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Optimal sensor placement for time fractional diffusion system via eigenvalue identification

Fudong Ge[1,2] and YangQuan Chen[3]

1. School of Computer Science, China University of Geosciences, Wuhan 430074, P. R. China
2. Hubei Key Laboratory of Intelligent Geo-Information Processing, China University of Geosciences, Wuhan 430074, P. R. China
E-mail: gefd@cug.edu.cn
3. School of Engineering (MESA-Lab), University of California, Merced, CA 95343, USA
E-mail: ychen53@ucmerced.edu

Abstract: The purpose of this paper is to explore the optimal placements of a finite number of fixed pointwise sensors for anomalous subdiffusion processes governed by time fractional diffusion system with a Riemann-Liouville time fractional derivative. Firstly, we express the solution of studied system using the spectral theory of system operator. Different from the usual method of designing a performance criterion to estimate some unknown system parameters with respect to the placements of sensors, we here introduce a new criterion of accuracy in system eigenvalue identification based on the spectrum decomposition assumption. It is shown that the optimal placements of sensors can be explicitly determined by the order of fractional derivative, the finite number of chosen eigenvalues and their corresponding eigenfunctions. An example is included in the end to confirm our results.

Key Words: Optimal sensor placement; Time fractional diffusion system; Averaged information matrix; Eigenvalue identification.

1 Introduction

Based on the continuous-time random walks theory [1], it is confirmed in [2, 3] that time fractional diffusion system can be used to well model the anomalous subdiffusive transport dynamics processes. Here the time fractional diffusion system is usually regarded as a generalization of conventional distributed parameter system (DPS), where the first-order time derivative is extended to a Riemann-Liouville fractional derivative of order $\alpha \in (0, 1]$. Moreover, recently an initial progress has been made in the investigation of observability for time fractional diffusion system with a finite number of sensors [4–6]. Considering the fact that sensor placements could significantly influence the quality of measurement and in many practical situations, however, it is impossible to obtain the full-state measurements due to the difficulties in measuring [7], studies on the optimal sensor placement problem for time fractional diffusion system with a finite number of sensors should be both interesting and challenging.

As cited in [8], the applications of time fractional diffusion system are rich in our real-life, which is usually used to model the transport processes in a spatially inhomogeneous environment. Typical examples include the chemical reaction processes in dispersive transport media [9], the diseases spreading control in agriculture lands [10], or the flow through porous media with a source [11], etc. For more information, we refer the reader to the monographs [12–14] and the references cited therein. Then the obtained results could provide some insights into the technical instructing for sensor configuration problem of these applications.

Besides, the past several decades have witnessed a big development in solving the optimal sensor placement problem for conventional non-fractional DPSs. Note that most of the available methods focus on optimizing a performance criterion of some unknown system parameters with respect to the sensor placements. To realize this, Ucinski in [15, 16] defined some general functional on the Fisher information matrix with respect to some unknown system parameters. A statistical methodology was introduced in [17, 18] by minimizing a information entropy based measure, which allows to make comparisons between sensor configurations involving a different number of sensors in each configuration. The authors in [19] offered two control strategies by achieving reduction of the noise field in an acoustic cavity. However, Yu and Seinfeld in [20] cited that the state of the considered system usually depends nonlinearly on the unknown system parameters. From a experimental design point of view, a new approach by identifying a finite number of system eigenvalues has been introduced in [21] to avoid the computational difficulties.

Realize that there is a need for further studies on the linear quadratic regulator problems and the stability theory determined by the norm of system matrix for fractional order systems. In this paper, we consider to study the optimal sensor placement problem for time fractional diffusion system by designing a averaged information matrix as the criterion of accuracy in system eigenvalue identification based on the spectrum decomposition assumption. This is motivated by the fact that the solution of considered system can be expressed in terms of the spectral theory of system operator. As a result, we explicitly express the optimal placements of a finite number of fixed pointwise sensors in terms of the order of fractional derivative, the finite number of chosen eigenvalues and their corresponding eigenfunctions. Moreover, the obtained results can be considered as an extension of the results in [21–23], which are easy to be used in experimental studies.

The rest of this paper is organized as follows: we formulate the problem to be studied in the next section. In Section 3, the necessary and sufficient condition for optimal sensor placements of studied systems is proposed based on the given averaged information matrix. Finally, an example is

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worked out.

2 Preliminary results

Let Ω be an open, bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Denote $L^2(\Omega)$ the usual Lebesgue integrable function space endowed with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, respectively. In what follows, we present some basic results to be used thereafter, introduce the basic idea of spectrum decomposition assumption and then formulate our considered problem.

