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Review

A new collection of real world applications of fractional calculus in science and engineering

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ABSTRACT

Fractional calculus is at this stage an arena where many models are still to be introduced, discussed and applied to real world applications in many branches of science and engineering where nonlocality plays a crucial role. Although researchers have already reported many excellent results in several seminal monographs and review articles, there are still a large number of non-local phenomena unexplored and waiting to be discovered. Therefore, year by year, we can discover new aspects of the fractional modeling and applications. This review article aims to present some short summaries written by distinguished researchers in the field of fractional calculus. We believe this incomplete, but important, information will guide young researchers and help newcomers to see some of the main real-world applications and gain an understanding of this powerful mathematical tool. We expect this collection will also benefit our community.

1. Introduction

Fractional calculus (FC) is an emerging field in mathematics with deep applications in all related fields of science and engineering. Some of the results were reported in various books or related review articles [1,2,4–17]. However, we are still at the beginning of applying this very powerful tool in many fields of research. At this moment, the fractional calculus has opened its wings even larger to cover the dynamics of complex real world and new ideas are starting to be implemented and tested on real data. In some cases, some patents were granted which make the tool of FC very promising. Though fractional calculus was introduced more than 300 years ago and applied into many fields of science and engineering, the promotion of applications is still an important task of the FC community. When we talk about FC with scientists and engineers outside of our community, two of the most frequently asked questions are about how FC has been applied and how scientists can apply it to their respective fields. Meanwhile, many FC researchers in theoretical fields are also not

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familiar with the application aspects. Therefore it is necessary to provide a brief introduction on successful applications of FC in science and engineering. Moreover, we should recognize that FC is not universal but has its own place in application; hence, providing some important existing successful applications of FC can offer a guide on application studies in the future.

To make this collection more comprehensive, we have invited several distinguished researchers in the application field of FC to contribute one or more application cases, and make a summary on a specific scientific/engineering area. However, there are still many experts in this field who have not been invited or contacted, due to the difficulty of email communication and the limitation of our knowledge. Furthermore, there are still many successful applications of FC which have not been included in this collection, due to the length limitation of this collection and the time limitation of submission.

This review is organized into nine sections. We begin with some important results of FC in physics, after that we briefly present some applications from the control theory and signal and image processing. The next main topics are from mechanics and dynamical systems, biology, environmental sciences, and materials. We end our review article by presenting some main results from applications of FC to multidisciplinary and other engineering fields.

Each section contains several contributions written by prestigious scientists. Each contribution contains some relevant references where the authors can find more information about the debated topics. In this review, we collected 46 contributions and we hope that the new information presented here will strongly contribute to the promotion and further development of fractional calculus and its applications.

2. Physics

2.1. Fractional Langevin equation description of viscoelastic anomalous diffusion in complex liquids

Many complex systems such as the crowded liquid inside biological cells, solutions and melts of polymeric materials, or lipid bilayer membranes are viscoelastic. Depending on the frequency with which these systems are probed, their response is more elastic or more viscous. Diffusion of tracer particles in these complex liquids is anomalous, with the mean squared displacement scaling like $\langle \mathbf{r}^2(t)\rangle \simeq t^{\alpha}$, where we speak of subdiffusion for $0<\alpha<1$ and superdiffusion for $\alpha>1$ [18]. Concurrently the increment correlation of the observed motion is antipersistent in the regime of subdiffusion: subsequent increments of the motion are likely to be directed in opposite directions. A slow power-law recovery to zero of the negative correlation is then observed at longer times. Mathematically, this motion is described by the *fractional Langevin equation*, in which the friction term involves a power-law memory, and the noise becomes power-law correlated [18]. Tracers inside biological cells or in crowded liquids and the constituents of biological membranes have been shown to exhibit this type of viscoelastic motion [19–21]. For superdiffusion, the increment correlations are always positive, a phenomenon that can also be observed in active biological systems [22]. (Contributed by Ralf Metzler, Anomalous diffusion).

2.2. Attenuation and dispersion in complex viscoelastic media

Fractional derivative models in the biomedical and underwater sediment fields are useful because they describe the power law attenuation encountered in these media better than other models.

In sediment acoustics, it has been shown that one of the most common models, the viscous grain shearing model [23], is based on the constitutive laws of a fractional Kelvin–Voigt and a fractional damper for the compressional and shear waves, respectively [24,25].

These models as well as the fractional Zener model have been proposed for modeling wave propagation in medical ultrasound imaging and elastography [26,27].

The models are useful for interpreting and simulating propagating waves. They may also give insight into mechanisms for absorption of energy. One example is that fractional models may be justified by the existence of many relaxation processes, e.g. as found from the properties of polymers [28]. A non-Newtonian material with time-dependent viscosity, which for the last decades has been used to describe the grain shearing process [23], will also give rise to the relaxation modulus which is similar to that of a fractional derivative element [29]. (Contributed by Sverre Holm, Fractional viscoelasticity).

2.3. Anomalous diffusion with internal states: functional distributions, escape probability, and first passage time

Normal diffusion describes the Brownian dynamics characterized by a large number of small events, e.g., the motion of pollen grains in water. However, in many cases, the (rare) large fluctuations result in the non-Brownian motion, anomalous diffusion, being carefully studied in physics, hydrology, finance and other fields (Fig. 1).

We derived the forward and backward fractional Feynman–Kac equations which describe the distribution of functionals of space and time-tempered anomalous diffusion, belonging to the continuous time random walk class. Several examples of the functionals are explicitly treated, including the first passage time, the occupation time in half-space, the maximal displacement, the fluctuations of the time-averaged position, and the fluctuations of the occupation fraction. For details, see [30].

We derived the nonlocal elliptic partial differential equations (PDEs) governing the mean first exit time and escape probability of the anomalous processes having the tempered Lévy stable waiting times with the tempering index $\mu>0$ and the stability index $0<\alpha\leq 1$. For details, see [31].

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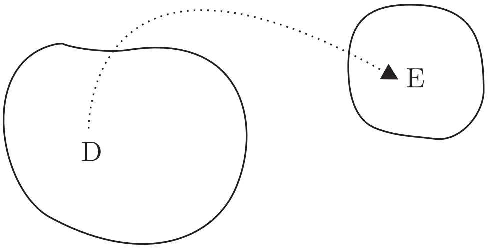

Fig. 1. Sketch map of the escape probability.

For the particles undergoing anomalous diffusion with different waiting time distributions for different internal states, we derive the Fokker–Planck and Feymann–Kac equations, respectively, describing positions of the particles and functional distributions of the trajectories of particles; in particular, the equations governing the functional distribution of internal states are also obtained. The dynamics of the stochastic processes are analyzed and the applications, calculating the distribution of the first passage time and the distribution of the fraction of the occupation time, of the equations are given. For the further application of the newly built models, we make very detailed discussions on the none-immediately repeated stochastic process, e.g., the random walk of smart animals. For details, see [32]. (Contributed by Weihua Deng, Anomalous diffusion).

2.4. Electrical spectroscopy impedance and fractional calculus

The electrical spectroscopy impedance technique plays an important role from the experimental point of view to obtain information about the electrical properties of many different materials, in particular, of liquids [33]. It has been investigated, from the theoretical point of view, by using the Poisson–Nernst–Planck diffusion model [34] and/or equivalent circuits. In the low frequency limit, these approaches with simple considerations (boundary conditions and/or circuit elements) are not able to describe the experimental behavior. These disagreements are especially remarkable in the low frequency limit [34,35]. However, by using the well-established features of the fractional calculus and performing suitable changes in the boundary conditions, in order to account the surface effects, it is possible to overcome this issue and describe the experimental behavior in all frequency range [36,37]. Furthermore, this approach can also be used to investigate the ion diffusion in an electrolytic cell through the electrical conductivity, which is directly related to the mean square displacement. In particular, for some systems [38], the diffusion manifested by the ions may not always depend on the frequency range considered and related to the surface effects. (Contributed by Ervin K. Lenzi, Rafael S. Zola, Haroldo V. Ribeiro and Luiz R. Evangelista, Complex fluids).

