(t_1) - H(t_1 - a) + \frac{\xi^{\gamma} K(t_1) a^{1+\lambda-\gamma}}{(t_1 - a)(1 + \lambda - \gamma)} \geq 0,$$

for sufficiently small $a > 0$. Letting $a \to 0$, we obtain $H^{'}(t_1) \geq 0$, which implies

$${}^{H}D_{t_0}^{\gamma} h(t) = \frac{1}{\Gamma(1-\gamma)} H^{'}(t_1) \geq 0.$$

The proof is completed. □

Lemma 3.2 *Let $h : [t_0, T) \to \mathbb{R}$ be a locally Hölder continuous such that for any $t_1 \in [t_0, T)$, we have $h(t_1) = 0$ and $h(t) \leq 0$ for $t_0 \leq t \leq t_1$. Then it follows that ${}_{C}^{H}D_{t_0}^{\gamma} h(t_1) \geq 0, \quad 0 < \gamma < 1$.*
Proof. *We know that*

$${}_{C}^{H}D_{t_0}^{\gamma} h(t) = {}^{H}D_{t_0}^{\gamma} h(t) - \frac{h(a)}{\Gamma(1-\gamma)} (\log \frac{t}{t_0})^{-\gamma}.$$

We shall employ the same method that used in the proof of Lemma 3.1. We have ${}_{C}^{H}D_{t_0}^{\gamma} h(t_1) \geq 0$.
The proof is complete. □

Theorem 3.1 *Assume the following conditions are satisfied:*
(1) *Let $h : [t_0, T] \times \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function.*
(2) *The inequality*

$${}_{C}^{H}D_{t_0}^{\gamma} h(t) \leq y(t, h(t)), \qquad t \geq t_0, \; t_0 \geq 1, \tag{3.4}$$

holds.
(3) *$Y(t) = Y(t, t_0, u_0)$ is the maximal solution of the initial value problem (3.2) and (3.3) existing on $[t_0, T]$.*
Then we have

$$h(t) \leq Y(t), \; t \in [t_0, T],$$

whenever $u_0 \geq h(t_0)$.

Proof. Let $\varepsilon > 0$ be an arbitrary number and $u_\varepsilon(t)$ be the solution of the following fractional differential equation

$$ {}^H_C D^\gamma_{t_0} u(t) = y(t, u(t)) + \varepsilon, \quad u_\varepsilon(t_0) = u_0 + \varepsilon, $$

Then $h(t_0) \le u_0 < u_\varepsilon(t_0)$ and ${}^H_C D^\gamma_{t_0} u_\varepsilon(t) > y(t, u_\varepsilon(t))$.
Assume that inequality $u_\varepsilon(t) \ge h(t),\ t \in [t_0, T]$ is not true. Then there exist a point $t_* \in (t_0, T)$ such that

$$ u_\varepsilon(t_*) = h(t_*),\ h(t) < u_\varepsilon(t),\ t \in [t_0, t_*). $$

Let $w(t) = h(t) - u_\varepsilon(t),\ t \in [t_0, t_*]$, then we have

$$ w(t_*) = 0,\ w(t) \le 0,\ t \in [t_0, t_*]. $$

Due to Lemma 3.2, we have ${}^H_C D^\gamma_{t_0} w(t_*) \ge 0$, which implies that

$$ {}^H_C D^\gamma_{t_0} h(t_*) \ge {}^H_C D^\gamma_{t_0} u_\varepsilon(t_*) = y(t_*, u_\varepsilon(t_*)) + \varepsilon. $$

So, we have ${}^H_C D^\gamma_{t_0} h(t_*) > y(t_*, u_\varepsilon(t_*))$, which is a contradiction in view of (3.4). Therefore $u_\varepsilon(t) \ge h(t),\ t \in [t_0, T]$. On the other hand, it's obvious $\lim_{\varepsilon \to 0} n_\varepsilon(t) = Y(t),\ (t \in [t_0, T])$. Then we have

$$ h(t) \le \lim_{\varepsilon \to 0} u_\varepsilon(t) = Y(t),\ t \in [t_0, T]. $$