2.1 System statement

Consider the anomalous subdiffusion transport dynamics in complex systems governed by time fractional diffusion system of the form:

$$\begin{cases} {}_0D_t^{\alpha} y(x,t) = Ay(x,t) \text{ in } \Omega\times[0,T], \\ y(x,t)=0 \text{ in } \partial\Omega\times[0,T], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} y(x,t) = y_0(x) \text{ in } \Omega, \end{cases} \tag{1}$$

where $\alpha\in(0,1]$, $y_0\in L^2(\Omega)$, ${}_0D_t^{\alpha}$ and ${}_0I_t^{\alpha}$ denote the Riemann-Liouville fractional derivative and integral on t, respectively. Here A is a symmetric operator with the domain

$$\mathscr{D}(A) = \left\{\varphi\in L^2(\Omega) : \varphi(x)=0,\ x\in\partial\Omega\right\} \tag{2}$$

satisfying the following hypothesis:

(H_1) There exists a sequence $(\lambda_n,\xi_n)_{n\geqslant 1}$ such that

- (λ_n,ξ_n) is the eigenvalue-eigenfunction pair of the problem

$$\begin{cases} A\varphi(x) = \lambda\varphi(t),\ x\in\Omega, \\ \varphi(x)=0,\ x\in\partial\Omega \end{cases} \tag{3}$$

  and eigenvalues $\{\lambda_n\}_{n\geqslant 1}$ are ordered such that

$$\lambda_{n+1}\leqslant\lambda_n,\ \forall n\geqslant 1 \text{ and } \lim_{n\to\infty}\lambda_n = -\infty; \tag{4}$$

- $\{\xi_n\}_{n\geqslant 1}$ forms a Riesz basis in $L^2(\Omega)$ and any function $\varphi\in L^2(\Omega)$ can be approximated by $\varphi_N(x) = \sum\limits_{n=1}^{N}(\varphi,\xi_n)\xi_n(x)$ such that

$$\lim_{N\to\infty}\|\varphi-\varphi_N\| = 0. \tag{5}$$

Remark 1. Note that the above hypothesis on operator A is general. For example, if $A := \Delta = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}$, by [24, 25], we get that $\lambda_n = -n^2\pi^2$, $\xi_n = \sqrt{2}\sin(n\pi x)$, $n = 1,2,\cdots$ and (H_1) holds.

To study the optimal sensor placement problem of system (1), we assume that the state $y(x,t)$ can be directly measured by q fixed pointwise sensors at $\sigma_1,\sigma_2,\cdots\sigma_q\in\Omega$. Then the measurements are described as follows

$$z(t) = (z_1(t),z_2(t),\cdots,z_q(t))^T, \tag{6}$$

where $z_k(t) = y(\sigma_k,t)$, $t\in[0,T]$ and $k=1,2,\cdots q$. Moreover, it is supposed that different sensors placed in same place have same coordinates. Let the elements of set $\sigma = \{\sigma_1,\sigma_2,\cdots\sigma_q\}$ be ordered in such a way that

$$\sigma_k\neq\sigma_l \text{ for } k\neq l,\ k,\ l=1,2,\cdots,\tilde{q}. \tag{7}$$

Here $\tilde{q}$ represents the number of distinct element in σ. Denote n_i the number of sensors which have the same coordinates as σ_i. Then $\sum\limits_{k=1}^{\tilde{q}} n_k = q$. Let $p_k = \frac{n_k}{q}$, $k=1,2,\cdots,\tilde{q}$. The measurements are designed as follows:

$$\begin{bmatrix} \sigma_1, & \sigma_2, & \cdots, & \sigma_{\tilde{q}} \\ p_1 & p_2 & \cdots & p_{\tilde{q}} \end{bmatrix}, \tag{8}$$

where $p_k\geqslant 0$ and $\sum\limits_{k=1}^{\tilde{q}} p_k = 1$.

Next, we recall some basic results to be used thereafter.

Definition 1. [26] The fractional integral of order $\alpha>0$ for a function $y(x,t):\Omega\times[0,T]\to\mathbf{R}$ on t is given by

$${}_0I_t^{\alpha} y(\cdot,t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(\cdot,s)ds \tag{9}$$

provided that the right side is pointwise defined on $[0,T]$.

Definition 2. [26] The Riemann-Liouville fractional derivative of order $\alpha\in(0,1]$ for a function $y(x,t):\Omega\times[0,T]\to\mathbf{R}$ on t is defined as

$${}_0D_t^{\alpha} y(\cdot,t) = \begin{cases} \frac{d}{dt}{}_0I_t^{1-\alpha} y(\cdot,t),\ \alpha\in(0,1), \\ \frac{\partial y(\cdot,t)}{\partial t},\ \alpha=1 \end{cases} \tag{10}$$

provided that the right side is pointwise defined on $[0,T]$.

