IMA Journal of Mathematical Control and Information Advance Access published January 28, 2016

IMA Journal of Mathematical Control and Information (2016) Page 1 of 18
doi:10.1093/imamci/dnw001

Regional boundary controllability of time fractional diffusion processes

Fudong Ge
College of Information Science and Technology, Donghua University, Shanghai 201620, PR China
gefd2011@gmail.com

YangQuan Chen*
Mechatronics, Embedded Systems and Automation Lab, University of California, Merced, CA 95343, USA
*Corresponding author. Email: ychen53@ucmerced.edu

and

Chunhai Kou
Department of Applied Mathematics, Donghua University, Shanghai 201620, PR China
kouchunhai@dhu.edu.cn

[Received on 27 September 2015; revised on 6 December 2015; accepted on 4 January 2016]

In this paper, we are concerned with the regional boundary controllability of the Riemann–Liouville time fractional diffusion systems of order $\alpha \in (0, 1]$. The characterizations of strategic actuators are established when the systems studied are regionally boundary controllable. The determination of control to achieve regional boundary controllability with minimum energy is explored. We also show a connection between the regional internal controllability and regional boundary controllability. Several useful results for the optimal control from an implementation point of view are presented in the end.

Keywords: regional boundary controllability; time fractional diffusion processes; strategic actuators; minimum energy control.

1. Introduction

In the past several decades, a lot of work has been carried out to deal with the problem of steering a system to a target state, especially after the introduction of the notions of actuators and sensors (El Jai & Pritchard, 1988; El Jai, 1991). However, in many real-world applications, we are only concerned with those cases where the target states of the problem studied are defined in a given subregion of the whole space domain. Then the regional idea emerges and we refer the reader to Sakawa (1974) and Zerrik *et al.* (1999, 2000) for more information on the concept of regional analysis for the Gaussian diffusion process. Besides, it should be pointed out that not only does the concept of regional analysis make sense closer to real-world problems, it also generalizes the results of existence contributions.

In addition, after the introduction of continuous time random walks by Montroll & Weiss (1965), the anomalous diffusion equation of fractional-order has attracted increasing interest and has been proved to be a useful tool in modelling many real-world problems (Adams & Gelhar, 1992; Metzler & Klafter, 2000, 2004; Meerschaert & Scalas, 2006; Cartea & Negrete, 2007). More precisely, the mean squared displacement of the anomalous diffusion process is described by a power law of fractional exponent, which is smaller (in the case of sub-diffusion) or bigger (in the case of super-diffusion) than that of the Brownian motion. It is confirmed that the time fractional diffusion system, where the

© The authors 2016. Published by Oxford University Press on behalf of the Institute of Mathematics and its Applications. All rights reserved.

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

traditional first-order time derivative is replaced by a Riemann–Liouville time fractional derivative of order $\alpha \in (0, 1]$, can be used to well characterize those sub-diffusion process (Metzler & Klafter, 2000, 2004). For example, the flow through porous media microscopic processes (Uchaikin & Sibatov, 2013), or swarm of robots moving through dense forest (Spears & Spears, 2012) etc. With regard to fractional calculus, as we all know, it has shown great potential in science and engineering applications and some phenomena such as self-similarity, non-stationary, non-Gaussian process and short- or long-memory process are all closely related to fractional calculus (Podlubny, 1999; Agrawal, 2002; Kilbas *et al.*, 2006). It is now widely believed that using fractional calculus in modelling can better capture the complex dynamics of natural and man-made systems, and fractional-order controls can offer better performance not achievable before using integer-order controls (Mandelbrot, 1983; Torvik & Bagley, 1984), which in fact raise important theoretical challenges and open new research opportunities.

Motivated by the argument above, the contribution of this present work is on the regional boundary controllability of the anomalous transport process described by time fractional diffusion systems. More precisely, for an open bounded subset $\Omega \subseteq \mathbf{R}^n$ with smooth boundary $\partial\Omega$, we consider:

- a subregion Γ of $\partial\Omega$ which may be unconnected.
- various kinds of actuators (zone, pointwise, internal or boundary) acting as controls.

The rest of this paper is organized as follows. The mathematical concept of regional boundary controllability and several preliminaries are presented in the next section; then we present an example which is regional boundary controllability but not globally boundary controllable. Section 3 is focused on the characterizations of Γ-strategic actuators and our main result on regional boundary controllability with the minimum energy problem is given in Section 4. In Section 5, a connection between internal and boundary regional controllability is established and, at last, we work out some useful results for the optimal control from an implementation point of view.

2. Regional boundary controllability

2.1 *Problem statement*

In this paper, we consider the following abstract time fractional diffusion system:

$$\begin{cases} {}_0D_t^{\alpha} z(t) = Az(t) + Bu(t), \quad t \in [0, b], \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} z(t) = z_0, \end{cases} \tag{2.1}$$

where A generates a strongly continuous semi-group $\{\Phi(t)\}_{t \geqslant 0}$ on the Hilbert space $Z := H^1(\Omega)$, $z \in L^2(0, b; Z)$ and the initial vector $z_0 \in Z$. It is supposed that $B : \mathbf{R}^p \to Z$ is the control operator and $u \in L^2(0, b; \mathbf{R}^p)$ depends on the number and structure of actuators. Moreover, the Riemann–Liouville fractional derivative ${}_0D_t^{\alpha}$ and the Riemann–Liouville fractional integral ${}_0I_t^{\alpha}$ are, respectively, given by Podlubny (1999) and Kilbas *et al.* (2006)

$$\begin{aligned} {}_0D_t^{\alpha} z(t) &= \frac{\mathrm{d}}{\mathrm{d}t} {}_0I_t^{1-\alpha} z(t), \quad \alpha \in (0, 1] \quad \text{and} \\ {}_0I_t^{\alpha} z(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, \mathrm{d}s, \quad \alpha > 0. \end{aligned} \tag{2.2}$$

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

Definition 2.1 (Ge *et al.*, 2015b) For any given $f \in L^2(0,b;Z)$, $\alpha \in (0,1]$, a function $v \in L^2(0,b;Z)$ is said to be a mild solution of the following system:

$$\begin{cases} {}_0D_t^{\alpha} v(t) = Av(t) + f(t), & t \in [0,b], \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} v(t) = v_0 \in Z, \end{cases} \tag{2.3}$$

if it satisfies

$$z(t) = K_\alpha(t) v_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) f(s)\, \mathrm{d}s, \tag{2.4}$$

where $K_\alpha(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi(t^\alpha \theta)\, \mathrm{d}\theta$, $\{\Phi(t)\}_{t \geqslant 0}$ is the strongly continuous semi-group generated by A, $\phi_\alpha(\theta) = (1/\alpha)\theta^{-1-1/\alpha} \psi_\alpha(\theta^{-1/\alpha})$ and ψ_α is a probability density function defined by

$$\psi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha), \quad \theta > 0. \tag{2.5}$$

In addition, we have Mainardi *et al.* (2000) and Zhou & Jiao (2010)

$$\int_0^\infty \psi_\alpha(\theta)\, \mathrm{d}\theta = 1 \quad \text{and} \quad \int_0^\infty \theta^\nu \phi_\alpha(\theta)\, \mathrm{d}\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \quad \nu \geqslant 0. \tag{2.6}$$

By Lemma 2.1, the mild solution $z(\cdot, u)$ of (2.1) can be given by

$$z(t,u) = t^{\alpha-1} K_\alpha(t) z_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) Bu(s)\, \mathrm{d}s. \tag{2.7}$$

Let $H : L^2(0,b;\mathbf{R}^p) \to Z$ be

$$Hu = \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s) Bu(s)\, \mathrm{d}s \quad \forall u \in L^2(0,b;\mathbf{R}^p). \tag{2.8}$$