2.5. Physical demonstration of iterated fractional order integrals

Fractional calculus theory predicts that iterated integrals of any order will result in an integral of order equal to the sum of the orders of the integrals as long as the integration interval is the same for each integral. That is:

$$ {}_aI_t^{\alpha}\, {}_aI_t^{\beta}\, {}_aI_t^{\gamma} f(t') = {}_aI_t^{\alpha+\beta+\gamma} f(t') $$

The validity of this rule was demonstrated in an analog circuit computing the solution to $\rho y^{(\delta)} = f(t) - \kappa y(t)$, where ρ and κ were determined by the scaling constants found from the impedance spectra of the factors used in the physical implementation. Fig. 2 shows the schematic map for the circuit implementing $\delta = 0.3 + 0.5 + 1.0$. Other cases with $\delta = 0.8$, 1.3 and 1.5 were also tested. In each case, the measured response for step function input matched that predicted by the Mittag–Leffler function. The amplitude and phase of the response for sinusoidal input signals matched that predicted by the Fourier description. Note that for the Fourier response, we had to wait for the transient response to dissipate.

For details, see Ref. [39]. (Contributed by Gary W. Bohannan and Brenda Knauber, Electromagnetism).

2.6. Frequency-dependent acoustic wave propagation in porous media

The frequency-dependent characteristic impedance and the propagation coefficient of acoustic wave were observed in the experimental study by Delany and Bazley. Various models were subsequently proposed to demonstrate the phenomenon without physical interpretation of the frequency-dependent indices. The fractional acoustic wave equation was proposed on the basis of characteristic impendence, continuity equation, and state equation. The two different indices were unified to be the fractional derivative order with clear physical meaning. The attenuation and dispersion functions of the presented acoustic wave model agreed well with the experimental results and obeyed the Kramers–Kronig relation. For details, see Ref. [40]. (Contributed by Wen Chen, Shuai Hu and Wei Cai, Acoustics).

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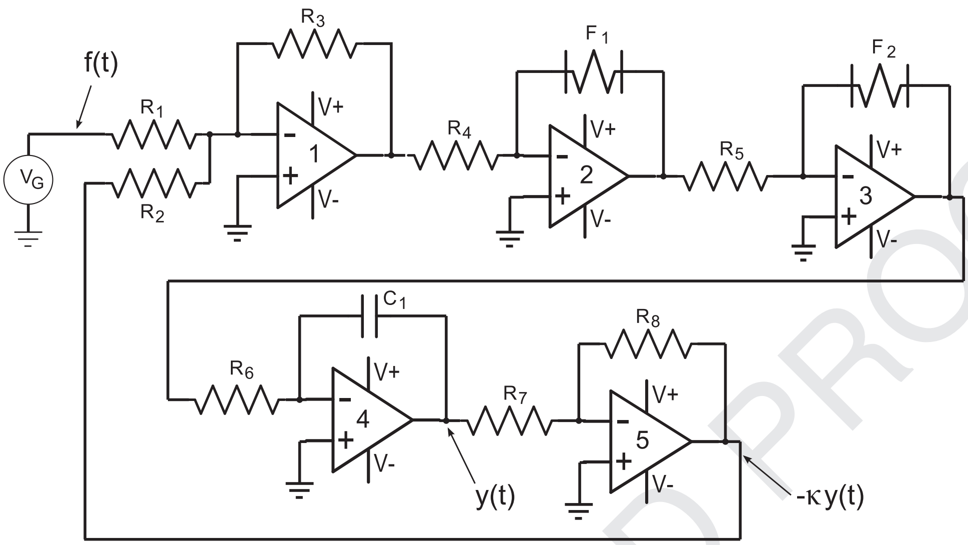

Fig. 2. A circuit diagram with $\alpha_{F1} = 0.3$ and $\beta_{F2} = 0.5$ and a low loss capacitor $\gamma_{C1} \approx 1.0$ to give a total fractional order of $\delta = 1.8$.

2.7. Fractional calculus technique in random optimal search

Random searches are ubiquitous because the locations of the specific targets are not known a priori in many situations. In this respect, the fundamental question is how to optimize the search for specific target scenarios. The key feature is that the scatterers have a power-law distribution of sizes, which motivates us to model the random optimal search problem using the fractional calculus technique. Precisely, the Continuous Time Random Walk (CTRW) optimal search framework was proposed to locate the optimum for both of search length's and waiting time's distributions by means of power-law function. The master equation was derived to describe the mechanism of such complex fractional dynamics. Many simulations were carried out to support the theoretical results. For details, see Ref. [41]. (Contributed by Caibin Zeng, Statistical physics).

2.8. Fractional diffusion equations for random walkers in an expanding medium

A well-known model for diffusion processes is the (uncoupled) CTRW model, in which each particle (random walker) takes a jump of size Δx after a waiting time Δt. These random variables are respectively drawn from the pdfs $\lambda(\Delta x)$ and $\varphi(\Delta t)$. If both $\lambda(\Delta x)$ and $\varphi(\Delta t)$ are "normal" (e.g., Gaussian), the probability density $P(x, t)$ of finding the walker at position x at time t obeys the standard diffusion equation. However, if $\varphi(\Delta t)$ [$\lambda(\Delta x)$] is heavy-tailed, $P(x, t)$ is governed by a generalized diffusion equation with a fractional temporal (spatial) derivative. Recently, these equations have been generalized to the case where the medium in which the random walk evolves is no longer static [42]. Instances of diffusion in expanding medium can be found, e.g., in biology [43] and cosmology [44]. In this case the corresponding diffusion equation, in comoving coordinates, preserves its form, albeit with an effective time-dependent diffusion coefficient induced by the medium expansion. For the case of an equation containing a fractional spatial derivative only, exact solutions for Green's function (propagator) have been obtained. In contrast, for the case of an equation containing a fractional time derivative alone, only the spatial moments are known. (Contributed by Felipe Le Vot, Enrique Abad and Santos B. Yuste, Statistical physics).

2.9. Thermal stresses in a solid with a heat source varying harmonically in time in the framework of fractional thermoelasticity

Classical thermoelasticity [45] starts from the standard parabolic heat conduction equation. Fractional thermoelasticity [13] is based on the heat conduction equation with differential operators of fractional order. Nowacki [45] considered an elastic space with a source of heat varying harmonically as a function of time and investigated associated thermal stresses. The analysis was based on the assumption that temperature can be represented as a product of a function of the spatial coordinates and the time-harmonic term. Such an assumption cannot be used in the case of fractional heat conduction equation, and the initial conditions should be used. The proposed approach allows studying harmonic impact also in the case of fractional thermoelasticity. For details, see Ref. [46]. (Contributed by Yuriy Povstenko, Thermoelasticity).

2.10. Nanoprecipitate growth in solid solutions

Clusterization of impurities and defects can substantially change mechanical, electrical and optical properties of materials. Kinetics of such a process is usually described by a model of diffusion-limited first-order transition. Evidences of anomalous transport of impurities and defects in disordered solids stimulate the development of generalized models. One of the ways is based on fractional calculus. Fractional approach avails to simplify essentially consideration of such phenomena as transport in inhomogeneous media, diffusion along grain boundaries, dislocations, etc. In [47], authors proposed the fractional model of subdiffusion-limited growth and dissolution of nanoprecipitates in alloys within the Ham approach. The fractional Stephan problem for spherical nanoprecipitates in an infinite matrix is considered in [48]. The fractional generalization of Ostwald ripening of multiple clusters is proposed in [49]. Observable power law kinetics of precipitation in some real systems (e.g. Cu clusters in Fe–Cu alloys) contradicts the normal diffusion-limited model and could be interpreted within the fractional approach. (Contributed by Renat Sibatov, Anomalous diffusion).

