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Asymptotical stability of fractional order systems with time delay via an integral inequality

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Abstract: In this paper, the asymptotical stability for several classes of fractional order differential systems with time delay is investigated. We firstly present an integral inequality by which the Halanay inequality is extended to fractional order case. Based on the generalized Halanay inequality, we establish several asymptotical stability conditions under which the fractional order systems with time delay are asymptotically stable. It is worth to note that these stability conditions are easy to check without resorting to the solution expression of the systems.

1 Introduction

Fractional calculus, more precisely, arbitrary order calculus, appears almost at the same time with the classical calculus. But it did not attract a great attention due to the lack of application background and the complexity of computation, until the last few decades when researchers found that there are many anomalies that can not be explained by classical calculus while fractional calculus is a useful tool to describe some abnormal diffusion process because of its memory and hereditary properties, then it is widely used in many fields such as blood flow phenomena, electrochemical processes, viscoelastic materials, etc.[1–3]. In 1991, Oustaloup proposes a CRONE control, and in 1999, Podlubny[4] introduces the $PI^{\lambda}D^{\mu}$ controller, fractional calculus has been used in control theory, and the researchers found fractional order controllers provide superior performances in both theory and practice, and many results are obtained, see[5–8].

Asymptotical stability analysis of fractional order systems, one of the fundamental issues in control theory, is to find some stability conditions under which systems are asymptotically stable. For linear fractional order systems, the first stability result is the well-known Matignon theorem [9]. This theorem enables us to determinate the stability of linear fractional order systems through the location in the complex plane of the dynamic matrix eigenvalues of the state. Recently, many mathematical tools have been applied to analyze the asymptotical stability in fractional order linear systems. These include LMI approach [10], the Laplace transform method [11], the Lyapunov method [12] and the Riesz basis approach and the semigroup method [13]. In [14], it introduces the characteristic equation of the system to give some simple sufficient asymptotical stability conditions for interval linear fractional order neutral system with time delay. Laplace transform method is developed in [11], where a n-dimensional linear fractional order differential equation with multiple time delays is addressed and several sufficient conditions of globally asymptotically stable are exploited.

For nonlinear fractional order systems, the Lyapunov method is a classical approach to deal with the stability problem. The key advantage of applying the Lyapunov method in obtaining the stability criteria for a given system is that it does not need to solve the system to obtain the explicit solution expression and the need is just the system structure. However, due to the memory effect and the weakly singular kernels of the fractional order derivative, the fractional Lyapunov method which is different from the classical Lyapunov method was not developed until 2009 in [12] and 2010 in [15], and the applicability of the fractional Lyapunov method was not available until 2014 in [16]. The Mittag-Leffler stability concept is introduced in [12] and this kind of stability implies the asymptotical stability. In [16], a simple but useful fractional order differential inequality is established, which proves the quadratic function, x^2, to be a good Lyapunov candidate function. Very recently, a class of convex and positive definite function as Lyapunov candidate function is developed in [17], which makes the fractional Lyapunov theorem more applicable. In [18], an indirect approach to obtain the stability for nonlinear fractional order systems is established by using the relation of the stability between the fractional order systems and the corresponding integer order systems.

For the stability of fractional order system with time delay (FSTD), Mittag-Leffler stability is addressed in [19]. In [20, 21], frequency domain method is used to obtain the asymptotical stability results. Razumikhin method was generalized to fractional order systems with delay in [22]. However, to the best of our knowledge, there are few easily verifiable asymptotical stability criteria for FSTD. In this paper, we are concerned with the asymptotical stability for FSTD via a new integral inequality. Based on this inequality, we generalize the Halanay inequality proposed by Aristide Halanay in [23]. With these inequalities and the result of [16], we establish some stability criteria that are easy to check.

The paper is organized as follows: Section 2 presented some basic concepts and lemmas about fractional calculus. A new integral inequality and a generalized Halanay inequality are introduced in Section 3. Section 4 is devoted to the asymptotical stability for several kinds of fractional order systems with time delay. Section 5 is a conclusion about the paper.

2 Preliminary

In this section, some basic definitions and preliminaries are given which are useful throughout this paper.

Definition 1. *[4, Page 79] The Caputo's fractional derivative of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by*

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\theta)}{(t-\theta)^{\alpha-m+1}} d\theta, \; t > 0 $$

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with $m = \min\{k \in \mathbb{N} : k > \alpha > 0\}$, where $f^{(m)}(t)$ is the m-order derivative of $f(t)$, and $\Gamma(\cdot)$ is the Gamma function.

In particular, when $0 < \alpha < 1$, we have

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\theta)}{(t-\theta)^\alpha} d\theta. $$

Definition 2. *[4, Page 16] The one-parameter Mittag-Leffler function and two-parameter Mittag-Leffler function are defined by*

$$ E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+1)} \text{ and } E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} $$

respectively, where $\alpha > 0, \beta > 0$.

Let $h > 0$ be a constant, we suppose that $x \in C([-h, +\infty); \mathbb{R}^n)$. For any $t \geq 0$, we denote by x_t an element of $C([-h, 0]; \mathbb{R}^n)$ defined by $x_t(\sigma) = x(t+\sigma), -h \leq \sigma \leq 0$. Consider the fractional order system with time delay.

$$ \begin{aligned} &{}_0^C D_t^\alpha x(t) = f(t, x_t), && t > 0, \\ &x(t) = \phi(t), && t \in [-h, 0], \end{aligned} \tag{1} $$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^+ \times C([-h, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is a nonlinear functional vector satisfies $f(0, 0) = 0$, ϕ is the continuous vector valued history function and $\|\phi\| = \sup_{\sigma\in[-h,0]} |\phi(\sigma)|$, where $|\cdot|$ denotes the Euclidean norm for vectors.