Suppose that $\{\Phi^*(t)\}_{t \geqslant 0}$, generated by the adjoint operator of A, is also a strongly continuous semi-group on the space Z. For any $v \in Z$, it follows from $\langle Hu, v\rangle = \langle u, H^* v\rangle$ that

$$H^* v = B^* (b-s)^{\alpha-1} K_\alpha^*(b-s) v, \tag{2.9}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing of space Z, B^* is the adjoint operator of B and

$$K_\alpha^*(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi^*(t^\alpha \theta)\, \mathrm{d}\theta.$$

Let $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ be the trace operator of order zero, which is linear continuous and surjective, γ^* denotes the adjoint operator. Moreover, if $\Gamma \subseteq \partial\Omega$, $p_\Gamma : H^{1/2}(\partial\Omega) \to H^{1/2}(\Gamma)$ defined by

$$p_\Gamma z := z|_\Gamma \tag{2.10}$$

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

and, for any $\bar{z} \in H^{1/2}(\Gamma)$, the adjoint operator p_Γ^* can be given by

$$p_\Gamma^* \bar{z}(x) := \begin{cases} \bar{z}(x), & x \in \Gamma, \\ 0, & x \in \partial\Omega \backslash \Gamma. \end{cases} \tag{2.11}$$

2.2 *Definition and characterizations*

Let $\omega \subseteq \Omega$ be a given region of positive Lebesgue measure. Denote the projection operator on ω by the restriction map

$$p_\omega : H^1(\Omega) \to H^1(\omega); \tag{2.12}$$

then we are ready to state the following definitions.

Definition 2.2 The system (2.1) is said to be exactly (respectively, approximately) regionally controllable on ω at time b if, for any $y_b \in H^1(\omega)$, given $\varepsilon > 0$, there exists a control $u \in L^2(0, b; \mathbf{R}^p)$ such that

$$p_\omega z(b, u) = y_b \quad \text{(respectively, } \|p_\omega z(b, u) - y_b\|_{H^1(\omega)} \leqslant \varepsilon). \tag{2.13}$$

Definition 2.3 The system (2.1) is said to be exactly (respectively, approximately) regionally boundary controllable on $\Gamma \subseteq \partial\Omega$ at time b if, for any $z_b \in H^{1/2}(\Gamma)$, given $\varepsilon > 0$, there exists a control $u \in L^2(0, b; \mathbf{R}^p)$ such that

$$p_\Gamma(\gamma z(b, u)) = z_b \quad \text{(respectively, } \|p_\Gamma(\gamma z(b, u)) - z_b\|_{H^{1/2}(\Gamma)} \leqslant \varepsilon). \tag{2.14}$$

Proposition 2.1 The following properties are equivalent:

(1) the system (2.1) is exactly regionally boundary controllable on Γ at time b;

(2) $\mathrm{Im}(p_\Gamma \gamma H) = H^{1/2}(\Gamma)$;

(3) $\mathrm{Ker}(p_\Gamma) + \mathrm{Im}(\gamma H) = H^{1/2}(\partial\Omega)$;

(4) for any $z \in H^{1/2}(\Gamma)$, there exists a positive constant c such that

$$\|z\|_{H^{1/2}(\Gamma)} \leqslant c \|H^* \gamma^* p_\Gamma^* z\|_{L^2(0,b;\mathbf{R}^p)}. \tag{2.15}$$

Proof. By Definition 2.3, it is not difficult to see that (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3) : For any $z \in H^{1/2}(\Gamma)$, let $\hat{z}$ be the extension of z to $H^{1/2}(\partial\Omega)$. Since $\mathrm{Im}(p_\Gamma \gamma H) = H^{1/2}(\Gamma)$, there exists $u \in L^2(0, b; \mathbf{R}^p)$, $z_1 \in \mathrm{Ker}(p_\Gamma)$ such that $\hat{z} = z_1 + \gamma H u$.

(3) $\Rightarrow$ (2) : For any $\tilde{z} \in H^{1/2}(\partial\Omega)$, $\tilde{z} = z_1 + z_2$, where $z_1 \in \mathrm{Ker}(p_\Gamma)$ and $z_2 \in \mathrm{Im}(\gamma H)$. Then there exists a control $u \in L^2(0, b; \mathbf{R}^p)$ such that $\gamma H u = z_2$. Hence, it follows from the definition of p_Γ that $\mathrm{Im}(p_\Gamma \gamma H) = H^{1/2}(\Gamma)$.

(1) $\Leftrightarrow$ (4) : the equivalence between (1) and (4) can be deduced from the following general result (Pritchard & Wirth, 1978): let E, F, G be reflexive Hilbert spaces and $f \in \mathcal{L}(E, G)$, $g \in \mathcal{L}(F, G)$. Then the following two properties are equivalent:

(1) $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$,

(2) $\exists\, \gamma > 0$ such that $\|f^* z^*\|_{E^*} \leqslant \gamma \|g^* z^*\|_{F^*}$, $\forall z^* \in G$.

By choosing $E = G = H^{1/2}(\Gamma)$, $F = L^2(0, b; \mathbf{R}^p)$, $f = \mathrm{Id}_{H^{1/2}(\Gamma)}$ and $g = p_\Gamma \gamma H$, then we complete the proof. □

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

PROPOSITION 2.2 There is an equivalence among the following properties:

⟨1⟩ The system (2.1) is approximately regionally boundary controllable on Γ at time b;

⟨2⟩ $\overline{\mathrm{Im}(p_\Gamma \gamma H)} = H^{1/2}(\Gamma)$;

⟨3⟩ $\mathrm{Ker}(p_\Gamma) + \overline{\mathrm{Im}(\gamma H)} = H^{1/2}(\partial\Omega)$;

⟨4⟩ the operator $p_\Gamma \gamma H H^* \gamma^* p_\Gamma^*$ is positive definite.

Proof. By Proposition 2.1, ⟨1⟩ ⇔ ⟨2⟩ ⇔ ⟨3⟩. Finally, we show that ⟨2⟩ ⇔ ⟨4⟩. In fact, since

$$\overline{\mathrm{Im}(p_\Gamma \gamma H)} = H^{1/2}(\Gamma) \Leftrightarrow (p_\Gamma \gamma H u, z)_{H^{1/2}(\Gamma)} = 0 \quad \text{for any } u \in L^2(0, b; \mathbf{R}^p) \text{ implies } z = 0,$$

where $(\cdot,\cdot)_{H^{1/2}(\Gamma)}$ is the inner product of $H^{1/2}(\Gamma)$. Let $u = H^* \gamma^* p_\omega^* z$. Then we see that

$$\overline{\mathrm{Im}(p_\Gamma \gamma H)} = H^{1/2}(\Gamma) \Leftrightarrow (p_\Gamma \gamma H H^* \gamma^* p_\Gamma^* z, z)_{H^{1/2}(\Gamma)} = 0 \text{ implies } z = 0, \quad z \in H^{1/2}(\Gamma),$$

i.e. the operator $p_\Gamma \gamma H H^* \gamma^* p_\Gamma^*$ is positive definite and the proof is complete. □

REMARK 2.1 (1) A system which is boundary controllable on Γ is boundary controllable on Γ_1 for every $\Gamma_1 \subseteq \Gamma$.

(2) The definitions (2.2) can be applied to the case where $\Gamma = \partial\Omega$ and there exist systems that are not boundary controllable but which are regionally boundary controllable. This is illustrated by the following example.