3. Control

3.1. Ubiquitous fractional order memory system

The future states of an integer order dynamic system depend on the current one (memoryless). Nevertheless, for a fractional order system, the current state depends on the whole history (long memory). This long memory is typically a nameplate of various fractional order systems [12,16]. Recall the first two successful applications of fractional calculus in the 1980s, i.e., fractional order viscoelasticity and fractional order quantum mechanics. Boltzmann superposition principle plays a crucial role and leads to an important byproduct “heavy tail”, which is a vivid expression of system memory. From superposition of exponential function, to stretched exponential function, and then to Mittag–Leffler function, the system structure becomes more concise and model accuracy improves to a great extent. However, time domain phenomena (model) may tell lies. Here we introduce two frequency domain tools, i.e. Prony technique and electrochemical workstation, that consider a system without knowledge of internal workings (black box). In other words, using these two tools, the structure of system (integer order or fractional order) is not required in advance. By doing so, the frequency responses of quite a few real systems, such as Bode plot, Nyquist plot, Cole–cole plot, etc, show non-ideal curves that point directly to fractional order phenomena and reveal the fractional order nature such as ubiquitous fractional order capacitors [50]. Lastly, one fractional order model-free discussion should be noted here, i.e. the scale-free patterns. Along with the improvement of nature’s complexity (long history), scale-free patterns have widely existed in both nature and human society, such as fractal patterns from atomic level to clouds, mountains and rivers as well as the internet world. The scale-free pattern permits infinite possibilities under finite conditions such as the huge inner surface area of small intestine in a very limited space. Particularly, the scale-free structure can be a source of fractional order dynamics such as the continuous time random walk in porous media. Besides, the scale-free network has also become very popular in bioinformatics mining. To show a big picture of the fractional order system is like “Six Blind Men and an Elephant”. Nevertheless, motivated by the above illustrations, telling the story of fractional order system is like telling the story of oneself, because fractional order system is ubiquitous. On the other side, fractional order systems are complicated, even if they are composed by several number of simple elements, and certain heritage mechanisms sustain such complexities [51,52]. Therefore, it is remarkable that fractional order systems are ubiquitous and have memory. (Contributed by Yan Li, YangQuan Chen, Control theory).

3.2. Application of D-decomposition technique in solving some control problems

The basic idea of D-decomposition technique, conceived by the Russian scientist Neimark during the 1950s, is now extended for the case of linear fractional order systems and gives powerful tool for the analysis of systems stability and performance. Its straightforward procedure makes this method easy to apply and applicable to a wide range of transfer functions: with or without time-delay, rational and non-rational ones, and those describing distributed parameter systems. One way to utilize this technique is by combining it with another useful procedure, named dominant pole placement, designed to deal with the problem of controlling a high order and complex systems. In order to control as many different processes as possible, a fractional order proportional-integral-derivative (PID) controller is introduced, as a generalization of classical PID controller. Another useful application of this technique is control of underactuated systems. Many systems in nature are inherently underactuated, with fewer actuators than degrees of freedom. However, even with a reduced number of actuators, these systems are able to produce complex movements. Classical benchmark examples for studying problems of this kind include inverted pendulum systems. Herein, the D-decomposition method can be successfully used to solve a problem of asymptotic stability of inverted pendulum systems controlled by a fractional order controller. For details, see Refs. [53,54]. (Contributed by Tomislav B. Š ekara, Petar D. Mandić, Control theory).

3.3. The application of fractional order control for an air-based precision positioning system

Precision, bandwidth (speed) and stability of motion are the most important performance indexes of any motion system. Fractional order PID has proven to be very effective to improve the performance. A recent work at TU Delft [55], utilizes the fractional order calculus to control a precision positioning stage. In this work, a contactless precision positioning system is designed by floating a silicon wafer on a thin film of air (see Fig. 3(a)). The system has been controlled as shown in Fig. 3(b) in which two cascade single-input/single-output (SISO) controllers are designed. It has been shown that, the bandwidth of a regular mass–spring system has been increased using fractional lead compensator. In addition, it has been demonstrated

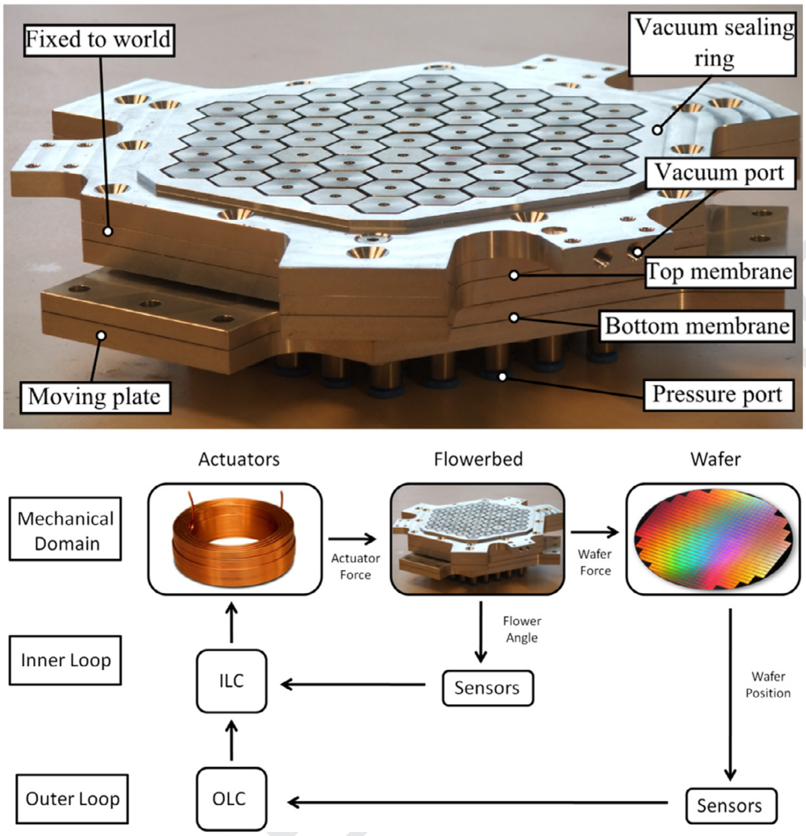

Fig. 3. (a) Overview of the air based precision positioning stage (so called the Flowerbed) designed at TU Delft. (b) Proposed Control Strategy, with an Inner Loop Controller (ILC) and an Outer Loop Controller (OLC).

that such a moving mass behaves fundamentally fractional. By using only the fractionality, the bandwidths are extended by 14.6% and 62%, for the inner and outer loops, respectively. Furthermore, a closed-loop positioning bandwidth of the wafer of 60 Hz is achieved, resulting in a positioning error of 104 nm, which is limited by sensor noise and pressure disturbances. (Contributed by S. Hassan HosseinNia, Fractional order control).

3.4. Application of fractional order calculus in active damping of flexible structures

In the past decades, research on Active Vibration Control (AVC) has found increasing interest in control of flexible links robots and thin-walled structures, mainly made of new advanced materials such as carbon fibre composites. Direct velocity control (DVC), Integral force control (IRC), Positive position feedback (PPF) and integral resonance control (IRC) are the methods which have been developed and used to actively control such flexible structures. Recently, authors from Spain and Netherlands have shown that fractional order calculus are effective tools to improve the active damping controllers compared to the integer order one [56–58]. In [56,57], a fractional order PPF compensator is proposed, implemented and compared to the standard integer-order PPF. The fractional-order controller is found to be more efficient in achieving the same performance as the integer order one with less actuation voltage (see Fig. 4). Moreover, it shows promising performance in reducing spillover effect due to uncontrolled modes. In [58], a fractional-order integral controller is proposed. This new methodology is compared with the most relevant controllers for smart structures. It is demonstrated that the proposed controller improves the robustness of the closed-loop system to changes in the mass of the payload at the tip. The previous controllers are robust in the sense of being insensitive to spillover and maintaining the closed-loop stability when changes occur in the plant parameters. However, the phase margin of such closed-loop systems (and, therefore, their damping) may change significantly as a result of these parameter variations. It has been proven and validated experimentally that the fractional order integral control with a very simple structure is an effective way to increase the phase margin robustness of the controlled system. (Contributed by S. Hassan HosseinNia, Active vibration control).

