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Extended Luenberger-type observer for a class of semilinear time fractional diffusion systems

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ABSTRACT

This paper is concerned with the design of an extended Luenberger-type observer to deal with the observation problem for a class of semilinear time fractional diffusion systems, which are usually used to well describe those sub-diffusion processes, such as water moving through grounds, or proteins diffusing across cell membranes etc. Mittag-Leffler stability for both the linearized and semilinear observer error systems are explored by using a backstepping-based technique. Moreover, a simulation example is provided to confirm the effectiveness of our results.

1. Introduction

In control theory, it is important to know the states of system. However, it usually fails to see full information of the states due to the fact that not all of the variables are measurable. Therefore, the study of state estimation problem is necessary, especially for those feedback control processes, which often rely on the availability of complete system state and provide the basis of many practical applications. Note that for linear lumped parameter systems (LPSs), there exists a well known state estimator called the Luenberger observer [1,2]. It is shown in [3–5] that Luenberger observer has been extended to study the exponential stability of linear distributed parameter systems (DPSs) and in [6–8] that it has been used to discuss the exponential stability of semilinear DPSs.

In addition, it is confirmed that the past decades have witnessed intense activity and progress in the study of time fractional diffusion system due to the great advantages they bring in describing those sub-diffusion processes [9–11], such as transport in groundwater flow [12], proteins diffusing across cell membranes [13], or pest or disease spreading process in agriculture lands [14,15], etc. What's more, for time fractional diffusion system, since its solution is usually characterized by Mittag-Leffler functions (see Lemma 2.1 below), together with the power-law decay property of Mittag-Leffler function, researchers are glad to consider their Mittag-Leffler stability/stabilization, which was first introduced in [16] by using fractional Lyapunov direct method. In addition, to the best of our knowledge, no results are available on the design of extended Luenberger observer and the analysis of Mittag-Leffler stabilization for semilinear time fractional diffusion systems. Here we try to fill up this gap.

In the following, similar to the LPSs setting [2] and DPSs setting [6], we first consider the Mittag-Leffler stability for the linearization of observer error systems by using a backstepping-based techniques. Based on the design of an invertible coordinate transform, the linearized observer error system is converted into a Mittag-Leffler stability system. In this case, to discuss the stabilisation problem of the studied system is transformed to explore the existence of the solution to a linear hyperbolic type partial differential equation. With this, Mittag-Leffler stabilizations of the observer error dynamics for the systems under consideration are presented.

The rest of this paper is organized as follows: the problem and its observer design are formulated in the next section. In Section 3, we shall explore the Mittag-Leffler stability for the linearization systems of system under consideration by using a backstepping-based techniques. Moreover, the Mittag-Leffler stabilizations for the semilinear cases is then analyzed in Section 4. This is illustrated in Section 5, where simulation results confirm our main results.

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2. Problem statement and observer design

Let $L^2[0,1]$ be the usual Lebesgue integrable function space endowed with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, respectively.

The system under study is described by a class of semilinear time fractional diffusion systems as follows

$$\begin{cases} {}_0^C D_t^\alpha y(x,t) = \triangle y(x,t) + f(x,t,y(x,t)) \text{ in } [0,1]\times[0,\infty), \\ y_x(0,t) = y_x(1,t) = 0 \ \text{ in } [0,\infty), \\ y(x,0) = y_0(x) \ \text{ in } [0,1], \end{cases} \tag{2.1}$$

where $\alpha \in (0,1]$, $y_0 \in L^2[0,1]$ and $\triangle$ is the Laplace operator defined as $\triangle = \frac{\partial^2}{\partial x^2}$. By [17], let $\lambda_n = n^2\pi^2$, $\xi_n(x) = \cos(n\pi x)$, $n = 1,2,\cdots$, we get that $-\lambda_n$ is the eigenvalue of $\triangle$, ξ_n is the eigenfunction corresponding to $-\lambda_n$ and $\{\xi_n, n=1,2,\cdots\}$ form a orthonormal basis in $L^2[0,1]$. This is, for any $\phi \in L^2[0,1]$, it can be expressed as $\phi = \sum_{n=1}^{\infty}(\phi,\xi_n)\xi_n$. In addition, here ${}_0^C D_t^\alpha$ given by [18,19]

$$ {}_0^C D_t^\alpha y(\cdot,t) = \begin{cases} {}_0 I_t^{1-\alpha} \frac{d}{dt} y(\cdot,t), \ \alpha\in(0,1), \\ \dfrac{d}{dt} y(\cdot,t), \ \ \alpha = 1 \end{cases} \tag{2.2}$$

is the Caputo fractional order derivative on t and

$$ {}_0 I_t^\alpha y(\cdot,t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(\cdot,s)ds, \ \alpha>0 \tag{2.3}$$

denotes the Riemann–Liouville fractional order integral on t.

