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ON THE EXISTENCE OF REGIONAL OPTIMAL CONTROL FOR A CLASS OF FRACTIONAL ORDER DIFFERENTIAL INCLUSION

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ABSTRACT

This paper is concerned with the investigation of the regional controllability of the fractional order differential inclusion(FODI). First, some preliminaries and definitions of regional controllability of the system are introduced. Then we obtained some equivalent conditions of regional controllable from the viewpoint of set relations and analyzed the regional controllability with minimum energy of a class of time fractional order differential inclusion.

1 INTRODUCTION

Recently, fractional order differential equations (FODEs) have attracted increasing attention and many monographs are obtained [15, 20, 22, 23]. Due to their nonlocal and heredity properties, they are widely applied in mechanics, electricity, economics, control theory and anomalous diffusion processes, et.al, such as [14, 16, 25, 27, 31].

Differential inclusions (DIs) are generalization of differential equations, which describe the maps from points to sets. It appears with practical applications and the development of control theories. In fact, in practical systems, single-valued maps are not adequate to describe the actual systems due to the existence of uncertainties, thus, the study of differential inclusions is meaningful and has attracted the attention of many mathematicians and physicists [1, 3–5, 24, 28].

Controllability is the basis of control theories. It is worth mentioning that, in general, not all states in the whole domain are reachable, and sometimes, the systems do not need to be controllable in the whole domain or state space. This is way we investigate the regional control, that is, the system is not necessary to be controllable in the whole domain, but in a given subregion. For regional controllability theories, El Jai and Zerrik have got some results on zone control, pointwise control and boundary control on integer order system, [2,9,29,30]. Recently, Ge et. al. extended the results on integer order to fractional order cases [10–13]. But for differential inclusions, there is no result so far to our best knowledge.

Based on the above, in this paper, we study the regional control of fractional order differential inclusion (FODI). First, we give some equivalent definitions of regionally exactly controllable and regionally approximately controllable from the viewpoint of set theory. Then, we assume that the fractional order differential inclusion is regionally approximately controllable, and establish conditions of the existence of optimal control.

The rest of this paper is organized as follows: some basic concepts on fractional order differential inclusions and regional controllability are presented in the next section. In section 3, the main results on regional controllability are given, together with a discussion on regional optimal control.

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2 PRELIMINARIES

In this section, some definitions and preliminaries are given, which will be used throughout this paper.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $Q = \Omega\times[0,T], \Sigma = \partial\Omega\times[0,T]$. Let $L^p(0,T;\Omega)(p\geq 1)$ be the space of Bochner integrable function on $[0,T]$ with the norm $\|x\|_{L^p(\Omega)} = (\int_0^T \|x(s)\|_{\mathbb{R}^n}^p ds)^{1/p}$ [19].

In this paper, we are concerned with the regional controllability of the fractional order differential inclusions

$$\begin{cases} {}_0^C D_t^\alpha z(t) \in Az(t) + Bu(t) + F(t)u(t), \\ z(0) = z_0 \in D(A) \end{cases} \tag{1}$$

where $t\in[0,T], 0<\alpha<1, z\in L^2(0,T;\Omega), {}^C D_t^\alpha$ denotes the Caputo fractional derivative of order α [22]. $D(A)$ holds for the domain of the operator A and A generates a strongly continuous semigroup $\{\Phi(t)\}_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$. $F:[0,T]\to\mathscr{P}(\mathbb{R}^{n\times m})$ is a set-valued map which maps $u(t)$ into $L^2(\Omega)$. In addition, $z_0\in L^2(\Omega), u\in L^2(0,T;\mathbb{R}^m)$ and $B:\mathbb{R}^m\to L^2(\Omega)$ is a linear operator and there exists a constant M_B such that $\|B\|\leq M_B$.

For a set-valued map $G:[0,T]\to\mathscr{P}(\mathbb{R}^{n\times m})$, G is said to be convex (closed), if $G(t)$ is convex (closed) for all $t\in[0,T]$. G is bounded on the bounded set if for any bounded subset J of $[0,T]$, $G(J) = \cup_{t\in J} G(t)$ is bounded in $\mathscr{P}(\mathbb{R}^{n\times m})$. In this paper, we suppose F is closed and convex.

