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Research article

Admissibility and robust stabilization of continuous linear singular fractional order systems with the fractional order α: The $0<\alpha<1$ case

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ABSTRACT

This paper presents three different necessary and sufficient conditions for the admissibility and robust stabilization of singular fractional order systems (FOS) with the fractional order α: $0 < \alpha < 1$ case. Two results are obtained in terms of strict linear matrix inequalities (LMIs) without equality constraint. The system uncertainties considered are norm bounded instead of interval uncertainties. The equivalence between quadratic admissibility and general quadric stability for FOS are derived. A condition is not only strict LMI condition without quality constraint but also avoid a singularity trouble caused by the superfluous solved variable. When $\alpha = 1$ and $E=I$, the three results reduce to the conditions of stability and robust stabilization of normal integer order systems. Numerical examples are given to verify the effectiveness of the criteria. With the approaches proposed in this technical note, we can analyze and design singular fractional order systems with similar way to the normal integer order systems.

1. Introduction

Singular systems are also referred to as descriptor systems, implicit systems, generalized state-space systems, differential-algebraic systems, or semi-state systems [1]. Singular systems are governed by the so-called singular differential equations, which endow the systems with many special features that are not found in classical systems. Among these are impulse terms and input derivatives in the state response, non-properness of transfer matrix, non-causality between input and state (or output), consistent initial conditions, etc., making the study of singular systems more sophisticated than that of classical linear systems. In recent years, researchers have noticed that the description of some phenomena are more accurate while the fractional derivative is introduced. A great number of fundamental notions and results of conventional integer order systems have been successfully extended to fractional order systems. The approaches of fractional order control systems are mainly borrowed from those of integer order control systems [1–7].

Stability is fundamental to all control systems, certainly including singular fractional order control systems. A great number of fundamental notions and results on stability of normal integer order systems have been successfully extended to singular integer order systems and normal fractional order systems, respectively [8–19]. In what concerns automatic control, I. N. Doyea [8] gives sufficient conditions for the robust asymptotical stabilization of uncertain descriptor fractional-order systems with the fractional order α satisfying $0 < \alpha < 2$, but the approach is restrictive since it involves the assumption that the system is normalizable. So the approach is essentially a normal system solution method instead of sheer singular system solution method. Podlubny [9] proposes a generalization of the PID controller, namely the $PI^{\lambda}D^{\mu}$ controller, involving an integrator of order λ and a differentiator of order μ. Yan and Chen [10] propose the definition of Mittag-Leffler stability and introduce the fractional Lyapunov direct method. Zhang and Chen [11] comment the fractional order derivatives and systems and point out the differences and relationships between fractional order systems and integer order systems. Sabatier [12] et al. discuss the LMI stability conditions for FOS but the results are limited only for certain and without disturbance FOS and some results are questionable since (76) in [12] involves a mismatched dimensional matrix θ_k. Matigon [13] presents stability properties for fractional differential systems which are fundamental in stability analysis for certain FOS. Lu and Chen et al. [14–16] study the robust stability and stabilization of fractional order interval systems. However, the uncertainties involved are not satisfied as so called norm bounded condition. Aguila-Camacho et al. [17] consider Lyapunov functions for fractional order systems. Farges [18] considers H_∞ analysis and control of commensurate fractional order systems. Recently, Zhang and Chen [19] develop D-stability based LMI criteria of stability and stabilization for fractional order systems and introduce new

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effective approaches to deal with FOS control issues.

Although, there are huge number of contributions in the above two areas, up to now, to the best of our knowledge, for norm bounded uncertainties, the robust stabilization of normal fractional order systems remain open problems, let alone the admissibility and robust stabilization of singular fractional order systems. In this technical note, we show that LMI based criteria of admissibility and robust stabilization for singular fractional order linear time systems are established in accord with those of continuous normal integer order linear time invariant systems. Even for singular integer order systems, many admissibility criteria involve non-strict linear matrix inequalities and equality constraints. The criteria developed in this technical note overcome the above weakness and work so effectively without the singularity trouble such as in Example 3.1 that it can be regarded as a natural extension of Lyapunov stability from singular fractional order systems to normal integer orders systems with the consistent format.

2. LMI criterion of admissibility of continuous linear singular FOS

Consider a continuous linear singular fractional order systems described by

$$ED^{\alpha}x(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the physical state of the system, $u(t) \in \mathbf{R}^l$, is the control input, and $E \in \mathbf{R}^{n\times n}$ is the system singular matrix. It may be singular, and we assume $0 \le \text{rank}(E) = m \le n$. $A \in \mathbf{R}^{n\times n}$ is the system matrix, and $B \in \mathbf{R}^{n\times l}$ is the system input matrix. The symbol $D^{\alpha}x(t)$ denotes the fractional order derivative of function $x(t)$ which fall into two main classes: Riemnn-Liouville derivative and Grunward-Letnikov derivative, on one hand, defined as [9,10]

$$D^{\alpha}x(t) = \frac{d^{\alpha}x(t)}{dt^{\alpha}} = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_0^t (t-\tau)^{k-\alpha-1}x(\tau)d\tau,$$

or Caputo derivative, on the other hand, defined as [9,10]

$$D^{\alpha}x(t) = \frac{d^{\alpha}x(t)}{dt^{\alpha}} = \frac{1}{\Gamma(k-\alpha)}\int_0^t (t-\tau)^{k-\alpha-1}x^{(k)}(\tau)d\tau,$$

where k is an integer satisfying $k-1 < \alpha \le k$, $\Gamma(\cdot)$ is Euler Gamma function.

Here and throughout the paper, only the Caputo definition is used since its Laplace transform allows using initial values of classical integer order derivatives with clear physical interpretations. In the rest of the paper, D^{α} is used to denote the Caputo frational derivative of order α.

For the unforced linear singular fractional order system

$$ED^{\alpha}x(t) = Ax(t). \tag{2}$$

We also denote (2) with the triple (E, A, α). If matrix $E(=I)$ is nonsingular, System (2) reduces normal fractional order system

$$D^{\alpha}x(t) = Ax(t). \tag{3}$$

Similar to the admissibility definition of singular integer order systems [1], we introduce the following admissibility definition of singular fractional order systems.

Definition 2.1. The triple (E, A, α) is said to be regular if $\det(s^{\alpha}E - A)$ is not identically zero. The triple (E, A, α) is said to be impulse-free if $\deg(\det(sE - A)) = \text{rank}(E)$. The triple (E, A, α) is said to be stable if all the roots of $\det(s^{\alpha}E - A) = 0$ satisfy $|\arg(\text{spec}(E, A, \alpha))| > \alpha\frac{\pi}{2}$, where $\text{spec}(E, A, \alpha)$ is the spectrum (set of all roots) of $\det(s^{\alpha}E - A) = 0$. The triple (E, A, α) is said to be admissible if it is regular, impulse-free and stable.

For notational simplicity, we write $\text{spec}(E, A) = \text{spec}(E, A, 1)$ and $\text{spec}(A, \alpha) = \text{spec}(I, A, \alpha)$ which are the spectrums for integer order singular systems and for fractional order normal systems respectively.

In order to develop the admissibility criterion for singular FOS, we firstly introduce and deduce the following lemmas.

Lemma 2.1 (19). *System (3) is asymptotically stable if and only if there exist two matrices $X, Y \in \mathbf{R}^{n\times n}$, such that*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{4}$$

$$aAX - bAY + aXA^{\mathrm{T}} + bYA^{\mathrm{T}} < 0. \tag{5}$$

where $a = \sin\left(\alpha\frac{\pi}{2}\right)$, $b = \cos\left(\alpha\frac{\pi}{2}\right)$.

