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A SURVEY OF FRACTIONAL-ORDER NEURAL NETWORKS

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ABSTRACT

In this paper, the literature of fractional-order neural networks is categorized and discussed, which includes a general introduction and overview of fractional-order neural networks. Various application areas of fractional-order neural networks have been found or used, and will be surveyed and summarized such as neuroscience, computational science, control and optimization. Recent trends in dynamics of fractional-order neural networks are presented and discussed. The results, especially the stability analysis of fractional-order neural networks, are reviewed and different analysis methods are compared. Furthermore, the challenges and conclusions of fractional-order neural networks are given.

1 INTRODUCTION

Fractional-order neural networks (FNN) have been an important topic in biology and computational science in the last two decades. Combining fractional calculus and neural networks, FNN brings the high efficiency and significant improvements. Compared with the classical integer-order neural networks, FNN is a better tool to describe the memory and hereditary properties of various processes in neuroscience. Besides, FNN owns a fundamental and general computation ability that can contribute to efficient information processing, stimulus anticipation and frequency-independent phase shifts of oscillatory neuronal firing. So, it has great potential applications in the original areas of neural networks. Then, an introduction to neural networks is given firstly.

A nerve cell (neuron) is the basic unit of the nervous system. Hundreds of neurons constitute the neural networks, whose researches began from the study the human brain before over thousands years old. The first step toward artificial neural networks came in 1943 when Warren McCulloch, a neurophysiologist, and a young mathematician, Walter Pitts, wrote a paper on how neurons might work [1]. Since Minsky and Papert showed the deficiencies of perceptron models in their published book Perceptrons in 1969 [2], most researchers left the field due to the redirected neural network funding. Only a few researchers continued their efforts, notably such as Teuvo Kohonen, Stephen Grossberg, James Anderson, and Kunihiko Fukushima.

In the beginning of the 1980s, J. Hopfield designed a new associative memory neural network, named Hopfield. He understood that some models of physical systems could be used to solve computational problems [3]. Such systems could be implemented in hardware by combining common and standard components such as capacitors and resistors. This feature is so significant especially in hardware implementation point of view. Neural networks have received much attention in recent years. Ranging from image processing, combinatorial optimization, associative memories, pattern recognition and other areas, neural networks have witnessed a large amount of successful applications in many fields. It is well known that the unique globally stable equilibrium is crucial to solve some optimization problems.

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Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator. In the past decade, engineers and scientists became aware of the fact that the description of some phenomena is more accurate when the fractional derivative is used. Nowadays, it has witnessed significant progress on fractional calculus, because the applications of fractional calculus were found in more and more scientific fields, covering mechanics, physics, engineering, informatics, and materials. So long list of the applications includes viscoelasticity [4, 5], colored noise, dielectric polarization [6], electrode-electrolyte polarization [7], electromagnetic waves [8], quantitative finance [9], quantum evolution of complex system [10], the control of fractional-order dynamic systems [11, 12], fractional kinetics, anomalous attenuation [13] and so on. The main advantage of fractional-order models in comparison with classical integer-order models is that fractional derivatives provides an excellent tool for the description of memory and hereditary properties of various processes. In fact, fractional-order systems have infinite memory. Taking into account these facts, it is easy to see that the incorporation of a memory term (in the form of a fractional derivative or integral) into a neural network model is an extremely important improvement.

The purpose of this paper is to provide a summary and review of the recent trends in FNN from both application point of view and the theoretical point of view. Section I is a general introduction to fractional calculus and neural networks. In Section II, we summarize the survey FNN that we used and present selected results from recent literature. Sections III and IV are the main parts of the paper, where we separate the literature into applications-focused results and theory-focused results respectively. Section V presents the challenges and issues being faced in FNN.