2.3 *An example*

Consider the following two-dimensional time fractional diffusion equation defined on $\Omega = [0,1] \times [0,1]$, which is excited by a zone actuator:

$$\begin{cases} {}_0D_t^\alpha z(x,y,t) = \dfrac{\partial^2}{\partial x^2} z(x,y,t) + \dfrac{\partial^2}{\partial y^2} z(x,y,t) + p_D u(t) & \text{in } \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} z(x,y,t) = 0 & \text{in } \Omega, \\ z(\xi,\eta,t) = 0 & \text{on } \partial\Omega \times [0,b], \end{cases} \tag{2.16}$$

where $\alpha \in (0,1]$, $D = \{0\} \times [d_1, d_2] \subseteq \Omega$, $A = \partial^2/\partial x^2 + \partial^2/\partial y^2$ with $\lambda_{ij} = -(i^2 + j^2)\pi^2, \xi_{ij}(x,y) = 2a_{ij}\cos(i\pi x)\cos(j\pi y)$, $a_{ij} = (1-\lambda_{ij})^{-1/2}$, $\Phi(t)z = \sum_{i,j=1}^\infty \exp(\lambda_{ij} t)(z,\xi_{ij})_Z \xi_{ij}$ and $K_\alpha(t)z(x) = \alpha\int_0^\infty \theta\phi_\alpha(\theta)\Phi(t^\alpha\theta)z(x)\,\mathrm{d}\theta = \sum_{i,j=1}^\infty E_{\alpha,\alpha}(\lambda_{ij}t^\alpha)(z,\xi_{ij})_Z \xi_{ij}(x)$. Furthermore, since

$$(H^*\gamma^* z)(t) = (b-t)^{\alpha-1}\sum_{i,j=1}^\infty E_{\alpha,\alpha}(\lambda_{ij}(b-t)^\alpha)(\gamma^* z, \xi_{ij})_Z (p_D, \xi_{ij})_Z$$

and $(p_D, \xi_{ij})_Z = (2a_{ij}/j\pi)[\sin(j\pi d_2) - \sin(j\pi d_1) + j\pi(\cos(j\pi d_2) - \cos(j\pi d_1))]$, there exists d_1, $d_2 \in [0,1]$ satisfying $\mathrm{Ker}(H^*) \neq \{0\}$ ($\overline{\mathrm{Im}(p_D H)} \neq L^2(\omega)$), i.e. the system (2.16) is not boundary controllable.

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

Moreover, let $d_1 = 0$, $d_2 = \frac{1}{2}$, $\Gamma = \{0\} \times [\frac{1}{4}, \frac{3}{4}]$ and $z_* = \xi_{ij}(0, y), (i, j = 4k, k = 1, 2, 3, \ldots)$. Obviously, z_* is not reachable on $\partial\Omega$. However, since

$$E_{\alpha,\alpha}(t) > 0 \quad (t \geqslant 0) \quad \text{and} \quad (p_D, \xi_{ij})_Z = \frac{2a_{ij}}{j\pi}[\sin(j\pi/2) + j\pi(\cos(j\pi/2) - 1)], \quad j = 1, 2, \ldots,$$

we see that

$$\begin{aligned} H^*\gamma^* p_\Gamma^* z_*)(t) &= \sum_{i,j=1}^{\infty} \frac{E_{\alpha,\alpha}(\lambda_{ij}(b-t)^{\alpha})}{(b-t)^{1-\alpha}} (\xi_{ij}, \gamma^* z_*)_{H^{1/2}(\Gamma)} (p_D, \xi_{ij})_Z \\ &= \sum_{i,j=1, j \neq 4k}^{\infty} \frac{2a_{ij} E_{\alpha,\alpha}(\lambda_{ij}(b-t)^{\alpha})}{j\pi(b-t)^{1-\alpha}} (\xi_{ij}, \gamma^* z_*)_{H^{1/2}(\Gamma)} \\ &\quad \times [\sin(j\pi/2) + j\pi(\cos(j\pi/2) - 1)] \\ &\neq 0. \end{aligned} \tag{2.17}$$

Hence z_* is regionally boundary controllable on $\Gamma = \{0\} \times [\frac{1}{4}, \frac{3}{4}]$.

To end this section, we finally recall a necessary lemma to be used afterwards.

LEMMA 2.1 (Dacorogna, 2008) Let $\Omega \subseteq \mathbf{R}^n$ be an open set and $C_0^\infty(\Omega)$ be the class of infinitely differentiable functions on Ω with compact support in Ω and $u \in L^1_{loc}(\Omega)$ be such that

$$\int_\Omega u(x)\psi(x)\,\mathrm{d}x = 0, \quad \forall \psi \in C_0^\infty(\Omega). \tag{2.18}$$

Then $u = 0$ almost everywhere in Ω.

3. Regional strategic actuators

The characteristic of actuators to achieve the regionally approximately boundary controllable of the system (2.1) will be explored in this section.

As cited in El Jai & Pritchard (1988), an actuator can be expressed by a couple (D, g) where $D \subseteq \Omega$ is the support of the actuator and g is its spatial distribution. To state our main results, it is supposed that the control is made by p actuators $(D_i, g_i)_{1 \leqslant i \leqslant p}$ and let $Bu = \sum_{i=1}^{p} p_{D_i} g_i(x) u_i(t)$, where $p \in \mathbf{N}$, $g_i(x) \in Z$, $u = (u_1, u_2, \ldots, u_p)$ and $u_i(t) \in L^2(0, b)$. As cited in Coopmans *et al.* (2015), all these distributed parameter systems with moving sensors and actuators form the so-called cyber-physical systems, which are rich in real-world applications. For instance, in the pest-spreading process, p is the number of the spreading machines and $u_i(\cdot)$ stands for the control input strategic of every spreading machine with respect to time t (Cao *et al.*, 2015). Then the system (2.1) can be rewritten as

$$\begin{cases} {}_0D_t^{\alpha} z(t, x) = Az(t, x) + \sum_{i=1}^{p} p_{D_i} g_i(x) u_i(t), & (t, x) \in [0, b] \times \Omega, \\ \lim_{t \to 0^+} {}_0I_t^{1-\alpha} z(t, x) = z_0(x). \end{cases} \tag{3.1}$$

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

Moreover, we suppose that $-A$ is a self-adjoint uniformly elliptic operator, by Courant & Hilbert (1966), we get that there exists a sequence $(\lambda_j, \xi_{jk}) : k = 1, 2, \ldots, r_j, j = 1, 2, \ldots$ such that

(1) For each $j = 1, 2, \ldots, \lambda_j$ is the eigenvalue of operator A with multiplicities r_j and

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_j > \cdots, \quad \lim_{j\to\infty} \lambda_j = -\infty.$$

(2) For each $j = 1, 2, \ldots, \xi_{jk} (k = 1, 2, \ldots, r_j)$ is the orthonormal eigenfunction corresponding to λ_j, i.e.,

$$(\xi_{jk_m}, \xi_{jk_n}) = \begin{cases} 1, & k_m = k_n, \\ 0, & k_m \neq k_n, \end{cases}$$

where $1 \leqslant k_m, k_n \leqslant r_j$, $k_m, k_n \in \mathbf{N}$ and $(\cdot, \cdot)$ is the inner product of space Z.

Hence, the sequence $\{\xi_{jk}, k = 1, 2, \ldots, r_j, j = 1, 2, \ldots\}$ is an orthonormal basis in Z, the strongly continuous semi-group $\{\Phi(t)\}_{t\geqslant 0}$ on Z generated by A is

$$\Phi(t)z(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} \exp(\lambda_j t)(z, \xi_{jk})\xi_{jk}(x), \quad x \in \Omega \tag{3.2}$$

and for any $z(x) \in Z$, it can be expressed as $z(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}(z, \xi_{jk})\xi_{jk}(x)$.

DEFINITION 3.1 A actuators (suite of actuators) is said to be Γ-strategic if the system under consideration is regionally approximately boundary controllable on Γ at time b.