Lemma 2.2 (22). *System (2) is regular if and only if there exist two nonsingular M and N such that*

$$MEN = \begin{bmatrix} I_m & 0 \\ 0 & J_{n-m} \end{bmatrix}, \quad MAN = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & I_{n-m} \end{bmatrix}, \tag{6}$$

where J_{n-m} is a nilpotent matrix.

Assume System (2) is regular, then based on Lemma 2.2, System (2) can be transformed into

$$\begin{cases} D^{\alpha}x_1(t) = \bar{A}_1x_1(t) \\ J_{n-m}D^{\alpha}x_2(t) = x_2(t) \end{cases},$$

where $\left[x_1(t)\ x_2(t)\right]^{\mathrm{T}} = N^{-1}x(t)$, the initial state response of System (2) is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = N\begin{bmatrix} E_{\alpha,1}(At^{\alpha})x_1(0) \\ -\sum_{k=1}^{n-m-1}\delta^{((k-1)\alpha)}(t)J_{n-m}^k x_2(0) \end{bmatrix}, \tag{7}$$

where $E_{\alpha,1}(At^{\alpha})$ is matrix Mittag-Leffler function defined as

$$E_{\alpha,1}(At^{\alpha}) = \sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

According to [22] and from (7), it is easy to obtain the following two lemmas.

Lemma 2.3. *Suppose that System (2) is regular, and two nonsingular matrices M and N are found such that (6) holds, then we have:*

(a) *System (2) is impulse-free if and only if $J_{n-m} = 0$.*
(b) *System (2) is stable if and only if $\left|\arg(\text{spec}(\bar{A}_1, \alpha))\right| > \alpha\frac{\pi}{2}$.*
(c) *System (2) is admissible if and only if $J_{n-m} = 0$ and $\left|\arg(\text{spec}(\bar{A}_1, \alpha))\right| > \alpha\frac{\pi}{2}$.*

When the regularity of the triple (E, A, α) in System (2) is not known. it is always possible to choose two nonsingular matrices M and N such that

$$MEN = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}. \tag{8}$$

For the above decomposition, from (7), we have the following result.

Lemma 2.4.

(a) *System (2) is impulse-free if and only if A_4 is nonsingular.*
(b) *System (2) is admissible if and only if A_4 is nonsingular and*

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$$\left|\arg\left(\mathrm{spec}\left(A_1 - A_2A_4^{-1}A_3, \alpha\right)\right)\right| > \alpha\frac{\pi}{2}.$$

Lemma 2.5 (*Lemma 2.8 in [5]*). *Let*

$$\hat{N} = \begin{bmatrix} P & X \\ Y & Z \end{bmatrix},$$

where P, X, Y *and* Z *are any real given matrices with appropriate dimensions such that* $\hat{N} + \hat{N}^{\mathrm{T}} < 0$. *Then,* Z *is nonsingular and*

$$P + P^{\mathrm{T}} - X Z^{-1}Y - Y^{\mathrm{T}}Z^{-\mathrm{T}}X^{\mathrm{T}} < 0.$$

Now we present the following admissibility criterion for System (2).

Theorem 2.1. *System (2) is admissible if and only if there exist matrices* $X, Y \in \mathbf{R}^{n\times n}$, *such that*

$$\begin{bmatrix} EX & EY \\ -EY & EX \end{bmatrix} = \begin{bmatrix} X^{\mathrm{T}}E^{\mathrm{T}} & -Y^{\mathrm{T}}E^{\mathrm{T}} \\ Y^{\mathrm{T}}E^{\mathrm{T}} & X^{\mathrm{T}}E^{\mathrm{T}} \end{bmatrix} \geq 0, \tag{9}$$

$$A(aX - bY) + (aX - bY)^{\mathrm{T}}A^{\mathrm{T}} < 0. \tag{10}$$

where $a = \sin\left(\frac{\alpha\pi}{2}\right)$, $b = \cos\left(\frac{\alpha\pi}{2}\right)$.

Proof. [Sufficiency] For System (2), it is easy to choose two arbitrary nonsingular matrices M and N satisfying (8). Let

$$N^{-1}XM^{\mathrm{T}} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad N^{-1}YM^{\mathrm{T}} = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}. \tag{11}$$

By (9), it can be shown that $X_2 = Y_2 = 0$ and $X_1 = X_1^{\mathrm{T}} \geq 0$, $Y_1 = -Y_1^{\mathrm{T}}$. Considering (10), and using the expressions in (8), we have

$$\begin{bmatrix} U_1 & U_2 \\ U_2^{\mathrm{T}} & U_3 \end{bmatrix} < 0, \tag{12}$$

where

$$U_1 = A_1(aX_1 - bY_1) + (aX_1 - bY_1)A_1^{\mathrm{T}} + A_2(aX_3 - bY_3) + (aX_3 - bY_3)^{\mathrm{T}}A_2^{\mathrm{T}},\ U_2 = A_2(aX_4 - bY_4) + (aX_1 - bY_1)A_3^{\mathrm{T}} + (aX_3 - bY_3)^{\mathrm{T}}A_4^{\mathrm{T}},\ U_3 = A_4(aX_4 - bY_4) + (aX_4 - bY_4)^{\mathrm{T}}A_4^{\mathrm{T}}.$$

Then, the 2-2 block in (12) gives

$$A_4(aX_4 - bY_4) + (aX_4 - bY_4)^{\mathrm{T}}A_4^{\mathrm{T}} < 0.$$

So, $A_4(aX_4 - bY_4)$ is nonsingular, which implies A_4 is nonsingular too. Hence, by Lemmas 2.4, it can be seen that the triple (E, A, α) is regular and impulse-free. Defining

$$\hat{N} = \begin{bmatrix} A_1(aX_1 - bY_1) + (aX_3 - bY_3)^{\mathrm{T}}A_2^{\mathrm{T}} & A_2(aX_4 - bY_4) \\ A_3(aX_1 - bY_1) + A_4(aX_3 - bY_3) & A_4(aX_4 - bY_4) \end{bmatrix}.$$

and considering (12), it is easy to see that

$$\hat{N} + \hat{N}^{\mathrm{T}} = \begin{bmatrix} U_1 & U_2 \\ U_2^{\mathrm{T}} & U_3 \end{bmatrix} < 0,$$

Therefore, by Lemma 2.5, it follows that

$$(A_1 - A_2A_4^{-1}A_3)(aX_1 - bY_1) + (aX_1 - bY_1)^{\mathrm{T}}(A_1 - A_2A_4^{-1}A_3)^{\mathrm{T}} < 0. \tag{13}$$

Noting (15) and employing Lemma 2.1, we have that the triple (E, A, α) is stable. This together with the regularity and non-impulsiveness of the triple (E, A, α) gives that System (2) is admissible.

