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Boundary feedback stabilisation for the time fractional-order anomalous diffusion system

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Abstract: In this study, the authors attempt to explore the boundary feedback stabilisation for an unstable heat process described by fractional-order partial differential equation (PDE), where the first-order time derivative of normal reaction–diffusion equation is replaced by a Caputo time fractional derivative of order $\alpha \in (0, 1]$. By designing an invertible coordinate transformation, the system under consideration is converted into a Mittag–Leffler stability linear system and the boundary stabilisation problem is transformed into a problem of solving a linear hyperbolic PDE. It is worth mentioning that with the help of this invertible coordinate transformation, they can explicitly obtain the closed-loop solutions of the original problem. The output feedback problem with both anti-collocated and collocated actuator/sensor pairs in one-dimensional domain is also presented. A numerical example is given to test the effectiveness of the authors' results.

1 Introduction

The heat process, also known as reaction–diffusion process, is used widely in science and engineering and a great deal of contributions have been given to them [1–6]. It is well known that the boundary stabilisation problem of integer-order unstable heat system is solved in [7–12], where the boundary control law, known as backstepping control law, is in the form of an integral operator with a known, continuous kernel function. However, to the best of our knowledge, there are very few results concerning the boundary feedback stabilisation of an unstable time fractional-order anomalous diffusion system. Based on the numerical simulation techniques, the boundary stabilisation of a one-dimensional (1D) fractional diffusion-wave equation was studied in [13, 14], the boundary control of a Caputo fractional wave equation via a fractional-order boundary controller was presented.

Moreover, as cited in [15], the heat conduction process in 1D system did not obey the Fourier law and a connection between anomalous heat conduction and anomalous diffusion in 1D systems was established. It is confirmed that many real-world life systems can be well characterised by utilising the notions of fractional order [16, 17], this is the reason why the fractional-order models are superior in comparison with the integer-order models. For the anomalous diffusion process in real world, as we all know, it is essentially distributed and the continuous time random walks (CTRWs) can be regarded as a useful tool to describe this phenomenon [18–21]. For example, when the particles are assumed to jump at fixed time intervals with a incorporating waiting times, the particles then undergo a sub-diffusion process and the time fractional-order anomalous diffusion system can be used to efficiently characterise them. Besides, we refer the readers to the monographs [22–24] and the references therein for more basic knowledge on fractional calculus and fractional-order partial differential equation (PDE).

Motivated by the arguments above, in this paper, we consider the following abstract fractional reaction–diffusion system with a destabilising linear term on the right-hand side

$$ {}_0^C D_t^{\alpha} z(x,t) = z_{xx}(x,t) + a(x) z(x,t) \quad \text{in } (0,b)\times(0,\infty) \qquad (1) $$

with the boundary conditions

$$ \begin{aligned} &z(0,t)=0, \quad z(b,t)=u(t), \quad t\geq 0 \ \text{ or}\\ &z_x(0,t)=0, \quad z_x(b,t)=u(t), \ \ t\geq 0, \end{aligned} \qquad (2) $$

where $0<\alpha\leq 1$, $a\in C^1[0,b]$, $u(t)$ is the control input, ${}_0^C D_t^{\alpha}$ and ${}_0 I_t^{\alpha}$ denote the Caputo time fractional-order derivative and Riemann–Liouville time fractional-order integral, respectively, given by [22, 23]

$$ {}_0^C D_t^{\alpha} z(x,t) = \begin{cases} \dfrac{\partial}{\partial t} z(x,t), & \alpha = 1,\\ {}_0 I_t^{1-\alpha} \dfrac{\partial}{\partial t} z(x,t), & 0<\alpha<1 \end{cases} $$

and

$$ {}_0 I_t^{\alpha} z(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(x,s)\,\mathrm{d}s, \quad \alpha>0. $$

The applications of (1) are rich in real world. In [25], (1) is usually introduced to better characterise those reaction–diffusion processes in the spatially inhomogeneous environment. For example, the reaction–diffusion processes in dispersive transport media [26], the flow through porous media with a source [27], or other diffusion-assisted dynamical processes occurring in disordered media [28]. Here (1) can be viewed as a model of a thin rod with not only the heat loss to a surrounding medium, but also the heat generation inside the rod in a spatially inhomogeneous environment. For more information on fractional reaction–diffusion equations, we refer the readers to [29–32] and the reference therein.