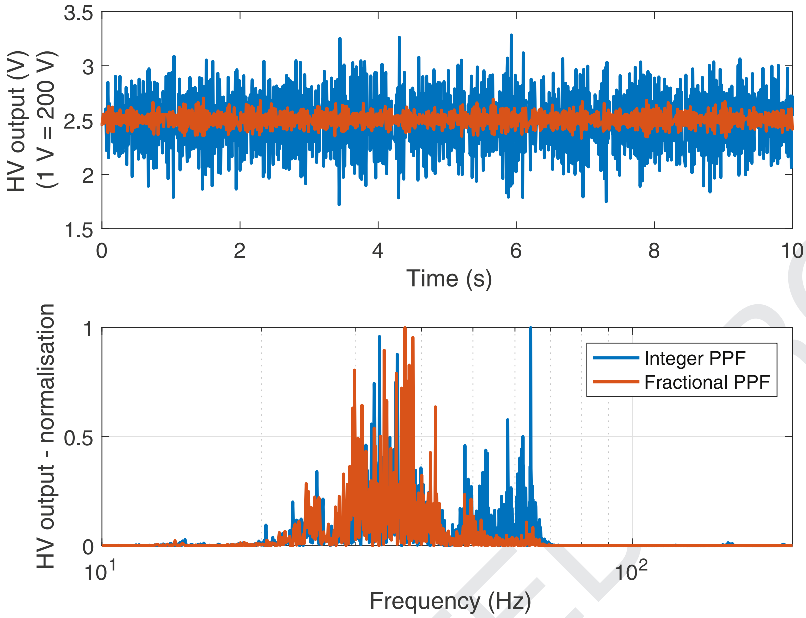

Fig. 4. Time history and corresponding FFT of the controlled system in case of integer-order PPF and fractional-order PPF control both tuned at 38.5 Hz.

3.5. New gray models by fractional calculus in system modeling and prediction

Gray prediction is an important branch in the gray system theory using small amount of data, while gray models are successfully applied in system modeling and prediction by many previous investigations. As a new topic, the fractional gray system was proposed as a general system, and its fractional gray models received great attention, which could be considered with more freedom and flexibility. At present, fractional gray model research mainly focuses on the following two types: one is the fractional accumulation of discrete gray model and the other is the continuous fractional-order gray model. The fractional calculus can mine the system or information more precisely than the classical model. However, the parameters estimation and optimization processes are different from the classical model. In our papers, the modeling process and parameter estimation were discussed for the fractional accumulation gray model [59], interval fractional accumulation gray model [60] and fractional derivative gray model [61], while, the optimal and modified models were also given for the models [59,61]. The applicability and accuracy of fractional gray models were checked by number of internet users and electricity data, respectively. For details, see Refs. [59–61]. (Contributed by Dingyü Xue and Yang Yang, System modeling).

3.6. Toolboxes for fractional-order control systems

Dedicated MATLAB toolboxes in fractional calculus and control category are very important in the relevant research and engineering practice. A review on MATLAB functions and toolboxes is given in [63], where the commonly used toolboxes are CRONE (French abbreviation for Non Integer Order Robust Control) [64], N-integer [65], FOTF (fractional order transfer function) [62,66] and FOMCON (fractional-order modeling and control) [67]. The newly updated version of FOTF Toolbox fully supports multivariable fractional-order control systems [68], with the high-precision algorithms [62] for fractional order differential equations. (Contributed by Dingyü Xue, Numerical implementation).

4. Signal and image processing

4.1. A study on fractional calculus applications in image processing

Fractional calculus is a fast developing mathematical discipline (that is, calculus of derivatives and integrals of any arbitrary real or complex order) has increased extensive notoriety and significance amid for more than four decades, mostly because of its applications in various apparently different and broad fields of science and engineering. It does surely give a few potentially valuable tools for solving integral, differential and integro-differential equations. Employing fractional differential to image processing is a prospering subject branch under discourse [69–81]. Recently, fractional calculus has been significantly examined in computer vision [76,77]. The principle purpose behind this advancement is the desire that the utilization of this theory will prompt a considerably more exquisite and viable method to treat problems of blocky effect and detail information protection. Particularly, the fractional order total variation (FOTV) models assume a vital role for image restoration, super-resolution, in-painting, image segmentation and motion estimation, etc. They can ease

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the contention between staircase elimination and edge preservation by selecting the order of derivative appropriately. Additionally, the fractional- order derivative operator has a non-local behavior because the fractional-order derivative at a point relies upon the characteristics of the entire function and not just the values in the vicinity of the point, which is helpful to enhance the performance of texture preservation. The numerical outcomes in published works show that the fractional-order derivative performs well in eliminating the staircase effect and preserving textures [77].

It has been demonstrated in [78] that the fractional-order derivative fulfills the lateral inhibition principle of the biological visual system better than the integer-order derivative. Pu et al. [74] considered the kinetic physical meaning of the fractional-order derivative and demonstrated that fractional differential-based methods can protect the low-frequency contour features in those smooth areas, and non-linearly keep high-frequency marginal feature in those regions where gray-level changes significantly, and furthermore preserve texture details in those areas that gray-level does not change obviously.

It is noted in [82] that for low-frequency signal, fractional differential lessens the signal not as much as the integer one and for high-frequency one, fractional differential improves signal not as much as the integer one. Hence, we get the conclusion that fractional differential can upgrade the high-frequency signals, and reinforce the medium frequency one, while non-linear retain the low-frequency one. In the digital image, weak edges and texture details relate to low frequency parts, and noise and boundaries correspond to high-frequency ones. If the sign is handled by integer derivative, weak edge and texture tend to be enormously debilitated, and then noise will be reinforced immensely. Favorably, the fractional differential is appropriate to overcome this disadvantage that is, the noise will not be strengthened enormously and weak edge and texture will be retained nonlinearly. These advantages are being employed to preserve weak edges and texture, and oppose noise to some degree. (Contributed by Asmat Ullah, Image processing).

4.2. Application of the GPCF and DGIs for improving the resolution and quality of nanoimages

We apply the generalized Pearson correlation function (GPCF) [83] POLS [84] and discrete geometrical invariants (DGI) for improving the quality and sharpness of nanoimages in the range of resolution (10–1000) nm. The GPCF helps to compare one piece of image with another one and the procedure of reduction to three incident points [85] allows finding “hidden” self-similar objects. The DGI based on the generalization of the Pythagoras theorem obtained by Babenko [86] allows comparing two randomly taken parts of images with each other and finding distinct differences expressed in terms of the integer moments. The quantitative parameters determined by the DGIs of the second and fourth orders, correspondingly allow monitoring the dynamics/changings of the chosen image in time. It can be applied for a wide set of random curves (experimental measurements) that are needed to be compared in terms of a limited number of the integer moments. The treatment of available images confirms the generality of this combined approach for a wide set of digital images obtained by different scanning microscopes. (Contributed by Raoul R. Nigmatullin, A. S. Vorobye, Image processing).

4.3. NAFASS in action: intermediate fractal model for the fitting of complex systems data

We essentially modernize the NAFASS (Non-orthogonal Amplitude Frequency Analysis of the Smoothed Signals) approach suggested earlier [87,88]. The NAFASS opens an alternative way for creation of new fluctuation spectroscopy when the segment of the Fourier series can fit any random signal with trend. However, the dispersion spectrum of the Fourier series $\omega_0 \cdot k(\omega_0 = 2\pi/T) \Rightarrow \Omega_k(k = 0, 1, 2, \ldots, K-1)$ is replaced by the specific dispersion law Ω_k calculated by the original algorithm. It implies that any finite signal will have a compact amplitude-frequency response (AFR), where the number of the modes is much less in comparison with the number of data points ($K \ll N$). The NAFASS approach can be applicable for quantitative description of a wide set of random signals/fluctuations and allows one to compare them with each other based on one general platform. We combine also the NAFASS with generalized Pearson correlation function [83,89] that allows to apply this combination for analysis of signals having self-similar origin with their subsequent fitting. New possibilities of the extended NAFASS approach are tested by available data. We suppose that the NAFASS approach can be applicable for description of different nonlinear random signals containing the hidden beatings in radioelectronics and acoustics. (Contributed by Raoul R. Nigmatullin, A. Morozov, Signal processing).