G is called upper semi-continuous (u.s.c.) at $t^*\in[0,T]$, if the set $G(t^*)$ is a nonempty, closed subset of $\mathscr{P}(\mathbb{R}^{n\times m})$, and for each open subset V of $\mathscr{P}(\mathbb{R}^{n\times m})$ containing $G(t^*)$, there exists an open neighbor J of t^* such that $G(J)\subset V$.

G is said to be lower semi-continuous (l.s.c.) at $t_*\in[0,T]$, if for any $x_*\in G(t_*)$ and any neighborhood $B(x_*)$ of x_*, there exists a neighborhood $B(t_*)$ of t_* such that

$$\forall t\in B(t_*),\ \ G(t)\cap B(x_*)\neq\emptyset.$$

G is said to be u.s.c. on $[0,T]$ if G is u.s.c. at each $t^*\in[0,T]$ and l.s.c. on $[0,T]$ if G is l.s.c. at each $t_*\in[0,T]$.

Definition 2.1. *The left-side Caputo fractional derivative of order $\alpha>0$ is defined by the operator*

$${}_0^C D_t^\alpha z(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} z^{(n)}(s)ds$$

provided that it exists almost everywhere on $[0,+\infty)$ where $n = [\alpha]+1$.

Let $\omega\subset\Omega$ be a given region of positive Lebesgue measure and $z_T\in L^2(\omega)$ (the target function) be a given element of the state space.

Based on the argument in [8] and [33], we can get the solution of (1).

Definition 2.2. *For any given $u\in L^2(0,T;\mathbb{R}^m)$, a set of functions $z\subset L^2(0,T;\Omega)$ is said to be a mild solution of the system (1) denotes by $z(\cdot,u)$, if it satisfies*

$$\begin{aligned} z(t,u) = \{S_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)Bu(s)ds \\ + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)F(s)ds\}, \end{aligned} \tag{2}$$

where

$$S_\alpha(t) = \int_0^\infty \phi_\alpha\Phi(t^\alpha\theta)d\theta, \tag{3}$$

and

$$K_\alpha(t) = \alpha\int_0^\infty \theta\phi_\alpha(\theta)\Phi(t^\alpha\theta)d\theta. \tag{4}$$

Here $\{\Phi(t)\}_{t\geq 0}$ is the strongly continuous semigroup generated by A, $\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1}\Psi_\alpha(\theta^{\frac{1}{\alpha}})$, where Ψ_α is a probability density function defined by

$$\Psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha-1)}{n!}\sin(n\pi\alpha), \theta\in(0,\infty),$$

it satisfies the following properties:

$$\int_0^\infty e^{-\lambda\theta}\Psi_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \int_0^\infty \Psi_\alpha(\theta)d\theta = 1, \alpha\in(0,1) \tag{5}$$

and

$$\int_0^\infty \theta^\nu\phi_\alpha(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \nu\geq 0.$$

We can see that, since $F(t)$ is a set valued map, the solution of system (1), written as $z(t,u)$, is a set. For any selection $f(t)\in F(t)$, we denote $z_f(t,u) = S_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)Bu(s)ds + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)f(s)ds$, it is easy to see that $z_f(t,u)$ is a selection of $z(t,u)$.

Lemma 2.1. *[17, 32] (1) For any $t\geq 0$, the operators $S_\alpha(t)$ and $K_\alpha(t)$ are linear and bounded, i.e., for any $x\in L^2(\Omega)$, we have*

$$\|S_\alpha(t)x\|_{L^2(\Omega)} \leq M\|x\|_{L^2(\Omega)}$$

and

$$\|K_\alpha(t)x\|_{L^2(\Omega)} \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\|_{L^2(\Omega)},$$

where M is defined in the inequality (7).

(2) Operators $\{S_\alpha(t)\}_{t\geq 0}$ and $\{K_\alpha(t)\}_{t\geq 0}$ are strongly continuous, that is, for any $x \in L^2(\Omega)$ and $0 \leq t_1 \leq t_2 \leq T$,

$$\|S_\alpha(t_1)x - S_\alpha(t_2)x\|_{L^2(\Omega)} \to 0$$

and

$$\|K_\alpha(t_1)x - K_\alpha(t_2)x\|_{L^2(\Omega)} \to 0 \text{ as } t_1 \to t_2$$

hold.

(3) For $t > 0, S_\alpha(t)$ and $K_\alpha(t)$ are compact operators if $\Phi(t)$ is compact.