Before to show our main result in this part, by Eq. (3.2), for any $z \in L^2(\Omega)$, we have

$$\begin{aligned}
K_\alpha(t)z(x) &= \alpha \int_0^\infty \theta\phi_\alpha(\theta)\Phi(t^\alpha\theta)z(x)\,\mathrm{d}\theta \\
&= \alpha \int_0^\infty \theta\phi_\alpha(\theta)\sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\exp(\lambda_j t^\alpha\theta)(z, \xi_{jk})\xi_{jk}(x)\,\mathrm{d}\theta \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{n=0}^{\infty}\frac{\alpha(\lambda_j t^\alpha)^n}{n!}(z, \xi_{jk})\xi_{jk}(x)\int_0^\infty \theta^{n+1}\phi_\alpha\,\mathrm{d}\theta \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{n=0}^{\infty}\frac{\alpha(n+1)!(\lambda_j t^\alpha)^n}{\Gamma(\alpha n+\alpha+1)n!}(z, \xi_{jk})\xi_{jk}(x) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\alpha E^2_{\alpha,\alpha+1}(\lambda_j t^\alpha)(z, \xi_{jk})\xi_{jk}(x),
\end{aligned}$$

where $E^{\mu}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty}((\mu)_n/\Gamma(\alpha n+\beta))(z^n/n!)$, $z \in \mathbf{C}$, $\alpha, \beta, \mu \in \mathbf{C}$, $\mathbf{Re}\,\alpha > 0$ is the generalized Mittag–Leffler function in three parameters and, here, $(\mu)_n$ is the Pochhammer symbol defined by (see

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

Erdèlyi *et al.*, 1953, Section 2.1.1)

$$(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1), \quad n \in \mathbf{N}. \tag{3.3}$$

If $\alpha, \beta \in \mathbf{C}$ such that $\mathbf{Re}\,\alpha > 0$, $\mathbf{Re}\,\beta > 1$, then (see Mathai & Haubold, 2008, Section 2.3.4 or Gorenflo *et al.*, 2014, Section 5.1.1)

$$\alpha E^2_{\alpha,\beta} = E_{\alpha,\beta-1} - (1+\alpha-\beta)E_{\alpha,\beta}. \tag{3.4}$$

It then follows that

$$K_\alpha(t)z(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} E_{\alpha,\alpha}(\lambda_j t^\alpha)(z,\xi_{jk})\xi_{jk}(x) \tag{3.5}$$

and

$$\int_0^t \tau^{\alpha-1}K_\alpha(\tau)Bu(t-\tau)\,\mathrm{d}\tau = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{p}\int_0^t g^i_{jk}u_i(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_j\tau^\alpha)\,\mathrm{d}\tau\,\xi_{jk}(x), \tag{3.6}$$

where $g^i_{jk} = (p_{D_i}g_i, \xi_{jk})$, $j = 1,2,\ldots$, $k = 1,2,\ldots,r_j$, $i = 1,2,\ldots,p$ and $E_{\alpha,\beta}(z) := \sum_{i=0}^{\infty} z^i/\Gamma(\alpha i+\beta)$, $\mathbf{Re}\,\alpha > 0$, $\beta, z \in \mathbf{C}$ is known as the generalized Mittag–Leffler function in two parameters.

THEOREM 3.1 For any $j = 1,2,\ldots$, define $p \times r_j$ matrices G_j as

$$G_j = \begin{bmatrix} g^1_{j1} & g^1_{j2} & \cdots & g^1_{jr_j} \\ g^2_{j1} & g^2_{j2} & \cdots & g^2_{jr_j} \\ \vdots & \vdots & \vdots & \vdots \\ g^p_{j1} & g^p_{j2} & \cdots & g^p_{jr_j} \end{bmatrix}, \tag{3.7}$$

where $g^i_{jk} = (p_{D_i}g_i, \xi_{jk})$, $j = 1,2,\ldots$, $k = 1,2,\ldots,r_j$, $i = 1,2,\ldots,p$. Then the suite of actuators $(D_i, g_i)_{1\leqslant i\leqslant p}$ is said to be Γ-strategic if and only if

$$p \geqslant r = \max\{r_j\} \quad \text{and} \quad \operatorname{rank} G_j = r_j \quad \text{for } j = 1,2,\ldots. \tag{3.8}$$

Proof. For any $z_* \in H^{1/2}(\Gamma)$, denote by $(\cdot,\cdot)_{H^{1/2}(\Gamma)}$ the inner product of space $H^{1/2}(\Gamma)$; we then see that

$$(p_\Gamma\gamma Hu, z_*)_{H^{1/2}(\Gamma)} = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{p}\int_0^b \tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_j\tau^\alpha)u_i(b-\tau)\,\mathrm{d}\tau\, g^i_{jk}z_{jk} = 0, \quad t \in [0,b], \tag{3.9}$$

where $z_{jk} = (p_\Gamma\gamma\xi_{jk}, z_*)_{H^{1/2}(\Gamma)}$, $j = 1,2,\ldots$, $k = 1,2,\ldots,r_j$. Further, Lemma 2.1 gives

$$\sum_{j=1}^{\infty}\sum_{k=1}^{r_j} t^{\alpha-1}E_{\alpha,\alpha}(\lambda_j t^\alpha)g^i_{jk}z_{jk} = \mathbf{0}_p := (0,0,\ldots,0) \in \mathbf{R}^p \quad \text{for } t > 0, i = 1,2,\ldots,p. \tag{3.10}$$

Then we conclude that the suite of actuators $(D_i, g_i)_{1\leqslant i\leqslant p}$ is Γ-strategic if and only if

$$\sum_{j=1}^{\infty} b^{\alpha-1}E_{\alpha,\alpha}(\lambda_j b^\alpha)G_j z_j = \mathbf{0}_p \Rightarrow z_* = 0, \tag{3.11}$$

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

where $z_j = (z_{j1}, z_{j2}, \ldots, z_{jr_j})^T$ is a vector in $\mathbf{R}^{r_j}$ and $j = 1, 2, \ldots$.

(a) If we assume that $p \geqslant r = \max\{r_j\}$ and rank $G_j < r_j$ for some $j = 1, 2, \ldots$, there exists a non-zero element $\tilde{z} \in H^{1/2}(\Gamma)$ with $\tilde{z}_j = (\tilde{z}_{j1}, \tilde{z}_{j2}, \ldots, \tilde{z}_{jr_j})^T \in \mathbf{R}^{r_j}$ such that

$$G_j \tilde{z}_j = \mathbf{0}_p. \tag{3.12}$$

It then follows from $E_{\alpha,\alpha}(\lambda_j t^\alpha) > 0$ $(t \geqslant 0)$ that we can find a non-zero vector $\tilde{z}$ satisfying

$$\sum_{j=1}^{\infty} b^{\alpha-1} E_{\alpha,\alpha}(\lambda_j b^\alpha) G_j \tilde{z}_j = \mathbf{0}_p. \tag{3.13}$$

This means that the actuators $(D_i, f_i)_{1 \leqslant i \leqslant p}$ are not Γ-strategic.

(b) However, on the contrary, if the actuators $(D_i, g_i)_{1 \leqslant i \leqslant p}$ are not Γ-strategic, i.e., $\overline{\mathrm{Im}(p_\Gamma \gamma H)} \neq H^{1/2}(\Gamma)$, then there exists a non-zero element $z \neq \mathbf{0}_n$ satisfying

$$(p_\Gamma \gamma H u, z)_{H^{1/2}(\Gamma)} = 0 \quad \text{for all } u \in L^2(0, b; \mathbf{R}^p). \tag{3.14}$$

Then we can find a non-zero element $z_{j^*} \in \mathbf{R}^{r_j}$ such that

$$G_{j^*} z_{j^*} = \mathbf{0}_p. \tag{3.15}$$

This allows us to complete the conclusion of the theorem. □

4. Regional boundary controllability with minimum energy control

In this section, we explore the possibility of finding a minimum energy control when the system (2.1) can be steered from a given initial vector z_0 to a target function z_b on the boundary subregion Γ. The method used here is an extension of those in El Jai & Pritchard (1988), El Jai (1991), Zerrik *et al.* (2000, 1999) and Sakawa (1974).