When $\alpha = 1$, the problem (1) is reduced to the classical integer-order unstable heat equation. The boundary feedback control law with any given continuously differentiable function a was studied in [7, 8] and its backstepping observers was investigated in [33]. For more basic theory on boundary feedback stabilisation and adaptive control, we refer the reader to the monographs [9, 10]. This paper is devoted to discussing the boundary feedback control law for the problem (1) and its application in observer design in two setups: the anti-collocated setup, when the sensor and actuator are placed at the opposite ends and the collocated setup, when the sensor and actuator are placed at the same end. We hope that the results here could provide some insights into the qualitative analysis of the design of fractional order controller and observer.

This paper is proceed as follows. In the following section, the boundary feedback stabilisation of an unstable heat equation described by a time fractional-order anomalous diffusion system with Dirichlet boundary conditions is presented and the system under consideration with Neumann boundary conditions is showed in Section 3. In Section 4, the observers for collocated and anti-collocated sensor/actuator pairs are designed. A numerical example is given to illustrate the effectiveness of our results.

2 Dirichlet boundary conditions

In what follows, we denote by $L^2(0,b)$ the usual Lebesgue integrable functions on $(0,b)$ with the norm $\|\cdot\|$ and $H^1(0,b)$ denotes the usual Sobolev space with the norm

$$\|z\|_{H^1(0,b)} = \|z\| + \|z_x\|, \quad z \in H^1(0,b).$$

Consider the following Dirichlet boundary value system

$$\begin{cases} {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t) + a(x)z(x,t) & \text{in } (0,b)\times(0,\infty), \\ z(0,t) = z(b,t) = 0 & \text{in } (0,\infty), \end{cases} \tag{3}$$

by Matignon [34], the stability of the system (3) is guaranteed if and only if the roots of some polynomial [the eigenvalues of $\partial^2/\partial x^2 + a(x)$ or the poles of its corresponding transfer function] lie outside the closed angular sector

$$\left|\arg\left(\text{spec}\left(\frac{\partial^2}{\partial x^2} + a(x)\right)\right)\right| \le \frac{\alpha\pi}{2}. \tag{4}$$

Moreover, since the eigenvalues of operator $\partial^2/\partial x^2$ are $\{\lambda_n\}_{n\ge 1}$ and satisfy

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots, \ \lim_{n\to\infty} \lambda_n = -\infty,$$

we conclude that the system (3) is unstable if a is positive and large enough. Moreover, since the term az is the source of the instability, the purpose here is to design a boundary feedback law and use the backstepping method to 'eliminate' it for any $a \in C^1[0,b]$.

Consider the following coordinate transformation

$$\omega(x,t) = z(x,t) + \int_0^x k(x,y)z(y,t)\,\mathrm{d}y, \tag{5}$$

along with the Dirichlet boundary feedback control

$$z(b,t) = -\int_0^b k(b,y)z(y,t)\,\mathrm{d}y, \tag{6}$$

the system

$$\begin{cases} {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t) + a(x)z(x,t) & \text{in } (0,b)\times(0,\infty), \\ z(0,t) = 0 & \text{in } (0,\infty), \\ z(x,0) = z_0(x) & \text{in } (0,b) \end{cases} \tag{7}$$

can be transformed into the target system

$$\begin{cases} {}_0^C D_t^\alpha \omega(x,t) = \omega_{xx}(x,t) - \lambda\omega(x,t) & \text{in } (0,b)\times(0,\infty), \\ \omega(0,t) = \omega(b,t) = 0 & \text{in } (0,\infty), \\ \omega(x,0) = \omega_0(x) & \text{in } (0,b), \end{cases} \tag{8}$$

where $\omega_0(x) = z_0(x) + \int_0^x k(x,y)z_0(y)\,\mathrm{d}y$ and $\lambda > 0$ is a positive constant depending on the control input u to describe the desired convergence speed. To state our results, the following compatible conditions for the initial data are introduced

$$z_0(0) = 0, \quad z_0(b) = -\int_0^b k(b,y)z_0(y)\,\mathrm{d}y. \tag{9}$$

By utilising the Dirichlet boundary feedback law (6), we have

$$\omega(b,t) = z(b,t) + \int_0^b k(b,y)z(y,t)\,\mathrm{d}y = 0 \tag{10}$$

and from the boundary condition of (7), we deduce that $\omega(0,t) = 0$.