Lemma 2.2. *[3] Let X be a metric space, Y a Banach space. Let F from X into the closed convex subsets of Y be lower semicontinuous. Then there exists $f : [0,T] \to \mathscr{P}(\mathbb{R}^{n\times m})$, a continuous selection of F.*

Definition 2.3. *The system (1) is said to be regional exactly controllable (or ω-exactly controllable), if for any $z_T \in L^2(\omega)$, there exists a control $u \in L^2(0,T;\mathbb{R}^m)$ such that, there exists a selection $z_f(t,u) \in z(t,u)$ satisfies*

$$P_\omega z_f(T,u) = z_T,$$

where $P_\omega : L^2(\Omega) \to L^2(\omega)$ defined by $P_\omega z = z|_\omega$ is a projection operator.

Remark 2.1. *From this definition, we see that, if system (1) is controllable, then for $\forall z_T \in L^2(\omega)$, there exists a control $u \in L^2(0,T;\mathbb{R}^m)$ such that $z_T \in P_\omega(z(T,u))$.*

Definition 2.4. *The system (1) is said to be regionally approximately controllable (or ω-approximately controllable), if for $\forall z_T \in L^2(\omega)$ and $\forall \varepsilon > 0$, there exists a control $u \in L^2(0,T;\mathbb{R}^m)$ such that, there exists a selection $z_f(t,u) \in z(t,u)$ satisfies*

$$\|P_\omega z_f(T,u) - z_T\|_{L^2(\omega)} \leq \varepsilon.$$

Remark 2.2. *From this definition, we see that, if system (1) is controllable, then for $\forall z_T \in L^2(\omega)$, there exists a control $u \in L^2(0,T;\mathbb{R}^m)$ such that*

$$d(z_T, P_\omega z(T,u)) < \varepsilon,$$

where d denotes the distance from a point to a set, defined as $d(x,E) = \inf_{y\in E} d(x,y)$.

3 MAIN RESULT

In this section, we first give some equivalent definitions of regional exactly controllable and regional approximately controllable, then we will explore the possibility of finding a minimum energy control which can steer the time FODI (1) from the initial z_0 to a target z_T on the region ω.

For $\forall u \in L^2(0,T;\mathbb{R}^m)$, for convenience, we define $H : L^2(0,T;\mathbb{R}^m) \to \mathscr{P}(L^2(\Omega))$ as

$$\begin{aligned} Hu = \{ &\int_0^T (T-s)^{\alpha-1} K_\alpha(T-s)Bu(s)ds \\ &+ \int_0^T (T-s)^{\alpha-1} K_\alpha(T-s)F(s)u(s)ds\}. \end{aligned} \tag{6}$$

Theorem 3.1. *Let H be defined as (6), the following properties are equivalent:*
(1) The system (1) is regionally exactly controllable in ω at time b;
(2) $L^2(\omega) \subset \mathrm{im}(P_\omega H)$;
(3) $L^2(\Omega) \subset \ker(P_\omega) + \mathrm{im}(H)$.

Proof. $(1) \Rightarrow (2)$: Since system (1) is regionally exactly controllable, for $\forall z_T \in L^2(\omega)$, there exists $z_f \in z$ such that $P_\omega z_f(T,u) = z_T$, that is, there exists $f(t) \in F(t)$, such that $z_f(t,u) = H_f u$, this implies $L^2(\omega) \subset \mathrm{im}(P_\omega H)$.
$(2) \Rightarrow (1)$: For any $y \in L^2(\omega)$, we know $y \in \mathrm{im}(P_\omega H)$, then, there exists $u \in L^2(0,T;\mathbb{R}^m)$ such that $y \in P_\omega Hu$, this means that there exists $f(t) \in F(t)$ satisfies $y = P_\omega H_f u$, which implies system (1) is regionally exactly controllable.
$(2) \Rightarrow (3)$: For any $y \in L^2(\omega)$, let $\bar{y}$ be the extension of y to $L^2(\Omega)$. Since $L^2(\omega) \subset \mathrm{im}(\mathrm{P}_\omega \mathrm{H})$, there exists $u \in L^2(0,T;\mathbb{R}^m)$, such that $y \in P_\omega Hu$, also, there exists $y_1 \in \ker(P_\omega)$, such that $\bar{y} \in y_1 + Hu$.
$(3) \Rightarrow (2)$: For any $\bar{y} \in L^2(\Omega)$, from (3), there exist $y_1 \in \ker(P_\omega), u \in L^2(0,T;\mathbb{R}^m)$, such that $\bar{y} \in y_1 + Hu$. Hence, it follows from the definition of P_ω that $L^2(\omega) \subset \mathrm{im}(P_\omega H)$.