Consider the following property of Laplace transform on Riemann-Liouville fractional derivative ${}_0D_t^{\alpha}$ of order $\alpha\in(0,1]$

$$\begin{aligned} &\mathscr{L}\{{}_0D_t^{\alpha} y\}(\cdot,s) = \mathscr{L}\left\{\tfrac{\partial}{\partial t}{}_0I_t^{1-\alpha} y\right\}(\cdot,s) \\ &= s\left[s^{\alpha-1}\mathscr{L}\{y\}(\cdot,s) - \lim_{t\to 0^+}{}_0I_t^{1-\alpha} y(\cdot,t)\right] \\ &= s^{\alpha}\tilde{y}(\cdot,s) - y_0(\cdot), \end{aligned} \tag{11}$$

let $E_{\alpha,\beta}(t) := \sum\limits_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}$, $\alpha,\ \beta>0$, $t\in\mathbb{C}$ be the Mittag-Leffler function in two parameters and write $E_{\alpha,1}(t) = E_{\alpha}(t)$ for short if $\beta=1$. The property [27]

$$\mathscr{L}\left\{t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}\mp\lambda},\ \mathbb{R}(s)\geqslant|\lambda|^{1/\alpha} \tag{12}$$

yields the following basic lemma.

Lemma 1. [28] Given $y_0\in L^2(\Omega)$, $y(x,t)$ is said to be the unique mild solution of system (1) on $\Omega\times[0,T]$ if it satisfies

$$y(x,t) = \sum_{n=1}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(\lambda_n t^{\alpha})(y_0,\xi_n)\xi_n(x). \tag{13}$$

2.2 Spectrum decomposition assumption

From the hypothesis (H_1), since operator A is only with a pure point spectrum, denote by $\sigma(A) := \{\lambda_n\}_{n\in\mathbb{N}^+}$, given a constant $c>0$, we assume that

(H_2) The set $\sigma(A)\cap\{\lambda_n,\lambda_n>-c\}$ contains only a finite number of elements of the point spectrum $\sigma(A)$;

(H_3) The spectrum decomposition assumption on operator A is true at the constant $c>0$ [29].

Under the above assumptions, it is possible to choose sufficiently large N such that

$$\lambda_n\leqslant -c \text{ for all } n>N. \tag{14}$$

Let $\{\lambda_n\}_{n\geqslant 1} = \theta_1(A) + \theta_2(A)$ be a decomposition of the eigenvalues of A with

$$\theta_1(A) = \{\lambda_1, \cdots, \lambda_N\} \text{ and } \theta_2(A) = \{\lambda_{N+1}, \lambda_{N+2}, \cdots\}. \quad (15)$$

Note that $\{\xi_n\}_{n\geqslant 1}$ forms a Riesz basis in $L^2(\Omega)$, let

$$Y_1 = \text{span}\{\xi_1, \xi_2, \cdots, \xi_N\} \text{ and } Y_2 = \text{span}\{\xi_{N+1}, \xi_{N+2}, \cdots\}.$$

They yield that $L^2(\Omega) = Y_1 + Y_2$. Moreover, define the orthogonal projections $P_i : L^2(\Omega) \to Y_i$, $i = 1,2$ by

$$y_1(x,t) := P_1 y(x,t) = \sum_{n=1}^{N} (y(\cdot,t), \xi_n)\xi_n(x) \quad (16)$$

and

$$y_2(x,t) := P_2 y(x,t) = \sum_{n=N+1}^{\infty} ((\cdot,t), \xi_n)\xi_n(x), \quad (17)$$

we have $y(x,t) = (P_1 + P_2)y(x,t) = y_1(x,t) + y_2(x,t)$. Consequently, system (1) can be decomposed into

$$\begin{cases} {}_0D_t^{\alpha} y_1(x,t) = A_1 y_1(x,t) \text{ in } \Omega \times [0,T], \\ y_1(x,t) = 0 \text{ in } \partial\Omega \times [0,T], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} y_1(x,t) = P_1 y_0(x) \text{ in } \Omega \end{cases} \quad (18)$$

and

$$\begin{cases} {}_0D_t^{\alpha} y_2(x,t) = A_2 y_2(x,t) \text{ in } \Omega \times [0,T], \\ y_2(x,t) = 0 \text{ in } \partial\Omega \times [0,T], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} y_2(x,t) = P_2 y_0(x) \text{ in } \Omega, \end{cases} \quad (19)$$

where $A_1 = P_1 A$ is a $N \times N$ matrix and $A_2 = P_2 A$ such that

$$\begin{aligned} \|y(\cdot,t)\| &\leqslant \|y_1(\cdot,t)\| + \|y_2(\cdot,t)\| \\ &= \left\| \sum_{n=1}^{N} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_n t^{\alpha})(y_0,\xi_n)\xi_n \right\| \\ &\quad + \left\| \sum_{n=N+1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_n t^{\alpha})(y_0,\xi_n)\xi_n \right\|. \end{aligned} \quad (20)$$