5. Mechanics and dynamic systems

5.1. Long-term control for discrete fractional systems

Many engineering problems hold the feature of discrete time or space structures, for example, images, economy series, signals and so on. Some efforts have been dedicated to the applications of the continuous fractional calculus to these topics, and researchers mainly adopted the numerical discretization of the fractional calculus. But it can readily result in tedious information or numerical errors due to the memory effect. Discrete fractional calculus can avoid this and it is a straightforward tool for discrete time systems. Stability theory of fractional difference equations is given. Long-term control for fractional systems becomes possible. For details, see Refs. [90–93]. (Contributed by Dumitru Baleanu and Guo-Cheng Wu, Dynamic system).

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5.2. Nonlocal elasticity and fractional viscoelasticity models of nanostructures

Many modified continuum theories, such as Eringen's nonlocal elasticity, have been widely employed to examine the dynamic behavior of nanostructures in order to consider size effects. However, these models lack to account for damping effects that are also present at small scales. For this purpose, we introduced a modified nonlocal fractional viscoelastic constitutive equation considering the nonlocality in the space domain and fractional viscoelastic behavior in the time domain. Derived governing equations enable one to examine dynamic behaviors of a wide range of nanostructure-based systems solely by solving fractional order partial differential equations. In spite of the fact that numerical analysis of time responses of nanostructure systems can bring to some important conclusions about damping and size effects, it can neglect some crucial features of fractional derivative models that are visible only in the complex domain. Therefore, application of well-known methods from complex analysis together with integral transforms is an important step in the analysis of linear fractional order models of nanostructures. Obtained responses in the time domain and investigation of poles in the complex domain are necessary to gain some qualitative conclusions about the application of nonlocal elasticity and fractional viscoelasticity models in dynamics of nanostructures. For details, see [94,95]. (Contributed by Mihailo Lazarević and Milan Cajić, Structural mechanics).

5.3. Microflows of viscoelastic fluids with fractional constitutive relationships

Recently the microflows of viscoelastic fluids have been studied extensively due to their importance in microfluidic systems. However, the application of fractional constitutive models in microchannel flow is still in early stages. Considering the successful applications of fractional constitutive models in the description of viscoelastic materials, we develop the mechanics models to study the electroosmotic slip flows of viscoelastic fluids under the mixed influence of electroosmosis and pressure gradient forcings. Then, the analytical/semi-analytical solutions of the corresponding fractional differential equations are derived and the corresponding numerical methods, such as the finite difference algorithm, are also presented. Finally, the combined effects of the slip boundary conditions, fluid rheology, electroosmotic and pressure gradient forcings on the fluid velocity distribution and the flow rate are discussed with graphics. Our results may be useful for viscoelastic fluids in the prediction of the flow behavior in microchannels and benefit the design of microfluidic devices. For details, see Refs. [96,97]. (Contributed by Haitao Qi, Non-Newtonian fluid mechanics and microflow).

5.4. Unsteady flow towards subsurface drains

Glover–Dumm equation (GDE), which is the most practical mathematical model to simulate water table profile between two parallel drainpipes under unsteady flow conditions, was obtained by analytically solving Boussinesq equation (BE). However, many previous investigations demonstrated that the GDE was not able to describe accurately the water table profile due to the heterogeneity of porous medium and scale effect on hydraulic conductivity. Fractional derivatives, because of having non-locality property, can reduce the scale effects on the parameters and, consequently, better simulate the hydrogeological processes. Hereby a fractional BE (FBE) was proposed and analytically solved for one-dimensional unsteady flow towards parallel subsurface drains. The applicability and accuracy of the resultant solution, called fractional Glover–Dumm equation (FGDE), were examined using both laboratory and field data measured at an experimental farm in Abadan, Iran. For detailed, see [98,99]. (Contributed by Behrouz Mehdinejadiani, Hossein Jafari and Dumitru Baleanu, Fluid dynamics).

5.5. Constitutive relation of non-Newtonian fluids in shear flow

Many contributions have been devoted to exploring the transport properties of non-Newtonian fluids in shear flow based on the traditional non-Newtonian constitutive equation, owing to successful use of the Herschel–Bulkley model in engineering. Based on the definition of viscosity, non-Newtonian fluids can be divided into two groups, namely, time-dependent and time-independent non-Newtonian fluids. However, some problems remain controversial. One of the issues on constitutive relation lies in the inaccurate description of time-dependent continuous variation of viscosity under shear (thixotropy and anti-thixotropy). The reversible effect implies that the variation of inner structure possesses the history-dependent feature, which can be well characterized by a time-variant fractional non-Newtonian model [100]. The other problem concerning time-independent non-Newtonian fluids is that empirical models lack a unified constitutive description for most non-Newtonian fluids. To tackle this deficiency, a fractional constitutive equation was proposed to capture the observed growth of shear stress for various velocity gradients [101]. These works provide the initial theoretical framework partially. (Contributed by Xu Yang and Wen Chen, Fluid mechanics).

5.6. Gas transport in heterogeneous media

Gas transport in heterogeneous media has an important influence on oil-gas exploitation and development [102]. Therefore, injecting gas into oil or gas reservoirs can significantly reduce oil viscosity, mitigate atmospheric emissions and control climate change to enhance oil or gas recovery efficiency and protect the environment. However, it is well-known that the random motion in gas transport in natural reservoirs deviates from the normal Brownian motion whose scaling limit cannot be properly described by classical models (such as Darcy's law and the advection-dispersion equation) due to heterogeneity and complexity of the medium structure. Anomalous transport of gas exhibits obvious path- and history-dependent behaviors. Subsequently, the fractional derivative models have been applied to explain the time memory and space non-locality in gas transport [103]. Applicability of the fractional derivative models have been efficiently verified by employing a set of experimental data in the literature which are compared well with the results of numerical simulation and analytical solution [104]. (Contributed by HongGuang Sun and Ailian Chang, Fluid mechanics).

5.7. A fractional order network model for ZIKA

Zika is a fast spreading epidemic. Here, we introduce a fractional order network model for Zika. Literature has shown that in most cases it is asymptomatic, and hence it is difficult to control. This paper studies direct (sexual) contact. Equilibrium states have been derived. Their stability has been studied. Numerical simulations for the model are given [105]. (Contributed by H. Elsaka and E. Ahmed, Dynamics).

5.8. Vibration analysis of the beam/plate resting on viscoelastic soil foundation

The interactions between the beam/plate and soil foundation were investigated under the hypothesis that the foundation was elastic. However, recent researchers have witnessed that soil behaves as viscoelastic materials. There is still a long way to investigate the vibration behavior of the beam/plate resting on the viscoelastic foundation, especially considering the existence of the shear layer. Hence, the three-parameter Pasternak model was proposed to characterize the reaction of the foundation. The softer foundation was found to be more time-dependent. Due to the existence of a constrained boundary, obvious wall effect was observed. For details, see [106]. (Contributed by Wei Cai, Wen Chen and Wenxiang Xu, Geomechanics).

5.9. Fractional description of time-dependent mechanical property evolution in materials with strain softening behavior

Based on the idea of using the variable fractional order to characterize the mechanical property evolution, a fractional model with variable-order is presented to describe the time-dependent deformation process. The developed model is applied to analyze the constant strain rate tension and compression results including the strain softening of ductile metals and soils [107], the viscoelastic behavior of rubber, and the large deformation response of glassy polymer. It is shown that the model can accurately characterize the stress-strain relationship of the above phenomenon during the constant strain rate test using only three material parameters. Assuming that the variable fractional order obeys a linear function, our model provides a way to obtain the entire curve even if we only have a narrow range of experimental data. This finding could effectively help in understanding and predicting the time-dependent deformation of these materials. Furthermore, the dependence of the order function on strain can reasonably exhibit the mechanical property change over different regions during deformation processes. For rubber and glassy polymer, a physical explanation is introduced based on the microstructure evolution of molecular chains to realize the essential physical meaning of the fractional order. It is then concluded that the change of the mechanical properties due to the evolution of microstructure is vividly captured by the variation of fractional order in our model during the time-dependent deformation processes. For details, see [107]. (Contributed by Deshun Yin and Ruifan Meng, Material mechanics).