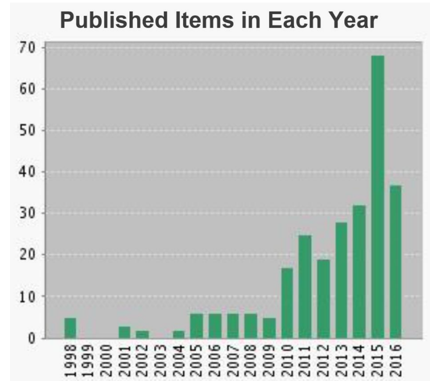


FIGURE 1: Published items of FNN in the latest 20 years

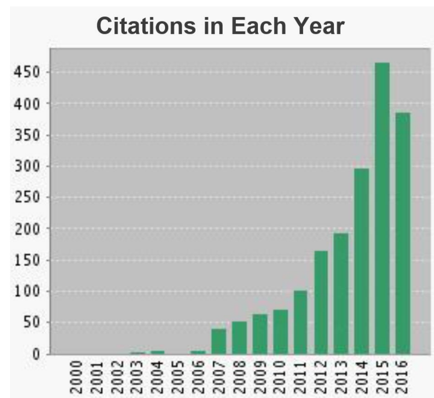


FIGURE 2: Citations of FNN in the latest 20 years

2 An Overview

In this section, we give an overview of the results obtained from our search for literature. The statistics presented in this section comes from the FNN search in "Web of Science". All databases up to August 2016, may vary because of the updating of new ones.

Our survey for FNN started from a search on "Web of Science", "IEEE Xplore", and "ScienceDirect" sites conducted on August 2016. We analyzed the statistics from searching the keywords "fractional-order neural network" in "Web of Science", and have a total of 267 publications. Among them, 179 publications come from "Web of Science Core Collection", and the oldest ones were published in 1998. Fig. 1 and Fig.2 give the numbers of FNN publications and its citations in the latest 20 years respectively. It is an exponential growth approximately for the increasing tendency of FNN publications, which shows that FNN has been a hot topic for researchers all around the world.

Overall, great majority of publications focused on the theoretical innovations of the dynamic analysis of FNN. Besides, the FNN advantages in applications of computational algorithms have attracted more attentions. As the development of the researches of FNN, more kinds of FNN models, appeared based on some practical situations, were studied such as delayed FNN, memristor-based FNN, impulsive FNN and so on. Therefore, FNN is an open and meaningful topic in both theory and applications.

3 Applications for Fractional-order Neural Networks

In this section, we list the main applications of fractional-order neural networks in many fields, such as neuroscience, computational science, control, optimization and so on. The current applications for FNN have involved various fields. However, many works for FNN are always in the stage of initial attempts, and the corresponding results are scattered and not deep enough.

3.1 Oculomotor Integrator

Oculomotor integrator is a network which consists of neurons in the nuclei prepositus hypoglossi and medial vestibular nuclei in the brainstem. Among it, premotor neurons mediate some oculomotor subsystems such as pursuit and saccades, and relay eye movement commands to extraocular motoneurons. They receive vestibular signals from canal afferents, whose frequency responses are able to be described by a fractional-order differentiation $\frac{d^{\alpha}x}{dt^{\alpha}}, \alpha > 0$. T. J. Auastasio first used fractional calculus to characterize the dynamics in motor and premotor neurons [14], which pointed out that the oculomotor integrator may belong to fractional order depending on eye velocity and eye position commands. This order is less than one, However the order of the velocity commands may be more than one. Since then, fractional differentiation has been combined into the eye position and given a better description on the output of motor and premotor neurons.

3.2 Fractional-order Hopfield Neural Networks (FHNN)

Hopfield Neural Networks (HNN) is a new associative memory neural network to solve computational problems, which was designed by J. Hopfield in the beginning of the 1980s [3]. HNN can be implemented in hardware by combining common and standard components such as capacitors and resistors, which is so significant especially in hardware implementation point of view. The topological structure of Hopfield net is shown in Fig. 3. In Fig. 3, the input and output of the net are analog signals. The resistance R_{i0} and capacitor C_i are parallel to simulate the time-delay characteristics of biologic neurons. The resistance $R_{ij}(i, j = 1, 2, \cdots, n)$ and the op-amps are used to simulate the synapse and the non-linear characteristic of biologic neurons, respectively. Then, the state and output equations of the Hopfield network with n neurons are obtained as:

$$C_i \frac{dP_i}{dt} = \sum_{j=1}^{n} W_{ij}V_j - \frac{P_i}{R_i} + I_i, \; P_i = \left(\frac{1}{\lambda}\right) \varphi_i^{-1}(V_i). \quad (1)$$

where $P_i(t)$ and $V_i(t)$ are the input and output of op-amp for the ith neuron at time t, respectively. I_i is the external input, λ is the learning rate and W_{ij} is the conductance between the ith and jth neuron with satisfying

$$W_{ij} = \frac{1}{R_{ij}}, \; \frac{1}{R_i} = \frac{1}{R_{i0}} + \sum_{j=1}^{n} W_{ij}.$$