The proof is completed. □

Theorem 3.2 *Assume:*
(1) *There exists a function* $V(t, x(t)) : [t_0, \infty) \times \mathbb{D} \to \mathbb{R}^+$ *be a continuously differentiable function and locally Lipschitz respect to* x *such that* $V(t, 0) = 0$.
(2) *The inequality*

$$ {}^H_C D^\gamma_{t_0} V(t, x(t)) \le y(t, V(t, x(t))), \quad (t, x) \in [t_0, \infty) \times \mathbb{D}. \tag{3.5} $$

holds.
(3) *The maximal solution* $Y(t, t_0, u_0)$ *of the IVP (3.2) and (3.3) exists on* $[t_0, \infty)$.
Then we have:
(i) *If there exists* $\varphi \in K$ *such that*

$$ V(t, x(t)) \ge \varphi(\|x\|), \tag{3.6} $$

then the stability of the zero solution of (3.2) implies the stability of the zero solution of (3.1); the asymptotic stability of the zero solution of (3.2) implies the asymptotic stability of the zero solution of (3.1);
(ii) *If*

$$ V(t, x(t)) \ge b\|x\|^\beta, \tag{3.7} $$

where $b > 0,\ \beta > 0$, *then the generalized Mittag-Leffler stability of the zero solution of (3.2) implies the generalized Mittag-Leffler stability of the zero solution of (3.1); the Mittag-Leffler stability of the zero solution of (3.2) implies the Mittag-Leffler stability of the zero solution of (3.1).*

Proof. (i) Since the zero solution of (3.2) is stable ,$\forall\varepsilon > 0$ $\exists\delta_1(\varepsilon)$, when $0 < u_0 < \delta_1$, we obtain $Y(t, t_0, u_0) < \varphi(\varepsilon)$. By the continuity of $V(t, x)$ and $V(t, 0) \equiv 0$, for the above $\delta_1(\varepsilon) > 0$, $\exists\delta(\varepsilon) > 0$ such that when $\|x(t_0)\| < \delta(\varepsilon)$, it holds

$$0 < V(t_0, x(t_0)) < \delta_1(\varepsilon).$$

Let $v(t) := V(t, x),\ v_0 := v(t_0) = V(t_0, x(t_0))$. Consider the following comparison equation:

$${}_{C}^{H}D_{t_0}^{\gamma}u(t) = y(t, u(t)),\ u(t_0) = u_0 = v_0.$$

By (3.6) and Theorem 3.1, we get

$$\varphi(\|x(t)\|) \le V(t, x(t)) \le Y(t, t_0, u_0) < \varphi(\varepsilon),$$

that is $\|x(t)\| < \varepsilon$. Thus, the zero solution of (3.1) is stable.
Then, choose $\sigma > 0$. when $\|u_0\| < \sigma$. Similar to the above proof and Theorem 3.1, we have

$$\varphi(\|x(t)\|) \le V(t, x(t)) \le Y(t, t_0, u_0) \to 0, \tag{3.8}$$

when $t \to \infty$, so the zero solution of (3.1) is asymptotically stable.
(ii) Taking $u_0 = v_0 = V(t_0, x(t_0))$ and applying Theorem 3.1 to (3.5), we have $V(t, x(t)) \le Y(t, t_0, u_0)$ when $t \ge t_0$. Since the zero solution of (3.2) is generalized Mittag-Leffler stable and by (3.7), we can get

$$\begin{aligned}\|x(t)\| \le& [\frac{Y(t, t_0, u_0)}{b}]^{\frac{1}{\beta}} \le [\frac{m(u_0)}{b^{\frac{1}{q}}}(\log\frac{t}{t_0})^{-\alpha}E_{\gamma,1-\alpha}(-\lambda(\log\frac{t}{t_0})^{\gamma})]^{\frac{q}{\beta}}\\ \le& [\frac{h_0u_0}{b^{\frac{1}{q}}}(\log\frac{t}{t_0})^{-\alpha}E_{\gamma,1-\alpha}(-\lambda(\log\frac{t}{t_0})^{\gamma})]^{\frac{q}{\beta}}\\ \le& [h_1(x(t_0))(\log\frac{t}{t_0})^{-\alpha}E_{\gamma,1-\alpha}(-\lambda(\log\frac{t}{t_0})^{\gamma})]^{\frac{q}{\beta}}\end{aligned} \tag{3.9}$$

where $h_1(x(t_0)) := \frac{h_0u_0}{b^{\frac{1}{q}}} = \frac{h_0}{b^{\frac{1}{q}}}V(t_0, x(t_0)) \ge 0,\ h(0) = 0,\ h(x) \ge 0$, and $h(x)$ is locally Lipschitz on $x \in \mathbb{D}$ with Lipschitz constant h_0. Since $V(t, x)$ is locally Lipschitz with respect to x and $V(t_0, x(t_0)) = 0$ iff $x(t_0) = 0$, it follows that $h_1(x(t_0))$ is Lipschitz with respect to $x(t_0)$ and $h_1(0) = 0$, which imply the generalized Mittag-Leffler stability of (3.1).