In addition, since $\frac{d}{dt}E_{\alpha,\alpha}(-t^{\alpha}) \leqslant 0$ for $\alpha \in (0,1]$, $t \geqslant 0$ [27] and $\lambda_n \leqslant -c$, $n > N$, by $(-\lambda_n)^{1/\alpha} \geqslant c^{1/\alpha}$, we get that

$$E_{\alpha,\alpha}(\lambda_n t^{\alpha}) = E_{\alpha,\alpha}\left(-\left((-\lambda_n)^{1/\alpha} t\right)^{\alpha}\right) \leqslant E_{\alpha,\alpha}(-ct^{\alpha}) \quad (21)$$

when $n > N$. This implies that

$$\begin{aligned} \|y_2(\cdot,t)\| &= \left\| \sum_{n=N+1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_n t^{\alpha})(y_0,\xi_n)\xi_n \right\| \\ &\leqslant \|y_0\| t^{\alpha-1} E_{\alpha,\alpha}(-ct^{\alpha}). \end{aligned} \quad (22)$$

With this, we could refer to the state $y_1(x,t)$ described by the finite dimensional system (18) as the estimated of system (1) and the state $y_2(x,t)$ governed by (19) as the residual parts.

2.3 Problem formulation

In this part, we define a averaged information matrix, the determinant of which can be considered as the criterion of accuracy in eigenvalue identification.

For eigenvalues $\lambda := [\lambda_1, \lambda_2, \cdots, \lambda_N]$, let

$$\nabla_{\lambda} = \left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2}, \cdots, \frac{\partial}{\partial \lambda_N} \right)^T. \quad (23)$$

By (6), we define a $N \times N$ matrix G_N as follows:

$$G_N = \sum_{k=1}^{\tilde{q}} n_k \int_0^T \left\{ \begin{array}{c} f(t)\left(\nabla_{\lambda}\, {}_0I_t^{1-\alpha} y_1(\sigma_k,t)\right) \times \\ \left(\nabla_{\lambda}\, {}_0I_t^{1-\alpha} y_1(\sigma_k,t)\right)^T \end{array} \right\} dt. \quad (24)$$

Here $f : [0,T] \to \mathbb{R}$ is a given positive function chosen to satisfy the convergent conditions to be specified later. It then follows from Lemma 1 that

$$\begin{aligned} &\frac{\partial}{\partial \lambda_n}\, {}_0I_t^{1-\alpha} y_1(\sigma_k,t) \\ &= \frac{\partial}{\partial \lambda_n}\, {}_0I_t^{1-\alpha} \sum_{n=1}^{N} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_n t^{\alpha})(y_0,\xi_n)\xi_n(\sigma_k) \\ &= \frac{\partial}{\partial \lambda_n} \sum_{n=1}^{N} \left\{ {}_0I_t^{1-\alpha} t^{\alpha-1} E_{\alpha,\alpha}(\lambda_n t^{\alpha}) \right\} (y_0,\xi_n)\xi_n(\sigma_k) \\ &= \frac{\partial}{\partial \lambda_n} \sum_{n=1}^{N} \left\{ \sum_{i=0}^{\infty} \frac{\lambda_n^i \int_0^t (t-s)^{-\alpha} s^{\alpha i+\alpha-1} ds}{\Gamma(1-\alpha)\Gamma(\alpha i+\alpha)} \right\} (y_0,\xi_n)\xi_n(\sigma_k) \\ &= \frac{\partial}{\partial \lambda_n} \sum_{n=1}^{N} \left\{ \sum_{i=0}^{\infty} \frac{\lambda_n^i t^{\alpha i}}{\Gamma(\alpha i+1)} \right\} (y_0,\xi_n)\xi_n(\sigma_k) \\ &= \sum_{i=1}^{\infty} \frac{i\lambda_n^{i-1} t^{\alpha i}}{\Gamma(\alpha i+1)} (y_0,\xi_n)\xi_n(\sigma_k) \\ &= \sum_{i=1}^{\infty} \frac{\lambda_n^{i-1} t^{\alpha i}}{\alpha\Gamma(\alpha i)} (y_0,\xi_n)\xi_n(\sigma_k) \\ &= \frac{1}{\alpha} \sum_{j=0}^{\infty} \frac{\lambda_n^j t^{\alpha j+\alpha}}{\Gamma(\alpha j+\alpha)} (y_0,\xi_n)\xi_n(\sigma_k) \\ &= \frac{t^{\alpha}}{\alpha} E_{\alpha,\alpha}(\lambda_n t^{\alpha})(y_0,\xi_n)\xi_n(\sigma_k). \end{aligned} \quad (25)$$

