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Robust stability analysis for fractional-order systems with time delay based on finite spectrum assignment

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Summary

In this paper, the robust stability of a fractional-order time-delay system is analyzed in the frequency domain based on finite spectrum assignment (FSA). The FSA algorithm is essentially an extension of the traditional pole assignment method, which can change the undesirable system characteristic equation into a desirable one. Therefore, the presented analysis scheme can also be used as an alternative time-delay compensation method. However, it is superior to other time-delay compensation schemes because it can be applied to open-loop poorly damped or unstable systems. The FSA algorithm is extended to a fractional-order version for time-delay systems at first. Then, the robustness of the proposed algorithm for a fractional-order delay system is analyzed, and the stability conditions are given. Finally, a simulation example is presented to show the superior robustness and delay compensation performance of the proposed algorithm. Moreover, the robust stability conditions and the time-delay compensation scheme presented can be applied on both integer-order and fractional-order systems.

KEYWORDS

fractional calculus, finite spectrum assignment, robust stability analysis, time-delay system

1 | INTRODUCTION

Time delay exists in most industry process control problems.[1] It may deteriorate closed-loop system performance and even make the control process complicated to analyze and synthesize. Conventionally, aiming at industrial processes, proportional-integral-derivative controllers are always the first choice due to ease of parameter tuning and robustness, to name a few.[2] However, the control loops in large delay systems controlled by traditional proportional-integral-derivative controllers usually have poor system performance and even have to be switched to manual mode.[3] Therefore, in order to improve control performance for such systems, dead-time compensators arose and became rather significant in the field. The most widely used dead-time compensation scheme in practical process control is the Smith Predictor, which was first proposed by Smith.[4] Nevertheless, one notable but not desirable feature of the Smith Predictor is that it retains the original system poles all the time[5]; hence, it cannot be used for unstable systems. This issue also limits its application to non–minimum phase systems because it may result in unsatisfactory control performance.[3]

An alternative time-delay compensation algorithm for integer-order systems is finite spectrum assignment (FSA).[3,6-8] The FSA algorithm was first proposed in the time domain by Manitius and Olbrot[6] and subsequently extended to the frequency domain by Ichikawa.[7] It does not have the restriction to open-loop stable systems. FSA can be regarded as an extension of the frequency-domain pole assignment developed by Wolovich,[8] which can arbitrarily assign the original finite closed-loop poles to desired positions. Furthermore, similar to the Smith Predictor, the algorithm is physically realizable and simple to implement.[3] The further development of frequency-domain FSA can be found in the works of Wang et al,[3,9,10] which show the effectiveness of this control algorithm applied to both stable and unstable systems.

However, the existing FSA scheme only aims at integer-order systems. For fractional-order systems, the tuning method should be modified as the fractional-order finite spectrum assignment (FFSA) algorithm. Fractional-order systems are much closer to most real-world physical processes compared with integer-order ones.[11] The applications of fractional-order calculus in many research fields have been widely explored and verified effectively.[12-18] Therefore, efficient control algorithms aiming at delayed fractional-order systems, both stable and unstable systems, are needed. In this paper, we focus on robust stability analysis based on the FFSA algorithm, which can be used on both fractional-order and integer-order systems. Note that we only take the poles, except for the delay term, into consideration. Moreover, we just consider a special class of fractional-order delay systems whose fractional orders are rational numbers. The stability and robustness analysis of the proposed FFSA algorithm has been discussed. The parameters of the FFSA algorithm can be tuned according to different control requirements. Both robustness and transient control performance can be guaranteed by applying the proposed FFSA algorithm to systems with time delay.

The rest of this paper is organized as follows. In Section 2, some preliminaries of fractional-order calculus are given. Detailed tuning rules of FFSA are presented in Section 3. Section 4 illustrates the stability and robustness analysis of the proposed control scheme. Simulation results to verify the effectiveness of the proposed FFSA algorithm are presented in Section 5. Finally, conclusions are made in Section 6.

2 | PRELIMINARIES

2.1 | Preliminaries

There are three widely accepted definitions of fractional-order differential operators, namely, the Grünwald-Letnikov definition, the Riemann-Liouville definition, and the Caputo definition.[19] Each of them has its own features and properties. However, they have no difference with null initial conditions. In this paper, we choose the Caputo definition, which has been frequently used in real physical situations and engineering application problems.[20]

The Caputo derivative of order α for a function $f(t) \in C^{n+1}([t_0, +\infty], \mathrm{R})$ is defined as[20]

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{1}$$

where t_0 and t are the lower and upper limits of the operator, $\alpha \in \mathrm{R}$ is the order of differentiation, and n is a positive integer, such that $n-1 < \alpha \leq n$.

The Laplace transform based on the Caputo definition is

$$ \mathscr{L}\{ {}_{t_0}D_t^{\alpha} f(t); s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(t_0), n-1 < \alpha < n, \tag{2}$$

where $\mathscr{L}\{\cdot\}$ represents the Laplace transform and s is the variable operator in the Laplace transform.

2.2 | Mittag-Leffler function

The Mittag-Leffler function is a generalization of the exponential function. It always appears in the solution of fractional-order differential equations.[21] A two-parameter Mittag-Leffler function can be defined according to a power series as[20]

$$ E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \tag{3}$$

where $\alpha > 0, \beta > 0$, and $x \in C$. Note that $E_{1,1}(x) = e^x$ when $\alpha = \beta = 1$. When $\beta = 1$, it reduces to a one-parameter form as

$$ E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)} = E_{\alpha,1}(x). \tag{4}$$

The Laplace transform of the two-parameter Mittag-Leffler function is

$$\mathscr{L}\left\{t^{\beta-1}E_{\alpha,\beta}\left(-\lambda t^{\alpha}\right)\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}, \left(\mathrm{Re}(s) > |\lambda|^{\frac{1}{\alpha}}\right), \tag{5}$$

where $t \geq 0$, s is the variable operator of the Laplace domain, and $\mathrm{Re}(s)$ is the real part of s.