Next, before we try to find out what conditions $k(x,y)$ has to satisfy, let us introduce the following notations

$$\begin{aligned} k_x(x,x) &= \frac{\partial}{\partial x}k(x,y)|_{y=x}, \\ k_y(x,x) &= \frac{\partial}{\partial y}k(x,y)|_{y=x}, \\ \frac{\mathrm{d}}{\mathrm{d}x}k(x,x) &= k_x(x,x) + k_y(x,x). \end{aligned}$$

The α-order derivative of the transformation (5) with respect to time is

$$\begin{aligned} {}_0^C D_t^\alpha \omega(x,t) &= {}_0^C D_t^\alpha z(x,t) + \int_0^x k(x,y)\,{}_0^C D_t^\alpha z(y,t)\,\mathrm{d}y \\ &= {}_0^C D_t^\alpha z(x,t) + \int_0^x k(x,y)\left[z_{yy}(y,t) + a(y)z(y,t)\right]\mathrm{d}y \\ &= {}_0^C D_t^\alpha z(x,t) + k(x,x)z_x(x,t) - k(x,0)z_x(0,t) \\ &\quad - k_y(x,x)z(x,t) + k_y(x,0)z(0,t) \\ &\quad + \int_0^x \left[k_{yy}(x,y)z(y,t) + k(x,y)a(y)z(y,t)\right]\mathrm{d}y \end{aligned}$$

and differentiating the transformation (5) on x gives

$$\begin{aligned} \omega_x(x,t) &= z_x(x,t) + k(x,x)z(x,t) + \int_0^x k_x(x,y)z(y,t)\,\mathrm{d}y, \\ \omega_{xx}(x,t) &= z_{xx}(x,t) + \frac{\mathrm{d}}{\mathrm{d}x}k(x,x)z(x,t) + k(x,x)z_x(x,t) \\ &\quad + k_x(x,x)z(x,t) + \int_0^x k_{xx}(x,y)z(y,t)\,\mathrm{d}y. \end{aligned}$$

Then we get that

$$\begin{aligned} &{}_0^C D_t^\alpha \omega(x,t) - \omega_{xx}(x,t) + \lambda\omega(x,t) \\ &\quad = {}_0^C D_t^\alpha z(x,t) + k(x,x)z_x(x,t) - k(x,0)z_x(0,t) - k_y(x,x)z(x,t) \\ &\qquad + k_y(x,0)z(0,t) + \int_0^x \left[k_{yy}(x,y)z(y,t) + k(x,y)a(y)z(y,t)\right]\mathrm{d}y \\ &\qquad - z_{xx}(x,t) - \frac{\mathrm{d}}{\mathrm{d}x}k(x,x)z(x,t) - k(x,x)z_x(x,t) - k_x(x,x)z(x,t) \\ &\qquad - \int_0^x k_{xx}(x,y)z(y,t)\,\mathrm{d}y + \lambda\left(z(x,t) + \int_0^x k(x,y)z(y,t)\,\mathrm{d}y\right) \\ &\quad = \left(a(x) - k_x(x,x) - k_y(x,x) - \frac{\mathrm{d}}{\mathrm{d}x}k(x,x) + \lambda\right)z(x,t) \\ &\qquad + k_y(x,0)z(0,t) - k(x,0)z_x(0,t) \\ &\qquad + \int_0^x \left[k_{yy}(x,y) - k_{xx}(x,y) + (a(y)+\lambda)k(x,y)\right]z(y,t)\,\mathrm{d}y. \end{aligned}$$

For the right-hand side to be zero for all z, it follows from the Dirichlet condition $z(0,t) = 0$ that the following three conditions

IET Control Theory Appl., pp. 1–8

have to be satisfied

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = (a(y)+\lambda)k(x,y), & 0 \le y \le x \le b, \\ k(x,0) = 0, & 0 \le x \le b, \\ k_x(x,x) + k_y(x,x) + \dfrac{\mathrm{d}}{\mathrm{d}x}k(x,x) = a(x) + \lambda, & 0 \le x \le b. \end{cases} \tag{11}$$

In fact, these three conditions of the system (11) are compatible and form a well-posed PDE of hyperbolic type. Besides, based on the argument in [7] or Chapter 4 in [8], we see that

Lemma 1 [7, 8]: The problem (11) has a unique solution which is twice continuously differentiable in $0 \le y \le x \le b$ provided that $a \in C^1[0,b]$.