Definition 3. *[24, Page 4] For system (1), the trivial solution is said to be:*

- stable, *if for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\phi\| < \delta$ implies $|x(t)| < \varepsilon$ for all $t \geq 0$.*
- asymptotically stable, *if it is stable and if in addition there exists a $\delta > 0$ such that if $\|\phi\| < \delta$, then $x(t) \to 0$ as $t \to +\infty$.*
- unstable, *if it is not stable.*

Lemma 1. *[16] Let $x(t) \in \mathbb{R}$ be a continuous and differentiable function, then, for any $t \geq 0$,*

$$ {}_0^C D_t^\alpha x^2(t) \leq 2x(t) {}_0^C D_t^\alpha x(t), \quad \forall \alpha \in (0, 1). $$

If $x(t) \in \mathbb{R}^n$, it holds that for $\forall \alpha \in (0, 1)$ and $t \geq t_0$,

$$ {}_0^C D_t^\alpha (x^\top(t)x(t)) \leq 2x^\top(t) {}_0^C D_t^\alpha x(t). $$

Lemma 2. *The Cauchy problem for fractional order time-delay system of order $0 < \alpha < 1$:*

$$ \begin{aligned} &{}_0^C D_t^\alpha x(t) = \lambda x(t) + f(t, x_t), && t \geq 0, \\ &x(t) = \phi(t), && t \in [-h, 0], \end{aligned} \tag{2} $$

is equivalent to the following integral equation:

$$ \begin{aligned} x(t) &= E_\alpha(\lambda t^\alpha)\phi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \\ &\qquad \times f(s, x_s)ds, \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{3} $$

where $\lambda \in \mathbb{R}$ and $f : \mathbb{R}^+ \times C([-h, 0]; \mathbb{R}) \to \mathbb{R}$ is a continuous function.

Proof: Taking the Laplace transform on both side of (2) gives

$$ s^\alpha \hat{x}(s) - s^{\alpha-1}\phi(0) = \lambda \hat{x}(s) + \hat{f}(s, x_s), \tag{4} $$

where

$$ \hat{x}(s) = \int_0^{+\infty} e^{-st} x(t)dt, \;\; \hat{f}(s, x_s) = \int_0^{+\infty} e^{-st} f(t, x_t)dt, $$

are respectively, the Laplace transform of functions $x(t), f(t, x_t)$. Therefore,

$$ \hat{x}(s) = \frac{s^{\alpha-1}\phi(0) + \hat{f}(s, x_s)}{s^\alpha + \lambda}. \tag{5} $$

Taking the inverse Laplace transform on both side of (5) yields

$$ x(t) = E_\alpha(-\lambda t^\alpha)\phi(0) + f(t, x_t) * t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), $$

where $*$ denotes the convolution operator, that is, for $t \geq 0$,

$$ \begin{aligned} x(t) = E_\alpha(-\lambda t^\alpha)\phi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \\ \times f(s, x_s)ds. \end{aligned} $$

For integral equation (3), it equals to

$$ x(t) = E_\alpha(-\lambda t^\alpha)\phi(0) + f(t, x_t) * t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), t \geq 0. \tag{6} $$

Taking Laplace transform on both side of (6) and then taking the inverse Laplace transform, we can transfer (3) to equation (2). This ends the proof. □

3 An integral inequality

In this section, we establish an integral inequality which can be used to deal with the asymptotical stability for fractional order systems with time delay. The proof is based on "inf − sup" method.

Theorem 1. *Let $\phi : [-h, +\infty) \to \mathbb{R}^+$ be bounded on $[-h, 0]$ and continuous on $[0, +\infty)$. Suppose that $a, K : \mathbb{R}^+ \to \mathbb{R}$ are two continuous functions which satisfy $\lim_{t\to+\infty} a(t) = 0$, $K(t) \geq 0$, $\lim_{t\to+\infty} K(t) = 0$ and $K \in L^1(\mathbb{R}^+)$. $\mu > 0$ is a constant and the following inequality holds:*

$$ \phi(t) \leq a(t) + \mu \int_0^t K(t-s) \sup_{s-h\leq\sigma\leq s} \phi(\sigma)ds, \; t \geq 0. \tag{7} $$

If $\mu\|K\|_{L^1(\mathbb{R}^+)} < 1$, then, $\lim_{t\to+\infty} \phi(t) = 0$.

Proof: This proof is divided into two steps.

Step 1: We show that $\phi(t)$ is bounded for all $t \geq -h$. Indeed, since for any $s \in [0, t]$, $[s-h, s] \subset [-h, t]$, then $\sup_{s-h\leq\sigma\leq s} \phi(\sigma) \leq \sup_{\sigma\in[-h,t]} \phi(\sigma)$, by (7), we obtain

$$ \begin{aligned} \phi(t) &\leq a(t) + \sup_{\sigma\in[-h,t]} \phi(\sigma)\mu \int_0^t K(t-s)ds \\ &= a(t) + \sup_{\sigma\in[-h,t]} \phi(\sigma)\mu \int_0^t K(s)ds \\ &\leq a(t) + \sup_{\sigma\in[-h,t]} \phi(\sigma)\mu\|K\|_{L^1(\mathbb{R}^+)}, \end{aligned} \tag{8} $$

which, together with the boundedness of $\sup_{s\geq0} a(s)$, gives

$$ \sup_{\sigma\in[0,t]} \phi(\sigma) \leq \sup_{s\geq0} a(s) + \sup_{\sigma\in[-h,t]} \phi(\sigma)\mu\|K\|_{L^1(\mathbb{R}^+)}. $$

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Noting that

$$\sup_{\sigma\in[-h,t]}\phi(\sigma)\le\sup_{\sigma\in[0,t]}\phi(\sigma)+\sup_{\sigma\in[-h,0]}\phi(\sigma),$$

we get

$$\sup_{\sigma\in[-h,t]}\phi(\sigma)\le\sup_{s\ge0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)+\sup_{\sigma\in[-h,t]}\phi(\sigma)\mu\|K\|_{L^1(\mathbb{R}^+)}. \tag{9}$$

Since $\mu\|K\|_{L^1(\mathbb{R}^+)}<1$, it follows from (9) that

$$\sup_{\sigma\in[-h,t]}\phi(\sigma)\le\frac{\sup_{s\ge0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}},\ t\ge0, \tag{10}$$

that is, $\phi(t)$ is bounded on $[-h,+\infty)$.