2.3 | Stability of the fractional-order system

Before discussing the tuning rules of FFSA, one important step is making clear how to distinguish whether a fractional-order system is stable or not. When a system is stable, both the fractional-order Smith Predictor and FSA can be used on the system. However, if the original system is unstable, only FFSA can be used on fixing the delay problem.

Several papers have discussed the stability problem of fractional-order systems.[19,20,22-24] In this paper, we used the fractional-order linear system stability conditions proposed in the work of Petrás et al[24] without loss of generality. Consider a fractional-order system described by the following transfer function:

$$G_0(s) = \frac{b_0 s^{\beta_0} + b_1 s^{\beta_1} + \cdots + b_m s^{\beta_m}}{a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \cdots + a_n s^{\alpha_n}}, \tag{6}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are constants that represent the coefficients of denominator and numerator; $\alpha_0, \alpha_1, \ldots, \alpha_n \quad (\alpha_0 < \alpha_1 < \cdots < \alpha_n)$ and $\beta_0, \beta_1, \ldots, \beta_m\,(\beta_0 < \beta_1 < \cdots < \beta_m)$ are arbitrary rational number orders of the denominator and numerator, respectively. The description of system (6) in the time domain is as follows: $\sum_{i=0}^{n} a_i \mathscr{D}^{\alpha_i} y(t) = \sum_{j=0}^{m} b_j \mathscr{D}^{\beta_j} u(t)$, where $\mathscr{D}^{\alpha_i}$ is the Caputo fractional-order differential operator. Due to the rational number orders, there exists an integer $\gamma > 1$, which is the least common multiple of the denominators of the fractional numbers $\alpha_i, (i = 0, \ldots, n)$ and $\beta_j, (j = 0, \ldots, m)$.[25] Then, a general transfer function can be obtained from the transformation of that in Equation (6) by a multivalued transfer function[26] as

$$G_0(s) = \frac{b_0 s^{v_0/\gamma} + b_1 s^{v_1/\gamma} + \cdots + b_m s^{v_m/\gamma}}{a_0 s^{\vartheta_0/\gamma} + a_1 s^{\vartheta_1/\gamma} + \cdots + a_n s^{\vartheta_n/\gamma}}, (\gamma > 1). \tag{7}$$

As it was said in the work of Petrás et al,[24] most FO systems could be expressed as Equation (7). The definition of $G_0(s)$ has one Riemann surface with γ Riemann sheets.[27] Then, through a transformation $w = s^{1/\gamma}$, the transfer function with operator s in the s-plane can be transformed into w in a complex w-plane with γ sheets in Riemann surface.[24] In the s-plane, the principal sheet of Riemann surface is described as $-\pi < \arg(s) < \pi$. After the transformation of $w = s^{1/\gamma}$, the principal sheet transfers into $-\pi/\gamma < \arg(w) < \pi/\gamma$ of the w-plane. That is to say, the unstable boundary in the w-plane is $\{w|\arg(w) = \pm\pi/2\gamma, \mathrm{Re}(w) \geq 0\}$ rather than the imaginary axis in the s-plane.

Consider an FO pseudo-polynomial, which is defined as

$$\begin{aligned} D(s) &= c_1 s^{q_1} + c_2 s^{q_2} + \cdots + c_k s^{q_k} \\ &= c_1 s^{\gamma_1/\gamma} + c_2 s^{\gamma_2/\gamma} + \cdots + c_k s^{\gamma_k/\gamma} \\ &= c_1 (s^{1/\gamma})^{\gamma_1} + c_2 (s^{1/\gamma})^{\gamma_2} + \cdots + c_k (s^{1/\gamma})^{\gamma_k}, \end{aligned} \tag{8}$$

where $q_i (i = 1, 2, \ldots, k) = \gamma_i/\gamma (i = 1, 2, \ldots, k)$ are rational orders, each γ_i is an integer, and γ is the least common multiple of the denominators of the fractional numbers q_i.[25] Then, the fractional degree (FDEG) of polynomial can be defined as follows: $\mathrm{FDEG}\{D(s)\} = \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$.[25]

Proposition 1. *Let $D(s)$ be a fractional-order polynomial with $FDEG\{D(s)\} = n$, then the equation $D(s) = 0$ has exactly n roots on the Riemann surface.[28] For a fractional-order linear time-invariant system whose poles are in general complex conjugate, the stability condition can be stated as follows.[29]*

Theorem 1. *A commensurate-order system is described by a rational transfer function*

$$G(w) = \frac{N(w)}{D(w)},$$

where $w = s^{1/\gamma}, \gamma \in R^{+} (0 < 1/\gamma < 1)$, is stable if and only if

$$|\arg(w_i)| > \frac{\pi}{2\gamma},$$

with $\forall w_i \in C (i \leq n, n = FDEG\{D(s)\})$, the ith root of $D(w) = 0$.

For the proof of Theorem 1, refer to the work of Matignon.[29]

3 | FRACTIONAL-ORDER FINITE SPECTRUM ASSIGNMENT

The FSA algorithm for integer-order delay systems, which was first mentioned in the frequency domain by Ichikawa[7] and Wang,[30] is essentially an extension of the original pole assignment for delay-free systems.[8] In this section, we will present the FSA algorithm for a single-input–single-output fractional-order delay system, whose delay is always single.[31] Consider a single-input–single-output fractional-order delay system