5.10. Fractional calculus in linear viscoelastic modeling

The classical viscoelastic models are consisted of parallel or series with elastic and viscous elements. The exponential material functions of these models encounter difficulties in characterizing the power-law phenomena, which are widely observed for various viscoelastic materials. Gemant justified the necessity of fractional differential operators to describe these phenomena for some viscoelastic fluids. Scott–Blair regarded the viscoelastic material as the intermediate state between elastic solid and viscous fluid and introduced the fractional derivative of strain to the constitutive equation, called the Scott–Blair model. The fractional viscoelastic model is validated to well predict the power- law phenomena. Thereafter, the fractional theory for linear viscoelasticity has been gradually improved by Rabotnov, Bagley, Caputo and Mainardi, et al. Fractional viscoelastic models have been widely used to describe the complex dynamics such as relaxation, oscillation, and wave for a variety of real materials. For details, see [108–110]. (Contributed by Xianglong Su, Engineering mechanics).

6. Biology

6.1. Fractional derivative models of diffusion in magnetic resonance imaging (MRI)

A common feature observed in diffusion-weighted MRI of the brain is anomalous diffusion. Hence, in white and gray matter, $S(b)$, the signal intensity decay is often characterized by a stretched exponential $S(b) = S_0 exp[-(bD_0)^{\alpha}]$, where b is the degree of diffusion-weighting, D_0 is the tissue water diffusion coefficient, and $0 < \alpha < 1$ [111]. Since normal, or Gaussian diffusion decays as a single exponential $(\alpha = 1)$, solutions to the anomalous diffusion problem proceed by neglecting important tissue compartments and components [112]. Fractional order models of diffusion capture this tissue complexity

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by incorporating fractional order time and space derivatives in the governing Bloch-Torrey equation [113]. The Caputo derivative solution can then be expressed in the form, $S(b) = S_0 E_\alpha[-(bD_f)^\alpha]$, where D_f is the fractional diffusion coefficient (mm^2/sec), and $E^\alpha[-x^\alpha]$, is the Mittag–Leffler function, which naturally exhibits multi-exponential $S(b)$ decay rates. As the fractional order α approaches 1 from below, the distribution of rates narrows and ultimately coalesces into a single exponential. This fractional order model captures the appearance of many exponential rates in both normal and diseased brain tissue [114,115]. (Contributed by Richard L. Magin, Bioengineering).

6.2. Abundant bursting patterns of a simple fractional Morris–Lecar neuron model

Neurons are believed to be key elements for signal processing in nervous systems, where the information is encoded, transmitted and decoded via firing activity of neurons. Bursting is the main mode of neuron activity alternating between quiescent state and repetitive spiking state. Neuron models, such as Hodgkin–Huxley model, Fitzhugh–Nagumo model, Hindmarsh–Rose model, Chay model and Morris–Lecar model, have been proposed for understanding the bursting patterns and complicated dynamics of nervous systems. Different models have different bursting patterns with special features. In order to characterize the memory effect and power law property of neuron membranes [116], a fractional-order Morris–Lecar neuron model is proposed. Using the bifurcation theory, numerical simulation shows that the new model exhibits not only the bursting patterns shown in the corresponding integer-order Morris–Lecar model, but also some bursting patterns that do not exist in the integer-order one but can be found in other common neuron models, such as the Chay neuron model and the Fitzhugh–Nagumo neuron model. Thus, the fractional Morris–Lecar model may help in understanding neuron activities, in efficient information processing, stimulus anticipation, as well as in frequency-independent phase shifts of oscillatory neuronal firing theoretically and experimentally. For details, see [117]. (Contributed by Zaihua Wang, Nonlinear Dynamics, Biology).

6.3. The HIV/TB coinfection severity in the presence of TB multi-drug resistant strains

Fractional order (FO) models have triggered a considerable amount of research in engineering, physics and biology. With respect to epidemiology, FO models fill the gap in the understanding of certain patterns, where the integer order models fail a full explanation. In this sense, a FO model is introduced for the coinfection of HIV and TB, in the presence of MDR-TB strains and treatment. The coinfection increases the severity of the disease and poses a significant threat to the public health care system. Coinfection is responsible for more infectious individuals, who are more prone to spread the epidemics. Moreover, the MDR-TB strains transmission, together with HIV infection, constitutes a major challenge for treatment, which requires anti- tuberculosis and ART to be administered unitedly. Altogether, the coinfection burden increases concerns of an extreme difficulty in TB control and elimination worldwide, and jeopardizes the ending of AIDS epidemic. The fractional derivative order, is a significant player in the epidemics theater. It may distinguish between individuals' immune system, age, treatment compliance, and other co-morbidities. The FO model may provide more "freedom" to adjust the model to real data from specific patients. For details, see [118,119]. (Contributed by Carlo Pinto, Epidemiology).

6.4. Fractional thermal wave model in spherical composite medium

Recently, many studies have shown that fractional calculus is very useful in the area of biorheology. In the work of Yu et al. [120], a fractional thermal wave model for the bi-layered spherical biological tissue during the hyperthermia treatments was set up. Implicit numerical method was constructed to solve the proposed fractional model. In the inverse analysis process, an efficient numerical method was proposed for simultaneously estimating multiple unknown fractional parameters. Based on the hyperthermia experimental data, the estimations of the order of the Caputo fractional derivative and the relaxation time parameters were obtained. By comparisons, one can obviously observe that the estimated temperature increase values agreed well with the measured temperature increase values in the experiment. The results demonstrated that the proposed fractional thermal wave model was efficient and accurate in modeling the heat transfer of the biological tissue during the hyperthermia treatments, and the proposed numerical method for simultaneously estimating multiple fractional parameters is effective. (Contributed by Bo Yu, Xiaoyan Jiang and Chu Wang, Biocenology).

6.5. Models of bone remodeling and bone tumors using variable order derivatives

Bone tissue is not static. Like every other part of our body, its cells are always dying and being replaced. The main actors of this process are the cells destroying bone tissue, called osteoclasts, and the cells that build bone back, called osteoblasts. The presence of osteoblasts influences the rate of increase of osteoclasts and the number of osteoclasts also influences their own evolution. The changes in dynamic behavior when there is a tumor can be modeled by tuning the parameters of autocrine and paracrine effects. Models found in the literature include intricate mathematical expressions for such variations. Our research has shown that the same effect can be obtained merely changing the order of the time derivative in the partial differential equations that model the involved diffusion phenomena. We studied the dynamic behavior of the resulting variable order partial differential equations and found in accord with the known qualitative behavior of healthy and tumorous bone remodeling. For details, see [121,122]. (Contributed by Duarte Valério, Susana Vinga, and Joana Neto, Bioengineering).

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7. Environmental science

7.1. Chloride ion anomalous diffusion in concrete structures

Chloride ion erosion is one of the main reasons to affect the durability of concrete structures, and the core issue in research is the chloride ion transport mechanism analysis and modeling. As a typical porous material, concrete is uneven and anisotropic, and hence the ideal Fick's law of diffusion is not applicable to describe the chloride ion diffusion behavior in concrete any more. In addition, due to the continuous hydration of cement binder, the geometrical, physical and chemical properties of concrete change over time, thus the chloride ion diffusion in concrete should also be time-dependent. The chloride binding effect of concrete makes the diffusion process exhibit some concentration dependence. Therefore, the traditional Fick's law cannot reflect these anomalous diffusion characteristics in such complex media. Fractional derivative is well known for featuring the non-local characteristics of complex systems, e.g., the temporal dependence and the spatial correlation. This type of derivative is often introduced to the differential equation models to describe the particle anomalous diffusion in complex media. Based on the transport characteristics and considering the advantage of the fractional derivative, several fractional derivative models for describing chloride ion anomalous diffusion were established, and checked against the field data. For details, see [123,124]. (Contributed by Wen Chen, Jianjun Zhang and Song Wei, Concrete corrosion).