Step 2. We show that $\lim_{t\to+\infty}\phi(t)=0$. Denote $\overline{\phi}(t):=\sup_{s\ge t}\phi(s)$. Since $\phi(t)$ is nonnegative and $\phi(t)$ is bounded on $[-h,+\infty)$ just proved in step 1, $\overline{\phi}(t)$ is well-defined and is non-increasing with respect to t, which implies that $\inf_{t\ge0}\overline{\phi}(t)$ exists. Moreover, $\inf_{t\ge0}\overline{\phi}(t)=\lim_{t\to+\infty}\overline{\phi}(t)$. Thus, to show $\lim_{t\to+\infty}\phi(t)=0$, it suffices to prove $\inf_{t\ge0}\overline{\phi}(t)=0$. Indeed, for any given $\varepsilon>0$, there exists $T\ge0$ such that $\overline{\phi}(t)\le\overline{\phi}(T)\le\inf_{t\ge0}\overline{\phi}(t)+\varepsilon$ for all $t\ge T$. It follows from (7) and (10) that

$$\begin{aligned}&\phi(t)\\&\le a(t)+\mu\int_0^t K(t-s)\overline{\phi}(s-h)ds\\&\le a(t)+\mu\int_0^{T+h}K(t-s)\overline{\phi}(s-h)ds\\&\quad+\mu\int_{T+h}^t K(t-s)\overline{\phi}(s-h)ds\\&\le a(t)+\frac{\sup_{s\ge0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\mu\int_0^{T+h}K(t-s)ds\\&\quad+\mu[\inf_{t\ge0}\overline{\phi}(t)+\varepsilon]\int_{T+h}^t K(t-s)ds\\&\le a(t)+\frac{\sup_{s\ge0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\mu\int_0^{T+h}K(t-s)ds\\&\quad+[\inf_{t\ge0}\overline{\phi}(t)+\varepsilon]\mu\|K\|_{L^1(\mathbb{R}^+)}.\end{aligned} \tag{11}$$

By the definition of $\overline{\phi}(t)$ and (11), we get for $t\ge T+h$,

$$\begin{aligned}&\overline{\phi}(t)\\&\le\sup_{s\ge t}\Big[a(s)+\frac{\sup_{s\ge0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\\&\quad\times\mu\int_0^{T+h}K(s-\sigma)d\sigma\Big]+[\inf_{t\ge0}\overline{\phi}(t)+\varepsilon]\mu\|K\|_{L^1(\mathbb{R}^+)}.\end{aligned} \tag{12}$$

From $\lim_{t\to+\infty}a(t)=0$ and $\lim_{t\to+\infty}K(t)=0$, we have

$$\begin{aligned}&\lim_{t\to+\infty}\sup_{s\ge t}\Big[a(s)+\frac{\sup_{s\ge0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\\&\quad\times\mu\int_0^{T+h}K(s-\sigma)d\sigma\Big]\\&\le\lim_{t\to+\infty}\sup_{s\ge t}a(s)+\frac{\sup_{s\ge0}a(s)+\sup_{\sigma\in[-h,0]}\phi(\sigma)}{1-\mu\|K\|_{L^1(\mathbb{R}^+)}}\\&\quad\times\mu\lim_{t\to+\infty}\sup_{s\ge t}\int_0^{T+h}K(s-\sigma)d\sigma=0.\end{aligned} \tag{13}$$

Taking the limitation on both side of (12) and noting that $\inf_{t\ge0}\overline{\phi}(t)=\lim_{t\to+\infty}\overline{\phi}(t)$, by (13), we derive

$$\inf_{t\ge0}\overline{\phi}(t)\le[\inf_{t\ge0}\overline{\phi}(t)+\varepsilon]\mu\|K\|_{L^1(\mathbb{R}^+)},$$

which, jointly with $\mu\|K\|_{L^1(\mathbb{R}^+)}<1$, yields $\inf_{t\ge0}\overline{\phi}(t)\le\varepsilon\mu\|K\|_{L^1(\mathbb{R}^+)}/(1-\mu\|K\|_{L^1(\mathbb{R}^+)})$. By the arbitrariness of ε, we get $\inf_{t\ge0}\overline{\phi}(t)=0$. Proof is completed. □

Several remarks about Theorem 1 are given in order:

Remark 1. *In general, the inequality (7) does not yield that $\phi(t)$ tends to zero, exponentially. For example, consider the following integral equation:*

$$\begin{aligned}\phi(t)=E_\alpha(-3t^\alpha)+\int_0^t\Big[(t-s)^{\alpha-1}E_{\alpha,\alpha}(-3(t-s)^\alpha)\\\times\frac{E_\alpha(-2s^\alpha)}{\psi(s)}\phi(s-1)\Big]ds,\ t\ge0,\end{aligned} \tag{14}$$

where $\phi(t)=1-t$ for $t\in[-1,0]$, and

$$\psi(t)=\begin{cases}2-t, & 0\le t\le1,\\ E_\alpha(-2(t-1)^\alpha), & t\ge1.\end{cases}$$

We claim that the solution of (14) satisfies the inequality (7). Indeed, by Lemma 2, it is seen that the integral equation (14) is actually equivalent to the following fractional order differential equation with time delay:

$$\begin{aligned}&{}_0^CD_t^\alpha\phi(t)=-3\phi(t)+\frac{E_\alpha(-2t^\alpha)}{\psi(t)}\phi(t-1),\quad t\ge0,\\&\phi_0=1-t\in C[-1,0].\end{aligned} \tag{15}$$