Since Mittag-Leffler stability of the zero solution of (3.2), then we guarantee that

$$\begin{aligned}\|x(t)\| \le& [\frac{Y(t, t_0, u_0)}{b}]^{\frac{1}{\beta}} \le [\frac{m(u_0)}{b^{\frac{1}{q}}}E_{\gamma}(-\lambda(\log\frac{t}{t_0})^{\gamma})]^{\frac{q}{\beta}}\\ \le& [\frac{h_0u_0}{b^{\frac{1}{q}}}E_{\gamma}(-\lambda(\log\frac{t}{t_0})^{\gamma})]^{\frac{q}{\beta}}.\end{aligned} \tag{3.10}$$

The following proof is similar to the above, so we omit it.

The proof is completed. $\square$

4 Example

Example 4.1 *For the Caputo-type Hadamard fractional order system*

$$ {}^{H}_{C}D^{\gamma}_{t_0}|x(t)| = -|x(t)| + f(t,x), \tag{4.1}$$

where $\gamma \in (0,1)$ and $f(t,x)$ satisfies Lipschitz condition, $f(t,0) = 0$ and $f(t,x) \leq 0$. Let the Lyapunov candidate be $V(t,x) = |x|$. Then

$$ {}^{H}_{C}D^{\gamma}_{t_0}V(t,x(t)) = -V(t,x(t)) + f(t,x) \leq -V(t,x(t)).$$

The solution of the Caputo-type Hadamard fractional differential equation

$$ {}^{H}_{C}D^{\gamma}_{t_0}u(t) = -u, \quad u(t_0) = V(t_0,x(t_0)) = |x(t_0)| \tag{4.2}$$

is given by $u(t) = u(t_0)E_{\gamma}(-(\log \frac{t}{t_0})^{\gamma})$. Thus, the zero solution $u = 0$ of (4.2) is Mittag-Leffler stable. By Theorem 3.2, the zero solution $x = 0$ of (4.1)is Mittag-Leffler stable.

References

[1] V. Kiryakova, Generalized Fractional Calculus and Applications. Pitman Res. Notes in Math. No. 301, Longman, Harlow, J. Wiley, N. York. 1994.

[2] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego. 1999.

[3] R.L. Magin, Fractional Calculus in Bioengineering, Begell House Publisher, Inc., Connecticut. 2006.

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam. 2006.

[5] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht. 2007.

[6] V. Lakshmikantham, S. Leela, J.V. Devi, Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, Cambridge (2009).

[7] D. Baleanu, K. Diethelm, E. Scalas, J. J.Trujillo, *Fractional calculus models and numerical methods. Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston. 2012.

[8] J, Hadamard, Essai sur I'etude des fonctions donnees par leur developpment de taylor, J Mat Pure Appl Ser. 8(1892),101-186.

[9] A.A.Kilbas: Hadamard-type fractional calculus. J. Korean Math. Soc. 38 (2001), 1191-1204.

[10] P.L.Butzer, A.A.Kilbas, J.J.Trujillo: Compositions of Hadamard-type fractional integration operators and the semigroup property. J. Math. Anal. Appl. 269 (2002), 387-400.

[11] A.A.Kilbas, J.J.Trujillo, Hadamard-type integrals as G-transforms, Integral Transform. Spec. Funct. 14 (2003), 413-427.

[12] B. Ahmad, S. Ntouyas, A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations, Fract. Calc. Appl. Anal. 17(2)(2014) 348-360.

[13] R. Garra, F. Polito, On some operators involving hadamard derivatives. Integral Transforms Spec Funct. 24(2013) 773-782.

[14] J. Wang, Y. Zhou, M. Medved, Existence and stability of fractional differential equations with hadamard derivative, Topol. Methods Nonlinear Anal. 41(2013) 113-133.

[15] Q. Ma, R. Wang, J. Wang, Y. Ma, Qualitative analysis for solutions of a certain more generalized two-dimensional fractional differential system with hadamard derivative. Appl. Math. Comput. 257(2015) 436-445.