Consider the following minimization problem:

$$\begin{cases} \inf\limits_u J(u) = \displaystyle\int_0^b \|u(t)\|^2_{\mathbf{R}^p}\, \mathrm{d}t, \\ u \in U_b = \{u \in L^2(0, b; \mathbf{R}^p) : p_\Gamma \gamma z(b, u) = z_b\}, \end{cases} \tag{4.1}$$

where, obviously, U_b is a closed convex set. We then show a direct approach to the solution of the minimum energy problem (4.1).

THEOREM 4.1 If the system (2.1) is regionally approximately boundary controllable on Γ, then, for any $z_b \in H^{1/2}(\Gamma)$, the minimum energy problem (4.1) has a unique solution given by

$$u^*(t) = (p_\Gamma \gamma H)^* R_\Gamma^{-1}(z_b - p_\Gamma \gamma b^{\alpha-1} K_\alpha(b) z_0), \tag{4.2}$$

where $R_\Gamma = p_\Gamma \gamma H H^* \gamma^* p_\Gamma^*$ and H^* is defined in Eq. (2.9).

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

Proof. To begin with, since the solution of (2.1) excited by the control u^* is given by

$$z(t,u^*) = t^{\alpha-1}K_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)Bu^*(s)\,\mathrm{d}s, \tag{4.3}$$

we get that

$$\begin{aligned}
p_\Gamma\gamma z(b,u^*) &= p_\Gamma\gamma\left[b^{\alpha-1}K_\alpha(b)z_0 + \int_0^b (b-s)^{\alpha-1}K_\alpha(b-s)Bu^*(s)\,\mathrm{d}s\right]\\
&= p_\Gamma\gamma b^{\alpha-1}K_\alpha(b)z_0 + p_\Gamma\gamma H(p_\Gamma\gamma H)^* R_\Gamma^{-1}(z_b - p_\Gamma\gamma b^{\alpha-1}K_\alpha(b)z_0)\\
&= z_b.
\end{aligned}$$

Next, we show that if the system (2.1) is regionally approximately boundary controllable on Γ at time b, then the operator R_Γ is coercive. In fact, for any $z_1 \in H^{1/2}(\Gamma)$, there exists a control $u \in L^2(0,b,\mathbf{R}^p)$ such that

$$z_1 = p_\Gamma\gamma[b^{\alpha-1}K_\alpha(b)z_0 + Hu] \tag{4.4}$$

and

$$\begin{aligned}
\langle R_\Gamma z_1, z_1\rangle_{H^{1/2}(\Gamma)} &= \|H^*\gamma^* p_\Gamma^* z_1\|^2_{L^2(0,b,\mathbf{R}^p)}\\
&= \|B^*(b-\cdot)^{\alpha-1}K_\alpha^*(b-\cdot)\gamma^* p_\Gamma^* z_1\|^2_{L^2(0,b,\mathbf{R}^p)}\\
&\geqslant \|z_1\|^2_{H^{1/2}(\Gamma)}.
\end{aligned}$$

Moreover, since $R_\Gamma \in \mathcal{L}(H^{1/2}(\Gamma), H^{1/2}(\Gamma))$, by Theorem 1.1 in Lions (1971), it follows that R_Γ is an isomorphism.

Finally, we prove that u^* solves the minimum energy problem (4.1). For this purpose, since $p_\Gamma\gamma z(b,u^*) = z_b$, for any $u \in L^2(0,b,\mathbf{R}^p)$ with $p_\Gamma\gamma z(b,u) = z_b$, one has

$$p_\Gamma\gamma[z(b,u^*) - z(b,u)] = 0, \tag{4.5}$$

which follows that

$$0 = p_\Gamma\gamma\int_0^b (b-s)^{\alpha-1}K_\alpha(b-s)B[u^*(s)-u(s)]\,\mathrm{d}s = p_\Gamma\gamma H[u^*-u].$$

Thus, by

$$\begin{aligned}
J'(u^*)(u^*-u) &= 2\int_0^b \big\langle u^*(s)-u(s), u^*(s)\big\rangle\,\mathrm{d}s\\
&= 2\int_0^b \big\langle u^*(s)-u(s), (p_\Gamma\gamma H)^* R_\Gamma^{-1}(z_b - p_\Gamma\gamma K_\alpha(b)z_0)\big\rangle\,\mathrm{d}s\\
&= 2\int_0^b \big\langle p_\Gamma\gamma H[u^*(s)-u(s)], R_\Gamma^{-1}(z_b - p_\Gamma\gamma K_\alpha(b)z_0)\big\rangle\,\mathrm{d}s\\
&= 0,
\end{aligned}$$

it follows that $J(u) \geqslant J(u^*)$, i.e., u^* solves the minimum energy problem (4.1) and the proof is complete. □

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

5. The connection between internal and boundary regional controllability

Based on an intension of the regional controllability of integer-order differential equations developed in Zerrik *et al.* (1999, 2000), we here give a transfer on the internal and boundary regional controllability of fractional-order sub-diffusion equations (2.1) and develop two types of controls, i.e., zone or pointwise.

5.1 *Internal and boundary regional controllability*

In this part, we present an internal and boundary regional controllability transfer of the problem (2.1). To this end, suppose that $z(b,u) \in Z$ and we first define an operator

$$T : H^{1/2}(\partial\Omega) \to H^1(\Omega) \text{ such that } \gamma Tg = g \quad \forall g \in H^{1/2}(\partial\Omega), \tag{5.1}$$

which is linear and continuous (Lions, 1988). Let $z_b \in H^{1/2}(\Gamma)$ with the extension $p_\Gamma^* z_b \in H^{1/2}(\partial\Omega)$ and consider the sets

$$\Omega_1 = \{Tp_\Gamma^* z_b \in Z \mid z_b \in H^{1/2}(\Gamma)\} \quad \text{and} \quad \Omega_2 = \bigcup_{z_b \in H^{1/2}(\Gamma)} \operatorname{Supp} Tp_\Gamma^* z_b. \tag{5.2}$$

For any $r > 0$ be arbitrary sufficiently small, consider

$$D_r = \bigcup_{z \in \Gamma} B(z,r) \quad \text{and let} \quad \omega_r = D_r \cap \Omega_2, \tag{5.3}$$

where $B(z,r)$ is a ball of radius r centred in z.

THEOREM 5.1 If the system (2.1) is exactly (respectively, approximately) controllable on ω_r, then it is also exactly (respectively, approximately) boundary controllable on Γ.

Proof. Let $z_b \in H^{1/2}(\Gamma)$ be the target function. By utilizing the trace theorem (Retherford, 1993), there exists $Tp_\Gamma^* z_b \in Z$ with a bounded support such that $\gamma(Tp_\Gamma^* z_b) = p_\Gamma^* z_b$. Then

(1) if the system (2.1) is exactly controllable on ω_r, for any $y_b \in H^1(\omega_r)$, there exists a control $u \in L^2(0,b;\mathbf{R}^p)$ such that

$$p_{\omega_r} z(b,u) = y_b. \tag{5.4}$$

Then $p_{\omega_r} Tp_\Gamma^* z_b \in H^1(\omega_r)$ and there exists a control $u \in L^2(0,b;\mathbf{R}^p)$ such that

$$p_{\omega_r} z(b,u) = p_{\omega_r} Tp_\Gamma^* z_b \quad \text{and} \quad \gamma p_{\omega_r} z(b,u) = p_\Gamma^* z_b. \tag{5.5}$$

Thus $p_\Gamma \gamma p_{\omega_r} z(b,u) = z_b$, i.e., the system (2.1) is exactly boundary controllable on Γ.