Theorem 3.2. *For regionally approximately controllable, we have the following equivalent conditions:*
(1) *The system (1) is regionally approximately controllable in ω at time b.*

$(2) L^2(\omega) \subset \overline{\mathrm{im}(P_\omega H)}$;
$(3) L^2(\Omega) \subset \ker(P_\omega) + \overline{\mathrm{im}(H)}$.

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1, so we omitted.

Now we prove the optimal result on the FODI (1), for this purpose, we make use the following hypotheses.

(H1) The semigroup $\{\Phi(t)\}_{t\geq 0}$ generated by operator A is uniformly bounded on $L^2(\Omega)$, i.e., there exists a constant $M > 0$ such that

$$\sup_{t\geq 0} \|\Phi(t)\| \leq M. \tag{7}$$

(H2) For any $t > 0$, $\Phi(t)$ is a compact operator.

(H3) The set-valued map F is closed and convex and lower-semicontinuous.

(H4) The set-valued map F is uniformly bounded, that is, there exists $M_F > 0$ such that

$$\sup_{t\geq 0} \|F(t)\| \leq M_F.$$

By (H3) and Lemma 2.2, we know that $F(t)$ has continuous selection $f(t)$. For a given $f(t) \in F(t)$, let

$$U_T = \{u \in L^2(0,T;\mathbb{R}^m) : \|P_\omega z_f(T,u) - z_T\|_{L^2(\omega)} \leq \varepsilon, \forall \varepsilon > 0\},$$

and $H_f u = \int_0^T (T-s)^{\alpha-1} K_\alpha(T-s)Bu(s)ds + \int_0^T (T-s)^{\alpha-1} K_\alpha(T-s) f(s)u(s)ds$. Consider the following minimization problem

$$\inf_u J(u) = \inf_u \{\int_0^T \|u(t)\|^2_{\mathbb{R}^m} dt : u \in U_T\}. \tag{8}$$

Since system (1) is linear, without loss of generality, we suppose $z_0 = 0$ in the following discussion.

Theorem 3.3. *Suppose hypotheses (H1)-(H4) hold and system (1) is ω-approximately controllable. Then, the minimization problem (8) admits at least one optimal solution.*

Proof. We can see that U_T is a closed and convex set. First, we prove that the operator H_f is strongly continuous [6], which admits the existence of optimal control for the minimization problem. According to the argument in [7], we only need to show that the operator H_f is compact since it is linear and continuous.

Define the operator $N : L^2(\mathbb{R}^m) \to L^2(\Omega)$ by

$$\begin{aligned} Nu(t) = &\int_0^t (t-s)^{\alpha-1} K_\alpha(t-s)Bu(s)ds \\ &+ \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) f(s)u(s)ds,\ t \in [0,T]. \end{aligned} \tag{9}$$

Let $\rho_r = \{u \in L^2(0,T;\mathbb{R}^m) : \|u\|_{L^2(0,T;\mathbb{R}^m)} \leq r\}$, now we show that N maps bounded set into relatively compact set. For any fixed $t \in [0,T], \varepsilon, \delta \in (0,t), u \in \rho_r$. Let

$$\begin{aligned} &N_{(\varepsilon,\delta)}u(t) \\ &= \alpha \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) Bu(s) d\theta ds \\ &\quad + \alpha \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) f(s)u(s) d\theta ds. \end{aligned} \tag{10}$$

Since $\Phi(\varepsilon^q \delta)$ is compact and

$$\begin{aligned} &N_{(\varepsilon,\delta)}u(t) \\ &= \Phi(\varepsilon^q \delta)\alpha \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta - \varepsilon^q \delta) \\ &\qquad \times Bu(s) d\theta ds \\ &\quad + \Phi(\varepsilon^q \delta)\alpha \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta - \varepsilon^q \delta) \\ &\qquad \times f(s)u(s) d\theta ds, \end{aligned} \tag{11}$$

we can see that $\{N_{(\varepsilon,\delta)}u(t) | u \in \rho_r\}$ is relatively compact. Consider $\|Bu(\cdot)\| \leq M_B r$ and $\|f(\cdot)u(\cdot)\| \leq M_F r$, for any $t \in [0,T]$, we have

