Construction of Chaotic Sensing Matrix for Fractional Bandlimited Signal Associated by Fractional Fourier Transform

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***Abstract*—Fractional Fourier transform (FrFT) is a powerful tool for the non-stationary signals because of its additional degree of freedom in the time-frequency plane. Due to the importance of the FrFT in signal processing, most of the bandlimited sampling theorems in traditional frequency domain have been extended to fractional Fourier bandlimited signals based on the relationship between the FrFT and regular integer order Fourier transform (FT). However, the implementations of those existing extensions are not efficient because of the high sampling rate which is related to the maximum fractional Fourier frequency of the signal. Compressed Sensing (CS) is a useful tool to collect information directly which reduces sampling pressure, computational load as well as saving the storage space. The construction of sensing matrix is the basic issue. Most of CS demand that the sensing matrix is constructed by random under-sampling which is uncontrollable and hard to be realized by hardware. This paper proposes a deterministic construction of sensing matrix for the multiband signals in the fractional Fourier domain (FrFD). We give the sparse basis of the signal and derive the sensing matrix based on the analog to information conversion technology. The sensing matrix is constructed by random sign matrix and Toeplitzed matrix. The sub-sampling method is used to obtain the structural signal. Theoretically, the matrix satisfies the incoherent condition and the entire structure of system is practical. We show in this paper that the sampling rate is much lower than the Nyquist rate. The signal reconstruction is studied based on the framework of compressed sensing. The performance of the proposed sampling method is verified by the simulation. The probability of the successful reconstruction and the mean squared error (MSE) are both analyzed. The numerical results suggest that proposed system is effective for a spectrum-blind sparse multiband signal in the FrFD and demonstrate its promising potentials.**

***Index Terms*—Fractional Fourier Transform; Fractional Bandlimited Signal; Compressed Sensing; Chaotic Sensing Matrix;**

I. Introduction

Fractional multiband signals consist of a relatively small number of narrowband across a wide spectrum rang in the fractional Fourier domain (FrFD). They are widely used in the radar and communication [1–4]. A typical signal of fractional bandlimited signal is linear frequency modulation (LFM or chirp) which is widely used in imaging radars [1]. Most of operations for these signals are conversion from analog to digital to get information or back to the analog for further transmission. As the development of electronic technology, the frequency of manual signal is becoming higher which leads to the wider spectral rang of multiband signals and higher requirement of the analog-digital converter (ADCs). The high sampling rate also means a big capacity of storage. It would be disappointing to put the Nyquist sampling law into practice. Recently, Compressed Sensing (CS) provides us a good solution. Compressed Sensing is an extension of Nyquist law which combines the compression and the sampling at the same time [5, 6]. In the signal sparse prior knowledge, the original signal can be sampled and accurately recovered from sub-Nyquist sampling frequency.

In CS theory, the original signal $x \in \mathbb{R}^{N\times 1}$ can be projected from a high-dimensional space to a low-dimensional space $\mathbb{R}^{M\times 1}$ through a linear projection matrix $\mathbf{\Psi}$, if the original signal is sparse or have the sparsity in a transform domain. The low-dimensional space projection vector $\mathbf{\Psi}$ contains all the information of the original signal. The original signal x can be recovered from a measurement vector y accurately. It can be expressed as following:

$$\begin{cases} y = \mathbf{\Phi} x \\ x = \mathbf{\Psi} \boldsymbol{a} \end{cases} \rightarrow y = \mathbf{\Theta}\boldsymbol{a} \text{ where } \mathbf{\Theta} = \mathbf{\Phi}\mathbf{\Psi}, \tag{1}$$

where $\mathbf{\Phi} \in \mathbb{C}^{M\times N}(M \ll N)$ is the observation matrix (or measurement matrix) for x. $\boldsymbol{a} \in \mathbb{C}^N$ is a linear sparse representation for x on an appropriate sparse matrix $\mathbf{\Psi} \in \mathbb{C}^{N\times N}$. $\mathbf{\Theta}$ is sensing matrix which combines $\mathbf{\Phi}$ and $\mathbf{\Psi}$. It's not easy to solve the equation $y = \mathbf{\Theta}\boldsymbol{a}$, because it is a NP problem [7]. It can be handled via ℓ_1 norm minimization $\min ||\boldsymbol{a}||_1$. Most of optimal methods have been proposed base on the convex optimization problem [8].