Moreover, f: $[0,1]\times[0,\infty)\times L^2([0,1]\times[0,\infty)) \to L^2([0,1]\times[0,\infty))$ is a continuous function and for any given $T \geq 0$, constant r, two positive constants $C = C(r,T)$ and $M = M(r,T)$ are found such that

$$\begin{aligned} &\|f(x,t,y_1) - f(x,t,y_2)\| \leq C\|y_1 - y_2\|, \\ &t\in[0,T], \ y_i \in L^2[0,1], \ \|y_i\| \leq r, \ i = 1,2 \end{aligned} \tag{2.4}$$

and

$$\sup_{x\in\Omega}\sup_{\|y\|\leq r} |\partial_y f(x,t,y)| \leq M, \ y\in L^2[0,1], \text{ respectively.} \tag{2.5}$$

The measurement is assumed subsequently to be given as follows

$$z(t) = Cy(x,t) := y(0,t). \tag{2.6}$$

This means that the measurements are obtained from a point sensor at $x = 0$ [20,21].

Based on the properties of Laplace transform, we then see the following basic lemma.

Lemma 2.1. *Given $y_0 \in L^2[0,1]$, $y(x,t)$ is said to be a solution of (2.1) on $[0,1]\times[0,\infty)$ if it satisfies*

$$\begin{aligned} y(x,t) = &\sum_{n=1}^{\infty} E_\alpha(-\lambda_n t^\alpha)(y_0,\xi_n)\xi_n(x) \\ &+ \sum_{n=1}^{\infty}\int_0^t \frac{E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)}{(t-\tau)^{1-\alpha}}(f(x,\tau,y(x,\tau)),\xi_n)d\tau\xi_n(x), \end{aligned} \tag{2.7}$$

where $E_{\alpha,\beta}(t) := \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}$, $\mathrm{Re}(\alpha) > 0$, $t \in \mathbb{C}$ is known as the Mittag-Leffler function in two parameter (when $\beta = 1$, write $E_{\alpha,1}(t) = E_\alpha(t)$ for short).

Proof. Since

$$\mathcal{L}\left\{{}_0^C D_t^\alpha y\right\}(\cdot,s) = s^\alpha\mathcal{L}\{y\}(\cdot,s) - s^{\alpha-1}y(\cdot,t)|_{t=0}, \ \alpha\in(0,1], \tag{2.8}$$

taking Laplace transform on time t for both sides of system (2.1), it then follows that

$$s^\alpha\mathcal{L}\{y\}(\cdot,s) - s^{\alpha-1}y_0 = \triangle\mathcal{L}\{y\}(\cdot,s) + \mathcal{L}\{f\}(s). \tag{2.9}$$

Hence

$$\mathcal{L}\{y\}(\cdot,s) = \frac{s^{\alpha-1}y_0 + \mathcal{L}\{f\}(s)}{s^\alpha - \triangle}. \tag{2.10}$$

Moreover, since $-\lambda_n$, ξ_n are respectively, the eigenvalue and the corresponding eigenfunction of operator $\triangle$ and the sequence $\{\xi_n\}_{n\geq 1}$ form a orthonormal basis in $L^2[0,1]$, similar to the argument in [17,22], from

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda}, \quad \mathrm{Re}(s) \geq |\lambda|^{1/\alpha} \tag{2.11}$$

and the properties of Laplace transform, we get the result and then finish the proof. □

Remark 2.1. Based on the properties of Mittag-Leffler functions, an equivalence is derived in [23] between Lemma 2.1 and the results in [24–27], where the solution of studied system is given in terms of some probability densities.

Definition 2.1. [16] The solution of system (2.1) is said to be Mittag-Leffler stable if there exist constants $C > 0$, $\lambda > 0$, $b > 0$ such that

$$\|z(\cdot,t)\| \leq C\{E_\alpha(-\lambda t^\alpha)\}^b\|z_0\|, \ t\geq 0, \ z_0\in L^2[0,1]. \tag{2.12}$$

Next, we design a semilinear distributed parameter Luenberger type observer as follows:

$$\begin{cases} {}_0^C D_t^\alpha \hat{y}(x,t) = \triangle\hat{y}(x,t) + f(x,t,\hat{y}(x,t)) \\ \quad + K(x,t)(z(t) - \hat{z}(t)) \text{ in } [0,1]\times[0,\infty), \\ \hat{y}_x(0,t) = \hat{y}_x(1,t) = 0 \ \text{ in } [0,\infty), \\ \hat{y}(x,0) = \hat{y}_0(x) \ \text{ in } [0,1], \end{cases} \tag{2.13}$$

where $\hat{z}(t) = C\hat{y}(\cdot,t) := \hat{y}(0,t)$. Let $e(x,t) := y(x,t) - \hat{y}(x,t)$ be the observer error. Here $K(x,t)$ is called to be the output feedback weight. Let $e_0(x) := y_0(x) - \hat{y}_0(x)$ in Ω. It then follows directly from systems (2.1) and (2.13) that $e(x,t)$ satisfies the following system

$$\begin{cases} {}_0^C D_t^\alpha e(x,t) = \triangle e(x,t) + f(x,t,e(x,t)+\hat{y}(x,t)) - f(x,t,\hat{y}(x,t)) \\ \quad - K(x,t)e(0,t) \text{ in } [0,1]\times[0,\infty), \\ e_x(0,t) = e_x(1,t) = 0 \ \text{ in } [0,\infty), \\ e(x,0) = e_0(x) \ \text{ in } [0,1]. \end{cases} \tag{2.14}$$