The solution is explicitly found to be

$$\phi(t)=\begin{cases}1-t, & -1\le t\le0,\\ E_\alpha(-2t^\alpha), & t\ge0,\end{cases} \tag{16}$$

which gives $\sup_{s-1\le\sigma\le s}\phi(\sigma)=\phi(s-1)$. On the other hand, since $E_\alpha(-2t^\alpha)$ is monotone decreasing on $t\in[0,+\infty)$ and $1-t\ge1=E_\alpha(-2\cdot0^\alpha)$ for $t\in[-1,0]$, we have $0<\frac{E_\alpha(-2s^\alpha)}{\psi(s)}\le1$. Thus, by (14), ϕ satisfies (7) with $h=1$, $\mu=1$, $a(t)=E_\alpha(-3t^\alpha)$, and $K(t)=t^{\alpha-1}E_{\alpha,\alpha}(-3t^\alpha)$. From the proof of Corollary 1 below, we get $\mu\|K\|_{L^1(\mathbb{R}^+)}=1/3<1$, which implies ϕ satisfies (7). However, by (16), ϕ is asymptotically convergent to zero and cannot decay exponentially as t goes to infinity.

Remark 2. *If $a(t)$ and $K(t)$ decay to zero exponentially, then $\phi(t)$ may tends to zero exponentially. To illustrate this, we consider the*

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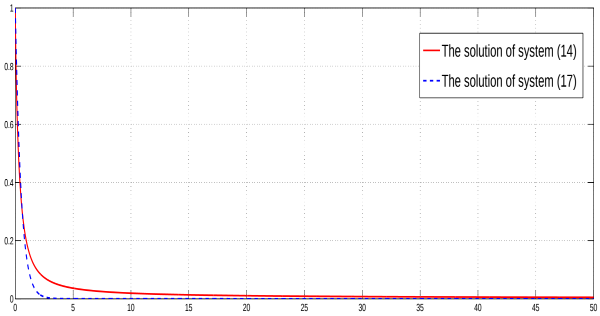

Fig. 1: The evolutions of the solutions of systems (15) and (18)

following integral equation

$$\phi(t) = e^{-3t} + e^{-2}\int_0^t e^{-3(t-s)}\phi(s-1)ds, \ \ t \geq 0, \tag{17}$$

where $\phi(t) = e^{-2t}$ *for* $t \in [-1, 0]$*. Similar to Remark 1, the solution of (17) satisfies the inequality (7) with* $h = 1$*,* $a(t) = K(t) = e^{-3t}$*. This system (17) is equivalent to the following first order differential equation with time delay*

$$\begin{aligned} &\dot{\phi}(t) = -3\phi(t) + e^{-2}\phi(t-1), \ \ t \geq 0, \\ &\phi_0 = e^{-2t} \in C[-1, 0]. \end{aligned} \tag{18}$$

A simple computation shows the solution of (18) is explicitly given by $\phi(t) = e^{-2t}$ *for all* $t \geq -1$*. We can see that* $\phi(t) > 0$ *and decays to zero exponentially.*

Figure 1 shows the evolutions of the solution of system (15) with $\alpha = 0.8$ and the solution of system (18). It is seen that the solution of system (18) decays quicker than the solution of system (15).

By using Theorem 1, we arrive at the following corollary.

Corollary 1. *Let* $V : [-h, +\infty) \to \mathbb{R}^+$ *be bounded on* $[-h, 0]$ *and continuous on* $[0, +\infty)$*. Assume that for some positive constants* $\lambda > \mu > 0$*, the following inequality holds:*

$${}_0^C D_t^\alpha V(t) \leq -\lambda V(t) + \mu \sup_{-h \leq \sigma \leq 0} V(t+\sigma), \ t \geq 0, \tag{19}$$

where $0 < \alpha < 1$*. Then,* $\lim_{t \to +\infty} V(t) = 0$*.*

Proof: By (19), we know that there exists a nonnegative function $M(t)$ satisfying

$${}_0^C D_t^\alpha V(t) + M(t) = -\lambda V(t) + \mu \sup_{-h \leq \sigma \leq 0} V(t+\sigma), \ t \geq 0. \tag{20}$$

Taking the Laplace transform on both sides of (20) gives

$$s^\alpha \widehat{V}(s) - V(0)s^{\alpha-1} + \widehat{M}(s) = -\lambda \widehat{V}(s) + \mu F(s), \ t \geq 0, \tag{21}$$

where

$$\begin{aligned} &\widehat{V}(s) := \int_0^{+\infty} e^{-st}V(t)dt, \quad \widehat{M}(s) := \int_0^{+\infty} e^{-st}M(t)dt, \\ &F(s) := \int_0^{+\infty} e^{-st} \sup_{-h \leq \sigma \leq 0} V(t+\sigma)dt, \end{aligned} \tag{22}$$

are respectively, the Laplace transform of the functions $V(t)$, $M(t)$ and $\sup_{-h \leq \sigma \leq 0} V(t+\sigma)$. Therefore, by (21), we have that

$$\widehat{V}(s) = \frac{V(0)s^{\alpha-1} - \widehat{M}(s) + \mu F(s)}{s^\alpha + \lambda}. \tag{23}$$

Taking the inverse Laplace transform on both sides of (23) yields

$$\begin{aligned} V(t) =& E_\alpha(-\lambda t^\alpha)V(0) - M(t) * [t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)] \\ &+ \mu[\sup_{-h \leq \sigma \leq 0} V(t+\sigma)] * [t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)], \end{aligned} \tag{24}$$

where $*$ denotes the convolution operator. Since $M(t)$, $t^{\alpha-1}$ and $E_{\alpha,\alpha}(-\lambda t^\alpha)$ are nonnegative functions, it follows that

$$\begin{aligned} V(t) \leq & E_\alpha(-\lambda t^\alpha)V(0) + \mu \int_0^t (t-s)^{\alpha-1} \\ & \times E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) \sup_{-h \leq \sigma \leq 0} V(s+\sigma)ds. \end{aligned} \tag{25}$$