$$\begin{aligned} &\|Nu(t) - N_{(\varepsilon,\delta)}u(t)\| \\ = \ &\alpha \| \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) Bu(s) d\theta ds \\ &+ \int_0^t \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) f(s)u(s) d\theta ds \\ &+ \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) f(s)u(s) d\theta ds \\ &+ \int_0^t \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) f(s)u(s) d\theta ds \\ &- \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) Bu(s) d\theta ds \\ &- \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) f(s)u(s) d\theta ds \| \\ \leq \ &\alpha \| \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) Bu(s) d\theta ds \| \\ &+ \alpha \| \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) Bu(s) d\theta ds \| \\ &+ \alpha \| \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \phi_\alpha(\theta) \Phi((t-s)^\alpha \theta) f(s)u(s) d\theta ds \| \end{aligned}$$

$$
\begin{aligned}
&+\alpha\| \int_{t-\varepsilon}^{t}\int_{\delta}^{\infty}(t-s)^{\alpha-1}\theta\phi_\alpha(\theta)\Phi((t-s)^\alpha\theta)f(s)u(s)d\theta ds\| \\
\le\ & MM_BrT^\alpha\int_0^\delta \theta\phi(\theta)d\theta+\frac{MM_Br\varepsilon^\alpha}{\Gamma(1+\alpha)} \\
&+MM_FrT^\alpha\int_0^\delta \theta\phi(\theta)d\theta+\frac{MM_Fr\varepsilon^\alpha}{\Gamma(1+\alpha)} \\
\to\ & 0
\end{aligned}
$$

as $\varepsilon,\delta\to 0$. We conclude that $N\rho_r$ is a relatively compact set in $L^2(\Omega)$ [21].

Next, we shall prove that $\{Nu|u\in\rho_r\}$ is equicontinuous on $[0,T]$. For any $u\in\rho_r, 0\le\sigma_1\le\sigma_2\le T$,

$$
\begin{aligned}
&\|Nu(\sigma_2)-Nu(\sigma_1)\| \\
\le\ & \|\int_0^{\sigma_1}[(\sigma_2-s)^{\alpha-1}-(\sigma_1-s)^{\alpha-1}]K_\alpha(\sigma_2-s)Bu(s)ds\| \\
&+\|\int_0^{\sigma_1}(\sigma_1-s)^{\alpha-1}[K_\alpha(\sigma_2-1)-K_\alpha(\sigma_1-s)]Bu(s)ds\| \\
&+\|\int_{\sigma_1}^{\sigma_2}(\sigma_2-s)^{\alpha-1}K_\alpha(\sigma_2-s)Bu(s)ds\| \\
&+\|\int_0^{\sigma_1}[(\sigma_2-s)^{\alpha-1}-(\sigma_1-s)^{\alpha-1}]K_\alpha(\sigma_2-s)f(s)u(s)ds\| \\
&+\|\int_0^{\sigma_1}(\sigma_1-s)^{\alpha-1}[K_\alpha(\sigma_2-s)-K_\alpha(\sigma_1-s)]f(s)u(s)ds\| \\
&+\|\int_{\sigma_1}^{\sigma_2}(\sigma_2-s)^{\alpha-1}K_\alpha(\sigma_2-s)f(s)u(s)ds\| \\
\le\ & \frac{MM_Br}{\Gamma(1+\alpha)}(\sigma_2^\alpha-\sigma_2^\alpha+(\sigma_2-\sigma_1)^\alpha)+\zeta \\
&+\frac{MM_Fr}{\Gamma(1+\alpha)}(\sigma_2^\alpha-\sigma_2^\alpha+(\sigma_2-\sigma_1)^\alpha)+\xi \\
&+\frac{MM_Br}{\Gamma(1+\alpha)}(\sigma_2-\sigma_1)^\alpha+\frac{MM_Fr}{\Gamma(1+\alpha)}(\sigma_2-\sigma_1)^\alpha,
\end{aligned}
$$

where

$$\zeta=\|\int_0^{\sigma_1}(\sigma_1-s)^{\alpha-1}[K_\alpha(\sigma_2-s)-K_\alpha(\sigma-s)]Bu(s)ds\|$$

and

$$\xi=\|\int_0^{\sigma_1}(\sigma_1-s)^{\alpha-1}[K_\alpha(\sigma_2-s)-K_\alpha(\sigma-s)]f(s)u(s)ds\|.$$