Let $g_{m,k}(t) = \frac{t^{\alpha}}{\alpha} E_{\alpha,\alpha}(\lambda_m t^{\alpha})(y_0,\xi_m)\xi_m(\sigma_k)$ and $g_{n,k}(t) = \frac{t^{\alpha}}{\alpha} E_{\alpha,\alpha}(\lambda_n t^{\alpha})(y_0,\xi_n)\xi_n(\sigma_k)$. We define the averaged information matrix as follows

$$\overline{AIM} = \lim_{T\to\infty} \frac{1}{qT} G_N. \quad (26)$$

Suppose that $\bar{a}_{mn}$ $(m,n = 1,2,\cdots,N)$ is the element of the $N \times N$ matrix $\overline{AIM}$, one has

$$\begin{aligned} \bar{a}_{mn} &= \lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{\tilde{q}} \frac{n_k}{q} \int_0^T f(t) g_{m,k}(t) g_{n,k}(t) dt \\ &= \frac{1}{\alpha^2}(y_0,\xi_m)(y_0,\xi_n) \sum_{k=1}^{\tilde{q}} p_k \xi_m(\sigma_k)\xi_n(\sigma_k) \times \\ &\quad \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) t^{2\alpha} E_{\alpha,\alpha}(\lambda_m t^{\alpha}) E_{\alpha,\alpha}(\lambda_n t^{\alpha}) dt. \end{aligned} \quad (27)$$

With this, let $\eta(x) = \sum_{k=1}^{\tilde{q}} p_k \delta(x - \sigma_k)$. Here δ denotes the Dirac delta function that is equal to zero everywhere except for zero. Then we obtain that (27) can be rewritten as

$$\begin{aligned} \bar{a}_{mn} = &\frac{1}{\alpha^2}(y_0,\xi_m)(y_0,\xi_n) \int_{\Omega} \xi_m(x)\xi_n(x)\eta(x) dx \times \\ &\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) t^{2\alpha} E_{\alpha,\alpha}(\lambda_m t^{\alpha}) E_{\alpha,\alpha}(\lambda_n t^{\alpha}) dt. \end{aligned} \quad (28)$$

This means that the determinant of the matrix $\overline{AIM}$, denoted by $\det\overline{AIM}$, is a function of η. Let

$$\overline{\eta} = \left\{ \begin{array}{c} \eta(x) : \eta(x) = \sum_{k=1}^{\tilde{q}} p_k \delta(x - \sigma_k) \geqslant 0 \text{ for} \\ \text{all } x \in \Omega \text{ such that } \int_{\Omega} \eta(x) dx = 1 \end{array} \right\}. \quad (29)$$

Hence, the optimal sensor placement problem for system (1) is to found a $\eta^* \in \overline{\eta}$ such that

$$\det\overline{AIM}(\eta^*) = \max_{\eta\in\overline{\eta}} \left\{ \det\overline{AIM}(\eta) \right\}. \quad (30)$$

Remark 2. Since $E_{\alpha,\alpha}(\lambda_n t^\alpha) \leqslant E_{\alpha,\alpha}(\lambda_1 t^\alpha)$ for all $t \geqslant 0$ [27], we get that

$$\begin{aligned}&\tfrac{1}{T}\int_0^T f(t)t^{2\alpha}E_{\alpha,\alpha}(\lambda_m t^\alpha)E_{\alpha,\alpha}(\lambda_n t^\alpha)dt\\ \leqslant\ &\tfrac{1}{T}\int_0^T f(t)t^{2\alpha}E_{\alpha,\alpha}(\lambda_1 t^\alpha)E_{\alpha,\alpha}(\lambda_1 t^\alpha)dt.\end{aligned} \tag{31}$$

Then function f can be chosen such as

$$f(t) = t^{-2\alpha}E^{-1}_{\alpha,\alpha}(\lambda_1 t^\alpha)E^{-1}_{\alpha,\alpha}(\lambda_1 t^\alpha) \tag{32}$$

to guarantee the convergence of G_N as $T \to \infty$.