For convenience, based on properties (2.4) and (2.5), let $\lambda(x) = \max_t \partial_y f(x,y)|_{y=\hat{y}(x,t)}$, we first consider the linearization system of (2.14):

$$\begin{cases} {}_0^C D_t^\alpha e(x,t) = \triangle e(x,t) + \lambda(x)e(x,t) - K(x,t)e(0,t), \\ e_x(0,t) = e_x(1,t) = 0 \ \text{ in } [0,\infty), \\ e(x,0) = e_0(x) \ \text{ in } \Omega. \end{cases} \tag{2.15}$$

According to [2,6], the observer system (2.13) with output feedback weight $K(x,t)$ determined by system (2.15) is subsequently called an extended Luenberger observer for the semilinear time fractional diffusion system (2.1). In the following section, we shall consider the backstepping-based technique, which was first introduced in [28] and then extended to time fractional diffusion system cases in [10], to compute $K(x,t)$ to guarantee the Mittag-Leffler stability of system (2.14).

3. Stability analysis of the linearized observer error systems

The purpose of this section is to determine the output feedback weight $K(x,t)$ to guarantee the Mittag-Leffler stability of observer error systems (2.14). For this, we first consider

$$\begin{cases} {}_0^CD_t^\alpha \omega(x,t) = \triangle\omega(x,t) - \mu\omega(x,t) \text{ in } [0,1]\times[0,\infty), \\ \omega_x(0,t) = \omega_x(1,t) = 0 \quad \text{in } [0,\infty), \\ \omega(x,0) = \omega_0(x) \quad \text{in } [0,\infty), \end{cases} \tag{3.1}$$

where μ is a given constant. By Lemma 2.1, we get that the solution of (3.1) is given by

$$\omega(x,t) = \sum_{n=1}^{\infty} E_\alpha[(-\lambda_n - \mu)t^\alpha](y_0, \xi_n)\xi_n(x). \tag{3.2}$$

So if $\lambda_1 + \mu \geq \varepsilon > 0$ for some $\varepsilon > 0$, since Mittag-Leffler function $E_\alpha(-t^\alpha)$ is a completely monotone for $\alpha \in (0, 1]$ and $t > 0$ (see [29,30]), i.e.,

$$\frac{d}{dt}E_\alpha(-t^\alpha) \leq 0, \quad t > 0, \ \alpha \in (0,1). \tag{3.3}$$

It then follows from $(\lambda_n + \mu)^{1/\alpha} \geq (\lambda_1 + \mu)^{1/\alpha} \geq \varepsilon^{1/\alpha} > 0$, $\alpha \in (0, 1]$, $n = 1, 2, \cdots$ that

$$\begin{aligned} \|\omega(\cdot,t)\| &= \left\| \sum_{n=1}^{\infty} E_\alpha[-((\lambda_n+\mu)^{1/\alpha}t)^\alpha](\omega_0,\xi_n)\xi_n \right\| \\ &\leq \left\| \sum_{n=1}^{\infty} E_\alpha[-((\lambda_1+\mu)^{1/\alpha}t)^\alpha](\omega_0,\xi_n)\xi_n \right\| \\ &\leq E_\alpha[-\varepsilon t^\alpha]\|\omega_0\|, \end{aligned} \tag{3.4}$$

which means that systems (3.1) is Mittag-Leffler stable in space $L^2(\Omega)$.

3.1. Backstepping-based transform

To transform the observer error system (2.14) into (3.1), let us consider the following transformation

$$\omega(x,t) = e(x,t) + \int_0^x g(x,\xi)e(\xi,t)d\xi. \tag{3.5}$$

Before exploring the existence of $g(x, \xi)$, denote $g_x(x,x) = \frac{\partial}{\partial x}g(x,\xi)|_{\xi=x}$, $\frac{d}{dx}g(x,x) = g_x(x,x) + g_\xi(x,x)$ and $g_\xi(x,x) = \frac{\partial}{\partial \xi}g(x,\xi)|_{\xi=x}$, it then follows that

$$\omega_x(x,t) = e_x(x,t) + g(x,x)e(x,t) + \int_0^x g_x(x,\xi)e(\xi,t)d\xi, \tag{3.6}$$

$$\begin{aligned} \omega_{xx}(x,t) = {} & e_{xx}(x,t) + \frac{d}{dx}g(x,x)e(x,t) + g(x,x)e_x(x,t) \\ & + g_x(x,x)e(x,t) + \int_0^x g_{xx}(x,\xi)e(\xi,t)d\xi \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} {}_0^CD_t^\alpha \omega(x,t) &= {}_0^CD_t^\alpha e(x,t) + \int_0^x g(x,\xi){}_0^CD_t^\alpha e(\xi,t)d\xi \\ &= {}_0^CD_t^\alpha e(x,t) + \int_0^x g(x,\xi) \\ &\quad \times [e_{\xi\xi}(\xi,t) + \lambda(\xi)e(\xi,t) - K(\xi,t)e(0,t)]d\xi \\ &= {}_0^CD_t^\alpha e(x,t) + g(x,x)e_x(x,t) \\ &\quad - g(x,0)e_x(0,t) - g_\xi(x,x)e(x,t) \\ &\quad + g_\xi(x,0)e(0,t) + \int_0^x g_{\xi\xi}(x,\xi)e(\xi,t)d\xi \\ &\quad + \int_0^x g(x,\xi)[\lambda(\xi)e(\xi,t) - K(\xi,t)e(0,t)]d\xi. \end{aligned} \tag{3.8}$$