Because of easy structure of the continuous HNN for implementation, they are often used for solving optimization problems.

In 2008, Fractional-order Hopfield Neural Networks (FHNN) was proposed by replacing the classical capacitors with fractance components [15]. Similar with FCNN, fractance is used in the continuous HNN instead of common capacitor and proposes a new continuous network, named Fractional-order Hopfield Neural Network (FHNN), which can obtained by (1):

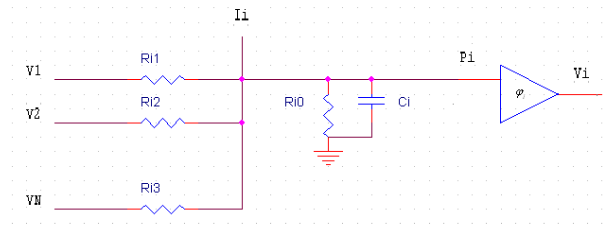


FIGURE 3: The dynamic neuron of continuous Hopfield

$$C_i \frac{d^{\alpha}P_i}{dt^{\alpha}} = \sum_{j=1}^{n} W_{ij}V_j - \frac{P_i}{R_i} + I_i, \quad (2)$$

where $0 < \alpha < 1$ and the corresponding parameters are same with (1).

FHNN has infinite memory and shows good significations comparing with the common Hopfield in various fields such as optimization problems. Due to the applications of FHNN to parallel computation and signal processing, the dynamics of FHNN became a hot topic. E. Kaslik and S. Sivasundaram firstly investigated several dynamical behaviors of FHNN, such as stability, multi-stability, bifurcations and chaos [16–19]. Then, more relational results appeared in the stability [20–26] and synchronization [25–28] of FHNN. These above theoretical results will be discussed detailed in Sections IV and V.

Fractional-order PID control in backpropagation diagrams

In theory, neural networks own the ability of learn arbitrarily complex functions for classification, decision, and similar tasks. For example, the weights between nodes in a fixed topology are trained by iteratively adjusting, such that the resulting values determine the performance of the network. Define a thresholding function at the output of each node as

$$\sigma = \frac{1}{1+e^{-net}}.$$

The input

$$net = \sum_{i \,\in\, Upstream(j)} x_i \cdot w_{ji},$$

where x_i denotes each upstream node i's output and w_{ji} is the weight of its connection to the node of interest j.

Backpropagation algorithm based on stochastic gradient descent shown in Fig. 4.a, can be used to train a neural network [29]. A set of training samples is provided along with the network's target outputs. For each output node i, by comparing the resulting outputs o and the target values t, an error is calculated by

$$\delta_i = \dot{o}\,(t - o)\,.$$

With respect to each of its downstream connections j, the error at hidden node i is

$$\delta_i = \dot{o} \sum_{j \,\in\, Downstream(i)} w_{ji} \cdot \delta_j .$$

For basic neural network (BNN), the weight w_{ji} is updated as

$$\Delta w_{ji} = \eta \cdot \delta_j \cdot x_i ,$$

where η is the learning constant.