(2) if the system (2.1) is approximately controllable on ω_r, for and $\varepsilon > 0$ and any $y_b \in H^1(\omega_r)$, there exists a control $u \in L^2(0,b;\mathbf{R}^p)$ such that

$$\|p_{\omega_r} z(b,u) - y_b\|_{H^1(\omega_r)} \leqslant \varepsilon. \tag{5.6}$$

Then for any $\varepsilon > 0$, there exists a control $u \in L^2(0,b;\mathbf{R}^p)$ such that

$$\|p_{\omega_r} z(b,u) - p_{\omega_r} Tp_\Gamma^* z_b\|_{H^1(\omega_r)} \leqslant \varepsilon. \tag{5.7}$$

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

Moreover, by the continuity of the trace mapping γ on $H^1(\omega_r)$, one has

$$\|\gamma(p_{\omega_r}z(b,u)) - \gamma(p_{\omega_r}Tp^*_{\Gamma}z_b)\|_{H^1(\partial\omega_r)} \leqslant \varepsilon, \tag{5.8}$$

therefore $\|p_{\Gamma}\gamma(p_{\omega_r}z(b,u)) - z_b\|_{H^1(\Gamma)} \leqslant \varepsilon$, Thus (2.1) is approximately boundary controllable on Γ and the proof is complete. □

5.2 *Regional boundary target control*

This part is concerned with the approach for the control which drives the problem (2.1) from z_0 to z_b on Γ. Let $z_b \in H^{1/2}(\Gamma)$ with the extension $p^*_{\Gamma}z_b \in H^{1/2}(\partial\Omega)$. By Theorem 5.1, the problem may be solved by driving the system (2.1) from z_0 to $y_b \in H^1(\omega_r)$ on ω_r.

The following two sets will be used in our discussion:

$$G = \{g \in H^1(\Omega) : g = 0 \text{ in } \Omega\backslash\omega_r\} \quad \text{and} \quad E = \{e \in H^1(\Omega) : e = 0 \text{ in } \omega_r\}. \tag{5.9}$$

5.2.1 *Case of zone actuator*

Let us consider the system (2.1) with a zone actuator (D,f) where $D \subseteq \Omega$ is the support of the actuator and f is its spatial distribution. Then the system can be written in the form

$$\begin{cases} {}_0D^{\alpha}_t z(x,t) = Az(x,t) + p_D f(x)u(t) & \text{in } \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I^{1-\alpha}_t z(x,t) = z_0(x) & \text{in } \Omega, \\ z(x,t) = 0 & \text{on } \partial\Omega \times [0,b]. \end{cases} \tag{5.10}$$

For any $g \in G$, consider the system

$$\begin{cases} Q_tD^{\alpha}_b\varphi(x,t) = A^*Q\varphi(x,t) & \text{in } \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} Q_tI^{1-\alpha}_b\varphi(x,t) = p^*_{\omega_r}g(x) & \text{in } \Omega, \\ \varphi(x,t) = 0 & \text{on } \partial\Omega \times [0,b]. \end{cases} \tag{5.11}$$

where Q is a reflection operator on interval $[0,b]$ such that

$$Qf(t) := f(b-t). \tag{5.12}$$

By the argument in Małgorzata (2009), we see that the following properties on operator Q hold:

$$Q_0I^{\alpha}_t f(t) = {}_tI^{\alpha}_b Qf(t), \qquad Q_0D^{\alpha}_t f(t) = {}_tD^{\alpha}_b Qf(t) \tag{5.13}$$

and

$${}_0I^{\alpha}_t Qf(t) = Q_tI^{\alpha}_b f(t), \qquad {}_0D^{\alpha}_t Qf(t) = Q_tD^{\alpha}_b f(t). \tag{5.14}$$

Then system (5.11) can be rewritten as

$$\begin{cases} {}_0D^{\alpha}_t Q\varphi(x,t) = A^*Q\varphi(x,t) & \text{in } \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I^{1-\alpha}_t Q[(b-t)^{1-\alpha}\varphi(x,t)] = p^*_{\omega_r}g(x) & \text{in } \Omega, \\ \varphi(x,t) = 0 & \text{on } \partial\Omega \times [0,b] \end{cases} \tag{5.15}$$

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

and its unique mild solution is $\varphi(x,t)=(b-t)^{\alpha-1}K_\alpha^*(b-t)p_{\omega_r}^*g(x)$. Moreover, we define the semi-norm

$$g\in G\to \|g\|_G^2=\int_0^b (f,\varphi(\cdot,t))_{L^2(D)}^2\,\mathrm{d}t \tag{5.16}$$

on G and obtain the following result.

LEMMA 5.1 (5.16) defines a norm on G if the system (5.10) is regionally approximately controllable on ω at time b.

Proof. For any $g\in G$, if the system (5.10) is regionally approximately controllable on ω, we have

$$\operatorname{Ker}(H^*p_\omega^*)=\operatorname{Ker}[(b-s)^{\alpha-1}(f,K_\alpha^*(b-s)p_{\omega_r}^*g)_{L^2(D)}]=\operatorname{Ker}[(f,\varphi(\cdot,t))_{L^2(D)}]=\{0\}.$$

It then follows from

$$\|g\|_G^2=\int_0^b (f,\varphi(\cdot,t))_{L^2(D)}^2\,\mathrm{d}t=0\Leftrightarrow (f,\varphi(\cdot,t))_{L^2(D)}=0$$

that $\|\cdot\|_G$ is a norm of space G and the proof is complete. $\square$

Moreover, let $u(t)=(f,\varphi(\cdot,t))_{L^2(D)}$ and decompose the system (5.10) into an autonomous system and a homogeneous initial condition one:

$$\begin{cases} {}_0D_t^\alpha\psi_1(x,t)=A\psi_1(x,t)+p_Df(x)(f,\varphi(\cdot,t))_{L^2(D)} & \text{in } \Omega\times[0,b],\\ \lim\limits_{t\to0^+} {}_0I_t^{1-\alpha}\psi_1(x,t)=0 & \text{in } \Omega,\\ \psi_1(x,t)=0 & \text{on } \partial\Omega\times[0,b] \end{cases} \tag{5.17}$$

and

$$\begin{cases} {}_0D_t^\alpha\psi_2(x,t)=A\psi_2(x,t) & \text{in } \Omega\times[0,b],\\ \lim\limits_{t\to0^+} {}_0I_t^{1-\alpha}\psi_2(x,t)=z_0(x) & \text{in } \Omega,\\ \psi_2(x,t)=0 & \text{on } \partial\Omega\times[0,b]. \end{cases} \tag{5.18}$$

Let $\wedge$ be the operator $\wedge: G\to E^\perp$ given by

$$\wedge g=p_{\omega_r}\psi_1(\cdot,b)\quad \forall g\in G. \tag{5.19}$$

Then for any $z_b\in H^1(\omega_r)$, the regional control problem on ω_r is equivalent to the resolution of the equation

$$\wedge g=z_b-p_{\omega_r}\psi_2(\cdot,b) \tag{5.20}$$

and we have the following result.

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

THEOREM 5.2 Assume that the system (5.10) is regionally approximately controllable on ω_r at time b; then (5.20) admits a unique solution $g \in G$ and the control

$$u^*(t) = (f, \varphi(\cdot, t))_{L^2(D)} \tag{5.21}$$

steers the problem (5.10) to z_b on ω_r. Moreover, u^* solves the minimum energy problem

$$\inf_u J(u) = \int_0^b \|u(t)\|^2_{\mathbf{R}^p}\, \mathrm{d}t. \tag{5.22}$$

Proof. From Lemma 5.1, if the system (5.10) is regionally approximately controllable on ω_r at time b, then $\|\cdot\|_G$ is a norm of space G. Let the completion of G with respect to the norm $\|\cdot\|_G$ again by G.

Next, we show that (5.20) admits a unique solution in G. For any $g \in G$, by Eq. (5.19), it follows that

$$\begin{aligned}
\langle g, \wedge g\rangle &= \left\langle g, p_{\omega_r}\psi_1(\cdot, b)\right\rangle \\
&= \left\langle g, p_{\omega_r}\int_0^b (b-s)^{\alpha-1}K_\alpha(b-s)p_D f(\cdot)(f, \varphi(\cdot, s))_{L^2(D)}\, \mathrm{d}s\right\rangle \\
&= \int_0^b \|(f, \varphi(\cdot, t))_{L^2(D)}\|^2\, \mathrm{d}s = \|g\|^2_G.
\end{aligned}$$

Hence, it follows from Theorem 1.1 in Lions (1971) that (5.20) admits a unique solution in G.