Hence,

$$\begin{aligned} & {}_0^CD_t^\alpha \omega(x,t) - \omega_{xx}(x,t) + \mu\omega(x,t) \\ &= {}_0^CD_t^\alpha e(x,t) + g(x,x)e_x(x,t) - g(x,0)e_x(0,t) \\ &\quad - g_\xi(x,x)e(x,t) + g_\xi(x,0)e(0,t) \\ &\quad + \int_0^x g_{\xi\xi}(x,\xi)e(\xi,t)d\xi + \int_0^x g(x,\xi) \\ &\quad \times [\lambda(\xi)e(\xi,t) - K(\xi,t)e(0,t)]d\xi - e_{xx}(x,t) \\ &\quad - \frac{d}{dx}g(x,x)e(x,t) - g(x,x)e_x(x,t) - g_x(x,x)e(x,t) \\ &\quad - \int_0^x g_{xx}(x,\xi)e(\xi,t)d\xi \\ &\quad + \mu\left(e(x,t) + \int_0^x g(x,\xi)e(\xi,t)d\xi\right) \\ &= \left(\mu + \lambda(x) - g_x(x,x) - g_\xi(x,x) - \frac{d}{dx}g(x,x)\right)e(x,t) \\ &\quad - g(x,0)e_x(0,t) + \int_0^x \{g_{\xi\xi}(x,\xi) - g_{xx}(x,\xi) \\ &\quad + [\mu + \lambda(\xi)]g(x,\xi)\}e(\xi,t)d\xi \\ &\quad + \left(g_\xi(x,0) - K(x,t) - \int_0^x g(x,\xi)K(\xi,t)d\xi\right)e(0,t). \end{aligned} \tag{3.9}$$

Suppose that the output feedback weight $K(x, t)$ is determined according to

$$g_\xi(x,0) - K(x,t) - \int_0^x g(x,\xi)K(\xi,t)d\xi = 0. \tag{3.10}$$

To make sure that the right-hand side of (3.9) to be zero for all e, we get that the following three conditions have to be satisfied

$$\begin{cases} g_{xx}(x,\xi) - g_{\xi\xi}(x,\xi) = (\mu + \lambda(\xi))g(x,\xi), \quad 0 \leq \xi \leq x \leq 1, \\ g(x,0) = 0, \quad 0 \leq x \leq 1, \\ g_x(x,x) + g_\xi(x,x) + \dfrac{d}{dx}g(x,x) = \mu + \lambda(x), \quad 0 \leq x \leq 1. \end{cases} \tag{3.11}$$

It is not difficult to see that system (3.11) is compatible and forms a well-posed partial differential equation of hyperbolic type and besides, based on [28] (see also [10]), we obtain that

Lemma 3.1. *[10, 28] For any $0 \leq \xi \leq x \leq 1$, if $\lambda \in C^1[0, 1]$, then systems (3.11) has a unique solution.*

Lemma 3.2. *[10, 28] Let G: $L^2[0, 1] \to L^2[0, 1]$ be the linear bounded operator defined as*

$$\omega(x,\cdot) = (Ge)(x,\cdot) := e(x,\cdot) + \int_0^x g(x,\xi)e(\xi,\cdot)d\xi. \tag{3.12}$$

If $g(x, \xi)$ solves problem (3.11), then G converts the system (2.1) into (3.1) and besides, the inverse $G^{-1} : L^2[0,1] \to L^2[0,1]$ exists and is bounded.

3.2. Mittag-Leffler stability of system (2.15)

Now we give the following lemma, which plays a key role to obtain our main results.

Lemma 3.3. *[31] For any given $t \geq 0$, we see*

$$\frac{1}{2}{}_0^CD_t^\alpha x^2(t) \leq x(t){}_0^CD_t^\alpha x(t), \quad \forall\alpha \in (0,1], \quad t \in [0,\infty) \tag{3.13}$$

provided that x is continuous and differentiable.

Theorem 3.1. *Let $\lambda \in C^1[0, 1]$, $\omega(\cdot, t)$ be continuous and differentiable on $[0, \infty)$ and the Laplace transform of $\omega(\cdot, t)^2$ exists. If $K(x, t)$ satisfies (3.10), then the following estimation*

$$\|e(\cdot,t)\|^2 \leq \nu^2\|e_0\|^2 E_\alpha(-2\mu t^\alpha), \quad t \in [0,\infty) \tag{3.14}$$

holds for some $\nu > 0$.