We can see that (25) is the form of (7) in Theorem 1 with $a(t) = E_\alpha(-\lambda t^\alpha)V(0)$ and $K(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$. It is obvious that $\lim_{t \to +\infty} a(t) = 0$, $K(t) \geq 0$ and $\lim_{t \to +\infty} K(t) = 0$. To show $\lim_{t \to +\infty} V(t) = 0$, it suffices to prove that $\mu\|K\|_{L^1(\mathbb{R}^+)} < 1$. Indeed, it follows from [27, Page 50, formula 1.10.7] that

$$\frac{d}{dt}[t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha)] = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha),$$

which gives us

$$\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)ds = t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha). \tag{26}$$

It follows from [27, Page 43, formula 1.8.28] that

$$E_{\alpha,\alpha+1}(-\lambda t^\alpha) = \frac{1}{\lambda t^\alpha} + \mathcal{O}(\frac{1}{\lambda^2 t^{2\alpha}})$$

with $t \to +\infty$, which yields

$$\lim_{t \to +\infty} t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) = \frac{1}{\lambda}.$$

On the other hand, from (26), since $\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)ds$ is non-decreasing with respect t, so does for $t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha)$. Thus,

$$\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)ds \leq \frac{1}{\lambda}, \ \forall t \geq 0, \tag{27}$$

which, jointly with $\lambda > \mu$, implies

$$\mu\|K\|_{L^1(\mathbb{R}^+)} = \mu \int_0^{+\infty} s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha)ds \leq \frac{\mu}{\lambda} < 1. \tag{28}$$

Thus, it follows from Theorem 1 that $\lim_{t \to +\infty} V(t) = 0$. □

Remark 3. *The above corollary can be regarded as a generalization of Halanay inequality [23, Page 378, Lemma], where* $\alpha = 1$ *and* $V(t)$ *tends to zero, exponentially, as t goes to infinity.*

However, when $\alpha \in (0, 1)$*, we generally cannot expect that* $V(t)$ *decay exponentially because there is a memory effect in the equation due to the tail of time. A typically example can be seen in Remark 1, system (15) satisfies the conditions in Corollary 1, the solution of (15) satisfies* $\lim_{t \to +\infty} \phi(t) = 0$ *but* $\phi(t)$ *cannot decay exponentially.*

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4 Asymptotical stability

In this section, we investigate the asymptotical stability for five classes of fractional order systems with time delay.

Example 4.1. *Consider the following fractional order differential system:*

$$ {}_0^C D_t^{\alpha} x(t) = -ax(t) + bx(t - \tau(t)), \tag{29} $$

where $0 < \alpha < 1$, $x \in \mathbb{R}$ is the state, a, b are two constants, and $\tau(t)$ is a continuous function and satisfies $0 < \tau(t) \le h$ for $t \ge 0$.

In order to obtain the stability condition of (29), we denote $V(t) = x^2(t)$ and choose two constants $\lambda > \mu > 0$. Finding Caputo's derivative of $V(t)$ with respect to t along the solution of (29) yields

$$ \begin{aligned} & {}_0^C D_t^{\alpha} V(t) + \lambda V(t) - \mu \sup_{-h \le \sigma \le 0} V(t+\sigma) \\ \le & 2x(t) {}_0^C D_t^{\alpha} x(t) + \lambda x^2(t) - \mu x^2(t - \tau(t)) \\ = & [x(t)\ x(t-\tau(t))] \begin{pmatrix} -2a+\lambda & b \\ b & -\mu \end{pmatrix} \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} < 0 \end{aligned} \tag{30} $$

if

$$ \begin{pmatrix} -2a+\lambda & b \\ b & -\mu \end{pmatrix} < 0, \tag{31} $$

which is equivalents to $-2a + \lambda < 0$ and $-(-2a+\lambda)\mu - b^2 > 0$, that is, $\lambda \in (0, 2a)$ and $(2a - \lambda)\mu > b^2$. Since $\lambda > \mu$, LMI (31) reduces to $(2a - \lambda)\lambda > b^2$. Therefore, if $a > |b|$, then LMI (31) is feasible, and by Corollary 1, the solution of (29) is asymptotically stable.

In system (29), taking $\alpha = 0.8, a = 8, b = 3, x_0 = 1 \in [-1, 0]$, the numerical simulations of this system with delays $\tau = 1$ and $\tau(t) = \sin^2(t)$, respectively, are plotted in Figure 2, from which, we can see that system (29) with both $\tau = 1$ and $\tau(t) = \sin^2(t)$ are asymptotically stable and the solution of the system (29) with time-varying delay decays faster than that of the system with constant delay. Taking $b = -3$ and the rest of parameters as before, the numerical simulations plotted in Figure 3 show that system (29) with both $\tau = 1$ and $\tau(t) = \sin^2(t)$ are still asymptotically stable. The solution of the system (29) with time-varying delay decays faster than that of the system with constant delay.

The stability for system (29) is considered in [22], where the author used the Lyapunov-like theorem to prove the asymptotical stability result. Figure 4 shows the regions of $a > |b|$ that make the system (29) to be asymptotically stable.

Proposition 1. *Suppose that $a > |b|$, then given $h > 0$, system (29) with $x_0 = \phi(t) \in C[-h, 0]$ is asymptotically stable for all fast-varying delay $\tau(t) \in [0, h]$.*

Example 4.2. *Now we consider the following fractional order differential system*

$$ {}_0^C D_t^{\alpha} x(t) = Ax(t) + A_1 x(t - \tau(t)), \tag{32} $$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$ is the state, A, A_1 are constants matrices, and $\tau(t)$ is a continuous function satisfies $0 < \tau(t) \le h$ for $t \ge 0$.

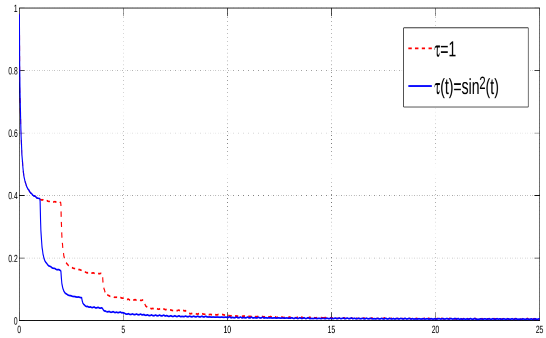

Fig. 2: System (29) wiht $\alpha = 0.8, a = 8, b = 3, x_0 = 1 \in [-1, 0]$, and with delay $\tau = 1$ and $\tau(t) = \sin^2(t)$.