Proportional, integral, derivative (PID) control shown in Fig. 4. b is a popular strategy for designing a simple feedback control system. For the PID neural network (PNN), the update law of the weight w_{ji} becomes

$$\Delta w_{ji} = \left(K_p \delta_j + K_i \int \delta_j dt + K_d \frac{d}{dt} \delta_j \right) \cdot x_i ,$$

where K_p, K_i and K_d are the corresponding PID control parameters. Combined with fractional calculus, the PID can be extended to fractional-order PID ($\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$) [30, 31]. For FNN, the update law of the weight w_{ji} becomes

$$\Delta w_{ji} = \left(K_p \delta_j + K_i \int \delta_j (dt)^{\lambda} + K_d \frac{d^{\mu}}{dt^{\mu}} \delta_j \right) \cdot x_i ,$$

where λ and μ are the orders of the fractional integral and derivative respectively. In [30], the authors discussed the above FNN and compared FNN with PNN and BNN, and showed that FNN can reach a lower mean squared error in both long-term and short-term.

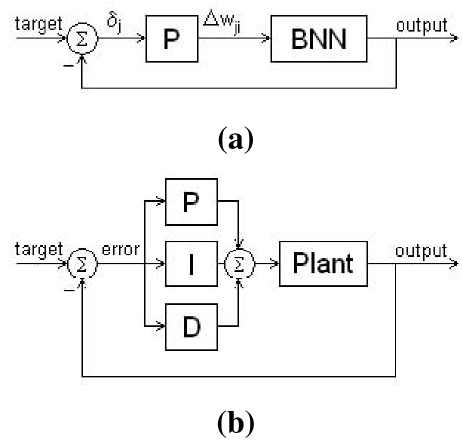


FIGURE 4: (a) Typical PID, (b) Basic Backpropagation Control System Diagrams

4 Dynamic Analysis of FNN

Due to the wide applications of FNN, the theoretical developments of FNN have been also paid attention to, especially, the dynamic analysis of the continuous FNN. It was seen that the literature related to the dynamics of FNN was broadly classified into chaos, limit cycle and bifurcation, stability of FNN, and stability of fractional-order delayed neural networks (FDNN).

Masses of papers discussed the above dynamical properties of FCNN and FHNN (2). In this section, we present a brief survey of these dynamical results.

4.1 Chaos of FNN

Chaos is a kind of nonlinear dynamical systems, which is sensitive to the initial conditions and looks random. It has been applied to numerous areas such as biology, physiology, mathematics, physics, electronics, information sciences, economics. With the presentation of FNN, the chaos phenomenon in FNN was discovered under some parameter conditions, such as FCNN in [32].

Since then, many scholars focused on the researches to discover or construct chaotic FNN [16, 19, 33–35]. The common analysis method is to vary the fractional order and observe the corresponding dynamics performance in numerical simulation, then it could obtain the some intervals of fractional order to distinguish different dynamics in studied systems.

Detailed, we provide an example to show this method. The dynamics of a ring neural network with 3 neurons were analyzed in [16] described by

$$\begin{cases} {}_0^C D_t^{\alpha} x_1 = -a x_1 + T_0 \sin x_1 + T_1 \sin x_2 + T_2 \sin x_3 , \\ {}_0^C D_t^{\alpha} x_2 = -a x_2 + T_2 \sin x_1 + T_0 \sin x_2 + T_1 \sin x_3 , \\ {}_0^C D_t^{\alpha} x_3 = -a x_3 + T_1 \sin x_1 + T_2 \sin x_2 + T_0 \sin x_3 . \end{cases} \quad (3)$$

To discuss the connections between the dynamics of system (3) and its fractional order α, the other parameters in [16] are chosen

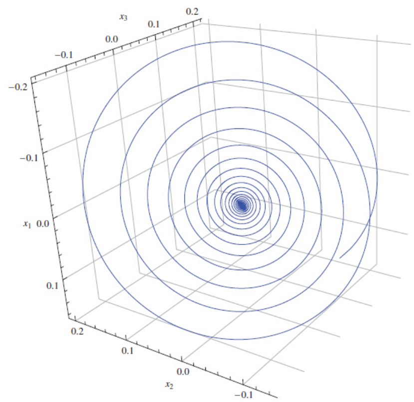


FIGURE 5: The trajectory of system (3) with $\alpha = 0.65$ (asymptotically stable)