Proof. Based on the argument above, if $K(x, t)$ satisfies (3.10), then the system (2.1) is transformed into (3.1). Moreover, it follows from Lemma 3.2 that

$$\|e(\cdot,t)\| = \|G^{-1}\omega(\cdot,t)\| \le \nu\|\omega(\cdot,t)\|, \quad \|\omega_0\| \le \nu\|e_0\| \tag{3.15}$$

holds for some $\nu > 0$.

Let $W(t) = \frac{1}{2}\int_0^1 \omega(x,t)^2 dx$. By Lemma 3.3,

$$\begin{aligned} {}_0^C D_t^\alpha W(t) &\le \int_0^1 \omega(x,t) {}_0^C D_t^\alpha \omega(x,t) dx \\ &= \int_0^1 \omega(x,t)\omega_{xx}(x,t)dx - \mu\int_0^1 \omega(x,t)^2 dx \\ &= -\int_0^1 \omega_x(x,t)^2 dx - \mu\int_0^1 \omega(x,t)^2 dx \\ &\le -2\mu W(t). \end{aligned} \tag{3.16}$$

Moreover, let

$$M(t) = -2\mu W(t) - {}_0^C D_t^\alpha W(t). \tag{3.17}$$

Obviously, $M(t) \ge 0$ on $[0, \infty)$. Since the Laplace transform of $\omega(\cdot, t)^2$ exists, applying the Laplace transform on (3.17), one has

$$\hat{M}(s) = -2\mu\hat{W}(s) - s^\alpha \hat{W}(s) + s^{\alpha-1}W(0), \tag{3.18}$$

where $W(0) = \frac{1}{2}\int_0^1 \omega_0(x)^2 dx = \frac{1}{2}\|\omega_0\|^2$, $\hat{W}(s) = \int_0^\infty e^{-st}W(t)dt$ and $\hat{M}(s) = \int_0^\infty e^{-st}M(t)dt$. Hence,

$$\hat{W}(s) = \frac{s^{\alpha-1}W(0) - \hat{M}(s)}{s^\alpha + 2\mu}. \tag{3.19}$$

It then follows from the nonnegativity of $M(t)$, $E_{\alpha,\alpha}(-2\lambda t^\alpha)$ and the uniqueness, existence theorem in [18] that the unique solution of (3.17) is

$$\begin{aligned} W(t) &= E_\alpha(-2\mu t^\alpha)W(0) - M(t) * [t^{\alpha-1}E_{\alpha,\alpha}(-2\mu t^\alpha)] \\ &\le E_\alpha(-2\mu t^\alpha)W(0), \end{aligned} \tag{3.20}$$

where * denotes the convolution operator. Finally, from (3.15), we have

$$\|e(\cdot,t)\|^2 \le \nu\|\omega_0\|^2 E_\alpha(-2\mu t^\alpha) \le \nu^2\|e_0\|^2 E_\alpha(-2\mu t^\alpha) \tag{3.21}$$

and finish the proof. □

Remark 3.1. Since (3.14) is equivalent to

$$\|e(\cdot,t)\| \le \nu\{E_\alpha(-2\mu t^\alpha)\}^{1/2}\|e_0\|, \ t \in [0,\infty), \tag{3.22}$$

then if $\mu \ge \varepsilon > 0$ for some $\varepsilon > 0$, system (2.15) is Mittag-Leffler stable.

4. Stability analysis of the semilinear observer error systems

As shown in Theorem 3.1, the presented design with $K(x, t)$ determined by (3.10) implies the Mittag-Leffler stability of system (2.15). In the following, we shall consider the semilinear system (2.14).

4.1. Backstepping-based transform

Similarly, consider the transformation

$$\omega(x,t) = e(x,t) + \int_0^x m(x,\xi)e(\xi,t)d\xi \tag{4.1}$$

and denote $m_x(x,x) = \frac{\partial}{\partial x}m(x,\xi)|_{\xi=x}$, $m_\xi(x,x) = \frac{\partial}{\partial \xi}m(x,\xi)|_{\xi=x}$, $\frac{d}{dx}m(x,x) = m_x(x,x) + m_\xi(x,x)$, it then follows that

$$\omega_x(x,t) = e_x(x,t) + m(x,x)e(x,t) + \int_0^x m_x(x,\xi)e(\xi,t)d\xi, \tag{4.2}$$

$$\begin{aligned} \omega_{xx}(x,t) &= e_{xx}(x,t) + \frac{d}{dx}m(x,x)e(x,t) + m(x,x)e_x(x,t) \\ &\quad + m_x(x,x)e(x,t) + \int_0^x m_{xx}(x,\xi)e(\xi,t)d\xi \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} {}_0^C D_t^\alpha \omega(x,t) &= {}_0^C D_t^\alpha e(x,t) + \int_0^x m(x,\xi) {}_0^C D_t^\alpha e(\xi,t)d\xi \\ &= {}_0^C D_t^\alpha e(x,t) + \int_0^x m(x,\xi)e_{\xi\xi}(\xi,t)d\xi \\ &\quad + \int_0^x m(x,\xi)[f(\xi,e(\xi,t)+\hat{y}(\xi,t)) \\ &\quad - f(\xi,\hat{y}(\xi,t)) - K(\xi,t)e(0,t)]d\xi \\ &= {}_0^C D_t^\alpha e(x,t) + m(x,x)e_x(x,t) \\ &\quad - m(x,0)e_x(0,t) - m_\xi(x,x)e(x,t) \\ &\quad + m_\xi(x,0)e(0,t) + \int_0^x m_{\xi\xi}(x,\xi)e(\xi,t)d\xi \\ &\quad + \int_0^x m(x,\xi)[f(\xi,e(\xi,t)+\hat{y}(\xi,t)) \\ &\quad - f(\xi,\hat{y}(\xi,t)) - K(\xi,t)e(0,t)]d\xi. \end{aligned} \tag{4.4}$$