Since $\varepsilon>0$ small enough, we have

$$
\begin{aligned}
\zeta &\le \int_0^{\sigma_1-\varepsilon}(\sigma_1-s)^{\alpha-1}\|K_\alpha(\sigma_2-s)-K_\alpha(\sigma_1-s)\|\|Bu(s)\|ds \\
&\quad+\int_{\sigma_1-\varepsilon}^{\sigma_1}(\sigma_1-s)^{\alpha-1}\|K_\alpha(\sigma_2-s)-K_\alpha(\sigma_1-s)\|\|Bu(s)\|ds \\
&\le \big[\frac{M_Br}{\alpha}(\sigma_1^q-\varepsilon^q)\big]_{s\in[0,\sigma_1-\varepsilon]}\|K_\alpha(\sigma_2-s)-K_\alpha(\sigma_1-s)\| \\
&\quad+\frac{2MM_Br}{\Gamma(1+\alpha)}\varepsilon^q \\
&\to 0
\end{aligned}
\tag{12}
$$

as $\sigma_2\to\sigma_1$ (due to the continuity of $K_\alpha(t)(t>0)$ in the uniform operator topology). Similarly, we have $\xi\to 0$ as $\sigma_2\to\sigma_1$. By the Arzela-Ascoli theorem [6], the operator N is compact. Thus, H_f is strongly continuous, which guarantees the existence of optimal control to the minimization problem (8) under the fact that U_T is a closed and convex set.

By assumption, system (1) is ω-approximately controllable, for any $z_T\in\omega$, suppose that $J(u^*)=\inf_u J(u)=\varepsilon<\infty$, by the definition of infimum, we deduce that there exists a sequence $\{u_i\}_{i=1,2,\cdots}$ such that for $\forall\varepsilon>0$,

$$\|P_\omega z_f(T,u_i)-z_T\|_{L^2(\omega)}\le\varepsilon, u_i\in U_T\subset L^2(0,T;\mathbb{R}^m), (i=1,2,\cdots),$$

and $J(u_i)\to J(u^*)$. Then we have $u_i\to_w u^*$ in $L^2(0,T;\mathbb{R}^m)$ ($\to_w$ denotes weak convergent). For any $t\in[0,T]$, by Definition 2.2 and Lemma 2.1, we get

$$
\begin{aligned}
&\|P_\omega z_f(t,u^*)-P_\omega z_f(t,u_i)\|_{L^2(\Omega)} \\
&=\|P_\omega\int_0^t(t-s)^{\alpha-1}K_\alpha(t-s)(B+f(s))(u^*(s)-u_i(s))ds\| \\
&\le\|\int_0^t(t-s)^{\alpha-1}K_\alpha(t-s)(B+f(s))(u^*(s)-u_i(s))ds\| \\
&\le\frac{\alpha M(M_B+M_F)}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}\|u^*(s)-u_i(s)\|_{L^2(\mathbb{R}^m)}ds,
\end{aligned}
\tag{13}
$$

which yields that

$$P_\omega z_f(t,u_i)\to P_\omega z_f(t,u^*)\ \text{ in }\ C(0,T;\omega)\ \text{ as }\ i\to\infty.$$

Since U_T is closed and convex, from Marzur Lemma [18] we see that $u^*\in U_T$. Thus it follows from the Balder's theorem [26] that

$$\varepsilon=J(u^*)=\lim_{i\to\infty}J(u_i)\ge J(u^*)\ge\varepsilon,$$

which means that u^* is the optimal solution of the minimization problem (8). This completes the proof.

4 Conclusions

In this paper, we investigated a class of time fractional order differential inclusion, First, we present some equivalent definitions of regional exactly controllable and regional approximately controllable. Then, base on the existing results [7, 10], we explore the existence of regional optimal control for a class of fractional order differential inclusion assuming that the system (1) is regional approximately controllable. The results we present here are expected to be extended to nonlinear case, such as, consider the following affine-in-control fractional order differential inclusion

$$\begin{cases} {}_0^C D_t^{\alpha} z(t) \in Az(t) + Bu(t) + F(z)u(t), \\ z(0) = z_0 \in D(A), \end{cases} \tag{14}$$

and investigate whether there exists a regional optimal control for the minimization problem

$$\inf_u J(u) = \{ \int_0^T (\|z(t)\|^2 + \|u(t)\|^2) dt : u \in U_T \}.$$

Motivated by this, in the future, we will try to consider the regional controllability criteria and their optimal control results.

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