Lemma 2 [7, 8]: If $k(x,y)$ is the solution of problem (11) and define the linear bounded operator $K : H^i(0,b) \to H^i(0,b) (i = 0,1,2)$ by

$$\omega(x,t) = (Kz)(x) := z(x,t) + \int_0^x k(x,y)z(y,t)\,\mathrm{d}y. \tag{12}$$

Then we get that

i. K has a linear bounded inverse K^{-1}: $H^i(0,b) \to H^i(0,b)$ $(i = 0,1,2)$, and
ii. K converts the system (1) with the Dirichlet boundary feedback control (6) into the target system (8).

Then we are ready to state the following result:

Theorem 1: Suppose that $a \in C^1[0,b]$, $\lambda > 0$ is any positive constant, $\omega(\cdot,t)$ is a continuous and differentiable function on $[0,\infty)$ and the Laplace transform of $\omega(\cdot,t)^2$ exists.

i. For arbitrary initial value $z_0 \in L^2(0,b)$ with the compatible conditions (9), given $t \ge 0$, (1) with Dirichlet boundary feedback control (6) has a unique solution satisfying the following L^2 Mittag–Leffler stability estimate

$$\|z(\cdot,t)\|^2 \le c_1\|z_0\|^2 E_\alpha(-2\lambda t^\alpha), \quad t \in [0,\infty) \tag{13}$$

for some positive constant c_1, where

$$E_\alpha(t) := \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + 1)}, \quad \mathbf{Re}\,\alpha > 0,\ t \in \mathbf{R}$$

is known as the Mittag–Leffler function in one parameter.
ii. For arbitrary initial value $z_0 \in H^1(0,b)$ with the compatible conditions (9), given $t \ge 0$, then (1) with Dirichlet boundary feedback control (6) admits a unique solution satisfying the following H^1 Mittag–Leffler stability estimate

$$\|z(\cdot,t)\|^2_{H^1(0,b)} \le c_2\|z_0\|^2_{H^1(0,b)} E_\alpha(-2\lambda t^\alpha), \quad t \in [0,\infty) \tag{14}$$

for some positive constant c_2.

The following lemma plays a central role in the proof of Theorem 1.

Lemma 3 [35]: Suppose that $x : [0,\infty) \to \mathbf{R}$ is a continuous and differentiable function, Then, for any given $t \ge 0$, we have

$$\frac{1}{2}{}_0^C D_t^\alpha x^2(t) \le x(t){}_0^C D_t^\alpha x(t), \quad \forall \alpha \in (0,1]. \tag{15}$$

Proof of Theorem 1: First, we see that the problem (1) with Dirichlet boundary condition (6) can be transformed to the target system (8) via the isomorphism (5) and both systems (1) and (8) are well defined. Moreover, by Lemma 2, there exists a positive constant ν_1 such that

$$\|z(\cdot,t)\| \le \nu_1\|\omega(\cdot,t)\|, \quad \|\omega_0\| \le \nu_1\|z_0\| \tag{16}$$

and there exists a positive constant ν_2 such that

$$\|z(\cdot,t)\|_{H^1(0,b)} \le \nu_2\|\omega(\cdot,t)\|_{H^1(0,b)}, \quad \|\omega_0\|_{H^1(0,b)} \le \nu_2\|z_0\|_{H^1(0,b)}. \tag{17}$$

i. Let $W(t) = 1/2\int_0^b \omega(x,t)^2\,\mathrm{d}x$. Since $\omega(\cdot,t)$ is a continuous and differentiable function on $[0,\infty)$, by Lemma 3, we have

$$\begin{aligned} {}_0^C D_t^\alpha W(t) &= \frac{1}{2}\int_0^b {}_0^C D_t^\alpha \omega(x,t)^2\,\mathrm{d}x \le \int_0^b \omega(x,t){}_0^C D_t^\alpha \omega(x,t)\,\mathrm{d}x \\ &= \int_0^b \omega(x,t)\omega_{xx}(x,t)\,\mathrm{d}x - \lambda\int_0^b \omega(x,t)^2\,\mathrm{d}x \\ &= -\int_0^b \omega_x(x,t)^2\,\mathrm{d}x - \lambda\int_0^b \omega(x,t)^2\,\mathrm{d}x \\ &\le -2\lambda W(t). \end{aligned}$$