Hence,

$$\begin{aligned} &{}_0^C D_t^\alpha \omega(x,t) - \omega_{xx}(x,t) + \mu\omega(x,t) \\ &= {}_0^C D_t^\alpha e(x,t) + m(x,x)e_x(x,t) \\ &\quad - m(x,0)e_x(0,t) - m_\xi(x,x)e(x,t) \\ &\quad + m_\xi(x,0)e(0,t) + \int_0^x m_{\xi\xi}(x,\xi)e(\xi,t)d\xi \\ &\quad + \int_0^x m(x,\xi)[f(\xi,e(\xi,t)+\hat{y}(\xi,t)) \\ &\quad - f(\xi,\hat{y}(\xi,t)) - K(\xi,t)e(0,t)]d\xi - e_{xx}(x,t) \\ &\quad - \frac{d}{dx}m(x,x)e(x,t) - m(x,x)e_x(x,t) - m_x(x,x)e(x,t) \\ &\quad - \int_0^x m_{xx}(x,\xi)e(\xi,t)d\xi \\ &+ \mu\left(e(x,t) + \int_0^x m(x,\xi)e(\xi,t)d\xi\right) \\ &= f(x,e(x,t)+\hat{y}(x,t)) - f(x,\hat{y}(x,t)) - m(x,0)e_x(0,t) \\ &\quad + \left(\mu - m_x(x,x) - m_\xi(x,x) - \frac{d}{dx}m(x,x)\right)e(x,t) \\ &\quad + \int_0^x \{m_{\xi\xi}(x,\xi) - m_{xx}(x,\xi) + \mu m(x,\xi)\}e(\xi,t)d\xi \\ &\quad + \int_0^x m(x,\xi)[f(\xi,e(\xi,t)+\hat{y}(\xi,t)) - f(\xi,\hat{y}(\xi,t))]d\xi \\ &\quad + \left[m_\xi(x,0) - K(x,t) - \int_0^x m(x,\xi)K(\xi,t)d\xi\right]e(0,t). \end{aligned}$$

Moreover, let

$$\begin{aligned} &\varphi(x,t,e(x,t),\hat{y}(x,t)) \\ &= f(x,t,e(x,t)+\hat{y}(x,t)) - f(x,t,\hat{y}(x,t)) - \lambda(x)e(x,t) \\ &\quad + \int_0^x m(x,\xi)[f(\xi,t,e(\xi,t)+\hat{y}(\xi,t)) \\ &\quad - f(\xi,t,\hat{y}(\xi,t)) - \lambda(\xi)e(\xi,t)]d\xi. \end{aligned} \tag{4.5}$$