$$Y(s) = G_0(s)U(s) = \frac{n_0(s)}{m_0(s)} e^{-Ls} U(s), \tag{9}$$

where $L > 0$ is the dead time; $Y(s)$, $U(s)$, $G_0(s)$ are Laplace transformation forms of system output signal $y(t)$, input signal $u(t)$, and the system transfer function, respectively; and $n_0(s)$ and $m_0(s)$ are pseudo-polynomials with highest order α and β, respectively, in which α and β are arbitrary rational numbers and $0 \leq \alpha < \beta$, so that the system is proper. Moreover, $n_0(s)$ and $m_0(s)$ are also supposed to be co-prime and monic without loss of generality; there are no further restrictions on $m_0(s)$, which means that the process can be unstable or non–minimum phase. In addition, all the powers of $n_0(s)$ and $m_0(s)$ are assumed nonnegative rational numbers; $d_0(s)$, which has the same highest order as $m_0(s)$, is the desirable characteristic polynomial of system (9). The powers of $d_0(s)$ are also assumed nonnegative rational numbers. In this case, there exists an integer $\gamma > 1$, which is the lowest common multiple of the denominators of all the powers in $n_0(s), m_0(s), d_0(s)$.[25] In other words, γ is the least positive integer such that these three functions are polynomials in $s^{1/\gamma}$, ie,

$$\begin{aligned} n_0(s) &= c_{n1}(s^{1/\gamma})^{\gamma_{n1}} + c_{n2}(s^{1/\gamma})^{\gamma_{n2}} + \cdots + c_{nk_n}(s^{1/\gamma})^{\gamma_{nk_n}}, \\ m_0(s) &= c_{m1}(s^{1/\gamma})^{\gamma_{m1}} + c_{m2}(s^{1/\gamma})^{\gamma_{m2}} + \cdots + c_{mk_m}(s^{1/\gamma})^{\gamma_{mk_m}}, \\ d_0(s) &= c_{d1}(s^{1/\gamma})^{\gamma_{d1}} + c_{d2}(s^{1/\gamma})^{\gamma_{d2}} + \cdots + c_{dk_d}(s^{1/\gamma})^{\gamma_{dk_d}}, \end{aligned}$$

where $\gamma_{n1}, \gamma_{n2} \ldots \gamma_{nk_n}, \gamma_{m1}, \gamma_{m2} \ldots \gamma_{mk_m}, \gamma_{d1}, \gamma_{d2} \ldots \gamma_{dk_d}$ are positive integers, $\gamma_{nk_n}/\gamma = \alpha$, and $\gamma_{mk_m}/\gamma = \gamma_{dk_d}/\gamma = \beta$. Then, $n_0(s), m_0(s), d_0(s)$ can be simplified by transforming the transfer function in Equation (9) into a general one as introduced in Section 2.3 with transformation operator $w = s^{1/\gamma}$, ie,

$$\begin{aligned} n(w) &= n(s^{1/\gamma}) = n_0(s) = c_{n1}w^{\gamma_{n1}} + c_{n2}w^{\gamma_{n2}} + \cdots + c_{nk_n}w^{\gamma_{nk_n}}, \\ m(w) &= m(s^{1/\gamma}) = m_0(s) = c_{m1}w^{\gamma_{m1}} + c_{m2}w^{\gamma_{m2}} + \cdots + c_{mk_m}w^{\gamma_{mk_m}}, \\ d(w) &= d(s^{1/\gamma}) = d_0(s) = c_{d1}w^{\gamma_{d1}} + c_{d2}w^{\gamma_{d2}} + \cdots + c_{dk_d}w^{\gamma_{dk_d}}. \end{aligned}$$

Hence, after the transformation, $n_0(s), m_0(s), d_0(s)$ are transformed into the w-plane as $n(w), m(w), d(w)$ with integer degrees denoted by a, b, b, respectively, where $a = \alpha\gamma$ and $b = \beta\gamma$. In this way, the transfer function in Equation (9) will be transformed into the w-plane as

$$Y(s) = G(w)U(s) = \frac{n(w)}{m(w)} e^{-Lw^{\gamma}} U(s). \tag{10}$$

Denote $\varphi(w) = m(w) - d(w)$; hence, $\varphi(w)$ is a polynomial with $b - 1$ degree at most. Introduce a $b - 1$ degree monic polynomial $q(w)$, which is asymptotically stable. It will not be explicitly addressed in the system transfer function after the FFSA algorithm is applied in the system, but it will be physically treated as the characteristic polynomial of a reduced-order Luenberger observer.[30] Now, a polynomial equation is considered as

$$k(w)m(w) + h(w)n(w) = q(w)\varphi(w), \tag{11}$$

where $k(w), h(w)$ are polynomials that will be achieved from Equation (11) uniquely with, at most, $b - 2, b - 1$ degrees, respectively.

Consider the fraction $\varphi(w)/m(w)$. Since $\varphi(w)$ is of degree $b - 1$ at most and $m(w)$ is of degree b, the partial fraction expansion of $\varphi(w)/m(w)$ in Equation (10) will be updated into

$$\frac{\varphi(w)}{m(w)} = \sum_{i=1}^{M}\sum_{j=1}^{V_i} \frac{C_{ij}}{(w - \lambda_i)^j} = \sum_{i=1}^{M}\sum_{j=1}^{V_i} \frac{C_{ij}}{(s^{\mu} - \lambda_i)^j}, \tag{12}$$

where $\mu = 1/\gamma$, M is the number of distinct poles of the system, and V_i are the multiplicities of the distinctive poles λ_i.