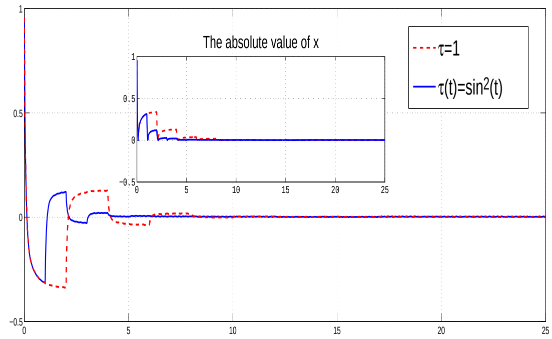

Fig. 3: System (29) with $\alpha = 0.8, a = 8, b = -3, x_0 = 1 \in [-1, 0]$, and with delay $\tau = 1$ and $\tau(t) = \sin^2(t)$.

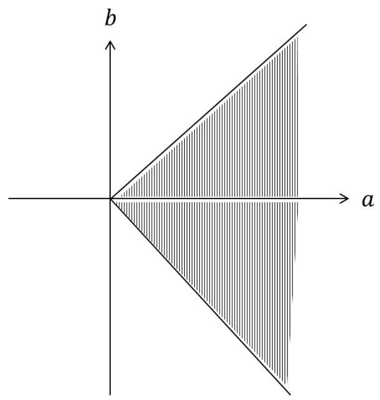

Fig. 4: The regions of $a > |b|$ which make the system (29) to be asymptotically stable

Similar to Example 4.1, let Lyapunov function $V(t) = x^\top(t)x(t)$. Then

$$ \begin{aligned} & {}_0^C D_t^{\alpha} V(t) + \lambda V(t) - \mu \sup_{-h \le \sigma \le 0} V(t+\sigma) \\ \le & 2x^\top(t) {}_0^C D_t^{\alpha} x(t) + \lambda x^\top(t)x(t) - \mu x^\top(t-\tau(t))x(t-\tau(t)) \\ \le & 2x^\top(t)[Ax(t) + A_1 x(t-\tau)] + \lambda x^\top(t)x(t) \\ & - \mu x^\top(t-\tau(t))x(t-\tau(t)) \\ \le & \left[x^\top(t)\ x^\top(t-\tau(t))\right] W \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} < 0 \end{aligned} $$

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if

$$W := \begin{pmatrix} A^\top + A + \lambda I & A_1 \\ A_1^\top & -\mu I \end{pmatrix} < 0. \qquad (33)$$

From Corollary 1, we have the following proposition.

Proposition 2. *Suppose that there exist two positive constants $\lambda > \mu > 0$ such that the LMI (33) is feasible. Then, for any given $h > 0$, system (32) with $x_0 = \phi \in C[-h, 0]$ is asymptotically stable for all fast-varying delay $\tau(t) \in [0, h]$.*

Remark 4. *When the delay is independent of time, i.e.,$\tau(t) \equiv h$ for some constant $h > 0$, the asymptotical stability of (32) is considered in [11], where the Laplace transform method is used. Note that when the delay is dependent of time, the asymptotical stability cannot follow from the Laplace transform method.*

Example 4.3. *Consider the following fractional order differential system with distributed delay*

$$\begin{cases} {}_0^C D_t^\alpha x(t) = -x(t) + y(t), \\ {}_0^C D_t^\alpha y(t) = -g(t, y(t)) - f(x(t)) \\ \qquad\qquad + \int_{-\tau}^{0} h(x(t+\theta)) y(t+\theta) d\theta, \end{cases} \qquad (34)$$

where $0 < \alpha < 1$, $x, y \in \mathbb{R}$ are the states, $\tau > 0$ is a constant, and $f, h : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Proposition 3. *Suppose that the function h satisfies $|h(x)| \le L$ with some $L \ge 0$ and there exist three positive constants ε, λ and μ with $\lambda > \mu > 0$ such that*

$$\tau^2 L^2 \le \varepsilon\mu \qquad (35)$$

and

$$\begin{aligned} &(\lambda - \mu - 2)x^2 + (\lambda + \varepsilon)y^2 \\ &\quad + 2y\big(x - g(t, y) - f(x)\big) \le 0. \end{aligned} \qquad (36)$$

Then, the system (34) with $x_0 = \phi \in C([-h, 0], \mathbb{R})$, $y_0 = \varphi \in C([-h, 0], \mathbb{R})$ is asymptotically stable.

Proof: Let $V(t) = x^2(t) + y^2(t)$ be a Lyapunov function. Finding Caputo's derivative of $V(t)$ with respect to t along the solution of (34) gives

$$\begin{aligned} &{}_0^C D_t^\alpha V(t) \\ \le& 2x(t) {}_0^C D_t^\alpha x(t) + 2y(t) {}_0^C D_t^\alpha y(t) \\ =& -2x^2(t) + 2x(t)y(t) - 2y(t)g(t, y) - 2y(t)f(x) \\ &+ 2y(t) \int_{-\tau}^{0} h(x(t+\theta)) y(t+\theta) d\theta. \end{aligned} \qquad (37)$$