Defining $K(x, t)$ as follows

$$K(x,t) = m_\xi(x,0) - \int_0^x m(x,\xi)K(\xi,t)d\xi, \tag{4.6}$$

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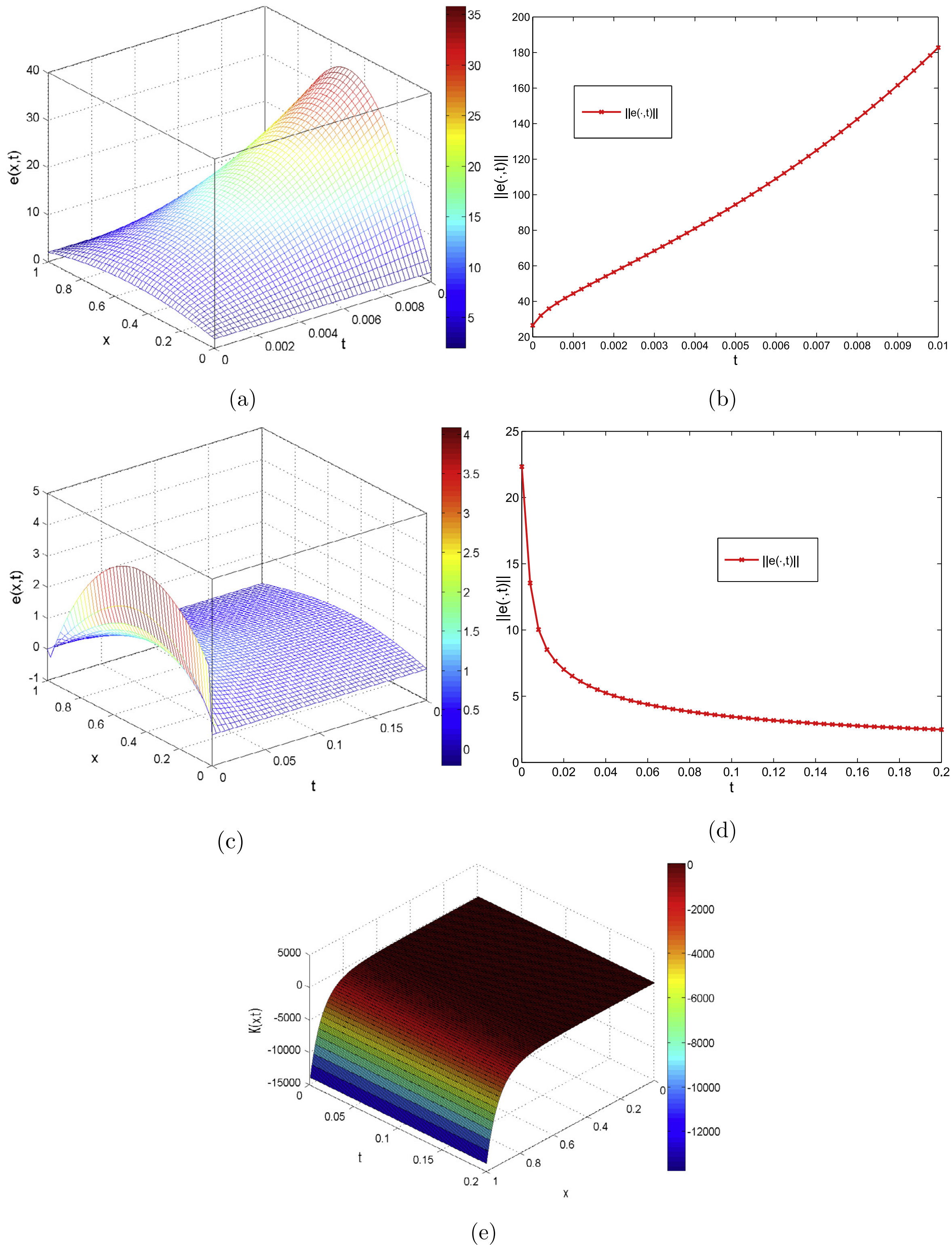

Fig. 1. Simulation results for system (2.1). (a) Solution $e(x, t)$ without control. (b) L^2-norm of observer error $e(x, t)$ without control. (c) Solution $e(x, t)$ with control. (d) L^2-norm of observer error $\|e(\cdot, t)\|$ with control. (e) Weights $K(x, t)$.

where $m(x, \xi)$ satisfies (3.11), the time fractional diffusion system (2.14) is equivalent to

$$\begin{cases} {}_0^C D_t^{\alpha} \omega(x,t) - \triangle\omega(x,t) + \mu\omega(x,t) = \varphi(x,t,e(x,t),\hat{y}(x,t)) \\ \quad \text{in } [0,1]\times[0,\infty), \\ \omega_x(0,t) = \omega_x(1,t) = 0 \quad \text{in } [0,\infty), \\ \omega(x,0) = \omega_0(x) \quad \text{in } [0,\infty). \end{cases} \tag{4.7}$$

In order to Mittag-Leffler stabilize (4.7), consider first the lemma below.

Lemma 4.1. *Suppose that $f(x, t, y)$ is continuous and satisfies (2.4) and (2.5). Given $t \geq 0$, $r > 0$, then we can find a constant C^* satisfying*

$$\int_0^1 \omega(x,t)\varphi(x,t,e(x,t),\hat{y}(x,t))dx \leq C^* \|\omega(\cdot,t)\| \|e(\cdot,t)\|. \tag{4.8}$$

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Proof. Consider function $\varphi(x,t,e(x,t),\hat{y}(x,t))$ in Eq. (4.5), let

$$A(t) = \int_0^1 \omega(x,t)[f(x,e(x,t)+\hat{y}(x,t)) - f(x,\hat{y}(x,t)) - \lambda(x)e(x,t)]dx, \tag{4.9}$$

$$B(t) = \int_0^1 \omega(x,t)\int_0^x m(x,\xi)[f(\xi,e(\xi,t)+\hat{y}(\xi,t)) - f(\xi,\hat{y}(\xi,t)) - \lambda(\xi)e(\xi,t)]d\xi dx. \tag{4.10}$$