Normally, in fractional-order systems, the pole assignment algorithms will take the feedback of all the pseudostates with suitable coefficients into consideration.[32,33]

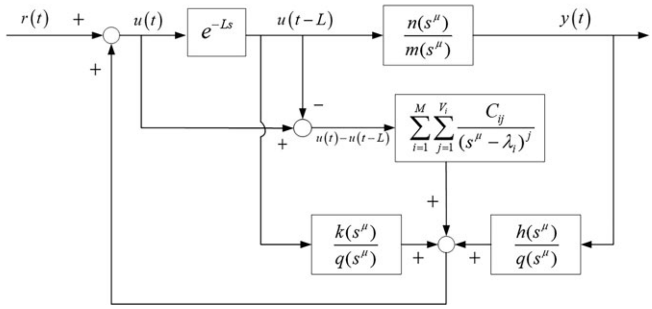

FIGURE 1 Structure of fractional-order finite spectrum assignment control for a delay system in the frequency domain

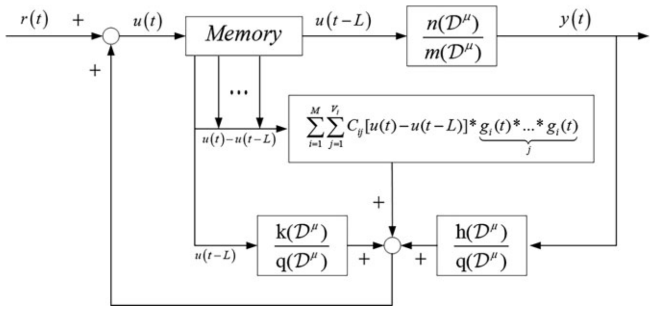

FIGURE 2 Fractional-order finite spectrum assignment control (14) for a delay system in the time domain

Then, the FFSA control of delay system (9) or (10) is designed, as shown in Figure 1. The structure of FFSA control in Figure 1 is described by several operators and modules in the frequency domain. According to Figure 1, the control law $U(s)$ is constituted by the sum of four parts and designed as

$$U(s)=\frac{k(s^\mu)}{q(s^\mu)}e^{-Ls}U(s)+\frac{h(s^\mu)}{q(s^\mu)}Y(s)+\sum_{i=1}^{M}\sum_{j=1}^{V_i}\frac{C_{ij}}{(s^\mu-\lambda_i)^j}[U(s)-U(s)e^{-Ls}]+R(s). \tag{13}$$

Taking the inverse Laplace transformation of system (13), it gives the control law $u(t)$ in the time domain, ie,

$$\begin{aligned}q(\mathscr{D}^\mu)u(t)&=k(\mathscr{D}^\mu)u(t-L)+h(\mathscr{D}^\mu)y(t)\\&+q(\mathscr{D}^\mu)\sum_{i=1}^{M}\sum_{j=1}^{V_i}C_{ij}\,[u(t)-u(t-L)]*\underbrace{g_i(t)*\cdots*g_i(t)}_{j}+q(\mathscr{D}^\mu)r(t),\end{aligned} \tag{14}$$

where $\mu=1/\gamma$; $\mathcal{D}^\mu$ is the Caputo fractional-order differential operator; $q(\cdot)$ is a chosen $b-1$ degree monic polynomial, which is asymptotically stable; $k(\cdot)$ and $h(\cdot)$ are the unique polynomial solutions, which are obtained from Equation (11); $g_i(t)=t^{\alpha-1}E_{\alpha,\alpha}(\lambda_i t^\alpha)$; $r(t),u(t),y(t)$ are the reference signal, control signal, and output signal, respectively; $R(s),U(s),Y(s)$ are the Laplace transforms of $r(t),u(t),y(t)$, respectively; the symbol $*$ denotes convolution multiplication; and scalar parameters $C_{ij},\ M,\ V_i,\ \lambda_i$ can be got from Equation (12). For FFSA control of delay system (9) or (10), the control law $u(t)$ (14) in the time domain is shown in Figure 2.

Theorem 2. *The control law in Equation (14) is realizable and achieves an arbitrary desired FSA for system (9).*

Proof. The polynomials $k(\mathscr{D}^\mu),\ h(\mathscr{D}^\mu),\ q(\mathscr{D}^\mu)$ are, at most, of $b-2, b-1, b-1$ degrees, respectively. The Laplace transformation of Equation (14) can be achieved as system (13). Due to Equation (12), it is obtained that

$$U(s)=\frac{k(w)}{q(w)}e^{-Ls}U(s)+\frac{h(w)}{q(w)}Y(s)+\frac{\varphi(w)}{m(w)}\left[U(s)-U(s)e^{-Ls}\right]+R(s). \tag{15}$$

Make multiplication on both sides of Equation (11) with $m^{-1}(w)U(s)e^{-Ls}$ and yield

$$k(w)U(s)e^{-Ls}+h(w)Y(s)=\frac{q(w)\varphi(w)}{m(w)}U(s)e^{-Ls}. \tag{16}$$

From Equations (15) and (16), it is obtained that

$$q(w)U(s) = \frac{q(w)\varphi(w)}{m(w)}U(s) + q(w)R(s). \tag{17}$$

Simplifying Equation (17), we have

$$[m(w) - \varphi(w)]\, q(w)U(s) = d(w)q(w)U(s) = m(w)q(w)R(s). \tag{18}$$

Substituting Equation (18) into Equation (10), the overall transfer function of the system will be achieved as

$$Y(s) = \frac{n(w)}{d(w)}e^{-Ls}R(s). \tag{19}$$

This completes the proof of Theorem 2. □

Next, we give some special cases for system (9).

If $m(w)$ in Equation (10) has no repeated zeros in it, the partial fraction expansion of $\varphi(w)/m(w)$ could become

$$\frac{\varphi(w)}{m(w)} = \sum_{i=1}^{N}\frac{C_i}{w-\lambda_i} = \sum_{i=1}^{N}\frac{C_i}{s^{\mu}-\lambda_i}, \tag{20}$$

where $\lambda_i, i = 1, 2, \ldots, N$ are distinct poles of the system. Moreover, consider the polynomial equation in Equation (11) and get the unique solution of $k(w), h(w)$ in the same way. Therefore, the control law $u(t)$ can be obtained from Equation (14) as

$$\begin{aligned} q\left(\mathscr{D}^{\mu}\right) u(t) = k\left(\mathscr{D}^{\mu}\right) u(t-L) + h(\mathscr{D}^{\mu})y(t) + q(\mathscr{D}^{\mu})\sum_{i=1}^{N} C_i \int_0^t (t-\tau)^{\mu-1} \\ E_{\mu,\mu}\left(\lambda_i(t-\tau)^{\mu}\right)[u(\tau) - u(\tau-L)]\,d\tau + q(\mathscr{D}^{\mu})r(t), \end{aligned} \tag{21}$$

where $r(t), u(t), y(t)$ are the reference signal, the control signal, and the output signal, respectively.