Noting the fact that $|h(x)| \le L$, by Young's inequality, (35), (36) and (37), we get

$$\begin{aligned} &{}_0^C D_t^\alpha V(t) + \lambda V(t) - \mu \sup_{-\tau \le \sigma \le 0} V(t+\sigma) \\ \le& -2x^2(t) + 2x(t)y(t) - 2y(t)g(t, y) - 2y(t)f(x) \\ &+ 2y(t)L\tau \sup_{-\tau \le \sigma \le 0} |y(t+\sigma)| + \lambda x^2(t) + \lambda y^2(t) \\ &- \mu \sup_{-\tau \le \sigma \le 0} (x^2(t+\sigma) + y^2(t+\sigma)) \\ \le& -2x^2(t) + 2x(t)y(t) + \lambda x^2(t) + \lambda y^2(t) - 2y(t)g(t, y) \\ &- 2y(t)f(x) + \frac{L^2\tau^2}{\mu} y^2(t) - \mu \sup_{-\tau \le \sigma \le 0} x^2(t+\sigma) \\ \le& -2x^2(t) + 2x(t)y(t) + \lambda x^2(t) + \lambda y^2(t) - 2y(t)g(t, y) \\ &- 2y(t)f(x) + \frac{L^2\tau^2}{\mu} y^2(t) - \mu x^2(t) \\ =& (\lambda - \mu - 2)x^2(t) + (\lambda + \frac{L^2\tau^2}{\mu}) y^2(t) + 2x(t)y(t) \\ &- 2y(t)g(t, y) - 2y(t)f(x) \\ \le& (\lambda - \mu - 2)x^2 + (\lambda + \varepsilon)y^2 + 2y(x - g(t, y) - f(x)) \\ \le& 0, \end{aligned}$$

which, jointly with Corollary 1, implies that $V(t) = x^2(t) + y^2(t) \to 0$ as $t \to +\infty$. Thus, the system (34) is asymptotically stable. □

Corollary 2. *Suppose that the functions f, g, h satisfy $f(x) \ge Mx, g(t, y) \ge Ny$, and $|h(x)| \le L$ with some $M > 0, N > 0, L \ge 0$ and there exist three positive constants ε, λ and μ with $\lambda > \mu > 0$ such that*

$$\tau^2 L^2 \le \varepsilon\mu \qquad (38)$$

and

$$\begin{pmatrix} \lambda - \mu - 2 & 1 - M \\ 1 - M & \lambda + \varepsilon - 2N \end{pmatrix} \le 0. \qquad (39)$$

Then, the system (34) with $x_0 = \phi \in C([-h, 0], \mathbb{R}), y_0 = \varphi \in C([-h, 0], \mathbb{R})$ is asymptotically stable.

Proof: By proposition 3, it suffices to prove that (36) holds. By conditions (38), we have

$$\begin{aligned} &(\lambda - \mu - 2)x^2(t) + (\lambda + \varepsilon)y^2(t) \\ &+ 2y(t)\big(x(t) - g(t, y) - f(x)\big) \\ =& (\lambda - \mu - 2)x^2(t) + (\lambda + \varepsilon)y^2(t) + 2x(t)y(t) \\ &- 2y(t)g(t, y) - 2y(t)f(x, y) \\ \le& (\lambda - \mu - 2)x^2(t) + (\lambda + \varepsilon - 2N)y^2(t) + (2 - 2M)x(t)y(t) \\ =& \begin{pmatrix} x(t) & y(t) \end{pmatrix} \begin{pmatrix} \lambda - \mu - 2 & 1 - M \\ 1 - M & \lambda + \varepsilon - 2N \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ \le& 0, \end{aligned}$$

which implies that (36) holds. This ends the proof. □

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Example 4.4. *Consider the following fractional order nonlinear system with distributed delay,*

$$ {}_a^C D_t^\alpha x(t) = f(x(t)) + \int_{t-\tau(t)}^{t} g(x(\zeta))d\zeta, \tag{40}$$

where $0 < \alpha < 1,\ 0 < \tau(t) \le h,\ x \in \mathbb{R}^n$, *the initial function is* $x_0 = \phi \in C([-h,0],\mathbb{R}^n)$. $f, g : \mathbb{R}^n \to \mathbb{R}^n$ *are two continuous vector-valued functions such that (40) has a global solution on* $[0,\infty)$.

Proposition 4. *Suppose that there exist two positive constants* $\lambda > \mu > 0$ *such that*

$$\begin{aligned} &2x^\top(t)\left[f(x(t)) + \int_{t-\tau(t)}^{t} g(x(\zeta))d\zeta\right] + \lambda x^\top(t)x(t) \\ &- \mu \sup_{t-\tau(t)\le\sigma\le t} x^\top(\sigma)x(\sigma) \le 0, \quad \forall x \in \mathbb{R}^n. \end{aligned} \tag{41}$$

Then the system (40) with $x_0 = \phi \in C([-h,0],\mathbb{R}^n)$ *is asymptotically stable.*

Proof: Let $V(t) = x^\top(t)x(t)$ be the Lyapunov function. By Lemma 1, computing the Caputo's derivative of $V(t)$ along the solution of (40) gives

$$\begin{aligned} &{}_0^C D_t^\alpha V(t) + \lambda V(t) - \mu \sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma) \\ \le & 2x^\top(t) {}_0^C D_t^\alpha x(t) + \lambda x^\top(t)x(t) - \mu \sup_{t-\tau(t)\le\sigma\le t} x^\top(\sigma)x(\sigma) \\ = & 2x^\top(t)\left[f(x(t)) + \int_{t-\tau(t)}^{t} g(x(\zeta))d\zeta\right] + \lambda x^\top(t)x(t) \\ & - \mu \sup_{t-\tau(t)\le\sigma\le t} x^\top(\sigma)x(\sigma) \le 0. \end{aligned} \tag{42}$$

Since $0 < \tau(t) \le h$, we have

$$\sup_{-h\le\sigma\le t} V(t+\sigma) \ge \sup_{-\tau(t)\le\sigma\le t} V(t+\sigma).$$

Thus, it follows from (42) that

$$\begin{aligned} &{}_0^C D_t^\alpha V(t) + \lambda V(t) - \mu \sup_{-h\le\sigma\le 0} V(t+\sigma) \\ \le & {}_0^C D_t^\alpha V(t) + \lambda V(t) - \mu \sup_{-\tau(t)\le\sigma\le 0} V(t+\sigma) \\ \le & 0, \end{aligned}$$

which, jointly with Corollary 1, implies that $V(t) \to 0$ as $t \to +\infty$. Thus, system (40) is asymptotically stable. □