CS has attracted lots of attention by suggesting that it is possible to breakthrough the limits of traditional Nyquist sampling theory in signal processing. Many research organizations have made substantial effort to promote the CS theory and try to put it into practice, for example: MRI (Magnetic

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Resonance Imaging) [9] in medical imaging field, compressed sensing SAR (Synthetic Aperture Radar) [10], tomographic SAR imaging in urban environment [11], MIMO (Multiple Input Multiple Output) radar [12] and wireless sensor network [13], UWB (ultra wide band) LFM [3, 4] and so on. Most of research about CS focused on three directions: observation matrix (measurement matrix $\mathbf{\Phi}$) construction, signal sparse representation and reconstruction algorithm design. Sensing matrix construction is the most active research area which must guarantee any sparse signal to be recovered from it. Tao [8] deeply analyzed the geometry of sensing matrix and proposed a well-known criterion named restricted isometry property (RIP). The RIP theorem can be expressed as: the observation matrix $\mathbf{\Phi}$ must satisfy the following condition for any K-sparse signal x.

$$(1-\delta_k)||x||_2^2 \leq ||\mathbf{\Phi}x||_2^2 \leq (1+\delta_k)||x||_2^2, \tag{2}$$

where $\delta_k \in (0,1)$. If there exist $\delta_{2k} \in (0,1)$ to make $\mathbf{\Phi}$ satisfying the $2K$-sparse RIP condition, the K-sparse x will be unique which can be recovered from y.

It is a complex work to judge whether the observation matrix $\mathbf{\Phi}$ meets the RIP condition. Fortunately, researchers introduce a statistical correlation between row vectors and column vectors to reduce the complexity of the observation matrix design, and it is called incoherence. Incoherence means there is small coherent in the sensing matrix. Some random matrixes whose elements satisfy *independent and identically distributed* (iid) [6, 14] have very high probability to satisfy the RIP condition, such as Gaussian distribution, Bernoulli distribution and sub-Gaussian distribution. Thus, the random observation matrixes become the first choice, such as random demodulation, *modulated wideband convertor* (MWC) [15] and other systems which take advantages of the random ± 1 matrix to construction.

In summary, a major issue for the practical compressed sensing is to construct a structurally observation matrix. Most of CS demand that the target signal should already be sampled completely. Otherwise, the random sampling cannot be reflected by common analog-to-digital converter (ADC). Thus, the sampling problem remains unsolved. Despite many previously mentioned efforts have been achieved on the sparse signal in the frequency domain, few contribute to the fractional order bandlimited signal. Sometimes, signals which are not sparse in the conventional frequency domain (FD) may be sparse in the fractional Fourier domain (FrFD), but the signals cannot have the same sparsity in both FrFD and FD [16] simultaneously. Liu etc. [1] proposed the dechirped method for high-speed target echo which is of sparsity in the FrFD, and gave an optimization search algorithm based on sparsity of the reconstructed range profiles, but it is based on the accurate estimation of the order which is not easy to be realized. In this paper, we extend the CS from the FD to the FrFD, and propose a structured chaotic sensing matrix for multiband signal in the FrFD.

The outline of this paper is organized as follows. In section II, some basis theorems and problem formulation are introduced, including the definition of α-bandlimited signal and its sampling method in the FrFD. In section III and IV, a constructed chaotic matrix and a Toeplitzed sensing matrix for the α-bandlimited signals is proposed. Detailed analysis of mutual coherence are presented. In section V, the result of the simulation and the potential application are discussed.

II. Preliminaries and Problem Formulation

A. Fractional Fourier Transform

The definition of fractional Fourier transform [17] is given as following:

$$F_\alpha(u) = \mathcal{F}^\alpha\{f(t)\} = \int_{+\infty}^{+\infty} K_\alpha(u,t)f(t)\mathrm{d}t, \tag{3}$$

where $\mathcal{F}^\alpha$ denotes FrFT operator. The kernel function $K_\alpha(u,t)$ is given by following:

$$K_\alpha(u,t) = \begin{cases} A_\alpha e^{j\frac{t^2+u^2}{2}\cot\alpha - jtu\csc\alpha}, & \alpha \neq k\pi, \\ \delta(u-t), & \alpha = 2k\pi, \\ \delta(u+t), & \alpha = (2k+1)\pi, \end{cases} \tag{4}$$

where $A_\alpha = \sqrt{\frac{1-j\cot\alpha}{2\pi}}$, $k \in \mathbb{Z}$.