Corollary 1. *The control law in Equation (21) is realizable and achieves an arbitrary desired FSA for system (9) with no repeated poles.*

The proof of Corollary 1 can be obtained from Theorem 1.

It is shown in Equation (21) that the arbitrary frequency-domain spectrum assignment is obtained by the proposed FFSA algorithm. Figure 3 illustrates visually the structure of the proposed control system.

Consider a delay-free fractional-order system as a special case $L = 0$ of the system in Equation (9), ie,

$$Y(s) = G_0(s)U(s) = \frac{n_0(s)}{m_0(s)}U(s). \tag{22}$$

With transformation operator $w = s^{1/\gamma}$, the transfer function in Equation (22) can become

$$Y(s) = G(w)U(s) = \frac{n(w)}{m(w)}U(s). \tag{23}$$

On the basis of Equation (14), the control law $u(t)$ can be given as

$$q(\mathscr{D}^{\mu})u(t) = k(\mathscr{D}^{\mu})u(t) + h(\mathscr{D}^{\mu})y(t) + q(\mathscr{D}^{\mu})r(t). \tag{24}$$

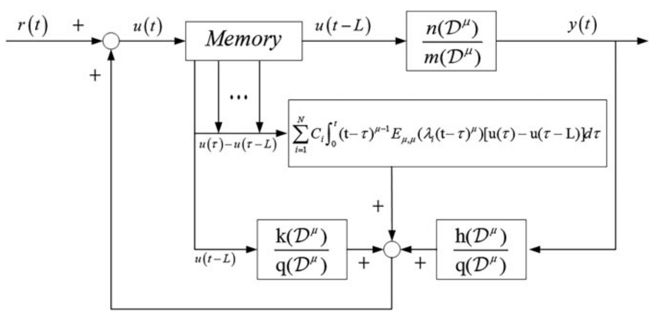

FIGURE 3 Fractional-order finite spectrum assignment control for a delay system without repeated poles in the time domain

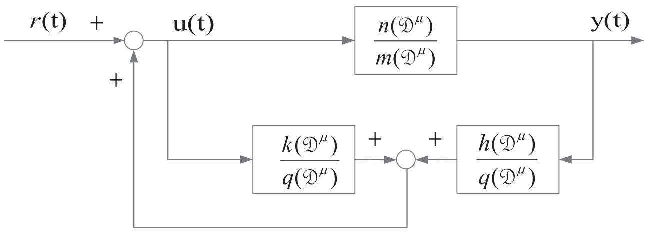

FIGURE 4 Fractional-order finite spectrum assignment control for a delay-free system in the time domain

The structure of the corresponding control system is illustrated in Figure 4.

Corollary 2. *The control law in Equation (24) is realizable and achieves an arbitrary desired FSA for system (22).*
The proof of Corollary 2 can be obtained from Theorem 1.

4 | ROBUSTNESS ANALYSIS AND DESIGN

In this section, the stability and robustness of the FFSA algorithm for a fractional-order delay system are analyzed. On the basis of the analysis result, the conditions for designing a robust controller are proposed. For simplicity, we will only take the systems without repeated poles into consideration here.

In Equation (10), we assume that $G(w) = \frac{n(w)}{m(w)}e^{-Lw^\gamma}$ is an actual process, but it is unknown. Only $\hat{G}(w) = \frac{\hat{n}(w)}{\hat{m}(w)}e^{-\hat{L}w^\gamma}$ is available and can be used in the design of the FFSA system. As a practical process, a small difference always exists in an FFSA system between $G(w)$ and $\hat{G}(w)$.

Due to Equations (11) and (20), we have

$$k(w)\hat{m}(w) + h(w)\hat{n}(w) = q(w)\varphi(w) \tag{25}$$

and

$$\frac{\varphi(w)}{\hat{m}(w)} = \sum_{i=1}^{N}\frac{C_i}{w-\lambda_i} = \sum_{i=1}^{N}\frac{C_i}{s^\mu-\lambda_i}, \tag{26}$$

where $\varphi(w) = \hat{m}(w) - d(w)$ and $\lambda_i, i = 1, 2, \dots, N$ are the poles of $\hat{G}(w)$. Then, the control law $u(t)$ is given as

$$\begin{aligned} q(\mathscr{D}^\mu)u(t) = k(\mathscr{D}^\mu)u(t-\hat{L}) + h(\mathscr{D}^\mu)y(t) + q(\mathscr{D}^\mu)\sum_{i=1}^{N}C_i\int_0^t (t-\tau)^{\mu-1} \\ E_{\mu,\mu}(\lambda_i(t-\tau)^\mu)[u(\tau)-u(\tau-\hat{L})]d\tau + q(\mathscr{D}^\mu)r(t). \end{aligned} \tag{27}$$

Theorem 3. *Using the model $\hat{G}(w)$ instead of $G(w)$ in Equation (10) and the control law satisfying Equations (25)-(27), it can be shown that the closed-loop transfer function in Figure 3 (with changing L by $\hat{L}$) is*

$$Y(s) = \frac{q(w)n(w)}{m(w)p(w)}e^{-Ls}R(s), \tag{28}$$

where

$$p(w) = \frac{q(w)d(w)}{\hat{m}(w)} + \frac{h(w)\hat{n}(w)}{\hat{m}(w)}e^{-\hat{L}s} - \frac{h(w)n(w)}{m(w)}e^{-Ls}.$$