Form (2.4) and (2.5), let $\lambda^* = \sup_{x\in[0,1]}\lambda(x)$ and $m^* = \sup_{x,\xi\in[0,1]}|m(x,\xi)|$, it then follows that

$$\begin{aligned} A(t) &\le |A(t)| \\ &\le \int_0^1 |\omega(x,t)||f(x,e(x,t)+\hat{y}(x,t)) - f(x,\hat{y}(x,t)) - \lambda(x)e(x,t)|dx \\ &\le \|\omega(\cdot,t)\|(\|f(\cdot,e(\cdot,t)+\hat{y}(\cdot,t)) - f(\cdot,\hat{y}(\cdot,t))\| + \|\lambda(\cdot)e(\cdot,t)\|) \\ &\le (C+\lambda^*)\|\omega(\cdot,t)\|\|e(\cdot,t)\| \end{aligned} \tag{4.11}$$

and similarly,

$$\begin{aligned} B(t) &\le m^*\|\omega(\cdot,t)\|(\|f(\cdot,e(\cdot,t)+\hat{y}(\cdot,t)) - f(\cdot,\hat{y}(\cdot,t))\| + \|\lambda(\cdot)e(\cdot,t)\|) \\ &\le m^*(C+\lambda^*)\|\omega(\cdot,t)\|\|e(\cdot,t)\|, \end{aligned}$$

where $C = C(r,T)$ is introduced in Eq. (2.4). Taking $C^* = (1+m^*)(C+\lambda^*)$, we then finish the proof. □

Theorem 4.1. *Suppose that all conditions of Theorem 3.1 and Lemma 4.1 hold. Then the solutions of (2.14) with K(x, t) satisfying (3.10) such that*

$$\|e(\cdot,t)\|^2 \le \nu^2\|e_0\|^2 E_\alpha(-2(\mu - C^*(r,T)\nu)t^\alpha),\ t\in[0,\infty) \tag{4.12}$$

for some positive constant ν.

Proof. Let $W(t) = \frac{1}{2}\int_0^1 \omega(x,t)^2dx$. By Lemma 3.3, we have

$$\begin{aligned} {}_0^CD_t^\alpha W(t) &\le \int_0^1 \omega(x,t){}_0^CD_t^\alpha \omega(x,t)dx \\ &= \int_0^1 \omega(x,t)\omega_{xx}(x,t)dx - \mu\int_0^1 \omega(x,t)^2dx \\ &\quad + \int_0^1 \omega(x,t)\varphi(x,t,e(x,t)+\hat{y}(x,t))dx \\ &= -\int_0^1 \omega_x(x,t)^2dx - \mu\int_0^1 \omega(x,t)^2dx \\ &\quad + \int_0^1 \omega(x,t)\varphi(x,t,e(x,t),\hat{y}(x,t))dx. \end{aligned}$$

It then from Lemma 4.1 and Eq. (3.15) that

$$\begin{aligned} {}_0^CD_t^\alpha W(t) &\le -2\mu W(t) + \int_0^1 \omega(x,t)\varphi(x,t,e(x,t),\hat{y}(x,t))dx \\ &\le -2\mu W(t) + C^*(r,T)\|\omega(\cdot,t)\|\|e(\cdot,t)\| \\ &\le -2\mu W(t) + C^*(r,T)\nu\|\omega(\cdot,t)\|^2 \\ &= -2(\mu - C^*(r,T)\nu)W(t). \end{aligned} \tag{4.13}$$

Moreover, similar to the proof of Theorem 3.1, we get that

$$\|e(\cdot,t)\|^2 \le \nu^2\|e_0\|^2 E_\alpha(-2(\mu - C^*(r,T)\nu)t^\alpha),\ t\in[0,\infty). \tag{4.14}$$

The proof is finished. □

Remark 4.1. Assume that $\mu - C^*(r,T)\nu \ge \varepsilon$ for some $\varepsilon > 0$ with $\nu > 0$ satisfying (3.15). Then we get that system (2.14) with $K(x,t)$ satisfying (3.10) is Mittag-Leffler stable.

Remark 4.2. In particular, when $\alpha = 1$, we refer the readers to [6] and those references cited therein.

5. Numerical simulation

In this section, a simulate example is worked out to confirm our results.

In system (2.1), take $\alpha = 0.5$, $y_0(x) = 10x(1-x)+2$, $x\in[0,1]$ and

$$f(x,t,y) = 10y(y+2\sin(t))e^x. \tag{5.1}$$

We get that the properties (2.4) and (2.5) hold. Moreover, let $\hat{y}_0(x)\equiv 0$. We have $e_0(x) = y_0(x) = 10x(1-x)+2$, $x\in[0,1]$. As seen from Fig. 1 (*a*) and (*b*), the solution of system (2.14) with $K \equiv 0$ is unstable.

Let $\mu = 1$ in target system (3.1). By using the coordinate transformation (3.5), (*c*) and (*d*) of Fig. 1 imply that system (2.14) with the output feedback weight $K(x,t)$ defined by (3.10) converges smoothly. Moreover, the corresponding output feedback weight $K(x,t)$ is shown in (*e*) of Fig. 1 with $g(x,\xi)$ determined by system (3.11). The simulation results show that the extended Luenberger-type observer designed by us yields satisfactory performance in dealing with semilinear time fractional diffusion systems.

6. Conclusion

In this paper, we present an extended Luenberger state observer for a class of semilinear time fractional diffusion systems. The Mittag-Leffler stability of both the linearized and semilinear observer error systems are guaranteed based on a backstepping-based technique. Moreover, the results studied here can also be extended to complex fractional distributed parameter systems (DPSs) and various open questions are still under consideration. For instance, the Mittag-Leffler stabilization of nonlinear fractional DPSs as well as their controller configurations are of great interest.

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