The FrFT operator $\mathcal{F}^\alpha$ satisfies the following properties.

1) Reversibility: the inverse FrFT operator is $-\alpha$ fractional Fourier transform, denoted by:

$$K_{-\alpha}(u,t) = A_{-\alpha}e^{-j\frac{t^2+u^2}{2}\cot\alpha + jtu\csc\alpha}. \tag{5}$$

2) Boundary: the relationship between the FrFT and FT as following:

$$\begin{aligned} &\mathcal{F}^\alpha\{f(t)\}(u) \\ =&\sqrt{2\pi}A_\alpha e^{j\frac{u^2}{2}\cot\alpha}\mathscr{F}\left[f(t)e^{j\frac{t^2}{2}\cot\alpha}\right](u\csc\alpha), \end{aligned} \tag{6}$$

where $\mathscr{F}$ is the integer order Fourier transform operator. The α order FrFT is simplified to be the traditional Fourier transform $\mathcal{F}^\alpha\{f(t)\}(u) = \mathscr{F}[f(t)]$ when $\alpha = 2n\pi + \frac{\pi}{2}$.

3) Additivity: the operator $\mathcal{F}^\alpha$ is additive.

$$\mathcal{F}^\alpha\mathcal{F}^\beta = \mathcal{F}^{\alpha+\beta}.$$

B. Definition of Fractional Bandlimited Signal

A Ω_α fractional bandpass signal $f(t)$ satisfies that its energy is finite. $F_\alpha(u)$ is zero outside the region $(-\Omega_h, -\Omega_l) \cup (\Omega_l, \Omega_h)$.

$$F_\alpha(u) = 0, \text{ for } |u| \geq \Omega_h \text{ and } |u| \leq \Omega_l, 0 \leq \Omega_l \leq \Omega_h, \tag{7}$$

and its Parseval expression as following:

$$\langle x(t), x^*(t)\rangle = \langle X_\alpha(u), X_\alpha^*(u)\rangle, \tag{8}$$

where $\Omega_\alpha = \Omega_h - \Omega_l$ denotes bandwidth. $\Omega_0 = (\Omega_h + \Omega_l)/2$ denotes fractional carrier "frequency". If the fractional carrier "frequency" Ω_0 was known, the signal $f(t)$ can be restored as following [18, 19]:

$$f(t) = A_\alpha e^{-\frac{j}{2}t^2 \cot\alpha} \sum_{n=-\infty}^{+\infty} f(nT_s) e^{\frac{j}{2}(nT_s)^2 \cot\alpha} \times \frac{\sin[(t-nT_s)\Omega_\alpha \csc\alpha]}{(t-nT_s)\Omega_\alpha \csc\alpha} e^{j\Omega_0 \csc\alpha (t-nT_s)}. \tag{9}$$

If Ω_l is unknown, $f(t)$ can be restored as following:

$$f(t) = \sqrt{\frac{1+j\cot\alpha}{2\pi}} e^{-\frac{j}{2}t^2 \cot\alpha} \sum_{n=-\infty}^{+\infty} f(nT_s) \times e^{\frac{j}{2}(nT_s)^2 \cot\alpha} \frac{\sin[(t-nT_s)\Omega_h \csc\alpha]}{(t-nT_s)\Omega_h \csc\alpha}, \tag{10}$$

where $T_s = \pi \sin\alpha / \Omega_h$. This basic result is the well-known sampling theorem by Xia [16] in the FrFD. It is observed that the sampling rate depends on its maximum fractional Fourier frequency Ω_h when the signal is bandlimited in the region $(-\Omega_h, -\Omega_l) \cup (\Omega_l, \Omega_h)$ with Ω_l unknown. The analog to digital converter must work at very high rate in case of spectrum aliasing, it is difficult to be realized in practice.

C. Problem Formulation

A sparse multiband signal in αth order FrFD contains N-nonzero narrow fractional Fourier bandlimited signals. Its energy concentrates in limited disjoint fractional frequency bands. There is no prior information of position in advance. Taking the number of active bands of signal as a prior, the FrFT of the sparse multiband signal is as Fig. 1, N is even due to the conjugate symmetry of the signal.