Proof. Making Laplace transformation on both sides of Equation (27), it gives the following based on Equation (26):

$$\begin{aligned} U(s) &= \frac{k(s^\mu)}{q(s^\mu)}e^{-\hat{L}s}U(s) + \frac{h(s^\mu)}{q(s^\mu)}Y(s) + \sum_{i=1}^{N}C_i\frac{1}{s^\mu-\lambda_i}[U(s)-U(s)e^{-\hat{L}s}] + R(s) \\ &= \frac{k(s^\mu)}{q(s^\mu)}e^{-\hat{L}s}U(s) + \frac{h(s^\mu)}{q(s^\mu)}Y(s) + \frac{\varphi(w)}{\hat{m}(w)}[U(s)-U(s)e^{-\hat{L}s}] + R(s). \end{aligned}$$

Then, it has

$$q(w)U(s) = k(w)e^{-\hat{L}s}U(s) + \frac{h(w)n(w)}{m(w)}e^{-Ls}U(s) + \frac{q(w)\varphi(w)}{\hat{m}(w)}[U(s)-U(s)e^{-\hat{L}s}] + q(w)R(s).$$

Due to Equations (11) and (25), we obtain

$$\left[\frac{q(w)d(w)}{\hat{m}(w)} + \frac{h(w)\hat{n}(w)}{\hat{m}(w)}e^{-\hat{L}s} - \frac{h(w)n(w)}{m(w)}e^{-Ls}\right] U(s) = q(w)R(s)$$

and

$$Y(s) = \frac{q(w)n(w)}{m(w)p(w)}e^{-Ls}R(s).$$

This completes the proof of Theorem 3. □

Let Σ_w and Σ_s denote the sets of all the zeros of $\bar{p}(w) = m(w)p(w)$ and $\bar{p}_0(s) = \bar{p}(s^\mu)$, ie,

$$\Sigma_w := \{z|\bar{p}(z) = 0\},$$

$$\Sigma_s := \{z|\bar{p}_0(z) = 0\}.$$

Definition 1 (See Bellman and Cooke[34]).
The closed-loop system is said to be stable if and only if set $\{\mathrm{Re}(z)|z \in \Sigma_s\}$ has a negative upper bound.

Theorem 4. *When Σ_s and Σ_w are finite sets, the closed-loop system is stable, if*

$$|\arg(z_s)| > \frac{\pi}{2} \quad or \quad |\arg(z_w)| > \frac{\mu\pi}{2},$$

for any $z_s \in \Sigma_s$ or any $z_w \in \Sigma_w$.

Proof. Theorem 4 can be obtained obviously based on Theorem 1 and Definition 1. This completes the proof of Theorem 4. □

On the basis of the definition of robust stability,[24] we give the FFSA one as follows.

Definition 2. Let a closed-loop system be finite spectrum assigned, with the nominal process $\hat{G}_0(s) = \hat{G}(s^\mu) = \frac{\hat{n}(s^\mu)}{\hat{m}(s^\mu)}e^{-\hat{L}s}$. The closed-loop system is said to be robust stable if and only if there exists a positive number δ, such that the system is stable for each process $G_0(s) = G(s^\mu) = \frac{n(s^\mu)}{m(s^\mu)}e^{-Ls}$, satisfying

$$\left|\frac{G_0(j\omega)}{\hat{G}_0(j\omega)} - 1\right| < \delta, \quad \forall\omega. \tag{29}$$

Then, define a family of processes Π, which owns norm-bounded uncertainty as

$$\Pi = \left\{ G_0 : \left|\frac{G_0(j\omega)}{\hat{G}_0(j\omega)} - 1\right| = |l_m((j\omega)^\mu)| \le \tilde{l}_m(\omega) \right\},$$

where $\tilde{l}_m(\omega)$ is the bound of the multiplicative uncertainty $l_m((j\omega)^\mu)$.

Theorem 5. *When $\hat{L} = 0$, $0 < \mu \le 1$, $G_0 \in \Pi$, and $\frac{\hat{n}(w)}{\hat{m}(w)}$ and each $\frac{n(w)}{m(w)}$ is strictly proper with owning the same number of unstable poles, then the FFSA system is robust stable if*

$$\left|G^*_{yr}((j\omega)^\mu)\right| \tilde{l}_m(\omega) < 1, \quad \forall\omega.$$

Proof. When $\hat{L} = 0$, we obtain

$$\bar{p}(w) = m(w)p(w) = m(w)\frac{\phi_0(w)}{\hat{m}(w)} - h(w)n(w)e^{-Lw^\gamma},$$

where

$$\phi_0(w) = q(w)d(w) + h(w)\hat{n}(w).$$

From Equation (25), for any zero λ_i of $\hat{m}(w)$, we have

$$h(\lambda_i)\hat{n}(\lambda_i) = q(\lambda_i)\varphi(\lambda_i), \text{ and } \phi_0(\lambda_i) = q(\lambda_i)d(\lambda_i) + q(\lambda_i)\varphi(\lambda_i) = q(\lambda_i)\hat{m}(\lambda_i).$$

Hence, each zero λ_i of $\hat{m}(w)$ is also the zero of $\phi_0(w)$. In other words, $\bar{p}(w)$ has no unstable poles, ie,

$$P^+(\bar{p}) = 0,$$

where $P^+(\bar{p})$ is the number of unstable poles of $\bar{p}$. Due to $w = s^\mu$ $(0 < \mu \le 1)$ and their relationship in Figure 4, it implies

$$P^+(\bar{p}_0) = 0.$$

Define that $N(k, \bar{p}_0)$ is the net number of clockwise encirclements of the point $(k, 0)$ by the image of the Nyquist D contour under $\bar{p}_0$, then we gain

$$N(k, \bar{p}_0) = Z^+(k, \bar{p}_0) - P^+(k, \bar{p}_0),$$

where $Z^+(k, \bar{p}_0)$ is the number of non–minimum phase zeros of $\bar{p}_0(s) = 0$. From (28), we have

$$\bar{p}(w) = \frac{m(w)}{\hat{m}(w)} d(w)q(w)\tilde{p}(w), \tag{30}$$

where

$$\tilde{p}(w) = 1 + G^*_{yr}(w)l_m(w),$$

and $G^*_{yr}(w) = -\frac{\hat{n}(w)h(w)}{d(w)q(w)}$, $l_m(w) = \frac{G(w)}{\hat{G}(w)} - 1$. Because $m(w)$ has the same number of non–minimum phase zeros as $\hat{m}(w)$ and $d(w)$, $q(w)$ are user-specified Hurwitz polynomials, from Equation (30), it has

$$N(0, \bar{p}) = N(0, \tilde{p})$$

and

$$N(0, \bar{p}_0) = N(0, \tilde{p}_0) = N(-1, \tilde{p}_0 - 1),$$

where $\tilde{p}_0(s) = \tilde{p}(s^\mu)$.