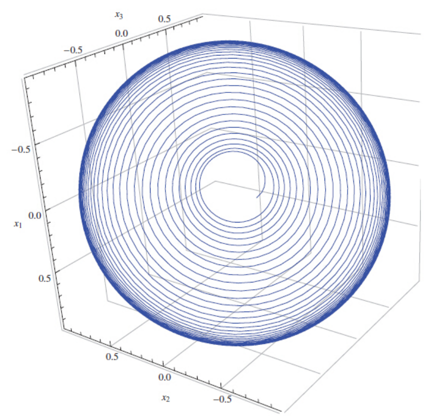


FIGURE 6: The trajectory of system (3) with $\alpha = 0.67$ (an asymptotically stable limit cycle)

as $a = 1$, $T_0 = 2$, $T_1 = 1$ and $T_2 = -9$. By increasing the fractional order α from 0.65 to 1, system (3) shows different dynamics around the original point $(0,0,0)^T$ such as asymptotical stability, limit cycle and chaos. When $\alpha \in [0.65, \frac{2}{3})$, the null solution of system (3) is asymptotically stable. In Fig. 5, the trajectory of system (18) with $\alpha = 0.65$ converges to zero solution asymptotically. As α increases to $\frac{2}{3} \approx 0.667$, a Hopf bifurcation occurs, then an asymptotically stable limit cycle appears in a neighborhood of the origin with $\alpha \in (\frac{2}{3}, 0.78]$. When $\alpha \in (0.78, 0.8)$, two symmetrical asymptotically stable limit cycles can be found. Finally, the chaotic attractor appears as $\alpha \in [0.8, 1)$.

4.2 Limit Cycle and Hopf Bifurcation of FNN

Limit cycle belongs to a special periodic solution of dynamic system, which plays an important role in neuroscience and biology [36] due to its good descriptions of stable periodic variation.

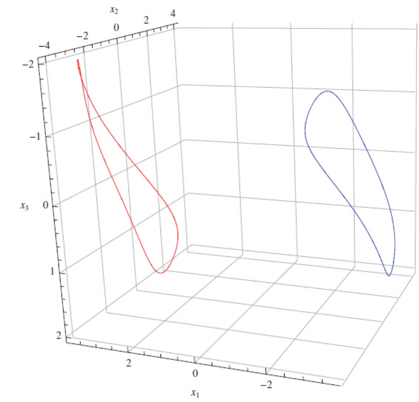


FIGURE 7: The trajectory of system (3) with $\alpha = 0.79$ (two asymptotically stable limit cycle)

Since fractional calculus was found in biological, economic, social, and neural systems, limit cycle of FNN was frequently discussed based on numerical evidence [16, 37, 38], such as Figs. 6 and 7 in last subsection.

In 2009, H.A. El-Saka et. al., presented a Hopf bifurcation condition for fractional-order system [39]. For an *n*-dimensional fractional-order systems,

$$ {}_0^C D_t^{\alpha} x = f(x,a), $$

the corresponding Hopf bifurcation condition at $a = a^*$ is

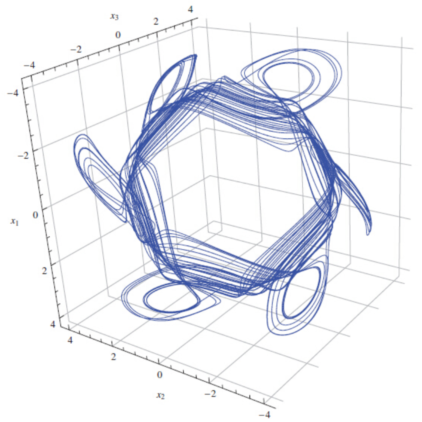
1. $|\arg[\lambda\,(a^*)]| = \frac{\alpha\pi}{2}$,
2. $|\lambda\,(a^*)| = 1$,
3. $\frac{d[\lambda(a^*)]}{da} \neq 0$.

Based on above condition, the Hopf bifurcation for FNN was studied in [19, 40, 41], which was a standard to discuss Hopf bifurcation of FNN in theory and simulation until a reference [42] in 2012 broke all above results.