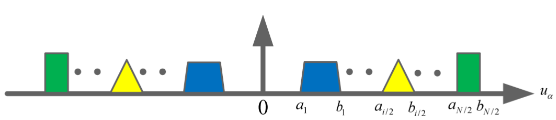


Fig. 1. The illustration of a multiband signal in α-order FrFT domain

1) The fractional Nyquist sampling rate is consider to be $2b_{N/2}$ where $b_{N/2}$ is the maximum fractional "frequency" of the signal.
2) The valid fractional "frequency" components of signal $x(t)$ are the set of non-zero frequency $\mathcal{F}_\alpha = U_{i=1}^{N/2} \mathcal{F}_{\alpha,i}$, and the maximum width of the signal is $B_\alpha = \max(b_i - a_i), i \in [1, N/2]$, and $2B_\alpha \csc\alpha$ is considered to be the minimum sampling frequency for the multiband signal [16].

In this situation, the high sampling rate may be not necessary. It is necessary to propose a practical sampling method for fractional Fourier bandlimited signal.

III. Construction of Sensing Matrix

A. Matrix Design

Suppose $x(t)$ is a α-order fractional Fourier bandlimited signal with the maximum bandwidth not exceed B_α. It is easy to prove $\exp(j\pi t^2 \cot\alpha)x(t)$ is also a bandlimited signal in the FD. The basis of the fractional bandlimited signal $\exp(j\pi t^2 \cot\alpha)x(t)$ is as following:

$$\begin{aligned} \mathcal{X}_\alpha(u) = [& e^{-\frac{j}{2}(u-L_0B_\alpha)^2 \cot\alpha} \mathcal{X}_\alpha(u - L_0B_\alpha), \\ & e^{-\frac{j}{2}(u-lB_\alpha)^2 \cot\alpha} \mathcal{X}_\alpha(u - lB_\alpha), \\ & \cdots, \mathcal{X}_\alpha(u), \cdots, \\ & e^{-\frac{j}{2}(u+lB_\alpha)^2 \cot\alpha} \mathcal{X}_\alpha(u + lB_\alpha), \\ & e^{-\frac{j}{2}(u+L_0B_\alpha)^2 \cot\alpha} \mathcal{X}_\alpha(u + L_0B_\alpha)]. \end{aligned} \tag{11}$$

The original signal $x(t)$ can be recovered from the sparse basis $\mathcal{X}_\alpha(u)$. We propose a sensing matrix of $\exp(j\pi t^2 \cot\alpha)x(t)$ with a cascade of three matrices, and it is denoted by following:

$$\boldsymbol{\Theta} = \boldsymbol{SFD}, \tag{12}$$

where $\Theta_i(\cdot)$ is the DFT of ith sign pattern. $\boldsymbol{F} \in \mathbb{R}^{M \times M}$ is an orthogonal matrix which can be derived by the FFT (fast Fourier transform), the DCT (discrete cosine transform) or the WHT (Walsh Hadamard transform). The elements of DFT kernel matrix are defined as follow:

$$\boldsymbol{F}_{lk} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi lk}{M}}, l \geq 0, k \leq M-1, \tag{13}$$

where $M = 2b_i/B_\alpha + 1$

$\boldsymbol{D} = diag(d_{L_0}, \cdots, d_l, \cdots, d_{-L_0}) \in \mathbb{R}^{M \times M}$ is a diagonal matrix, where d_l is denoted by following:

$$d_l = \begin{cases} \frac{1}{M}, & l = 0, \\ \frac{1-\theta^l}{2j\pi l}, & l \neq 0. \end{cases} \tag{14}$$

where $\theta = e^{-j\frac{2\pi}{M}}$

$\boldsymbol{S} \in \mathbb{R}^{m \times M}$ is the sub-sampling operator which randomly selects subset of the rows of the $\boldsymbol{FD}$ matrix. $\boldsymbol{S}$ is a chaotic-based matrix which is a uniform random permutation matrix (i.e. stochastic vector with entries taking the values ± 1 with probability 1/2). m is the number of rows can be interpreted as the number of sampling channels.