3 Main results

Based on the results in the previous section, we get that $0 < \det\overline{AIM}(\eta^*) < \infty$ when $\eta \in \overline{\eta}$ and

$$\begin{aligned}&\det\overline{AIM}(\eta) = \sum_{m=1}^N\sum_{n=1}^N \bar{a}_{mn}(\eta)\\ &= \sum_{m=1}^N\sum_{n=1}^N \tfrac{1}{\alpha^2}(y_0,\xi_m)(y_0,\xi_n)\int_\Omega \xi_m(x)\xi_n(x)\eta(x)dx\times\\ &\quad\lim_{T\to\infty}\tfrac{1}{T}\int_0^T f(t)t^{2\alpha}E_{\alpha,\alpha}(\lambda_m t^\alpha)E_{\alpha,\alpha}(\lambda_n t^\alpha)dt.\end{aligned} \tag{33}$$

By the Hadamard's determinant inequality (see [30], Theorem II.3.17), it yields that

$$\det\overline{AIM}(\eta) \leqslant \Phi(\eta)\prod_{n=1}^N\left\{\lim_{T\to\infty}\tfrac{1}{T}\int_0^T f(t)t^{2\alpha}E^2_{\alpha,\alpha}(\lambda_n t^\alpha)dt\right\}$$

and the equality holds if and only if $\det\overline{AIM}(\eta)$ is a diagonal matrix. Here $\Phi(\eta) = \prod_{n=1}^N \frac{1}{\alpha^2}(y_0,\xi_n)^2\int_\Omega \xi_n^2(x)\eta(x)dx$.

Note that the part

$$\prod_{n=1}^N\left\{\lim_{T\to\infty}\tfrac{1}{T}\int_0^T f(t)t^{2\alpha}E^2_{\alpha,\alpha}(\lambda_n t^\alpha)dt\right\} \tag{34}$$

does not depend on $\eta \in \overline{\eta}$. Based on the strictly increasing property of natural logarithm function $\ln(x)$, we have

$$\ln\Phi(\eta) = \sum_{n=1}^N \ln\left(\tfrac{1}{\alpha^2}\int_\Omega \xi_n^2(x)\eta(x)dx\right) + \sum_{n=1}^N \ln(y_0,\xi_n)^2 \tag{35}$$

and here $\sum_{n=1}^N \ln(y_0,\xi_n)^2$ can be regarded as a constant. Then these above considerations show that the considered problem (30) is equivalent to find a $\eta^* \in \overline{\eta}$ such that

$$\phi(\eta^*) = \max_{\eta\in\overline{\eta}}\phi(\eta), \tag{36}$$

where $\phi(\eta) := \sum_{n=1}^N \ln\left(\frac{1}{\alpha^2}\int_\Omega \xi_n^2(x)\eta(x)dx\right)$.

Let $\phi_n(\eta) := \frac{1}{\alpha^2}\int_\Omega \xi_n^2(x)\eta(x)dx$, $n = 1,2,\cdots,N$. It follows that $\phi(\eta) = \sum_{n=1}^N \ln(\phi_n(\eta))$. Now we state the following main theorem, which gives the solution of problem (30).

Theorem 1. If the hypotheses $(H_1)-(H_3)$ hold, then $\eta^* = \sum_{k=1}^{\tilde{q}} p_k\delta(x-\sigma_k^*)$ is said to be the optimal placements of pointwise sensors to solve the problem (30) if and only if the following relationship holds

$$\alpha^2 N = \sum_{n=1}^N \frac{\sup_{\sigma\in\Omega}\xi_n^2(\sigma)}{\phi_n(\eta^*)}. \tag{37}$$

Proof. From the definition of $\eta(x)$, we obtain that

$$\phi(\eta) = \sum_{n=1}^N \ln(\phi_n(\eta)) = \sum_{n=1}^N \ln\left(\tfrac{1}{\alpha^2}\int_\Omega \xi_n^2(x)\eta(x)dx\right). \tag{38}$$

Then we finish our proof in two steps.