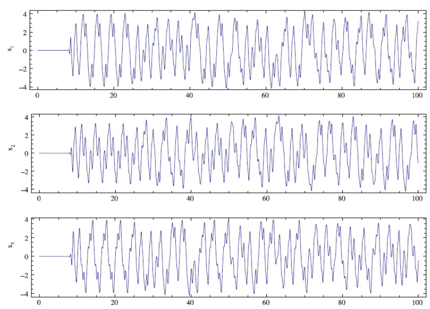
In [42], E. Kaslik and S. Sivasundaram pointed that periodic solutions in fractional-order autonomous dynamical systems do not exist. That means that FNN, modeled as fractional-order autonomous dynamical systems, can not own limit cycle and Hopf bifurcation in theory. Thus, the previous simulation results which seems to obtain the existence of limit cycle in FNN may be not effective. Since then, few research focused on the limit cycle and hopf Bifurcation, and more scholars paid their attentions on stability or chaos of FNN.

4.3 Stability of FNN

Due to the lack of theoretical results, more works for FNN were to explore the chaos and limit cycle via numerical simulations. However, as the non-existence of limit cycle in FNN was



(a)



(b)

FIGURE 8: The trajectory of system (3) with $\alpha = 0.9$ (chaotic)

provided in [42], more researches return to theoretical innovation at the stability of FNN [43–48].

The FHNN was introduced in Section III, and its modeling process was shown as system (1). According to system (1), a common FHNN model is modified and described by

$$ {}_0^C D_t^{\alpha} x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + I_i, \tag{4} $$

where $i = 1, 2, \cdots, n$ and n denotes the number of units in a neural network, $x_i(t)$ is the state of the ith unit at time t, f_j is the activation function of the jth neuron, $c_i > 0$ is the rate with which the ith neuron resets its potential to the resting state when disconnected from the network, a_{ij} represents the constant connection weight of the jth neuron on the ith neuron, and I_i is the constant external inputs.

Then system (4) can be transformed as the vector form as follows:

$$ {}_0^C D_t^{\alpha} x(t) = -Cx(t) + Af(x(t)) + I, \tag{5} $$

where $C = diag\{c_1, c_2, \cdots, c_n\}$, $A = (a_{ij})_{n\times n}$, $f(x) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n))^T$, $I = (I_1, I_2, \cdots, I_n)^T$.

To study the stability of FNN, the original method is to linearize system (5) and obtain its local stability conditions. However the linearized stability method is just to obtain the local stability. To gain the global stability conditions, Mittag-Leffler stability method (Lyapunov direct method) became popular and promoted more stability results. In addition, some other methods were proposed in last few years. Detailed, they are all list as follows.

4.3.1 Linearized stability method We first introduce the widely accepted necessary and sufficient condition for the stability of linear fractional-order systems.

Consider an n-dimensional linear fractional-order system,

$$ {}_0^C D_t^{q} x = Ax, $$

where $0 < q < 2$ is fractional order and $A \in R^{n\times n}$ is constant. Above linear autonomous fractional-order system is asymptotically stable if and only if

$$ |\arg(\lambda)| > \frac{q\pi}{2}, \ \forall \lambda \in \varphi(A), $$

where $\varphi(A)$ is the set of all eigenvalues of the matrix A. When $0 < q < 1$, equivalently, another necessary and sufficient condition of above inequality is

$$ |\mathrm{Im}(\lambda)| > \mathrm{Re}(\lambda) \tan\frac{q\pi}{2}, \ \forall \lambda \in \varphi(A). $$

Its corresponding stability region is shown in Fig. 9.

For FNN (5), the linearized stability method can be described by

1. Calculate the steady states $\bar{x}$s of FNN (5), which is the solutions of

$$ -C\bar{x} + Af(\bar{x}) + I = 0. $$

2. For an $\bar{x}$, calculate its Jacobian matrix by

$$ J(\bar{x}) = -C + ADf(\bar{x}), $$

where $Df(\bar{x}) = diag\{\frac{f_1(\bar{x}_1)}{dx_1}, \frac{f_2(\bar{x}_2)}{dx_2}, \cdots, \frac{f_n(\bar{x}_n)}{dx_n}\}$.