B. Incoherence Analysis

Incoherence implies that the maximum magnitude of the entries of the measurement matrix $\boldsymbol{\Theta}$ is relatively small. Generally speaking, the smaller the mutual coherence, the better quality of reconstruction is achieved. Incoherence is much easier to verify than the RIP.

Theorem 1 (Incoherent Measurement [20])**.** *$\boldsymbol{\Phi} \in \mathbb{R}^{m \times M}$ is the observation matrix, $\boldsymbol{\Psi} \in \mathbb{R}^{M \times M}$ is the sparse basis matrix, and suppose all the column vector of matrix are normalized. The definition of μ mutual coherent for $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ as following:*

$$\mu(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = \sqrt{M} \max_{i,j} \langle \phi_i \psi_j \rangle, \tag{15}$$

where ϕ_i is the ith row of the $\boldsymbol{\Phi}$. ψ_j is the jth column of the $\boldsymbol{\Psi}$. For any K-sparse signal x,

$$m > a\mu K \log M, \tag{16}$$

where a is positive number. There is a big probability for reconstruction of original signal x by $y = \mathbf{\Phi}x$.

The parameter μ is denoted as following [21]:

$$1 < \mu(\mathbf{\Phi\Psi}) < \sqrt{M}. \quad (17)$$

It is observed that the smaller coherent μ between the observation matrix $\mathbf{\Phi}$ and sparse basis $\mathbf{\Psi}$, the less number of observation points. This method is also called the incoherent measurement. The Gaussian sensing matrix is highly incoherent with all matrices $\mathbf{\Psi}$. The incoherence between $\boldsymbol{S}$ and $\boldsymbol{FD}$ is small which has been proved in Ref. [14].

Random observation matrix has following advantages:

1) Every observed value in the observation matrix $\boldsymbol{S}$ has the same power for the result, so the loss of a small amount of observations does not affect the signal recovery;
2) If signal x is sparse in some transform domain $\mathbf{\Psi}$ rather than in the actual application environment. At this situation, the sensing matrix $\mathbf{\Theta}$ needs to meet the RIP condition. It is easy to prove $\mathbf{\Theta} = \mathbf{\Phi\Psi}$ satisfying RIP, if $\mathbf{\Phi}$ is random matrix and $\mathbf{\Psi}$ is orthogonal matrix [14]. $\mathbf{\Theta}$ is also a random matrix which has the same feature with the $\mathbf{\Phi}$.
3) A complete random matrix is hard to realize practically. Some results like random demodulation and *modulated wideband converter* (MWC) have proved that the partial random matrix also satisfy the RIP.

IV. Optimal Sensing Matrix

Most of CS matrixes are completely random whose elements are compliance with the *independent and identically distributed*. There are some disadvantages. Firstly, putting the completely stochastic into practice means high computational complexity; Secondly, the random matrix must be stored before used, so the high degree of freedom in the matrix equals to the high design complexity, and big capacity of storage. Some researchers make effort to improve the complexity of the complete random unstructured matrix. Tropp etc. [22] introduce a random filter system, the proposal makes convolution with the input signal and a random filter, then the product can be sub-sampled. The relationship between sampling value and the input signal can be expressed as Toeplitz random matrix. Romberg [23] made the input signal circular convolution with a specially constructed random vector. Haupt etc. [24] used random Toeplitzed matrix to construct observation matrix, and gave the RIP constraint, and utilized it to identify the discrete linear time invariant system. In this part, we use a circulant measurement matrix to construct the sensing matrix.

A. Circulant Sensing Matrix

The circulant matrix $\boldsymbol{S}$ can be constructed by following:

$$\boldsymbol{V} = \frac{1}{M} G^* P G, \quad (18)$$

where the factor $\frac{1}{M}$ is used to keep the columns of $\boldsymbol{S}$ have the normalized norm.