(*Necessity*). Suppose System (2) is admissible. According to Lemma 2.4, from the full rank decomposition of descriptor matrix E, for the two arbitrary chosen nonsingular matrices M_1 and N_1 satisfying (6) and

$$\left|\arg(\mathrm{spec}(\bar{A}_1, \alpha))\right| > \alpha\frac{\pi}{2}.$$

Noting the above inequality and applying Lemma 2.1, it can be seen that there exist matrices X_1, Y_1 such that (4) holds and

$$\bar{A}_1(aX_1 - bY_1) + (aX_1 - bY_1)^{\mathrm{T}}\bar{A}_1^{\mathrm{T}} < 0$$

Let

$$X = N_1\begin{bmatrix} X_1 & 0 \\ 0 & -I_{n-m} \end{bmatrix}M_1^{-\mathrm{T}}, \quad Y = N_1\begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix}M_1^{-\mathrm{T}}.$$

Therefore the matrices X and Y satisfy (9) and (10), respectively. This completes the proof. □

It is noted that the condition in (9) is non-strict LMI, which contains equality constraint and may be of little problem theoretically, but may cause a big trouble in checking the conditions numerically. Because of round-off errors in digital computation, the equality constraints are fragile and in usual not satisfied perfectly. Therefore, strict LMI conditions are more desirable than non-strict ones from the numerical point of view. To this end, the following theorem introduces a matrix $S \in \mathbf{R}^{n\times(n-m)}$ which is of full column rank and composed of bases of null space of E. With the help of the matrix S, a strict LMI without equality constraint condition is derived.

Theorem 2.2. *System (2) is admissible if and only if there exist matrices* $X, Y \in \mathbf{R}^{n\times n}$, $Q \in \mathbf{R}^{(n-m)\times n}$ *such that (4) holds and*

$$A(aXE^{\mathrm{T}} - bYE^{\mathrm{T}}) + ASQ + (aEX + bEY)A^{\mathrm{T}} + Q^{\mathrm{T}}S^{\mathrm{T}}A^{\mathrm{T}} < 0. \tag{14}$$

where $a = \sin\left(\frac{\alpha\pi}{2}\right)$, $b = \cos\left(\frac{\alpha\pi}{2}\right)$, *and* $S \in \mathbf{R}^{n\times(n-m)}$ *is any matrix with full column rank and satisfies ES=0.*

Proof. [Sufficiency] Assume that there exist matrices X, Y and Q such that (4) and (14) hold. Let

$$\bar{X} = XE^{\mathrm{T}} + a^{-1}SQ,\ \bar{Y} = YE^{\mathrm{T}}.$$

Then by (4) and (14), it is easy to verify $\bar{X}$ and $\bar{Y}$ satisfy (9) and (10). Therefore by Theorem 2.1, it follows that System (2) is admissible.

(*Necessity*). Suppose System (2) is admissible. According to Lemma 2.4, from the full rank decomposition of descriptor matrix E, for the two arbitrary chosen nonsingular matrices M_1 and N_1 satisfying (6) and

$$\left|\arg(\mathrm{spec}(\bar{A}_1, \alpha))\right| > \alpha\frac{\pi}{2}.$$

Considering the definition of matrix S, we can write

$$S = N_1\begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}H,$$

where H is any nonsingular matrix. Noting the above inequality and applying Lemma 2.1, it can be seen that there exist matrices X_1, Y_1 such that (4) holds and

$$\bar{A}_1(aX_1 - bY_1) + (aX_1 - bY_1)^{\mathrm{T}}\bar{A}_1^{\mathrm{T}} < 0.$$

Let

$$X = N_1\begin{bmatrix} X_1 & 0 \\ 0 & I_{n-m}\end{bmatrix}N_1^{\mathrm{T}}, \quad Y = N_1\begin{bmatrix} Y_1 & 0 \\ 0 & 0\end{bmatrix}N_1^{\mathrm{T}}, \quad Q = H^{-1}\begin{bmatrix} 0 & -I_{n-m}\end{bmatrix}M_1^{-\mathrm{T}}.$$

Therefore the matrices X, Y and Q satisfy (4) and (14), respectively. This completes the proof. □

The criteria proposed in Theorem 2.2 introduces the basis of null space of singular matrix E and involves more solved variables than Theorem 2.1, i.e, the additional Q. The following theorem just involves the simple full rank decomposition of singular matrix E, without the decompositions of system matrix A and input matrix B and include less solved variables of LMIs than the those in Theorem 2.2 and can be effectively solved with any LMI toolbox.

Theorem 2.3. *System (2) is admissible if and only if there exist matrices* $X_1, X_2 \in \mathbf{R}^{m\times m}$, $X_3 \in \mathbf{R}^{(n-m)\times m}$ *and* $X_4 \in \mathbf{R}^{(n-m)\times(n-m)}$ *such that*

$$\begin{bmatrix} X_1 & X_2 \\ -X_2 & X_1\end{bmatrix} > 0, \tag{15}$$

$$aMANX - bMANY + aX^{\mathrm{T}}N^{\mathrm{T}}A^{\mathrm{T}}M^{\mathrm{T}} - bY^{\mathrm{T}}N^{\mathrm{T}}A^{\mathrm{T}}M^{\mathrm{T}} < 0. \tag{16}$$

where

$$X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4\end{bmatrix}, \quad Y = \begin{bmatrix} X_2 & 0 \\ 0 & 0\end{bmatrix}, \tag{17}$$

$m = \mathrm{rank}(E)$, $a = \sin\left(\frac{\alpha\pi}{2}\right)$, $b = \cos\left(\frac{\alpha\pi}{2}\right)$, $M, N \in \mathbf{R}^{n\times n}$ *are two arbitrary chosen nonsingular matrices satisfying (8).*

Proof. [Sufficiency] For System (2), it is easy to choose two arbitrary nonsingular matrices M and N satisfying (8). Assume that there exist matrices X_1, X_2, X_3, and X_4 such that (15), (16) and (17) hold. Then, considering (16), and using the expression in (17), we have

$$\begin{bmatrix} U_1 & U_2 \\ U_2^{\mathrm{T}} & U_3\end{bmatrix} < 0, \tag{18}$$

where

$$U_1 = A_1(aX_1 - bX_2) + (aX_1 + bX_2)A_1^{\mathrm{T}} + aA_2X_3 + aX_3^{\mathrm{T}}A_2^{\mathrm{T}},$$
$$U_2 = aA_2X_4 + (aX_1 + bX_2)A_3^{\mathrm{T}} + aX_3^{\mathrm{T}}A_4^{\mathrm{T}}, \; U_3 = aA_4X_4 + aX_4^{\mathrm{T}}A_4^{\mathrm{T}}.$$

Then, with $0 < \alpha \le 1$, $a = \sin\left(\frac{\alpha\pi}{2}\right) > 0$, the 2-2 block in (18) gives

$$A_4X_4 + X_4^{\mathrm{T}}A_4^{\mathrm{T}} < 0.$$

So, X_4A_4 is nonsingular, which implies A_4 is nonsingular too. Hence, by Lemmas 2.4, it can be seen that the triple (E, A, α) is regular and impulse-free. Defining

$$\hat{N} = \begin{bmatrix} A_1(aX_1 - bX_2) + aX_3^{\mathrm{T}}A_2^{\mathrm{T}} & aA_2X_4 \\ A_3(aX_1 - bX_2) + aA_4X_3 & aA_4X_4\end{bmatrix},$$

and considering (18), it is easy to see that

$$\hat{N} + \hat{N}^{\mathrm{T}} = \begin{bmatrix} U_1 & U_2 \\ U_2^{\mathrm{T}} & U_3\end{bmatrix} < 0.$$

Therefore, by Lemma 2.5, it follows that

$$(A_1 - A_2A_4^{-1}A_3)(aX_1 - bX_2) + (aX_1 + bX_2)(A_1 - A_2A_4^{-1}A_3)^{\mathrm{T}} < 0. \tag{19}$$

Noting (15) and employing Lemma 2.1, we have that the triple (E, A, α) is stable. This together with the regularity and non-impulsiveness of the triple (E, A, α) gives that System (2) is admissible.