Let $u = u^*$ in problem (5.10); then $p_{\omega_r}z(b, u^*) = z_b$. Finally, we show that the u^* minimize the const functional (5.22). For any $u_1 \in L^2(0, b, \mathbf{R}^p)$ with $p_{\omega_r}z(b, u_1) = z_b$, we have

$$p_{\omega_r}[z(b, u^*) - z(b, u_1)] = 0. \tag{5.23}$$

Then

$$0 = p_{\omega_r}\int_0^b (b-s)^{\alpha-1}K_\alpha(b-s)p_D f(x)[u^*(s) - u_1(s)]\, \mathrm{d}s.$$

Moreover, since

$$\begin{aligned}
J'(u^*)(u^* - u_1) &= 2\int_0^b (u^*(s) - u_1(s))u^*(s)\, \mathrm{d}s \\
&= 2\int_0^b (u^*(s) - u_1(s))(f, \varphi(\cdot, t))_{L^2(D)}\, \mathrm{d}s \\
&= 2\int_0^b (p_D f[u^*(s) - u_1(s)], (b-t)^{\alpha-1}K^*_\alpha(b-t)p^*_{\omega_r}g)\, \mathrm{d}s \\
&= 2\left(p_{\omega_r}\int_0^b (b-s)^{\alpha-1}K_\alpha(b-s)p_D f(x)[u^*(s) - u_1(s)]\, \mathrm{d}s, g\right) \\
&= 0,
\end{aligned}$$

one has $J(u) \geqslant J(u^*)$, i.e., u^* solves the minimum energy problem (5.22) and the proof is complete. □

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

5.2.2 *Case of pointwise actuator* Consider the system (2.1) with a pointwise internal actuator, which can be written in the form

$$\begin{cases} {}_0D_t^{\alpha} z(x,t) = Az(x,t) + \delta(x-\sigma)u(t) & \text{in } \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} z(x,t) = z_0 & \text{in } \Omega, \\ z(x,t) = 0 & \text{on } \partial\Omega \times [0,b], \end{cases} \tag{5.24}$$

where σ is the actuator support. For any $g \in G$, consider (5.11) and define the semi-norm

$$g \to \|g\|_G^2 = \int_0^b \|\varphi(\sigma,s)\|^2 \,\mathrm{d}s, \tag{5.25}$$

which defines a norm on G if (5.24) is regionally approximately controllable.

Similar to the argument in Section 5.2.1, let $u(t) = \varphi(\sigma,t)$ and we consider the following system:

$$\begin{cases} {}_0D_t^{\alpha} \psi_1(x,t) = A\psi_1(x,t) + \delta(x-\sigma)\varphi(\sigma,t) & \text{in } \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} \psi_1(x,t) = 0 & \text{in } \Omega, \\ \psi_1(x,t) = 0 & \text{on } \partial\Omega \times [0,b], \end{cases} \tag{5.26}$$

and

$$\begin{cases} {}_0D_t^{\alpha} \psi_2(x,t) = A\psi_2(x,t) & \text{in } \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} \psi_2(x,t) = z_0(x) & \text{in } \Omega, \\ \psi_2(x,t) = 0 & \text{on } \partial\Omega \times [0,b]. \end{cases} \tag{5.27}$$

Then the regional control problem on ω_r is equivalent to the resolution of the equation

$$\wedge\, g = z_b - p_{\omega_r}\psi_2(\cdot,b) \tag{5.28}$$

and we see the following result.

THEOREM 5.3 Assume that the system (5.24) is regionally approximately controllable on ω_r at time b; then (5.28) admits a unique solution $g \in G$ and the control

$$u^*(t) = \varphi(\sigma,t) \tag{5.29}$$

steers (5.10) to z_b on ω_r. Moreover, this control minimizes the cost functional (5.22).

5.2.3 *Simulation* The resolution of the regional boundary control problem may be seen via the following simplified steps (see the case of pointwise actuator for an example).

(1) Initial data Ω, Γ, z_b and the actuator;

(2) Solve the problem (5.28) ($\to g$);

(3) Solve the problem (5.11) ($\to \varphi(\sigma,t)$);

(4) Apply the control $u^*(t) = \varphi(\sigma,t)$.

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

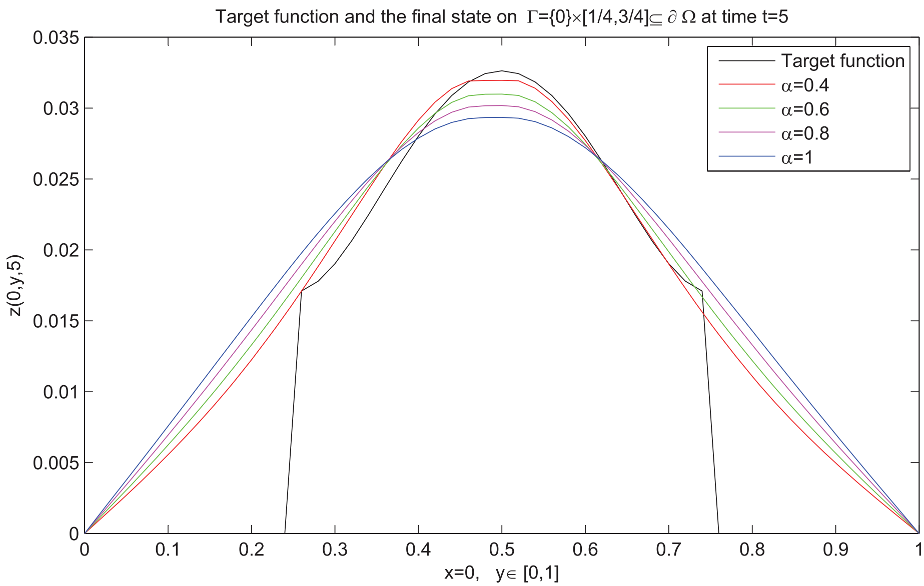

FIG. 1. Final reached state and target function on $\Gamma \subseteq \partial\Omega$ at time $t = 5$.

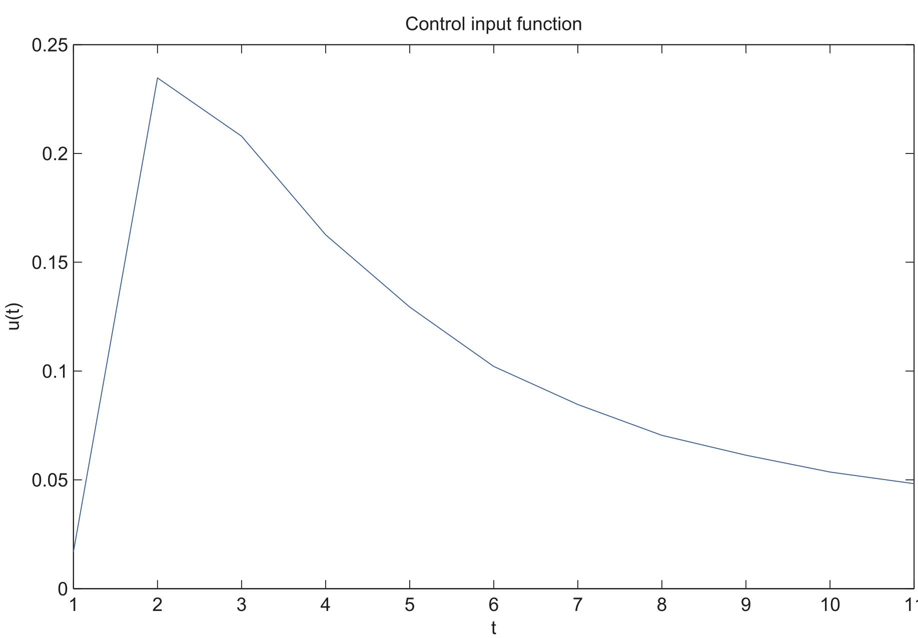

FIG. 2. Control input function, which is calculated by the formula (5.29).