Similar to the arguments in [36, 37], let

$$M(t) = -2\lambda W(t) - {}_0^C D_t^\alpha W(t). \tag{18}$$

Since the Laplace transform of $\omega(\cdot,t)^2$ exists and $\omega(x,t)$ is the solution of system (8), it follows that both ${}_0^C D_t^\alpha W(t)$ and $M(t)$ are continuous and differentiable on $[0,\infty)$ and their Laplace transforms on t exist. Moreover, we see that $M(t)$ is a non-negative function on $[0,\infty)$.

Taking the Laplace transform on both sides of (18) gives

$$\hat{M}(s) = -2\lambda\hat{W}(s) - s^\alpha\hat{W}(s) + s^{\alpha-1}W(0), \tag{19}$$

where $W(0) = 1/2\int_0^b \omega(x,0)^2\,\mathrm{d}x = 1/2\int_0^b \omega_0(x)^2\,\mathrm{d}x \ge 0$

$$\hat{W}(s) := \int_0^\infty e^{-st}W(t)\,\mathrm{d}t \quad \text{and} \quad \hat{M}(s) := \int_0^\infty e^{-st}M(t)\,\mathrm{d}t \tag{20}$$

are, respectively, the Laplace transform of the functions $W(t)$ and $M(t)$. Hence, we see that

$$\hat{W}(s) = \frac{s^{\alpha-1}W(0) - \hat{M}(s)}{s^\alpha + 2\lambda}. \tag{21}$$

Since $M(t)$ is a non-negative continuous and differentiable function on $[0,\infty)$, and its Laplace transform exits, given $t \ge 0$, it follows from the uniqueness and existence theorem [22], and the inverse Laplace transform that the unique solution of (18) is

$$\begin{aligned} W(t) &= E_\alpha(-2\lambda t^\alpha)W(0) \\ &\quad - M(t) * \left[t^{\alpha-1}E_{\alpha,\alpha}(-2\lambda t^\alpha)\right], \quad t \ge 0, \end{aligned} \tag{22}$$

where $*$ denotes the convolution operator. Moreover, since $t^{\alpha-1}$ and $E_{\alpha,\alpha}(-2\lambda t^\alpha)$ are non-negative function, it follows that

$$W(t) \le E_\alpha(-2\lambda t^\alpha)W(0), \quad t \ge 0. \tag{23}$$

This, together with (16), implies that

$$\begin{aligned} \|z(\cdot,t)\|^2 &\le \nu_1\|\omega(\cdot,t)\|^2 \le \nu_1\|\omega_0\|^2 E_\alpha(-2\lambda t^\alpha) \\ &\le \nu_1^2\|z_0\|^2 E_\alpha(-2\lambda t^\alpha), \quad t \in [0,\infty). \end{aligned} \tag{24}$$

ii. Define $V(t)=\int_0^b \omega_x(x,t)^2\,\mathrm{d}x$. Multiplying the first equation of (8) by ω_{xx} and integrating from 0 to b by parts we obtain

$$\int_0^b \omega_{xx}(x,t)\,{}_0^C D_t^\alpha \omega(x,t)\,\mathrm{d}x=\int_0^b \omega_{xx}(x,t)^2\,\mathrm{d}x+\lambda V(t).$$