(*Necessity*). Suppose System (2) is admissible. According to Lemma 2.4, from the full rank decomposition of descriptor matrix E, for the two arbitrary chosen nonsingular matrices M and N satisfying (8), where A_4 is nonsingular, and

$$\left|\arg(\mathrm{spec}(A_1 - A_2A_4^{-1}A_3, \alpha))\right| > \alpha\frac{\pi}{2}.$$

Noting the above inequality and applying Lemma 2.1, it can be seen that there exist matrices X_1, X_2 such that (15) and (19) hold. Let

$$M_1 = \begin{bmatrix} I_m & -A_2A_4^{-1} \\ 0 & A_4^{-1}\end{bmatrix},$$
$$N_1 = \begin{bmatrix} I_m & 0 \\ -A_4^{-1}A_3 & I_{n-m}\end{bmatrix},$$
$$\bar{X} = \begin{bmatrix} aX_1 - bX_2 & 0 \\ 0 & -I_{n-m}\end{bmatrix}.$$

Then, M_1 and N_1 are nonsingular and we have

$$M_1MENN_1 = M_1\begin{bmatrix} I_m & 0 \\ 0 & 0\end{bmatrix}N_1 = \begin{bmatrix} I_m & 0 \\ 0 & 0\end{bmatrix},$$
$$M_1MANN_1 = M_1\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4\end{bmatrix}$$
$$N_1 = \begin{bmatrix} A_1 - A_2A_4^{-1}A_3 & 0 \\ 0 & I_{n-m}\end{bmatrix}.$$

Therefore, it is easy to see

$$M_1MANN_1\bar{X} + \bar{X}^{\mathrm{T}}N_1^{\mathrm{T}}N^{\mathrm{T}}A^{\mathrm{T}}M^{\mathrm{T}}M_1^{\mathrm{T}} < 0. \tag{20}$$

Pre-multiplying and post-multiplying (20) by M_1^{-1} and $M_1^{-\mathrm{T}}$, respectively, we have

$$MANN_1\bar{X}M_1^{-\mathrm{T}} + M^{-1}\bar{X}^{\mathrm{T}}N_1^{\mathrm{T}}N^{\mathrm{T}}A^{\mathrm{T}}M^{\mathrm{T}} < 0.$$

Consider

$$N_1\bar{X}M_1^{-\mathrm{T}} = \begin{bmatrix} I_m & 0 \\ -A_4^{-1}A_3 & I_{n-m}\end{bmatrix}\begin{bmatrix} aX_1 - bX_2 & 0 \\ 0 & -I_{n-m}\end{bmatrix}\begin{bmatrix} I_m & 0 \\ A_4^{-\mathrm{T}}A_2^{\mathrm{T}} & I_{n-m}\end{bmatrix}$$
$$= \begin{bmatrix} aX_1 - bX_2 & 0 \\ -A^{-1}A_3(aX_1 - bX_2) - A_4^{-\mathrm{T}}A_2^{\mathrm{T}} & -I_{n-m}\end{bmatrix}$$
$$= \begin{bmatrix} aX_1 & 0 \\ -A^{-1}A_3(aX_1 - bX_2) - A_4^{-\mathrm{T}}A_2^{\mathrm{T}} & -I_{n-m}\end{bmatrix} + \begin{bmatrix} -bX_2 & 0 \\ 0 & 0\end{bmatrix}.$$

Let

$$X = \begin{bmatrix} X_1 & 0 \\ -a^{-1}A^{-1}A_3(aX_1 - bX_2) - a^{-1}A_4^{-\mathrm{T}}A_2^{\mathrm{T}} & -a^{-1}I_{n-m}\end{bmatrix}, \quad Y = \begin{bmatrix} X_2 & 0 \\ 0 & 0\end{bmatrix}.$$

Therefore the matrices X_1, X_2, X and Y satisfy (15) and (16), respectively. This completes the proof. □

Remark 2.1. The matrices M and N in Theorem 2.3 can be obtained in the process of deducing the matrix E as its diagonal standard form. The function reff in MATLAB is effective on obtaining them. The function SVD in MATLAB is also helpful to do it. The matrices M and N constituted with the conditions in Theorem 2.3 depend only on the matrix E and do not need to involve system matrix A.

Remark 2.2. In the case when $\alpha = 1$, the singular fractional order system in (2) reduces to a singular integer order system. In the case when $E=I$, the singular fractional order system in (2) reduces to a normal fractional order system. Therefore, Theorems 2.1–2.3 can be regarded as extensions of Lyapunov stability theory from continuous normal integer systems to continuous singular fractional order systems.

Remark 2.3. The conditions of Theorems 2.2 and 2.3 are strict LMIs conditions, the matrix X in (16) and (17) is easily calculated with standard feasible solution problem in MATLAB LMI box. Even in the case when $\alpha = 1$, the conditions of Theorems 2.2 and 2.3 are much more tractable and reliable in numerical computation than the those of Theorem 2.1.

Remark 2.4. The conditions of Theorems 2.2 and 2.3 do not need equality constraint conditions, i.e. they have not the equality constraint like (9). Theorems 2.3 do not need to introduce the bases of null space of singular matrix E and involves the less solved variables. Theorems 2.3 can avoid the singularity trouble caused by the superfluous solved variable Q such as in Example 3.1.

3. LMI criterion of stabilization of continuous linear singular FOS

For a singular fractional order system, it is important to develop conditions which guarantee that the given closed loop singular fractional order system is not only stable but also regular and impulse-free. Consider System (1), we use the following state feedback controller:

$$u(t) = Kx(t),\ K \in \mathbf{R}^{l\times n}. \tag{21}$$

Applying Controller (21) to the system in (1), we obtain the closed-loop system as follows:

$$ED^{\alpha}x(t) = (A + BK)x(t). \tag{22}$$

Now, we are in a position to give the LMI criteria of stabilization of continuous singular fractional order systems.

Considering the closed-loop linear singular fractional order system (22) and involving in Theorem 2.2, we have

$$(A + BK)(aXE^{T} - bYE^{T} + SQ) + (aXE^{T} - bYE^{T} + SQ)^{T}(A + BK)^{T} < 0.$$

If denote $Z = K(aXE^{T} - bYE^{T} + SQ)$, we have the following theorem.

Theorem 3.1. *For the continuous singular fractional order System (1), there exists a state feedback controller (21) such that the closed-loop system (22) is admissible if and only if there exist matrices $X, Y \in \mathbf{R}^{n\times n}$, $Q \in \mathbf{R}^{(n-m)\times n}$, and $Z \in \mathbf{R}^{l\times n}$ such that (4) holds and*

$$A(aXE^{T} - bYE^{T} + SQ) + BZ + (aXE^{T} - bYE^{T} + SQ)^{T}A^{T} + Z^{T}B^{T} < 0. \tag{23}$$

Then a stabilizing state feedback controller gain matrix can be chosen as

$$K = Z(aXE^{T} - bYE^{T} + SQ)^{-1}. \tag{24}$$

Example 3.1. Consider System (22) with parameters as follows:

$$\alpha = 0.5,\ E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\quad A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix},\quad B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

It is easy to verify the system is not regular, has impulse mode and is unstable. In order to use Theorem 3.1, we first choose

$$S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which is with full column rank and satisfies $ES=0$. By solving the LMIs (4) and (23) in Theorem 3.1, we can obtain

$$X = \begin{bmatrix} 322.1838 & 0 & 0 \\ 0 & 322.4185 & 0 \\ 0 & 0 & 322.1838 \end{bmatrix},$$

$$Y = 0,\quad Q = \begin{bmatrix} 0 & 0 & 300.0299 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & -183.2459 \\ 241.9080 & 402.9289 & 0 \end{bmatrix},$$

$$\Omega = aXE^{T} - bYE^{T} + SQ = \begin{bmatrix} 0 & 0 & 227.8183 \\ 227.9843 & 227.9843 & 0 \\ 0 & 0 & 300.0299 \end{bmatrix}.$$

Theorem 3.1 provides a criterion to judge the admissibility for singular fractional order systems which is expressed in terms of strict definite LMIs without equality constraint. However, it involves an additional solved variable Q which may cause the singularity trouble. The following example is also a counterexample which conflicts with Theorem 3.1.