[16] S. Aljoudi, B. Ahmad, J.J. Nieto, A. Alsaedi, A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions, Chaos Solitons Fractals, 91(2016) 39-46.

[17] G. Wang, K. Pei, D. Baleanu, Explicit iteration to Hadamard fractional integro-differential equations on infinite domain, Adv. Difference Equ. 299 (2016), 1-11.

[18] G. Wang, K. Pei, R.P. Agarwal, L. Zhang, B. Ahmad, Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line, J. Comput. Appl. Math., 343 (2018) 230-239.

[19] T. Wang, G. Wang, X. Yang, On a Hadamard-type fractional turbulent flow model with deviating arguments in a porous medium, Nonlinear Analysis: Modelling and Control, 22(2017), 765-784.

[20] K. Pei, G. Wang, Y. Sun, Successive iterations and positive extremal solutions for a Hadamard type fractional integro-differential equations on infinite domain, Appl. Math. Comput. 312(2017), 158-168.

[21] D. Matignon, Stability results for fractional differential equations with applications to control processing,in:IMACS-SMC Proceedings, Lille, France, 1996, pp.963-968.

[22] Y.Q. Chen, K.L. Moore, Analytical stability bound for a class of delayed fractional-order dynamic systems, Nonlinear Dyn. 29 (2002), 191-200.

[23] V. Lakshmikantham, S. Leela, M. Sambandham, Lyapunov theory for fractional differential equations. Commun. Appl. Anal. 12, No 4 (2008), 365-376.

[24] M.P. Lazarevic, A.M. Spasic, Finite-time stability analysis of fractional order time-delay systems: Gronwalls approach, Math. Comput. Model. 49 (2009), 475-481.

[25] Y. Li, Y.Q. Chen, I. Podlubny, Mittag-Leffler stability of fractional order nonlinear dynamic systems, Automatica 45(8)(2009), 1965-1969.

[26] Y. Li, Y.Q. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, Comput. Math. Appl. 59 (2010), 1810-1821.

[27] J. Sabatier, M. Moze, C. Farges, Lmi stability conditions for fractional order systems, Comput. Math. Appl. 59(2010), 1594-1609.

[28] K.W. Liu, W. Jiang, Finite-time stability of linear fractional order neutral systems, Math. Appl. 24(4)(2011), 724-730.

[29] C.P. Li, F.R. Zhang, A survey on the stability of fractional differential equations, Eur. Phys. J. Spec. Top. 193(2011), 27-47.

[30] S. Liu, X.Y. Li, W. Jiang, X.F. Zhou, Mittag-Leffler stability of nonlinear fractional neutral singular systems, Commun. Nonlinear. Sci. Numer. Simulat. 12(2012), 3961-3966.

[31] J. Devi, F. Mc Rae, Z. Drici, Variational Lyapunov method for frac- tional differential equations. Comput. Math. Appl. 64 (2012), 2982-2989.

[32] S.J. Sadati, R. Ghaderi, A.N. Ranjbar, Some fractional comparison results and stability theorem for fractional time delay systems, Rom. Rep. Phys. 65 (1)(2013), 94-102.

[33] M. Rivero, S.V. Rogosin, J.A.T. Machado, J.J. Trujillo, Stability of fractional order systems, Math. Probl. Eng. 2013 (2013).

[34] R. Agarwal, D. O'Regan, S. Hristova, Stability of Caputo fractional differential equations by Lyapunov functions. Appl. Math. 60(6)(2015), 653-676.

[35] R. Agarwal, D. O'Regan, S. Hristova, Stability and Caputo fractional Dini derivative of Lyapunov functions for Caputo fractional differential equations. In: Intern. Workshop QUALITDE, December 27-29, 2015, Tbilisi, Georgia, 3-6.

[36] R. Agarwal, S. Hristova, D. O'Regan, Practical stability of Caputo fractional differential equations by Lyapunov functions. Diff. Eq. Appl. 8(1)(2016), 53-68.

[37] R. Agarwal, S. Hristova, D. O'Regan, A survey of Lyapunpv functions, stability and impulsive Caputo fractional equations, Fract. Calc. Appl. Anal. 19(2)(2016), 290-318.

[38] K.W. Liu, W. Jiang, Stability of nonlinear Caputo fractional differential equations. Appl. Math. Model. 40(2016), 3919-3924.