7.2. Simulation of solute transport through porous media

The traditional advection-dispersion equation (ADE), which is built for the process of Brownian motion, has been widely used to simulate solute transport in porous media. The numerous laboratory and field studies demonstrated that the ADE cannot well describe the solute transport process in porous media, especially in the heterogeneous ones, due to the scale-dependency of dispersion coefficient. According to laboratory and field studies, the solute transport process in the porous media was found to be a spatial nonlocal process. Hereby a spatial fractional ADE (FADE) was proposed to describe the non-local transport process of solute in the porous media and overcome the drawbacks of ADE. The applicability and accuracy of FADE were studied under both laboratory and field conditions. For details Refs [125–127]. (Contributed by Behrouz Mehdinejadiani, Environmental science).

7.3. Water flow across the earth surface: apply the fractional calculus to interpret the hydrology cycle

Hydrologic cycle in the Earth system involves a wide range of flow processes within and across natural geologic media (i.e., overland, open channel, soil, and aquifers) with multi-scale heterogeneity, challenging the standard modeling approaches and providing an ideal testbed for the application of fractional partial differential equations (PDE) in stochastic hydrology. One example is the hillslope subsurface stormflow which exhibits complex flow patterns when natural soils with multiscale heterogeneity impart a spatiotemporally nonlocal memory on flow dynamics. To reliably capture real-world stormflow through various slopes using the well-known Dupuit–Forchheimer (D–F) equation, local variations in soil property, slope geometry, and hydraulic conditions must be mapped for details, resulting in a strongly nonlinear PDE with prohibitive computational burden. A fully subordinated linear flow model was then proposed, using fractional calculus and generalizing the D-F equation, to efficiently capture the impact of both preferential flow paths and low-conductivity zones on flow response due to system heterogeneity, without the burden to map detailed medium properties. The fractional PDEs can also be applied to quantify other flow processes in the hydrologic cycle, including overland surface runoff. For details, see Refs. [128,129]. (Contributed by Yong Zhang and HongGuang Sun, Hydrology).

7.4. Application of fractional derivative model to sediment bed-load transport

In recent years, it has been found that anomalous diffusion exists in the process of bed-load transport. Bed sediment transport in rivers is scale dependent, with anomalous and Fickian scaling dominate at different scales. However, it is difficult to capture such phenomena using traditional empirical models. On this basis, various models based on fractional derivatives are proposed. The definition of fractional derivative includes memory property that can well capture the time non-locality and the spatial non-locality of bed-load transport. Such models explain the inherent mechanism of anomalous diffusion of bed-load transport by assuming the heavy tailed distribution of the waiting time and jump step. A series of experiments verify the accuracy of the fractional derivative model for capturing anomalous diffusion at the laboratory scale. For details, see [130,131]. (Contributed by ZhiPeng Li and HongGuang Sun, Sediment transport).

7.5. Variable-order fractional-derivative model to describe transient dispersion in heterogeneous media

Many numerical experiments and field observations of solute transport indicate that the growth of contaminant plumes may not exhibit a constant scaling through heterogeneous porous and fractured media, but can rather transition between diffusive states at various transport scales, and the diffusion process changes with time and space [132]. The transition is usually attributed to physical properties of the medium, e.g. spatial variations in medium heterogeneity. The solute transport equation model based on Fick's law is difficult to characterize the dynamic process accurately. Hereby a variable order fractional derivative model was proposed to describe transient dispersion from sub-diffusion to super-diffusion [133,134]. The variable order fractional model can well characterize above transition, with the scale parameter being a linear function of time or space. The applicability and accuracy of VFDM were checked against both numerical theory and a set of published experimental data. (Contributed by Shiqian Nie and HongGuang Sun, Solute transport).

7.6. Chemical reactions in underground water

The chemical reactions in aquifers are not isolated from the surrounding systems but related to the water dynamics in subsurface. Many previous investigations show the dispersion of contaminants does not display Fickian scaling. It is worth a try to apply anomalous transport. Fractional dispersion equations are proposed to solve the problem while the traditional advection–diffusion equation could not describe the transport process accurately. Thus, the fractional reaction–diffusion equation was presented based on the anomalous transport process of the reactive contaminants. Meanwhile, laboratory experiments show that the tracer breakthrough curves exhibit subdiffusive behavior with a heavy tail, supporting the use of power-law memory function in time. One could notice that on one hand, the memory has a great effect on the evolution of reactants; on the other hand, the history of the reaction also affects the memory. Thus, it is necessary to propose and develop fractional reaction-diffusion equations that are consistent with the system. For details, see [135,136]. (Contributed by HongGuang Sun and XiaoTing Liu, Environmental chemistry).

8. Materials

8.1. Fractional derivative model for shape memory polymers

A shape-memory polymer (SMP) is a polymeric material that is capable of memorizing its original shape, and can acquire a temporary shape upon deformation and returns to its permanent shape in response to an external stimulus such as a temperature change. SMPs have been widely used from industrial to medical applications and even everyday life [137,138].

Since the properties of SMPs are temperature dependent and often very sensitive to an external temperature change, their accurate modeling has been a very challenging issue. The previously developed integer-order differential equation models often have a very complicated form and typically contain a large number of parameters to be determined. In recent years, fractional differential equation (FDE) models have been used to model these problems, and have shown to be capable of describing complex viscoelastic behaviors using only a few parameters.

However, SMPs can have significant changes of their shapes depending on whether an external stimulus temperature change exceeds their prescribed temperature, which in turn have significant impact on their microscopic network structure. In the process the temperature can change significantly which in turn has significant impact on the physical properties of the SMP materials. Consequently, the constant-order fractional differential equation model cannot fully model the entire process well. Li et al. [139] accordingly proposed a data-driven variable-order FDE model, which was shown to better describe the shape-memory behaviors of amorphous polymers than its constant-order analogue. (Contributed by Hong Wang, Shape-memory polymer).

8.2. Fractional viscoelastic-plastic constitutive model

In the recent works, we have developed several fractional viscoelastic-plastic models to describe the thermomechanical behaviors of amorphous polymers. Specifically, a fractional viscoelastic model is developed for amorphous thermoplastics with two parallel fractional Maxwell elements, which aims to describe the glass transition and viscous flow, respectively. The model is able to describe the stress relaxation, dynamic properties and stress response at various temperatures and strain rates. We also develop a 3D finite deformation fractional viscoplastic model, which is an extension of the fractional Zener model. The Eyring model is adopted for stress activated viscous flow. The model is able to describe the stress response of amorphous thermosets across the glass transition. For details, see [140–142]. (Contributed by Rui Xiao, Viscoelastic-plastic materials).

9. Economic

9.1. Basic concepts of economic processes with memory

All previous investigations on the economic processes with memory were considered within the discrete-time approach. In economics the fractional differencing and integrating have been suggested in the works of Granger and Joyeux, and Hosking, using the discrete time approach only. These fractional differencing and integrating are used in economics without direct connection with the fractional calculus and the well-known finite differences of non-integer orders. We demonstrate that the fractional differencing and integrating, which are used in economic papers, are the well-known Grunwald–Letnikov fractional differences, which have been suggested one hundred and fifty years ago. Recently the fractional calculus has been applied to the continuous-time finance. These papers consider only the financial processes. The basic economic concepts for economic processes with memory are not considered. We consider economic processes with power-law fading memory in the framework of the continuous time approach. To describe the economic processes with power-law memory, we generalize the basic concepts of economic theory. Using the fractional calculus to describe the power-law memory, we proposed generalizations of some basic economic notions, such as the elasticity of fractional order, the accelerator with memory, the marginal value of non-integer by employing the fractional calculus as a powerful tool to describe the power-law memory. We have suggested the marginal value of non-integer order, the elasticity and measures of risk aversion, the concepts of accelerator and multiplier with memory for power-law memory processes and deterministic factor analysis. For details, see [143–145] and related references. (Contributed by Valentina V. Tarasova and Vasily E. Tarasov, Mathematical economics).