Let the current $i_C(t)$ in the supercondensator with the capacity C be the α order derivative of its charge $q(t)$ [23]

$$i_C(t) = \frac{d^{\alpha}q(t)}{dt^{\alpha}}.$$

Taking into account that $q(t) = Cu_C(t)$ we obtain

$$i_C(t) = C\frac{d^{\alpha}u_C(t)}{dt^{\alpha}},$$

where $u_C(t)$ is the voltage on the supercondensator.

Example 3.2. Consider electrical circuit shown on Fig. 1 with given resistance R, capacitances C_1, C_2, C_3 in the supercondensator with the capacity C_i be the α order derivative of its charge $q_i(t)$ satisfying the above fractional order equations [23] and source voltages e_1 and e_2. Using the Kirchhoffs laws, we can write for the electrical circuit the equations

$$\begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}\frac{d^{\alpha}}{dt^{\alpha}}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Let $\alpha = 0.8$, $R = C_1 = C_2 = C_3 = 1$. It is easy to verify the system is regular, impulse free and unstable. In order to use Theorem 3.1, we use the command $S=\text{null}(E)$ to get S which is with full column rank and satisfies $ES=0$. By solving the LMIs (4) and (23) in Theorem 3.1, one obtains the following information, which also means Theorem 3.1 is invalid in this case.

Result: best value of t: 1.101422e-012
f-radius saturation: 73.734% of $R=1.00\text{e}+009$
Marginal infeasibility: these LMI constraints may be feasible but are not strictly feasible.

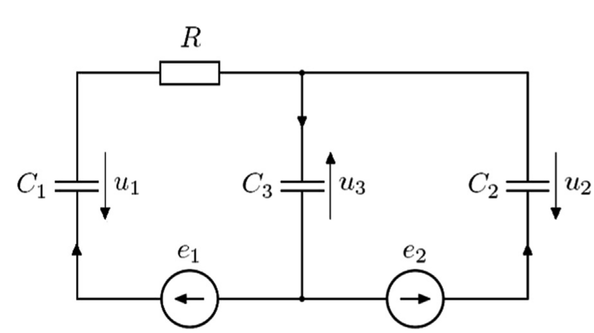

Fig. 1. Electrical circuit Illustration to Example 3.2.

When the singularity of Ω in Example 3.2 occurs, Theorem 3.1 is disable to give a consequence for the admissibility of closed-loop System (22). To this end, Although the feasible solutions of LMIs (4) and (23) are not unique, but whenever a singularity of Ω arises in the computer simulation, the approach in Theorem 3.1 is incapable of stabilizing state feedback controller for System (22). In this example, as Ω is singular, Theorem 3.1 does not work and fails to conclude whether closed-loop System (22) is admissible or not. If we change $\alpha = 0.5$ in Example 3.1, then solving LMIs (4) and (23) by MATLAB, one obtains the following information, which also means Theorem 3.1 is invalid in this case.

Result: best value of t: 2.710051e-011
f-radius saturation: 0.039 % of R=1.00e+009
Marginal infeasibility: these LMI constraints may be feasible but are not strictly feasible.

The following theorem can overcome the above weakness.

Considering the closed-loop linear singular fractional order system (22) and involving in Theorem 23, we have

$$aMANX - bMANY + MBKN(aX - bY) + aX^TN^TA^TM^T - bY^TN^TA^TM^T + (aX - bY)^TN^TB^TK^TM^T < 0.$$

If denote $Z = KN(aX - bY)$, we have the following results desired immediately.

Theorem 3.2. *For the continuous singular fractional order System (1), there exists a state feedback controller (21) such that the closed-loop system (22) is admissible if and only if there exist appropriate dimensional matrices X_1, X_2, X_3, X_4, and Z such that (15) and (17) hold and*

$$MAN(aX - bY) + MBZ + (aX - bY)^TN^TA^TM^T + Z^TN^TB^TM^T < 0. \tag{25}$$

Where X, Y, M, N and m have the same meaning as in the case of Theorem 2.1 . Then a stabilizing state feedback controller gain matrix can be chosen as

$$K = Z(aX - bY)^{-1}N^{-1}. \tag{26}$$

The detailed explanations on how to construct the matrices X_1, X_2, X_3, X_4, X and Y in Theorems 2.1 and 3.1 are listed in Appendix A.

Solving the LMIs (15) and (25) in Theorem 3.2, we obtain the feasible solutions as follows:

$$X = 10^8\begin{bmatrix} 0.0006 & 0 & 0 \\ 0 & 0.0006 & 0 \\ -4.8952 & 0.4499 & 0.6174 \end{bmatrix},$$
$$Y = \begin{bmatrix} 0 & 0.0373 & 0 \\ -0.0373 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$Z = 10^8\begin{bmatrix} -4.6552 & 0.4275 & -2.9974 \\ -5.7267 & -2.7294 & 1.1741 \end{bmatrix},$$
$$K = 10^4\begin{bmatrix} -5.0939 & 0.4676 & -0.4681 \\ 0.6421 & -0.6423 & 0.6425 \end{bmatrix}.$$

4. LMI criterion of robust stabilization of continuous linear singular FOS

Consider a continuous linear uncertain singular fractional order system with norm bounded described by

$$ED^\alpha x(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) \tag{27}$$

where ΔA and ΔB are the two unknown time-invariant matrices representing norm-bounded parameter uncertainties, which are assumed to be the form

$$\begin{bmatrix} \Delta A & \Delta B \end{bmatrix} = PF(\sigma)\begin{bmatrix} Q_1 & Q_2 \end{bmatrix},$$

where P, Q_1 and Q_2 are known real constant matrices with appropriate dimension. The uncertain matrix $F(\sigma)$ satisfies

$$F^T(\sigma)F(\sigma) \le I,$$

where $\sigma \in \Theta$, and Θ is a compact set in $\mathbf{R}$.

The triple $(E, A + \Delta A, \alpha)$ is used to denote the following unforced uncertain singular fractional order system

$$D^\alpha Ex(t) = (A + \Delta A)x(t). \tag{28}$$

Similar to the study of stability robustness of singular integer order systems, we introduce the following definitions of quadratic admissibility [21] and generalized quadratic stability [2] of singular fractional order systems.

Definition 4.1. The uncertain singular fractional order system in (28) is said to be quadratically admissible if there exist matrices $X, Y \in \mathbf{R}^{n\times n}$ such that (9) holds and

$$(A + \Delta A)(aX - bY) + (aX - bY)^T(A + \Delta A)^T < 0. \tag{29}$$

for all allowable uncertainty ΔA.

Definition 4.2. The uncertain singular fractional order system in (28) is said to be generalized quadratically stable if there exist matrices $X, Y \in \mathbf{R}^{n\times n}$ and $Q \in \mathbf{R}^{(n-m)\times n}$ such that (9) holds and

$$(A + \Delta A)(aXE^T - bYE^T + SQ) + (aXE^T - bYE^T + SQ)^T(A + \Delta A)^T < 0. \tag{30}$$

for all allowable uncertainty ΔA.