Necessity: Assume that $\eta^* \in \overline{\eta}$ is the optimal solution of problem (30), for any $\eta \in \overline{\eta}$, it follows that

$$\left[\tfrac{d}{d\beta}\phi(\beta\eta+(1-\beta)\eta^*)\right]_{\beta=0} \leqslant 0,\ \ \forall\eta\in\overline{\eta}. \tag{39}$$

Moreover, since $\overline{\eta}$ is a convex set, we obtain that

$$\begin{aligned}0 &\geqslant \left[\tfrac{d}{d\beta}\phi(\beta\eta+(1-\beta)\eta^*)\right]_{\beta=0}\\ &= \left[\tfrac{d}{d\beta}\sum_{n=1}^N \ln(\phi_n(\beta\eta+(1-\beta)\eta^*))\right]_{\beta=0}\\ &= \left[\sum_{n=1}^N \frac{\frac{d}{d\beta}\left(\frac{1}{\alpha^2}\int_\Omega \xi_n^2(x)(\beta\eta(x)+(1-\beta)\eta^*(x))dx\right)}{\phi_n(\beta\eta+(1-\beta)\eta^*)}\right]_{\beta=0}\\ &= \sum_{n=1}^N \frac{\phi_n(\eta)-\phi_n(\eta^*)}{\phi_n(\eta^*)}\\ &= \sum_{n=1}^N\left(\frac{\phi_n(\eta)}{\phi_n(\eta^*)}-1\right),\ \ \forall\eta\in\overline{\eta}.\end{aligned} \tag{40}$$

Then for any sensor located at $\sigma = \{\sigma_1,\sigma_2,\cdots\sigma_q\}$, one has

$$N \geqslant \sum_{n=1}^N \frac{\phi_n(\eta)}{\phi_n(\eta^*)} = \sum_{n=1}^N \frac{\frac{1}{\alpha^2}\sum_{k=1}^{\tilde{q}} p_k\xi_n^2(\sigma_k)}{\phi_n(\eta^*)}, \tag{41}$$

which yields that

$$\alpha^2 N \geqslant \sum_{n=1}^N \frac{\sup_{\sigma\in\Omega}\xi_n^2(\sigma)}{\phi_n(\eta^*)}. \tag{42}$$

On the other hand, let $\eta^*(x) = \sum_{k=1}^{\tilde{q}} p_k\delta(x-\sigma_k^*)$. We have

$$\alpha^2 N = \sum_{n=1}^N \frac{\alpha^2\phi_n(\eta^*)}{\phi_n(\eta^*)} = \sum_{n=1}^N \frac{\sum_{k=1}^{\tilde{q}} p_k\xi_n^2(\sigma_k^*)}{\phi_n(\eta^*)} \leqslant \sum_{n=1}^N \frac{\sup_{\sigma\in\Omega}\xi_n^2(\sigma)}{\phi_n(\eta^*)}. \tag{43}$$

It then follows from (42) and (43) that the relationship (37) is satisfied.

Sufficient: Suppose that (37) holds for some $\eta^* \in \overline{\eta}$. Given any $\eta = \sum_{k=1}^{\tilde{q}} p_k\delta(x-\sigma_k) \in \overline{\eta}$, it follows that

$$\phi_n(\eta) = \tfrac{1}{\alpha^2}\sum_{k=1}^{\tilde{q}} p_k\xi_n^2(\sigma_k) \geqslant 0. \tag{44}$$

By [6], when $\phi_n(\eta) = 0$, the studied system would be unobservable by the sensors located at $\{\sigma_k\}_{1\leqslant k\leqslant q}$. Then we here assume that $\phi_n(\eta) = \frac{1}{\alpha^2}\sum_{k=1}^{\tilde{q}} p_k\xi_n^2(\sigma_k) > 0$ for all $\eta \in \overline{\eta}$, i.e., we only focus on the cases when the studied system is observable with the measurements (6).

Consider the function $h(x) = x - \ln x$, $x > 0$. Since $h'(x) = 1 - 1/x$ and $h(1) = 1$, we get that $h(x) \geqslant h(1) = 1$ for all

$x > 0$. This is, $\ln(x) \leqslant x-1$ for all $x > 0$. With this, one has

$$\begin{aligned}\phi(\eta)-\phi(\eta^*) &= \sum_{n=1}^{N} \ln\left(\frac{\phi_n(\eta)}{\phi_n(\eta^*)}\right) \\ &= \sum_{n=1}^{N} \ln\left(\frac{\frac{1}{\alpha^2}\sum_{k=1}^{\tilde{q}} p_k \xi_n^2(\sigma_k)}{\phi_n(\eta^*)}\right) \\ &\leqslant \sum_{n=1}^{N}\left(\frac{\frac{1}{\alpha^2}\sum_{k=1}^{\tilde{q}} p_k \xi_n^2(\sigma_k)}{\phi_n(\eta^*)}-1\right) \\ &\leqslant \frac{1}{\alpha^2}\sum_{n=1}^{N}\frac{\sup_{\sigma\in\Omega}\xi_n^2(\sigma)}{\phi_n(\eta^*)}-N \\ &= 0, \ \forall \eta \in \overline{\eta}.\end{aligned} \tag{45}$$

It then follows from the arbitrariness of $\eta \in \overline{\eta}$ that η^* is the solution of optimal problem (30). This finishes our proof.