Moreover, since $\omega(0,t)=\omega(b,t)=0$ for all $t\ge 0$, we see that ${}_0^C D_t^\alpha \omega(0,t)={}_0^C D_t^\alpha \omega(b,t)=0$ for all $t\ge 0$. Together with $w\in H^1(0,b)$, we deduce that

$$\begin{aligned}\int_0^b \omega_{xx}(x,t)\,{}_0^C D_t^\alpha \omega(x,t)\,\mathrm{d}x&=\Big[\omega_x(x,t)\,{}_0^C D_t^\alpha \omega(x,t)\Big]\Big|_{x=0}^b\\&\quad-\int_0^b \omega_x(x,t)\,{}_0^C D_t^\alpha \omega_x(x,t)\,\mathrm{d}x\\&=-\int_0^b \omega_x(x,t)\,{}_0^C D_t^\alpha \omega_x(x,t)\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}{}_0^C D_t^\alpha V(t)&=\int_0^b {}_0^C D_t^\alpha \omega_x(x,t)^2\,\mathrm{d}x\\&\le 2\int_0^b \omega_x(x,t)\,{}_0^C D_t^\alpha \omega_x(x,t)\,\mathrm{d}x\le -2\lambda V(t).\end{aligned}$$

Similarly, we see that

$$V(t)\le E_\alpha(-2\lambda t^\alpha)V(0),\quad t\ge 0 \tag{25}$$

and the proof of Theorem 1 is completed.

□

3 Neumann boundary conditions

In this section, we continue the discussion of boundary feedback control of the following time fractional-order anomalous diffusion problem

$$\begin{cases}{}_0^C D_t^\alpha z(x,t)=z_{xx}(x,t)+a(x)z(x,t) & \text{in } (0,b)\times(0,\infty),\\ z_x(0,t)=z_x(b,t)=0 & \text{in } (0,\infty)\end{cases} \tag{26}$$

with the Neumann boundary feedback law

$$z_x(b,t)=-k(b,b)z(b,t)-\int_0^b k_x(b,y)z(y,t)\,\mathrm{d}y \text{ in } (0,\infty). \tag{27}$$

Similarly, the system

$$\begin{cases}{}_0^C D_t^\alpha z(x,t)=z_{xx}(x,t)+a(x)z(x,t) & \text{in } (0,b)\times(0,\infty),\\ z_x(0,t)=0 & \text{in } (0,\infty),\\ z(x,0)=z^0(x) & \text{in } (0,b)\end{cases} \tag{28}$$

can be converted into the target system

$$\begin{cases}{}_0^C D_t^\alpha \omega(x,t)=\omega_{xx}(x,t)-\lambda\omega(x,t) & \text{in } (0,b)\times(0,\infty),\\ \omega_x(0,t)=\omega_x(b,t)=0 & \text{in } (0,\infty),\\ \omega(x,0)=\omega^0(x) & \text{in } (0,b),\end{cases} \tag{29}$$

where $\lambda>0$ is a positive constant, $\omega^0(x)=z^0(x)+\int_0^x k(x,y)z^0(y)\,\mathrm{d}y$ and $k(x,y)$ satisfies the system

$$\begin{cases}k_{xx}(x,y)-k_{yy}(x,y)=(a(y)+\lambda)k(x,y), & 0\le y\le x\le b,\\ k_y(x,0)=0, & 0\le x\le b,\\ k_x(x,x)+k_y(x,x)+\dfrac{\mathrm{d}}{\mathrm{d}x}k(x,x)=a(x)+\lambda, & 0\le x\le b.\end{cases} \tag{30}$$

To state our results, the following compatible conditions for the initial data are introduced

$$z_x^0(0)=0,\quad z_x^0(b)=-k(b,b)z^0(b)-\int_0^b k_x(b,y)z^0(y)\,\mathrm{d}y. \tag{31}$$

By utilising the Neumann boundary feedback law (27), we have

$$\omega_x(b,t)=z_x(b,t)+k(b,b)z(b,t)+\int_0^b k_x(b,y)z(y,t)\,\mathrm{d}y=0 \tag{32}$$

and from the boundary condition of (28), we deduce that $\omega_x(0,t)=0$.

Next, before to show our main results in this part, we state the following lemma first.

Lemma 4 [7]: If $a\in C^1[0,b]$, then the problem (30) has a unique solution which is twice continuously differentiable in $0\le y\le x\le b$.