Lemma 4.1 ([5] *Schur Complement*). *For given real matrices S_1, S_2 and S_3 where, $S_1 = S_1^T$, $S_3 > 0$.*

$$S_1 + S_2 S_3^{-1}S_2^T < 0,$$

if and only if

$$\begin{bmatrix} S_1 & S_2 \\ S_2^T & -S_3 \end{bmatrix} < 0.$$

Lemma 4.2 ([20]). *For given appropriate dimensional complex matrices Ω, Γ and Ξ, where Ω is symmetric, then,*

$$\Omega + \Gamma F(\sigma)\Xi + \Xi^TF^T(\sigma)\Gamma^T < 0,$$

for arbitrary $F(\sigma)$ satisfying $F^T(\sigma)F(\sigma) \le I$, if and only if there exists a scalar $\epsilon > 0$, such that

$$\Omega + \epsilon\Gamma\Gamma^T + \epsilon^{-1}\Xi^T\Xi < 0.$$

Substituting $(A + \Delta A)$ into A in Theorems 2.2 and 2.3, we obtain the following results of quadratic admissibility.

Theorem 4.1. *The uncertain singular system in (28) is quadratically admissible if and only if there exist matrices $X, Y \in \mathbf{R}^{n\times n}$ and a scalar $\epsilon > 0$ such that (9) holds and*

$$\begin{bmatrix} A(aX - bY) + (aX - bY)^TA^T + \epsilon PP^T & (aX - bY)^TQ_1^T \\ Q_1(aX - bY) & -\epsilon I \end{bmatrix} < 0. \tag{31}$$

for all allowable $F(\sigma)$, where $S \in \mathbf{R}^{n\times(n-m)}$ is any matrix with full column rank and satisfies ES=0.

Theorem 4.2. *The uncertain singular fractional order system in (28) is quadratically admissible if and only if there exist matrices $X_1, X_2 \in \mathbf{R}^{m\times m}$, $X_3 \in \mathbf{R}^{(n-m)\times m}$ and $X_4 \in \mathbf{R}^{(n-m)\times(n-m)}$ such that (15) and (17) hold and*

$$M(A + \Delta A)N(aX - bY) + (aX - bY)^TN^T(A + \Delta A)^TM^T < 0. \tag{32}$$

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for all allowable uncertainty ΔA. *Where* X, Y, M, N, m, a, *and* b *have the same meanings as those in Theorem 2.1.*

By using Lemmas 4.1 and 4.2, it is easy to obtain the following results.

Theorem 4.3. *The uncertain singular fractional order system in (28) is generalized quadratically stable if and only if there exist matrices* $X, Y \in \mathbf{R}^{n\times n}$, $Q \in \mathbf{R}^{(n-m)\times n}$ *and a scalar* $\epsilon > 0$ *such that (4) holds and*

$$\begin{bmatrix} \prod & (aEX + bEY + Q^{\mathrm{T}}S^{\mathrm{T}})Q_1^{\mathrm{T}} \\ Q_1(aXE^{\mathrm{T}} - bYE^{\mathrm{T}} + SQ) & -\epsilon I \end{bmatrix} < 0. \tag{33}$$

where $\prod = A(aXE^{\mathrm{T}} - bYE^{\mathrm{T}} + SQ) + (aEX + bEY + Q^{\mathrm{T}}S^{\mathrm{T}})A^{\mathrm{T}} + \epsilon PP^{\mathrm{T}}$, S *is the same as that in Definition 4.2.*

Theorem 4.4. *System (28) is quadratically admissible if and only if it is generalized quadratically stable.*

Proof. [Sufficiency] Suppose that System (28) is generalized quadratically stable, then it follows from Theorem 4.3 that there exist matrices $X, Y \in \mathbf{R}^{n\times n}$ and $Q \in \mathbf{R}^{(n-m)\times n}$ such that (4) and (33) hold. Setting

$$\bar{X} = XE^{\mathrm{T}} + a^{-1}SQ,\ \bar{Y} = YE^{\mathrm{T}}.$$

Where $S \in \mathbf{R}^{n\times(n-m)}$ is any matrix with full column rank and satisfies $ES = 0$, Then by (4) and (33), it is easy to verify $\bar{X}$ and $\bar{Y}$ satisfy (9) and (31). Therefore by Theorem 4.1, it follows that System (28) is admissible.

(*Necessity*). Suppose System (28) is quadratically admissible. Then from Theorem Theorem 4.1 there exist matrices $\bar{X}, \bar{Y} \in \mathbf{R}^{n\times n}$ and a positive scalar ϵ such that (9) and (31) hold. And there exist two nonsingular matrices U and V such that

$$\bar{E} = UEV = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}.$$

Denote

$$\bar{X} = V^{-1}\tilde{X}U^{\mathrm{T}} = \begin{bmatrix} \bar{X}_1 & \bar{X}_2 \\ \bar{X}_3 & \bar{X}_4 \end{bmatrix},\quad \bar{Y} = V^{-1}\tilde{Y}U^{\mathrm{T}} = \begin{bmatrix} \bar{Y}_1 & \bar{Y}_2 \\ \bar{Y}_3 & \bar{Y}_4 \end{bmatrix}$$

and let

$$\bar{A} = UAV,\ \bar{P} = UP,\ \bar{Q}_1 = UQ_1V,\ \bar{I} = UU^{\mathrm{T}}.$$

Pre-multiplying and post-multiplying (9) and (31) by diag(U,U) and their transposes, respectively, we have

$$\begin{bmatrix} \bar{E}\bar{X} & \bar{E}\bar{Y} \\ -\bar{E}\bar{Y} & \bar{E}\bar{X} \end{bmatrix} = \begin{bmatrix} \bar{X}^{\mathrm{T}}\bar{E}^{\mathrm{T}} & -\bar{Y}^{\mathrm{T}}\bar{E}^{\mathrm{T}} \\ \bar{Y}^{\mathrm{T}}\bar{E}^{\mathrm{T}} & \bar{X}^{\mathrm{T}}\bar{E}^{\mathrm{T}} \end{bmatrix} \geq 0, \tag{34}$$

$$\begin{bmatrix} \bar{A}(a\bar{X} - b\bar{Y}) + (a\bar{X} - b\bar{Y})^{\mathrm{T}}\bar{A}^{\mathrm{T}} + \epsilon\bar{P}\bar{P}^{\mathrm{T}} & (a\bar{X} - b\bar{Y})^{\mathrm{T}}\bar{Q}_1^{\mathrm{T}} \\ \bar{Q}_1(a\bar{X} - b\bar{Y}) & -\epsilon\bar{I} \end{bmatrix} < 0. \tag{35}$$

By (34), it can be shown that $\bar{X}_2 = \bar{Y}_2 = 0$ and

$$\begin{bmatrix} \bar{X}_1 & \bar{Y}_1 \\ -\bar{Y}_1 & \bar{X}_1 \end{bmatrix} > 0.$$

Setting

$$\Lambda_1 = \begin{bmatrix} \bar{X}_1 & 0 \\ 0 & I_{n-m} \end{bmatrix},\quad \Lambda_2 = \begin{bmatrix} \bar{Y}_1 & 0 \\ 0 & 0 \end{bmatrix},\quad \bar{S} = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} a\bar{X}_3 - b\bar{Y}_3 & a\bar{X}_4 - b\bar{Y}_4 - aI_{n-m} \end{bmatrix}.$$