Due to $|G^*_{yr}((j\omega)^\mu)|\tilde{l}_m(\omega) < 1$ for $\forall\omega$, it has

$$|\tilde{p}_0(j\omega) - 1| < 1, \quad \forall\omega.$$

From the stability of $G^*_{yr}(s^\mu) = -\frac{\hat{n}(s^\mu)h(s^\mu)}{d(s^\mu)q(s^\mu)}$, it has

$$\left|G^*_{yr}(s^\mu)\right| < M, \quad \forall\omega,$$

for some $M > 0$. A $\delta > 0$ can be chosen such that $M\delta < 1$; then, for any process $G_0(s) = G(s^\mu)$, it satisfies

$$\left|\frac{G_0(j\omega)}{\hat{G}_0(j\omega)} - 1\right| < \delta, \quad \forall\omega, \tag{31}$$

and there holds

$$|\tilde{p}_0(j\omega) - 1| < 1, \quad \forall\omega.$$

Therefore, we have $|\tilde{p}_0(s) - 1| < 1$ for each point of the Nyquist contour. That means the Nyquist curve of $\tilde{p}_0(s)$ does not encircle the origin. Thus, it implies $N(0, \bar{p}_0) = N(0, \tilde{p}_0) = N(-1, \tilde{p}_0 - 1) = 0$ and

$$Z^+(k, \bar{p}_0) = N(k, \bar{p}_0) + P^+(k, \bar{p}_0) = 0.$$

On the basis of Theorem 4, the closed-loop system is stable, and inequality (29) holds in the proof of inequality (31). Thus, the closed-loop system is robust stable.

This completes the proof of Theorem 5. □

5 | SIMULATION

In this section, a simulation example is presented to verify the effectiveness and applicability of the proposed robust FFSA delay compensation algorithm.

Consider a process with the transfer function

$$G_0(s) = \frac{K}{(s^\alpha + C)(s^\alpha + 1)} e^{-Ls},$$

where $\alpha = 0.9$, and the parameters K, C, and L are uncertain, ie,

$$0.5 \le K \le 1.5, \quad 0.4 \le C \le 0.8, \quad 0 \le L \le 2.$$

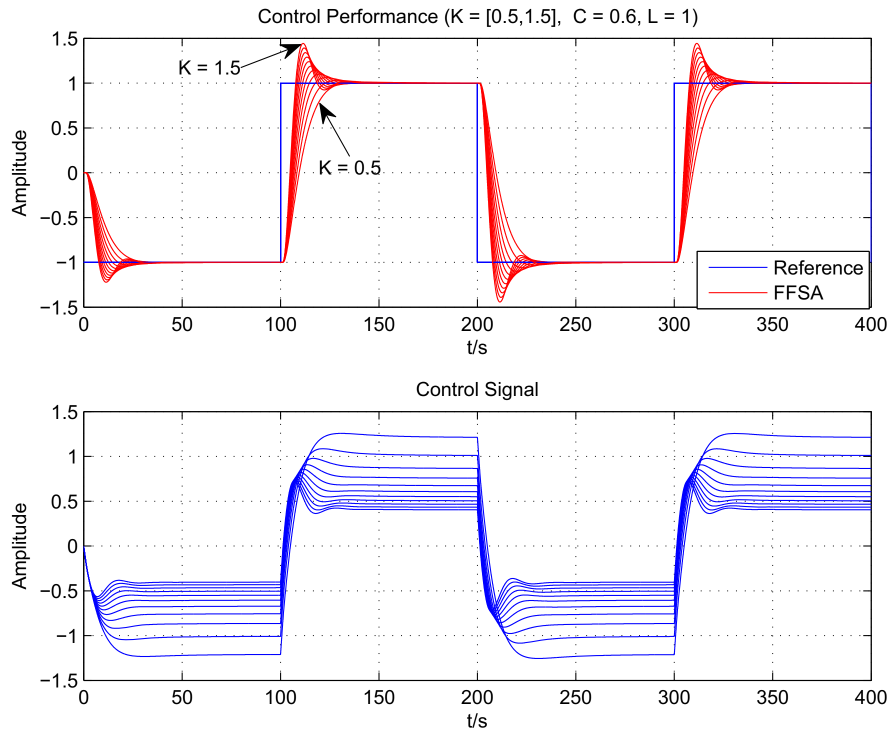

FIGURE 5 Control performance for the robustness test with uncertain K. FFSA, fractional-order finite spectrum assignment [Colour figure can be viewed at wileyonlinelibrary.com]