3. If all eigenvalues λs of the Jacobian matrix $J(\bar{x})$ satisfy

$$ |\arg(\lambda)| > \frac{q\pi}{2}, \ \forall \lambda \in \varphi(J(\bar{x})), $$

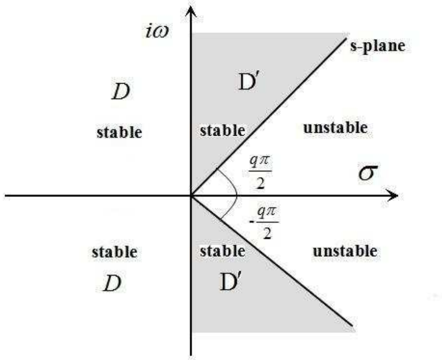


FIGURE 9: Stability region of the linear fractional-order system with order q.

the steady state $\bar{x}$ is locally asymptotically stable.

The linearized stability method is simple and effective to analyze the local stability of a steady state. However, by using this method, the global stability can not be gained due to the nonlinearity of FNN (5).

4.3.2 Mittag-Leffler stability method

To obtain the the global stability of FNN (5), one of the most frequently used methods is Mittag-Leffler stability method (or called Lyapunov direct method). If the nonlinear fractional-order system

$$ {}_0^C D_t^{\alpha} x(t) = f(t, x(t)), $$

has a steady state $\bar{x} = 0$. Then its solution is called to be Mittag-Leffler stable, if

$$ \|x(t)\| \leq [m(x_0) E_{\alpha}(-\lambda (t)^{\alpha})]^b, $$

where $\lambda > 0$, $b > 0$, $\|\cdot\|$ denotes an arbitrary norm and $m(x) \geq 0$ ($m(0) = 0$) satisfies locally Lipschitz condition on $x \in R^n$ with Lipschitz constant m_0 $E_{\alpha}(\cdot)$ is a Mittag-Leffler function.

Mittag-Leffler stability implies asymptotic stability for fractional-order systems i.e., $\lim\limits_{t\to+\infty} \|x(t)\| = 0$. To obtain the Mittag-Leffler stability, Y. Li, Y.Q. Chen and I. Podlubny presented a remarkable sufficient condition called Lyapunov direct method shown as follow.

(Fractional-order) Lyapunov direct method [49]: The steady state $\bar{x} = 0$ of above nonlinear fractional-order system is Mittag-Leffler stable if there exist positive constants α_1, α_2, α_3, a, b, and a continuously differentiable function $V(t, x(t))$ satisfying

$$ \begin{aligned} &\alpha_1 \|x\|^a \leq V(t, x(t)) \leq \alpha_2 \|x\|^{ab}, \\ &{}_0D_t^q V(t, x(t)) \leq -\alpha_3 \|x\|^{ab}, \end{aligned} $$

where $t \geq 0$, $q \in (0,1)$, $V(t, x(t)) : [0, \infty) \times D \to R$ satisfies locally Lipschitz condition on x; $D \subset R^n$ is a domain containing the origin. If the assumptions hold globally on R^n, $\bar{x} = 0$ is globally Mittag-Leffler stable.

In 2014, two inequalities were proposed, which provided powerful tool to choose qualified Lyapunov functions. The two inequalities [23, 50] are described by

$$ {}_0D_t^q |x(t^+)| \leq sgn(x(t))\, {}_0D_t^q x(t), \ \ \forall q \in (0,1), $$

$$ \frac{1}{2}\, {}_0D_t^q x^2(t) \leq x(t)\, {}_0D_t^q x(t), \ \ \forall q \in (0,1), $$

where $x(t) \in R$ be a continuous and derivable function. By using these two inequalities, $\|x\|_1$ (or $\sum\limits_{i=1}^{n} \beta_i |x_i|$, $\beta_i > 0$) and $\|x\|_2^2$ (or $x^T P x$, P is positive definite) can be effective Lyapunov functions to analyze the global stability of FNN (5). The corresponding stability results are given in [23, 26]. They always gave some conditions to ensure the existence and uniqueness of the steady state in FNN (5) based on contraction mapping theorem, and then obtained the global stability conditions by using Lyapunov direct method.