Corollary 3. *Suppose that the function* g *satisfies* $\|g(x)\| \le L\|x\|$ *with some* $L \ge 0$ *and there exist three positive constants* ε, λ *and* μ *with* $\lambda > \mu > 0$ *such that*

$$h^2L^2 \le \varepsilon\mu, \text{ and } 2x^\top f(x) + (\lambda+\varepsilon)x^\top x \le 0, \text{ for all } x \in \mathbb{R}^n. \tag{43}$$

Then the system (40) with $x_0 = \phi \in C([-h,0],\mathbb{R}^n)$ *is asymptotically stable.*

Proof: By Proposition 4, it suffices to show that (41) holds. Indeed, by Young's inequality

$$\begin{aligned} &\|x(t)\| \sup_{t-\tau(t)\le\sigma\le t} \|x(\sigma)\| \\ \le & \frac{1}{2\mu} x^\top(t)x(t) + \frac{\mu}{2} \sup_{t-\tau(t)\le\sigma\le t} x^\top(\sigma)x(\sigma), \end{aligned}$$

(43) and the fact that $\|g(x)\| \le L\|x\|$, we have

$$\begin{aligned} &2x^\top(t)\left[f(x) + \int_{t-\tau(t)}^{t} g(x(\sigma))d\sigma\right] + \lambda x^\top(t)x(t) \\ &- \mu \sup_{t-\tau(t)\le\sigma\le t} x^\top(\sigma)x(\sigma) \\ \le & 2x^\top(t)f(x) + 2L\|x\| \int_{t-\tau(t)}^{t} \|x(\sigma)\|d\sigma + \lambda x^\top(t)x(t) \\ &- \mu \sup_{t-\tau(t)\le\sigma\le t} x^\top(\sigma)x(\sigma) \\ \le & 2x^\top(t)f(x) + 2Lh\|x\| \sup_{t-\tau(t)\le\sigma\le t} \|x(\sigma)\| + \lambda x^\top(t)x(t) \\ &- \mu \sup_{t-\tau\le\sigma\le t} x^\top(\sigma)x(\sigma) \\ \le & 2x^\top(t)f(x) + \frac{L^2h^2}{\mu} x^\top(t)x(t) + \mu \sup_{t-\tau\le\sigma\le t} x^\top(\sigma)x(\sigma) \\ &+ \lambda x^\top(t)x(t) - \mu \sup_{t-\tau\le\sigma\le t} x^\top(\sigma)x(\sigma) \\ = & 2x^\top(t)f(x) + \frac{L^2h^2}{\mu} x^\top(t)x(t) + \lambda x^\top(t)x(t) \\ \le & 2x^\top(t)f(x) + (\lambda+\varepsilon)x^\top(t)x(t) \\ \le & 0, \end{aligned}$$

which proves that (41) holds. This ends the proof. □

Example 4.5. *Consider the fractional order nonlinear system with distributed delay governed by*

$$\begin{cases} {}_0^C D_t^\alpha x_1(t) = f_1(x(t)) + \int_{t-\tau_1(t)}^{t} g_1(x(\zeta))d\zeta, \\ {}_0^C D_t^\alpha x_2(t) = f_2(x(t)) + \int_{t-\tau_2(t)}^{t} g_2(x(\zeta))d\zeta, \\ \vdots \\ {}_0^C D_t^\alpha x_n(t) = f_n(x(t)) + \int_{t-\tau_n(t)}^{t} g_n(x(\zeta))d\zeta, \end{cases} \tag{44}$$

where $0 < \alpha < 1,\ x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^\top \in \mathbb{R}^n$, *the initial function is* $x_0 = \phi \in C([-h,0],\mathbb{R}^n)$. *The time delay functions* $\tau_i : \mathbb{R}^+ \to [0,h]$ *are continuous. Functions* $f_i(x), g_i(x) : \mathbb{R}^n \to \mathbb{R}$ *are continuous such that (44) has a global solution on* $[0,\infty)$.

Proposition 5. *For any* $x(t) \in \mathbb{R}^n$, *suppose that functions* g_i *satisfy* $|g_i(x)| \le L_i\|x\|$ *with some* $L_i \ge 0$, $i = 1,2,\dots,n$, *and there exist three positive constants* ε, λ *and* μ *with* $\lambda > \mu > 0$ *such that*

$$\begin{aligned} &h^2L^2 \le \varepsilon\mu, \quad \text{and} \\ &2x^\top f(x) + (\lambda+\varepsilon)x^\top x \le 0, \text{ for all } x \in \mathbb{R}^n, \end{aligned} \tag{45}$$

where $L = \max_i L_i$. *Then, given* $h > 0$, *the system (44) with* $\phi \in C[-h,0]$ *is asymptotically stable.*

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Proof: The Proposition can be established by mimicking the proof of Corollary 3. We leave details to the interested reader. □

Remark 5. *The key idea of Lyapunov method to get the stability is that we do not need to solve the differential systems but just use the structure of systems. Clearly, the conditions* (36)*,* (43) *and* (45) *do not involve the solution of* (34)*,* (40) *and* (44)*, respectively, thus* (36)*,* (43) *and* (45) *are easily to verify for application purpose. The results here could provide some insights into the qualitative analysis of fractional order systems with time delay.*

5 Concluding remarks

In this paper, we have obtained a new integral inequality, which is our novelty. Applying this integral inequality and Laplace transform, Halanay inequality is extended to Caputo fractional order case. Several examples validate that this generalized Halanay inequality can be easily applied to obtain the asymptotical stability conditions for fractional order systems with time delay, where the delay is bounded. If the delay is unbounded, a new technique to prove the integral inequality should be developed. Stability of fractional order systems with time-varying delay or time constant delay is a wide open and fertile area for future research. The presented integral inequality can be used to obtain the stability of Caputo fractional order systems with multiple time delays.

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