9.2. Macroeconomic models with dynamic memory

A generalization of the basic macroeconomic concepts has been proposed within the continuous time approach for economic processes with memory. A discrete-time accelerator for economic processes with the power-law memory also has been suggested for the case of periodic sharp splashes (kicks). Using the concepts of accelerator and multiplier with memory, we generalize the macroeconomic models by taking account the dynamic memory with power-law fading. The need to take into account memory effects in macroeconomics is based on the fact that economic agents remember the history of changes of economic processes. These changes can then be taken into account when making economic decisions, which changes the behavior of the agent. We proposed the generalization of the following macroeconomic models: the natural growth model, the Harrod–Domar model, the Keynes model, growth model with constant pace, logistic growth model, the dynamic intersectoral models, the Leontief (input Coutput) model and time-dependent dynamic economic models. The proposed economic growth models with power-law memory have shown that the memory effects can play an important role in economic phenomena and processes. For details, see [146–148] and related references. (Contributed by Valentina V. Tarasova and Vasily E. Tarasov, Mathematical economics).

10. Multidisciplinary in engineering fields

10.1. Anomalous dielectric properties

In Maxwell's equations, which govern the propagation of electromagnetic waves, the interaction between polarization and electric fields is described by the complex susceptibility. This is an empirical law derived by matching experimental data in some mathematical model. After the simpler Debye model, more involved models have been proposed [149]. In the Havriliak–Negami (HN) model, two real powers are introduced to fit the anomalous dielectric properties observed in disordered materials and heterogeneous systems; the normalized HN frequency-domain susceptibility is

$$\hat{\chi}_{\mathrm{HN}}(\mathrm{i}\omega) = \frac{1}{\left(1 + \left(\mathrm{i}\tau_\star\omega\right)^{\alpha}\right)^{\gamma}}, \quad 0 < \alpha \le 1, \quad 0 < \gamma \le 1,$$

with $\tau_\star$ the relaxation time. In the time-domain, the HN susceptibility can be described by pseudo-fractional differential operators obtained by inversion of the so-called Prabhakar integral [150]

$$\left({}_0J_t^{\alpha} + \tau_\star^{-\alpha}\right)^{\gamma} f(t) = \int_0^t (t-u)^{\alpha\gamma-1} E^{\gamma}_{\alpha,\alpha\gamma}\left(-(t-u)^{\alpha}\tau_\star^{-\alpha}\right) f(u)\,\mathrm{d}u, \qquad (1)$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is the three parameter Mittag–Leffler function, usually known as the Prabhakar function [151].

The use of fractional-order operators like (1) allows a more accurate investigation and simulation of electromagnetic fields in materials with anomalous dielectric properties; in the particular case $\gamma = 1$ (Cole-Cole model) standard fractional differential equations are involved. (Contributed by Roberto Garrappa, Anomalous dielectric materials).

10.2. Computation of supercapacitors parameters using fractional-order electrical modeling

Supercapacitors are electrochemical energy storage devices known for their high power performance, excellent reversibility, long-term cyclability, low maintenance, and ease of integration into electronic systems. Because of their nano-architected electrodes material and structure, and their electrochemical design, the spectral impedance of supercapacitors shows a clear deviation from the -90^{o} phase angle of an ideal capacitor [152–154]. Nonetheless, the evaluation of their electric behavior is usually described using classical conventional capacitors formulae. Supercapacitors have been modeled as the collection of many discrete resistive and capacitive elements representing the distribution of time constants in the device similar to a transmission-line. This is, however, rather an artificial view of the way these devices operate and is not entirely satisfactory when fitting experimental data. The constant phase element (CPE) model of impedance ($Z = 1/jQ\omega^{\alpha}$) and its fractional-order time-domain counterpart ($i_Q(t) = Qd^{\alpha}v_Q/dt^{\alpha}$) where Q is the CPE parameter and α ($0 < \alpha \le 1$) is the CPE fractional exponent, is another approach that reduces the number of variables while at the same time exhibit excellent goodness-of-fit [155]. With CPE modeling, we derived in [153,154] equations that estimate the effective capacitance and energy stored in super-capacitors which are very important for the successful deployment of these devices in their ever growing applications. (Contributed by Ahmed S. Elwakil (Electrical Engineering) and Anis Allagui (Materials Electrochemistry) and Todd Freeborn (Electrical engineering)).

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10.3. Anomalous diffusive transport in heterogeneously distributed nano-scale digital rock

Shale gas has become an important energy resource in the world. However, the mechanism in its recovery is far from being understood, because shale formation often has insufficient permeability due to the existence of nano-pores. Consequently, fluid flows in confined nano-scale heterogeneous structures exhibit physical behaviors that are not observed in large-scale structures. Molecular dynamics (MD) has proven to be a rigorous approach for modeling fluid flow in nano-scale materials, but is computationally very expensive and often intractable in applications. An example run of an MD simulation of a diffusive transport in a digital rock of a size of 25 nm^3over a time period of 3 ns on a cluster with 32 processors takes about 2 weeks of CPU times. Zhao et al. [159] developed an integrated fractional partial differential equation (FPDE)-MD upscale modeling of anomalous diffusive transport in heterogeneously distributed nano-scale digital rock. The reason is that for a heterogeneous porous medium with confined pore spaces, large quantity of gas molecules may get absorbed to the micropores in rock [157,158]. Thus, the travel time of the adsorbed gas molecules may deviate from that of the gas molecules in the bulk phase [160], leading to a subdiffusive transport [156]. The MD simulation is used to generate the diffusivity of the pores. Representative numerical experiments show that a simulation of diffusive transport in a digital rock of a size of $9 \times 10^4\,\mathrm{nm}^3$ over a time period of $1.8\mu s$ on a laptop takes about several hours of CPU time, leading to an improvement of computational efficiency of millions of times than the MD simulations. (Contributed by Hong Wang, Energy).

10.4. Evolution equation with fractional Laplacian: modeling, analyzing, and computing

In general, if the diffusion in $\mathbb{R}^n$ obeys Fick's law, then the classical Laplace operator (or Laplacian for simplicity)

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

can accurately characterize such diffusion. But if the diffusion in $\mathbb{R}^n$ obeys a power-law distribution rather than Fick's law, such anomalous diffusion can be genuinely reflected by the fractional Laplacian, which is defined below,

$$(-\Delta)^s v(x') = C_{n,s} \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{v(x') - v(\xi)}{|x' - \xi|^{n+2s}} \mathrm{d}\xi,$$

where the parameter $s \in (0, 1)$, $C_{n,s}$ is a normalizing constant and P.V. stands for the Cauchy principle value.

According to Caffarelli–Silvestre extension technique [161], the following elliptic equation with fractional Laplacian

$$\begin{cases} (-\Delta)^s v = g, & \text{in } \Omega, \\ v = 0, & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{2}$$

can be lifted to a mixed boundary value equation as follows,

$$\begin{cases} \nabla \cdot (z^\beta \nabla w) = 0, & \text{in } \Omega \times \{z | z \in \mathbb{R}^+\}, \\ \frac{\partial w}{\partial z^\beta} = d_s g, & \text{on } \Omega \times \{z = 0\}, \\ w = 0, & \text{on } \partial\Omega \times [0, \infty). \end{cases} \tag{3}$$

So $v(\mathrm{x}) = \lim_{z \to 0^+} w(\mathrm{x}, z)$. The main advantage of the extension described above is that it enables us to solve the local equation (3) instead of dealing with the nonlocal operator $(-\Delta)^s$ in Eq. (2). In other words, it restricts the volume constrained data to boundary data directly.

How to model anomalous diffusion and/or spatial heterogeneity in $\mathbb{R}^n$ using fractional Laplacian, and how to characterize history dependance in time using Caputo derivative are likely the key problems in fractional modeling. Once the mathematical models are available, the next step is to determine the existence, uniqueness, and regularity of the solution to the established mathematical equation. Solving these equations is by no means a facile problem due to the fractional operators. Similar to the integer-order partial differential equations, we can choose typical numerical methods such as finite difference methods and finite element methods [162] to solve them. Although there have existed a few works, for example [163,164] and limited references cited therein, there are still lots of unsolved problems [165]. In the end, we need to check whether or not the established mathematical models really and truly reflect the dynamical behaviors of real world. (Contributed by Changpin Li, Interdisciplinary).

Uncited reference

Q5 [3].

Acknowledgments

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