4.4 Stability of FDNN

As we know, time delay is unavoidable in practice and able to cause oscillations or instabilities in dynamic systems. Thus, in recent years, the stability of fractional-order delayed neural networks (FDNN) has become an hot topic with increasing interest [20–22, 24, 51–56]. The common model for FHNN with constant delay is described by

$$ \begin{cases} {}_0^C D_t^{\alpha} x(t) = -Cx(t) + Af(x(t)) + Bg(x(t-\tau)) + I, \\ x(t) = h(t), \ \ t \in [-\tau, 0]. \end{cases} \tag{6} $$

Similar with the stability methods for FNN, the current stability methods for FDNN include uniform stability method, linearized stability method, Lyapunov-like stability method and so on.

4.4.1 Linearized stability method for FDNN

Linearized stability method for FDNN [20, 24] is similar with that for FNN. Firstly, the steady states are calculated and the nonlinear FDNN is linearized under a steady state. Then, based on corresponding stability theorem of linear fractional-order delayed system, it can obtain the local stability condition for FDNN.

For FDNN (6), we assume $I=0$ and give its linearized equation with steady state $\bar{x}=0$,

$$ {}_0^C D_t^{\alpha} x(t) = -Cx(t) + \bar{A}x(t) + \bar{B}x(t-\tau). $$

Take the Laplace transform of above linearized equation and obtain a characteristic matrix $\Delta(s)$ as

$$ \Delta(s) = \begin{pmatrix} s^{\alpha}+c_1-a_{11}-b_{11}e^{-s\tau} & \cdots & -a_{1n}-b_{1n}e^{-s\tau} \\ \vdots & \ddots & \vdots \\ -a_{n1}-b_{n1}e^{-s\tau} & \cdots & s^{\alpha}+c_n-a_{nn}-b_{nn}e^{-s\tau} \end{pmatrix}. $$

Based on the stability theorem of linear fractional-order delayed system in [57], the steady state $\bar{x}=0$ is local Lyapunov asymptotically stable if one of following two conditions holds.

1. All the roots of the characteristic equation $det(\Delta(s))=0$ have negative real parts.

2. All the eigenvalues of $M=-C+\bar{A}+\bar{B}$ have negative real parts and the characteristic equation $det(\Delta(s))=0$ has no pure imaginary roots for any $\tau>0$.

Certainly, this method is only to the gain the local asymptotical stability condition. For global stability of FDNN, Lyapunov-like stability method is introduced next.

4.4.2 Lyapunov-like stability method for FDNN

Lyapunov direct method in [58] is just valid for FNN. The Lyapunov direct method was reformed in [22, 51], so that the new Lyapunov-like stability method can be used to analyze the global stability of FDNN. Besides, a comparison principle of fractional-order delay systems was proposed, which was an effective tool in the study of FDNN. The nonlinear FDNN could be transformed to a linear fractional-order inequality by choosing a suitable Lyapunov function. Then, due to the above comparison principle, the stability of linear fractional-order inequality can be obtained by analyze the corresponding linear fractional-order equation.

5 Challenges and Trends in FNN

As it shows in the above sections, the existing publications of FNN have been introduced briefly in both theory and application.

In recent few years, extreme learning machine [59] has become a hot topic, because of its higher speed than classical gradient-based learning algorithms. Different from the gradient-based update law for weights in neural networks, the weights' update law in extreme learning machine is an any continuous stochastic process, such as Brownian motion.

L*é*vy flight is a random walk whose step size is a kinds of heavy-tailed distributions and becomes large value occasionally [60]. The probability distribution function p of L*é*vy distribution is defined as

$$ p(x,\alpha,\beta) = \begin{cases} \dfrac{\sqrt{\frac{\beta}{2\pi}}}{e^{\frac{\beta}{2(x-\alpha)}}(x-\alpha)^{3/2}}, & x>\alpha, \\ 0, & x\le\alpha, \end{cases} $$

where α is the location parameter and β is the scale parameter. Thus, L*é*vy flight can be regarded as a fractional stochastic process, and has obtained some good computational results in the applications of optimization [61]. The connection of L*é*vy flight and extreme learning machine may gain better computational effects, and even bring an obvious improvement of machine learning.

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