Therefore, (35) becomes

$$\begin{bmatrix} \bar{A}(a\Lambda_1\bar{E}^{\mathrm{T}} - b\Lambda_2\bar{E}^{\mathrm{T}} + \bar{S}\bar{Q}) + (a\Lambda_1\bar{E}^{\mathrm{T}} - b\Lambda_2\bar{E}^{\mathrm{T}} + \bar{S}\bar{Q})^{\mathrm{T}}\bar{A}^{\mathrm{T}} + \epsilon\bar{P}\bar{P}^{\mathrm{T}} & (a\bar{X} - b\bar{Y})^{\mathrm{T}}\bar{Q}_1^{\mathrm{T}} \\ \bar{Q}_1(a\bar{X} - b\bar{Y}) & -\epsilon\bar{I} \end{bmatrix} < 0. \tag{36}$$

Considering the definition of the matrix S in Definition 2.2, we can write $S = V\bar{S}H$, where H is any $n-m$ order nonsingular matrix. Denoting

$$X = V\Lambda_1V^{\mathrm{T}},\ \ Y = V\Lambda_2V^{\mathrm{T}},\ \ Q = H^{-1}\bar{Q}U^{-\mathrm{T}},$$

and substituting

$$\bar{Q}_1 = Q_1U,\ \ \bar{Q} = HQU^{\mathrm{T}},\ \ \bar{S} = V^{-1}SH^{-1},\ \ \Lambda_1 = V^{-1}XV^{-T},\ \ \Lambda_2 = V^{-1}YV^{-T},$$
$$\bar{P} = UP,\ \ \bar{I} = UU^{\mathrm{T}}$$

into (36) gives

$$\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \prod & (aEX + bEY + Q^{\mathrm{T}}S^{\mathrm{T}})Q_1^{\mathrm{T}} \\ Q_1(aXE^{\mathrm{T}} - bYE^{\mathrm{T}} + SQ) & -\epsilon I \end{bmatrix}\begin{bmatrix} U^{\mathrm{T}} & 0 \\ 0 & U^{\mathrm{T}} \end{bmatrix} < 0.$$

Where $\prod = A(aXE^{\mathrm{T}} - bYE^{\mathrm{T}} + SQ) + (aEX + bEY + Q^{\mathrm{T}}S^{\mathrm{T}})A^{\mathrm{T}} + \epsilon PP^{\mathrm{T}}$. Therefore, matrices X, Y, Q, and scalar $\epsilon > 0$ satisfy (4) and (33). This completes the proof. □

Theorem 4.5. *The uncertain singular fractional order system in (28) is quadratically admissible if and only if there exist matrices* $X_1, X_2 \in \mathbf{R}^{m\times m}$, $X_3 \in \mathbf{R}^{(n-m)\times m}$, $X_4 \in \mathbf{R}^{(n-m)\times(n-m)}$ *and a scalar* $\epsilon > 0$ *such that (15) and (17) hold and*

$$\begin{bmatrix} MAN(aX - bY) + (aX - bY)^{\mathrm{T}}N^{\mathrm{T}}A^{\mathrm{T}}M^{\mathrm{T}} + \epsilon MPP^{\mathrm{T}}M^{\mathrm{T}} & (aX - bY)^{\mathrm{T}}N^{\mathrm{T}}Q_1^{\mathrm{T}} \\ Q_1N(aX - bY) & -\epsilon I \end{bmatrix} < 0, \tag{37}$$

where X, M, *and* N *have the same meanings as those in Theorem 2.1.*

Proof. The proof is similar to that in Theorem 4.4 and is omitted.□.

Applying Controller (21) to systems in (27), we obtain the closed-loop system as follows:

$$ED^{\alpha}x(t) = (A + \Delta A + (B + \Delta B)K))x(t). \tag{38}$$

Considering the closed-loop singular system (38), denoting $W = KA(aXE^{\mathrm{T}} - bYE^{\mathrm{T}} + SQ)$ and $Z = KN(aX - bY)$ borrowing by Theorems 4.3 and 4.4, respectively, we have the following results desired immediately.

Theorem 4.6. *For the continuous uncertain singular fractional order System (27), there exists a state feedback controller (21) such that the*

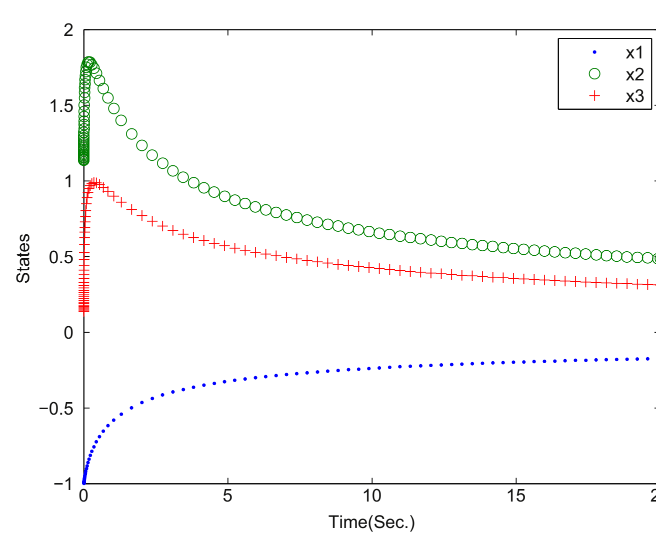

Fig. 2. The closed-loop singular fractional order system in Example 5.2.

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closed-loop system (38) is quadratically admissible if and only if there exist appropriate dimensional matrices X, Y, and W such that (4) holds and

$$\begin{bmatrix} \prod + BW + W^TB^T & (aEX + bEY + Q^TS^T)Q_1^T + W^TQ_2^T \\ Q_1(aXE^T - bYE^T + SQ) + Q_2W & -\epsilon I \end{bmatrix} < 0. \tag{39}$$

where $\prod$ is the same as that in Theorem 4.3. Then a stabilizing state feedback controller gain matrix can be chosen as (26).

Theorem 4.7. *For the continuous uncertain singular fractional order System (27), there exists a state feedback controller (21) such that the closed-loop system (38) is quadratically admissible if and only if there exist appropriate dimensional matrices X_1, X_2, X_3, X_4, and Z such that (15) and (17) hold and*

$$\begin{bmatrix} \Gamma + \epsilon MPP^TM^T & (aX - bY)^TN^TQ_1^T + Z^TQ_2^T \\ Q_1N(aX - bY) + Q_2Z & -\epsilon I \end{bmatrix} < 0. \tag{40}$$

where $\Gamma = MAN(aX - bY) + MBZ + (aX - bY)^TN^TA^TM^T + Z^TB^TM^T$, X, Y, M and N have the same meanings as those in Theorem 2.1. Then a stabilizing state feedback controller gain matrix can be chosen as (26).

5. Numerical examples

The there simulation examples are given to verify the availability of results presented in the paper.