The matrix $\boldsymbol{G}$ is denoted by:

$$\boldsymbol{G}_{t,w} = e^{j2\pi\frac{(b-1)(w-1)}{M}}, 1 < b, w < M, \quad (19)$$

where b, w are the index of the matrix. The nonzero entries of diagonal matrix P is defined by follows:

1) $w = 0$, $p_w = \{1, -1\}$ with the equal probability;
2) $0 < w < (M-1)/2$, $p_w = e^{j\theta(w)}$ where $\theta(w)$ is the random phase, drawn uniformly in $[0, 2\pi]$;
3) $(M-1)/2 < w < M-1$, $p_w = p^*_{M-w}$;

The random sampling matrix $\boldsymbol{S}$ is a full random matrix whose elements are selected in $\{\pm 1\}$ with equal probability. The proposed matrix $\boldsymbol{V}$ is constructed by circulant shift of ith vector to get another vector of the matrix. v_i is the ith column of circulant matrix $\boldsymbol{V}$ which is generated by i-order circulant shift of v_j, $v_i = v_{j+\tau_i}$. τ_i is defined by $\{\tau_i\}_{1\le i\le m} \subset \{0, 1, \cdots, M-1\}$ and τ_i is a random variable of discrete uniform distribution.

B. Incoherent Analysis

Theorem 2. *[23], Suppose $\mathbf{\Phi}$ is an orthogonal matrix. The matrix V is construct by the above circulant method, for $0 < \eta < 1$, then the correlation coefficient $\mu(\boldsymbol{V}, \mathbf{\Psi})$ between V and $\mathbf{\Psi}$ is as following:*

$$\mu(\boldsymbol{V}, \mathbf{\Psi}) \le \sqrt{2 \log \frac{M^2}{\eta}}. \quad (20)$$

The proof is as follows. Rewrite (18) as:

$$\boldsymbol{V} = (\frac{1}{\sqrt{M}}\boldsymbol{G})^* \boldsymbol{P} (\frac{1}{\sqrt{M}}\boldsymbol{G}), \quad (21)$$

where $\mathbf{\Omega} = \frac{1}{\sqrt{M}}\boldsymbol{G}$ is $M \times M$ discrete Fourier transform matrix.

The correlation coefficient is computed by finding the maximum factor of the product of the two matrixes as:

$$\begin{aligned} \mu(\mathbf{\Phi\Psi}) &= \sqrt{M} \max_{i,j} \langle \boldsymbol{\phi_i} \boldsymbol{\psi_j} \rangle \\ &= \sqrt{M} \max_{i,j} (\mathbf{\Phi\Psi})_{i,j}. \end{aligned} \quad (22)$$

The above function can be expressed as following based on the relationship between circulant matrix and discrete Fourier matrix.

$$\mu(\mathbf{\Phi\Psi}) = \sqrt{M} \max_{i,j} (\boldsymbol{V} \frac{1}{\sqrt{M}} \boldsymbol{G})_{i,j}, \quad (23)$$

where $\mathbf{\Omega}$ is conjugate, symmetry and orthogonal matrix. $\boldsymbol{V\Omega}$ is computed as:

$$\begin{aligned} \boldsymbol{VG} =& (\frac{1}{M}\boldsymbol{G}^* \boldsymbol{PG})\boldsymbol{G} \\ =& \frac{1}{M}\boldsymbol{G}^* \boldsymbol{P}(\boldsymbol{GG}) \\ =& \frac{1}{\sqrt{M}}\boldsymbol{G^*P} \begin{pmatrix} 1 & 0_{1\times(M-1)} \\ 0_{1\times(M-1)} & I^T \end{pmatrix}, \end{aligned} \quad (24)$$

where I is $(M-1)\times(M-1)$ identity matrix. $\boldsymbol{P}$ is diagonal matrix, then:

$$\begin{aligned}\boldsymbol{VG} &= \frac{1}{\sqrt{M}}\boldsymbol{G}^* \begin{pmatrix} 1 & 0_{1\times(M-1)} \\ 0_{1\times(M-1)} & 0_{(M-1)\times(M-1)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{M}} & 0_{1\times(M-1)} \\ 0_{1\times(M-1)} & 0_{(M-1)\times(M-1)} \end{pmatrix}.\end{aligned} \tag{25}$$

From the above equation, the matrix $\boldsymbol{V\Omega}$ only have a nonzero element, then we get

$$\mu(\boldsymbol{\Phi\Psi}) = 1. \tag{26}$$

There is the minimum correlation between $\boldsymbol{V}$ and $\boldsymbol{G}$, so circulant matrix $\boldsymbol{V}$ is optimal for signals sparse in fractional Fourier frequency domain and achieves the minimal measurements.