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

For example, consider the system (2.16) and let $\Omega = [0,1] \times [0,1]$, $\Gamma = \{0\} \times [1/4, 3/4]$, $b = 5$. For the target function z_b on $\Gamma \subseteq \partial\Omega$, we assume that

$$z_b(0,y) = \begin{cases} 0, & 0 \leqslant y < 1/4; \\ 0.017 + 4(y-1/4)^2(y-3/4)^2, & 1/4 \leqslant y \leqslant 3/4; \\ 0, & 3/4 < y \leqslant 1 \end{cases} \tag{5.30}$$

and the actuator is supposed to be located in $D = \{0\} \times \{0.5\} \subseteq \Omega$.

Figure 1 shows how the final reached state is very close to the target function on $\Gamma \subseteq \partial\Omega$ at time $t = 5$ when $\alpha = 0.4, 0.6, 0.8, 1.0$. This also implies that time fractional diffusion systems can offer better performance compared with those using integer-order distributed parameter systems. Moreover, when $\alpha = 0.4$, the corresponding control input, which is calculated by the formula (5.29), is presented in Fig. 2.

6. Conclusions

In this paper, the regional boundary controllability of the Riemann–Liouville time fractional diffusion systems of order $\alpha \in (0,1]$ is discussed, which is motivated by many realistic situation encountered in various applications. The results here provide some insights into the qualitative analysis of the design of fractional-order diffusion equations, which can also be extended to complex fractional-order distributed parameter dynamic systems. Various open questions are still under consideration. The problem of constrained control as well as the case of fractional-order distributed parameter dynamic systems with more complicated regional sensing and actuation configurations are of great interest. For more information on the potential topics related to fractional-order distributed parameter systems, we refer the readers to Ge *et al.* (2015a) and the references therein.

Funding

This work was supported by Chinese Universities Scientific Fund (No. CUSF-DH-D-2014061) and the Natural Science Foundation of Shanghai (No. 15ZR1400800).

References

Adams, E. E. & Gelhar, L. W. (1992) Field study of dispersion in a heterogeneous aquifer 2. Spatial moments analysis. *Water Resources Res.*, **28**, 3293–3307.

Agrawal, O. P. (2002) Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynam.*, **29**, 145–155.

Cao, J., Chen, Y. & Li, C. (2015) Multi-UAV-based optimal crop-dusting of anomalously diffusing infestation of crops. *Proc. of the American Control Conference* (ACC). pp. 1278–1283.

Cartea, À. & Negrete, D. (2007) Fluid limit of the continuous-time random walk with general Lèvy jump distribution functions. *Phys. Rev. E*, **76**, 1–8.

Coopmans, C., Stark, B., Jensen, A., Chen, Y. & McKee, M. (2015) *Cyber-physical systems Enabled by small unmanned aerial vehicles: a chapter in Handbook of Unmanned Aerial Vehicles* (K. P. Valavanis & G. J. Vachtsevanos eds). Berlin, Heidelberg, New York: Springer.

Courant, R. & Hilbert, D. (1966) *Methods of Mathematical Physics*, vol. I. New York: Interscience.

Dacorogna, B. (2008) *Direct methods in the calculus of variations*, 2nd edn. Appl. Math. Sci., vol. 78. New York: Springer.

El Jai, A. (1991) Distributed systems analysis via sensors and actuators. *Int. J. Sensors Actuators*, **29**, 1–11.

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016

El Jai, A. & Pritchard, A. J. (1988) *Sensors and Controls in the Analysis of Distributed Systems*. New York: Halsted Press.

Erdèlyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. (1953) *Higher Transcendental Functions*, vol. I. New York, Toronto, London: McGraw-Hill.

Ge, F., Chen, Y. & Kou, C. (2015a) Cyber-physical systems as general distributed parameter systems: three types of fractional order models and emerging research opportunities. *IEEE/CAA J. Autom. Sin.*, **2**, 353–357.

Ge, F., Chen, Y. & Kou, C. (2015b) Regional controllability of anomalous diffusion generated by the time fractional diffusion equations. *ASME IDETC/CIE 2015*, Boston, August 2–5, DETC2015-46697. Preprint arXiv:1508.00047.

Gorenflo, R., Kilbas, A. A., Mainardi, F. & Rogosin, S. V. (2014) *Mittag-Leffler Functions, Related Topics and Applications*. Berlin, Heidelberg: Springer.

Kilbas, A. A., Srivastava, H. M. & Trujillo, J. J. (2006) *Theory and Applications of Fractional Differential Equations*. Amsterdam, The Netherlands: Elsevier.

Lions, J. L. (1971) *Optimal Control of Systems Governed by Partial Differential Equations*, vol. 170. Berlin, Heidelberg, New York: Springer.

Lions, J. L. (1988) Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, **30**, 1–68.

Mainardi, F., Paradisi, P. & Gorenflo, R. (2000) Probability distributions generated by fractional diffusion equations. *Econophysics: An Emerging Science* (J. Kertesz & I. Kondor eds). Dordrecht: Kluwer.

Małgorzata, K. (2009) *On Solutions of Linear Fractional Differential Equations of a Variational Type*. Poland: Czestochowa University of Technology.

Mandelbrot, B. B. (1983) *The Fractal Geometry of Nature*, vol. 173. Macmillan.

Mathai, A. M. & Haubold, H. J. (2008) *Special Functions for Applied Scientists*. London: Springer Science Business Media, LLC.

Meerschaert, M. M. & Scalas, E. (2006) Coupled continuous time random walks in finance. *Phys. A*, **370**, 114–118.

Metzler, R. & Klafter, J. (2000) The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, **339**, 1–77.

Metzler, R. & Klafter, J. (2004) The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A*, **37**, 161–208.

Montroll, E. W. & Weiss, G. H. (1965) Random walks on lattices. II. *J. Math. Phys.*, **6**, 167–181.

Podlubny, I. (1999) *Fractional Differential Equations*. San Diego: Academic Press.

Pritchard, A. J. & Wirth, A. (1978) Unbounded control and observation systems and their duality. *SIAM J. Control Optim.*, **16**, 535–545.

Retherford, J. R. (1993) *Hilbert Space: Compact Operators and the Trace Theorem*. London Mathematical Society Student Texts, vol. 27. Cambridge: Cambridge University Press.

Sakawa, Y. (1974) Controllability for partial differential equations of parabolic type. *SIAM J. Control*, **12**, 389–400.

Spears, W. M. & Spears, D. F. (2012) *Physicomimetics: Physics-based Swarm Intelligence*. London: Springer Science & Business Media.

Torvik, P. J. & Bagley, R. L. (1984) On the appearance of the fractional derivative in the behavior of real materials. *J. Appl. Mech.*, **51**, 294–298.

Uchaikin, V. & Sibatov, R. (2013) *Fractional Kinetics in Solids: Anomalous Charge Transport in Semiconductors*. Singapore: World Scientific Publishing Co. Pte. Ltd.

Zerrik, E., Badraoui, L. & El Jai, A. (1999) Sensors and regional boundary state reconstruction of parabolic systems. *Sens Actuators*, **75**, 102–117.

Zerrik, E., Boutoulout, A. El Jai, A. (2000) Actuators and regional boundary controllability for parabolic systems. *Int. J. Syst. Sci.*, **31**, 73–82.

Zhou, Y. & Jiao, F. (2010) Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.*, **59**, 1063–1077.

Downloaded from http://imamci.oxfordjournals.org/ at University of California, Merced on April 17, 2016