5.1. Stability of singular FOS

Example 5.1. Consider System (2) with parameters as follows:

$$\alpha = \frac{1}{3}, E = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ -2 & -2 & 1 & -2 \\ -1 & -1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 3 & 2 & 0 \\ 2 & -5 & -1 & -5 \\ 1 & -2 & -1 & -1 \end{bmatrix}.$$

By Lemmas 2.1 and 2.1, it is easy to verify that $\det\left(s^{\frac{1}{3}}E - A\right) = -2\left(s - s^{\frac{2}{3}} + s^{\frac{1}{3}} + 2\right)$ is not identically zero. So System (2) is regular. It is easy to see that $\deg(\det(sE - A)) = \text{rank}(E) = 3$, i.e. System (2) is impulse-free. The three roots of the polynomial $\det\left(s^{\frac{1}{3}}E - A\right) = 0$ are $1 \pm i$ and -1. All their roots satisfy $\left|\arg\left(\text{spec}\left(E, A, \frac{1}{3}\right)\right)\right| > \frac{\pi}{6}$. It shows that System (2) is stable. So we can obtain that System (2) is admissible. Solving the LMIs (4) and (14) in Theorem 2.2, we obtain the feasible solutions as follows:

$$X = \begin{bmatrix} 0.7589 & 0.1248 & 0.0045 & 0.0773 \\ 0.1248 & 0.7936 & 0.0196 & -0.0909 \\ 0.0045 & 0.0196 & 0.6474 & 0.1614 \\ 0.0773 & -0.0909 & 0.1614 & 0.2295 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & -0.1940 & -0.1326 & 0.0712 \\ 0.1940 & 0 & -0.1934 & 0.1929 \\ 0.1326 & 0.1934 & 0 & 0.0609 \\ -0.0712 & -0.1929 & -0.0609 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} -0.1105 & 2.5768 & -2.4126 & -1.4127 \end{bmatrix}.$$

We can use the following the simple MATLAB commands Appendix B to obtain the matrices M and N in Theorem 2.3.

Then, we have

$$M = \begin{bmatrix} 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ -2 & 0 & 1 & 2 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Using MATLAB LMI Control Toolbox to solve the LMIs (15) and (16) in Theorem 2.3, we obtain the feasible solutions as follows:

$$X = \begin{bmatrix} 1.5087 & 0.0192 & -0.0230 & 0 \\ 0.0192 & 0.9356 & -0.0270 & 0 \\ -0.0230 & -0.0270 & 0.9046 & 0 \\ -0.2778 & -0.1101 & 0.1054 & -0.6081 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & -0.0464 & -0.2313 & 0 \\ 0.0464 & 0 & -0.7590 & 0 \\ 0.2313 & 0.7590 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5.2. Stabilization of Singular FOS

Example 5.2. Consider System (1) with parameters as follows:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \alpha = \frac{1}{2}.$$

It is easy to verify that System (1) is regular and impulse-free but is unstable. So we can obtain that System (1) is not admissible.Solving the LMI (33) in Theorem 3.1, we obtain the feasible solutions as follows:

$$X = \begin{bmatrix} 13.3730 & -8.8287 & 0 \\ -8.8287 & 31.0305 & 0 \\ 0 & 0 & 31.0305 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -8.8287 & 0 \\ 8.8287 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} -9.4562 & -37.4571 & 85.5465 \end{bmatrix},$$

$$Q = \begin{bmatrix} -3.0295 & 107.4884 & 15.5152 \end{bmatrix},$$

$$K = \begin{bmatrix} -17.2948 & -12.8952 & 2.2839 \end{bmatrix}$$

Solving the LMI (33) in Theorem 3.2, we obtain the less solved variables feasible solutions as follows:

$$X = \begin{bmatrix} 13.3730 & -8.8287 & 0 \\ -8.8287 & 31.0305 & 0 \\ -4.2844 & -37.1767 & 21.9419 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -8.8287 & 0 \\ 8.8287 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} -9.4562 & -37.4571 & -48.2298 \end{bmatrix},$$

$$K = \begin{bmatrix} -9.1674 & -5.4314 & -3.1085 \end{bmatrix}.$$

Involving in the control gain matrix K obtained in Example 5.2, we can draw the state figure of the closed-loop system in Example 4.2 shown as in Fig. 2 It is easy to see that although the open loop singular system is unstable, but by Fig. 2, the closed-loop singular system is admissible and it can be stabilized by the control law (26) in about 15 s.

5.3. Robust stabilization of singular FOS

Example 5.3. Consider System (27) with parameters as follows:

$$E = \begin{bmatrix} -1 & -1 & -1 & -2 \\ 4 & 3 & 2 & 5 \\ -6 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & -4 & -2 & -6 \\ 9 & 15 & 5 & 18 \\ -18 & -4 & 2 & -4 \\ 2 & 5 & 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 1 \\ -1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix},$$

$$\alpha = \frac{1}{2},\quad P = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad Q_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix},$$

$$F(\sigma) = \mathrm{diag}(e^{-\mathrm{rand}(1)}, \sin(0.1\mathrm{rand}(1)), \cos(0.1\mathrm{rand}(1)), \sin^2(0.1\mathrm{rand}(1))).$$

By Lemmas 2.1 and 2.1, it is easy to verify that $\det\left(s^{\frac{1}{2}}E - A\right) = -4\left(\left(s^{\frac{1}{2}} - 1\right)^3\right)$ is not identically zero. So System (27) is regular. It is easy to see that $\deg(\det(sE - A)) \neq \mathrm{rank}(E)$, i.e. System (27) is not impulse-free. The three roots of the polynomial $\det\left(s^{\frac{1}{2}}E - A\right) = 0$ are all 1. All their roots satisfy $\left|\arg\left(\mathrm{spec}\left(E, A, \frac{1}{3}\right)\right)\right| < \frac{\pi}{4}$. It shows that System (27) is unstable. So we can obtain that System (27) is not admissible. Solving the LMI (40) in Theorem 4.2, we obtain the feasible solution as follows:

$$X = \begin{bmatrix} 10.9397 & -17.7388 & 26.2621 & 0 \\ -17.7388 & 32.2082 & -37.4259 & 0 \\ 26.2621 & -37.4259 & 101.0052 & 0 \\ -47.2901 & 106.4228 & -245.6401 & 23.5736 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & -4.8307 & -15.6788 & 0 \\ 4.8307 & 0 & 34.8119 & 0 \\ 15.6788 & -34.8119 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} -11.5265 & 46.7537 & -40.3503 & -38.6179 \\ 28.3167 & -97.4039 & 97.1194 & -17.4671 \end{bmatrix},$$

$$K = \begin{bmatrix} 123.3969 & 55.5415 & -16.0583 & 41.7999 \\ -16.3450 & -8.3844 & 0.3515 & -6.9850 \end{bmatrix},$$

$$\epsilon = 67.1275.$$

6. Conclusions

This paper deals with necessary and sufficient conditions of admissibility and robust stabilization for linear time-invariant continuous singular fractional order systems with fractional order α belonging to $0 < \alpha < 1$. The strict LMI approaches have been developed to both judge admissibility and design robust stabilizing state feedback controller. It is worth noting that the criteria obtained in this technical note can be regarded as a natural extension of Lyapunov stability from singular fractional order systems to normal integer orders systems with the consistent format. The conditions of admissibility and robust stabilization have been giving without decomposing the system matrices and without including equality constraints and without involving non-strict inequality. The issues of H_∞ control and stabilization with time delay of linear time-invariant both normal and singular fractional order systems in terms of LMIs can also be expected to extend with the similar approaches in Theorems 3.1 and 4.1.

Appendix A

```
mm=size(B);m=rank(E);n=mm(1);
a=sin(k*pi/2);b=cos(k*pi/2);
[X1,n1,sX1]=lmivar(1,[m 1]);
[X2,n2,sX2]=lmivar(3,skewdec(m,n1));
[X3,n3,sX3]=lmivar(2,[n-m,m]);
[X4,n4,sX4]=lmivar(1,[n-m 1]);
bigX=lmivar(3,[sX1,sX2;-sX2,sX1]);
X=lmivar(3,[sX1,zeros(m,n-m); sX3,sX4]);
Y=lmivar(3,[sX2,zeros(m,n-m); zeros(n-m,n)]);
```

Appendix B

```
≫ n=4; op=rref([E,eye(n)]);
  M=op(:,n+1:2*n);
  y=op(:,1:n)';
  z=rref([y,eye(n)]);
  N=z(:,n+1:2*n)';
```

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