4 An example

Consider the following scalar, one dimensional time fractional diffusion system

$$\begin{cases} {}_0D_t^{\alpha} y(x,t) = y_{xx}(x,t) \text{ in } (0,1)\times[0,T], \\ y(0,t) = y(1,t) = 0 \ \text{ in } [0,T], \\ \lim_{t\to 0^+} {}_0I_t^{1-\alpha} y(x,t) = y_0(x) \ \text{ in } (0,1). \end{cases} \tag{46}$$

By [31], the eigenvalues and corresponding eigenfunctions are respectively,

$$\lambda_n = -n^2\pi^2 \text{ and } \xi_n(x) = \sqrt{2}\sin(n\pi x),\ n = 1,2,\cdots. \tag{47}$$

With this, the hypotheses $(H_1)-(H_3)$ are satisfied for some given $c > 0$. Suppose that there exists three sensors placed at $\sigma_i \in (0,1)$, $i = 1,2,3$. Then the Theorem 1 shows that the necessary and sufficient condition for the optimal sensor placements $\eta^* = \sum_{k=1}^{3} \frac{1}{3}\delta(x-\sigma_k^*)$ of example (46) is

$$2 = \alpha^2 N \sum_{n=1}^{N} \phi_n(\eta^*) = N\sum_{n=1}^{N}\sum_{k=1}^{3}\frac{2}{3}\sin^2(n\pi\sigma_k^*). \tag{48}$$

Case 1, If $c = 2\pi^2$, one has $N = 1$ and (48) deduces to $3 = \sum_{k=1}^{3} \sin^2(\pi\sigma_k^*)$. Then the optimal placements of the three pointwise sensors should be $\sigma_1^* = \sigma_2^* = \sigma_3^* = 1/2$.
Case 2, If $c = 5\pi^2$, then $N = 2$ and the optimal sensor placements σ_k^*, $k = 1,2,3$ should satisfy

$$3 = \sum_{k=1}^{3}\left(\cos(2\pi\sigma_k^*)+\cos(4\pi\sigma_k^*)\right). \tag{49}$$

Let $m(\tau) = \cos(2\pi\tau)+\cos(4\pi\tau)$. We get the following table 1, where $\tau_1 \in (0,1/6)$ and $\tau_2 \in (5/6,1)$. Then there exists more than one solution for the considered optimal sensor placements problem. For example, it can be $(0,1/3,1)$, $(0,2/3,1)$, $(0,\tau_1,5/6)$ or $(1/6,\tau_2,1)$, etc.

Table 1: Values of the function m

τ	0	τ_1	1/6	1/3	1/2	2/3	5/6	τ_2	1
$m(\tau)$	2	1	0	−1	0	−1	0	1	2

Case 3, If $c = 10\pi^2$, then $N = 3$ and the optimal sensor placements σ_k^*, $k = 1,2,3$ satisfy

$$7 = \sum_{k=1}^{3}\left(\cos(2\pi\sigma_k^*)+\cos(4\pi\sigma_k^*)+\cos(6\pi\sigma_k^*)\right). \tag{50}$$

Let $m_2(\tau) = \cos(2\pi\tau)+\cos(4\pi\tau)+\cos(6\pi\tau)$. We get the following table 2, where $\tau_1 \in (0,1/6)$ and $\tau_2 \in (5/6,1)$. Then similarly, the optimal sensor placements can be such as, $(0,\tau_1,1)$, $(0,0,\tau_2)$ or $(0,\tau_2,1)$, etc.

Table 2: Values of the function m

τ	0	τ_1	1/6	1/3	1/2	2/3	5/6	τ_2	1
$m_2(\tau)$	3	1	−1	0	−1	0	−1	1	3

5 Conclusions

In this paper, a new averaged information matrix is proposed based on the spectrum decomposition assumption to determine the optimal placements of sensors for the time fractional diffusion system by maximizing the accuracy in system eigenvalue identification. The necessary and sufficient conditions for optimal sensor placements of studied systems can be explicitly expressed by the order of fractional derivative, the finite number of chosen eigenvalues and their corresponding eigenfunctions.

Further investigations are desirable to for example, the actuators/sensors configuration problems as well as the cases with measurement noise and process noise for more complex fractional distributed parameter systems. Various open questions are still under consideration. For more information on the potential topics related to fractional order distributed parameter systems, we refer the readers to [32] and the references cited therein.

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