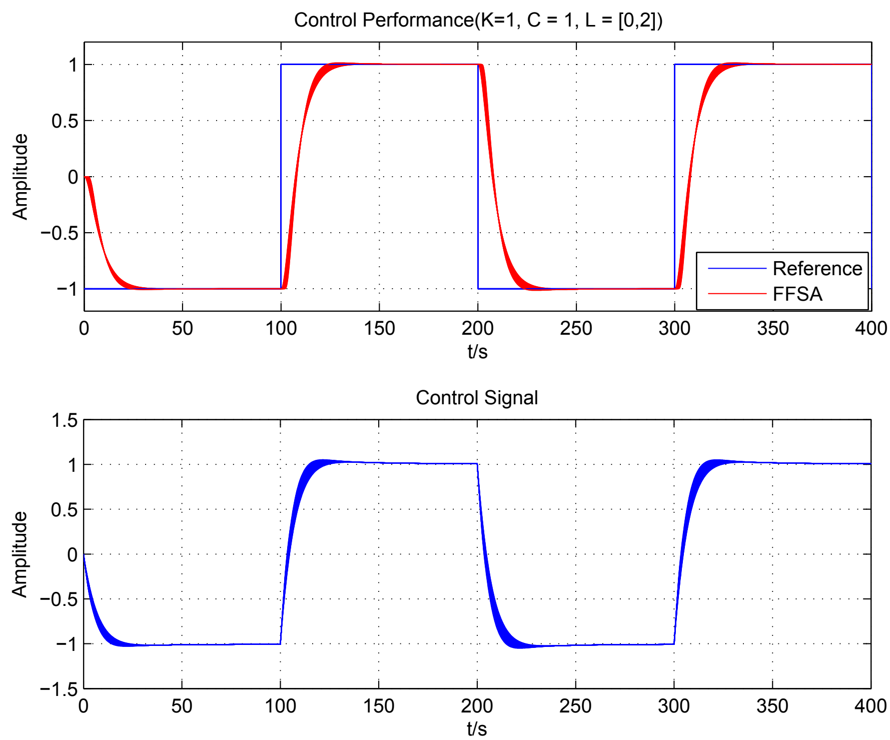

FIGURE 6 Control performance for the robustness test with uncertain L. FFSA, fractional-order finite spectrum assignment [Colour figure can be viewed at wileyonlinelibrary.com]

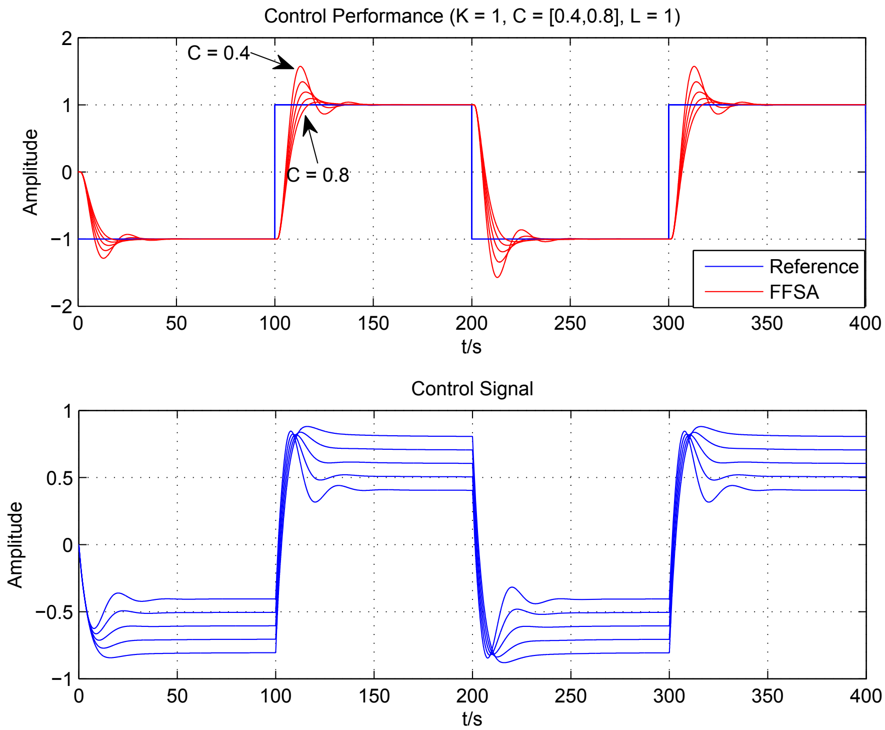

FIGURE 7 Control performance for the robustness test with uncertain C. FFSA, fractional-order finite spectrum assignment [Colour figure can be viewed at wileyonlinelibrary.com]

The corresponding nominal model is described by

$$\hat{G}_0(s) = \frac{1}{(s^\alpha + 0.6)(s^\alpha + 1)}.$$

Then, we obtain

$$\tilde{l}_m(\omega) < 4,$$

for any $\omega \geq 0$. The closed-loop polynomial $d(s^\alpha) = (s^\alpha + 1)^2$ and observer polynomial $q(s^\alpha) = s^\alpha + 1$ are chosen, such that

$$\left|G^*_{yr}((j\omega)^\alpha)\right| < \frac{1}{5},$$

for any $\omega \geq 0$. On the basis of Theorem 5, the closed-loop system is robust stable. Figures 5, 6, and 7 are the robustness test results with parameter uncertainties. The reference is a square signal whose frequency is 0.01 and whose amplitude is $[-1, 1]$. The simulation results verify the robustness analysis in Section 4. Moreover, the stability condition given in Section 4 only guarantees the stability of the controlled system. The parameter settings can be tuned to satisfy any specific performance requirement in the controlled system.

6 | CONCLUSION

A robust stability analysis algorithm for time-delay systems based on FSA has been proposed in the frequency domain. It is also a time-delay compensation scheme aiming at fractional-order systems with rational orders. Unlike other delay compensators, it can be used on all kinds of plants, including poorly damped and unstable ones. It changes the unsatisfactory pole locations into satisfactory ones. Therefore, the control performance can be improved significantly after the application of the proposed control scheme. The derivation processes for systems with and without time delay are presented, and the implementation can be simple and efficient. A simulation example shows that the proposed control scheme can achieve satisfactory robustness and control performance, and it can also be tuned to satisfy different control requirements. Furthermore, detailed guidance on how to choose a desirable spectrum and robust stability analysis based on the FFSA algorithm for linear time-invariant systems with multiple time delays may be discussed in our future work.

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