V. Numerical Simulation and Discussions

A. Signal Model

To evaluate the performance of the sensing matrix, we use chirp signal as the test subject to simulate the proposed sensing matrix. Chirp signal is a typical fractional bandlimited signal. The original signal is given by following:

$$\begin{aligned}x(t) &= \sum_{i=1}^{N/2} x_i(t) + n(t) \\ &= \sum_{i=1}^{N/2} E_i e^{j2\pi kt^2}\cos(2\pi f_i(t-\tau_i)) + n(t),\end{aligned} \tag{27}$$

where $x_i(t)$ is the ith signal component. $n(t)$ is zero-mean Gaussian noise. E_i is the amplitude of signal. $k = 1M$ is the signal modulated frequency, sampling time is $5s$, $B = kt = 10M$Hz is the fractional maximum bandwidth of signal. τ_i is the time delay between different signals, and f_i is random frequency carrier. In terms of the CS algorithm, the sparsity level is equal to N, which is the total number of band occupied by the signal during the observation interval time. The signal is bandlimited with order $\alpha = -1.11\times 10^{-6}$, and its fractional bandwidth is $B_\alpha = 1M$.

We assume signal's Nyquist sampling rate $f_{NYQ} = 1G$Hz, and the carrier frequency is f_i which is less than $0.95G$Hz. The sampling rate is $10M$Hz. The compressed rate can be computed as M/m. The OMP algorithm was run to recovery the original signal. A set of 500 trials were carried out for each value to ensure statistically stable results. We choose signal components with a fixed time delay. The SNR (signal to noise ratio) is defined by $10\log(||x||^2/||n||)$.

We can use successful recovery rate and the normalized mean squared error (NMSE) to measure the performance of the compressed sampling by proposed signal. The successful recovery rate is defined as the ratio of the number of empirical successful reconstruction and total trials, and successful recovery means that the recovered support coincides with the basis. The computation of NMSE is as following:

$$\text{NMSE} = \frac{\int_{-\infty}^{+\infty}|x(t)-\hat{x}(t)|^2\text{d}t}{\int_{-\infty}^{+\infty}|x(t)|^2\text{d}t}, \tag{28}$$

where $x(t)$ is the original signal and $\hat{x}(t)$ is denoted the recovered signal.

B. Sensing Performance

Fig. 2 gives the performance of successful recovery rate for the full random matrix and circulant matrix with different channels and number of bandlimited signals in the noise free situation. The experimental condition is set as follows: the number of channels is fixed to be 34; The horizontal axis is the number of channels rang from 10 to 60 with a 2-channel step; The number of sampling points is 91, the number of bands is $\{4, 6, 8\}$. It is observed that the successful recovery rate of the circulant matrix is little bit higher that the full random matrix in the noise-free.

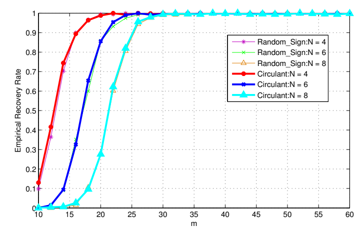


Fig. 2. Comparison between the random and circulant matrix with different Channels and Sparsity

The tradeoff between successful recovery rate and sparsity is shown in Fig. 3, the successful recovery rate decreases with the increasing of sparsity of the signal. The circulant matrix shows better robust compare with Fig. 2.

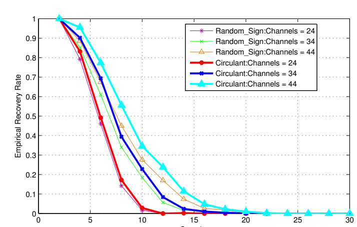
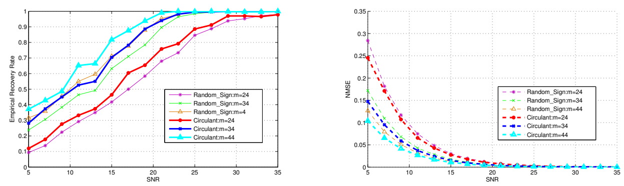


Fig. 3. Performance of the random and circulant matrix with different Sparsity

Fig. 4 shows the performance of robustness in the noisy condition by comparison between the full random matrix and circulant matrix. The experimental condition is as follows: The length of record time is 5s. The "SNR" is signal to noise ratio in statistics which range from 5dB to 35dB with a 2dB step. The number of channels is fixed to be $\{24, 34, 44\}$. The number of bands is 6. The compressed ratio can be compute as the ratio of the total sampling rate and the Nyquist sampling rate. In this situation, the compressed ration is $\{0.21, 0.34, 0.44\}$ for different number of channels $\{24, 34, 44\}$ respectively. The result of the successful recovery rate and the NMSE are shown in Fig. 4(a) and Fig. 4(b) respectively. It is observed that the circulant matrix with greater recovery rate and less NMSE has better performance than the full random sensing matrix.

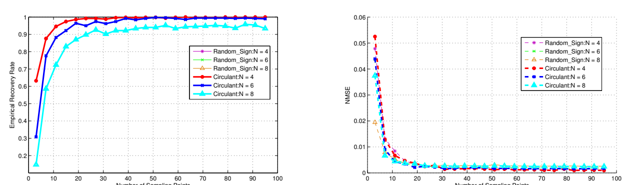
That means the circulant matrix has better robustness. The decrease of degree of freedom for circulant sensing matrix does not come with sacrifice of recovery accuracy. Full random measurement matrix is sufficient for the signal recovery but not necessary.



(a) Empirical recovery rate (b) Empirical NMSE

Fig. 4. Performance of the random and circulant sensing matrix with different SNR and Channel

Fig. 5 shows the relationship between number of sampling points and successful recovery rate with different number of sparse signals. The length of sampling time changes with the number of sampling points. Fig. 5(a) and Fig. 5(b) gives the performance of the successful recovery probability and NMSE respectively. The experimental condition is as follows: the SNR is fixed to be 35dB, the number of bands is fixed to $\{4, 6, 8\}$. The number of sampling points as a self-variable varies from 3 to 95 with a 4-point step. In Fig. 5(a), the performance of the circulant sensing matrix is almost coincide with the full random. They have the equal performance with the same number of sampling points. Fig. 5(b) shows the equal performance for NMSE. In some applications, it is possible to get part of information with short sampling time.



(a) Empirical recovery rate (b) Empirical NMSE

Fig. 5. Performance of the random and circulant with different compressed rate

The circulant random sensing matrix shows the better performance which is mainly due to the mutual incoherence provided by the corresponding matrices. Comparing Fig. 2, Fig. 3 and Fig. 4, the circulant matrix has better robustness in a noisy environment, which would broad more application prospects. The random sign construction is referred to as the global randomizer. The circulant matrix is referred to as the local randomizer. In both cases the sensing matrixes are randomly generated. The random sign matrix needs $m \times M$ bits to store the sensing matrix since it is comprised by ± 1. The circulant matrix needs $n \times m \times M/2$ bits where n stands for the number of bits used to represent each non-integer entry $\theta(w)$. To some extent, the implementation of the deterministic circulant matrices is more hardware demanding than the Gaussian matrix.

C. Potential Applications

Fractional Fourier transform is a generalization of the frequency Fourier transform which can effectively spread the frequency space. Some applications in frequency domain can be extend to the fractional Fourier domain as a potential candidate, just like fractional Fourier domain communication system [25], image compression and encryption [1, 26]. Those promising applications also have to face the bottleneck of sampling rate and storage space, so compressed sampling in fractional Fourier domain is necessary for those promising applications. Compressed sampling in FrFD opens up a potential orientation for some non-sparse signal processing in the FD.

VI. Conclusion

This paper extended the classical CS from frequency domain to the fractional Fourier domain, since the frequency domain is a special case of the FrFD. This paper takes the fractional bandlimited as priors and introduces construction of the sensing matrix for multi fractional bandlimited signals in the FrFD. We proposed two typical methods for the sensing matrix, and both of them can work well for the bandlimited signal. The proposed sensing matrixes have their own advantages respectively. The optimal method with circulant matrix has higher successful recovery rate for the noised signal, and the classic random matrix has the less capacity of storage requirement compared with the circulant matrix. It is hard to judge which on is more optimal, they both have their respective suitable applications. The circulant sensing matrix is more practical in some actual applications since it has more robustness in noisy circumstance. There is also a future work to do before the system can be put into the practise, including reducing the computational complexity of the algorithm.

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