Theorem 2: Suppose that $a\in C^1[0,b]$, $\lambda>0$ is any positive constant, $\omega(\cdot,t)$ is a continuous and differentiable function on $[0,\infty)$, and the Laplace transform of $\omega(\cdot,t)^2$ exists.

i. For arbitrary initial value $z_0\in L^2(0,b)$ with the compatible conditions (31), given $t\ge 0$, then (26) with Neumann boundary feedback law (27) admits a unique solution satisfying the following L^2 Mittag–Leffler stability estimate

$$\|z(\cdot,t)\|^2\le c_3\|z_0\|^2 E_\alpha(-2\lambda t^\alpha),\quad t\in[0,\infty) \tag{33}$$

for some positive constant c_3.

ii. For arbitrary initial value $z_0\in H^1(0,b)$ with the compatible conditions (31), given $t\ge 0$, then (26) with Neumann boundary feedback law (27) has a unique solution satisfying the following H^1 Mittag–Leffler stability estimate

$$\|z(\cdot,t)\|^2_{H^1(0,b)}\le c_4\|z_0\|^2_{H^1(0,b)} E_\alpha(-2\lambda t^\alpha),\quad t\in[0,\infty) \tag{34}$$

for some positive constant c_4.

Proof: Very similar to the proof of Theorem 1. □

4 Observer design for anti-collocated and collocated sensor/actuator pairs

In this section, we try to discuss the observer design with the available measurement being at the opposite/same end of the actuation in two setups, where the boundary conditions is Dirichlet boundary conditions or Neumann boundary conditions.

4.1 Observer design for anti-collocated sensor/actuator pairs

4.1.1 Dirichlet boundary conditions: We propose the following observer for the system (1) with the available measurement being at the opposite end of the actuation and the Dirichlet boundary conditions under the assumption that the observer error at $x = 0$ is zero

$$\begin{cases} {}_0^C D_t^\alpha \hat{z}(x,t) = \hat{z}_{xx}(x,t) + a(x)\hat{z}(x,t) + \rho(x)\left[z(0,t) - \hat{z}(0,t)\right], \\ \hat{z}(0,t) = 0, \quad \hat{z}(b,t) = u(t). \end{cases} \tag{35}$$

Here $\rho(x)$ is the output injection function to be designed. It then follows that the observer error $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \tilde{z}(x,t) = \tilde{z}_{xx}(x,t) + a(x)\tilde{z}(x,t) - \rho(x)\tilde{z}(0,t), \\ \tilde{z}(0,t) = 0, \quad \tilde{z}(b,t) = 0. \end{cases} \tag{36}$$

Consider the following invertible coordinate transformation

$$\tilde{\omega}(x,t) = \tilde{z}(x,t) + \int_0^x k(x,y)\tilde{z}(y,t)\,\mathrm{d}y, \tag{37}$$

together with the Dirichlet boundary feedback controller

$$\tilde{z}(b,t) = -\int_0^b k(b,y)\tilde{z}(y,t)\,\mathrm{d}y, \tag{38}$$

the system (36) is converted into the following system

$$\begin{cases} {}_0^C D_t^\alpha \tilde{\omega}(x,t) = \tilde{\omega}_{xx}(x,t) - \lambda\tilde{\omega}(x,t) + \left[k_y(x,0) - \rho(x)\right]\tilde{\omega}(0,t), \\ \tilde{\omega}(0,t) = \tilde{\omega}(b,t) = 0 \qquad . \end{cases} \tag{39}$$

where $\lambda > 0$ is a constant and is used to set the desired observer convergence speed. Moreover, the function $k(\cdot,\cdot)$ described by system (11) satisfies Lemma 1. Compared with the system (8), it follows that the observer gains should be chosen as

$$\rho(x) = k_y(x,0). \tag{40}$$

Moreover, by Lemma 1, if $a \in C^1[0,b]$, the function $k(x,y)$ exists uniquely in $0 \le y \le x \le b$ and the observer gains can be obtain from (40). The results can be formulated as follows.

Theorem 3: Let $k(x,y)$ be the solution of the system (11) and $\rho(x)$ be given by (40). Suppose that all conditions in Theorem 1 are satisfied. Then for any $\tilde{z}_0 \in L^2(0,b)$ or $\tilde{z}_0 \in H^1(0,b)$, the system (36) admits a unique solution $\tilde{z}(x,t)$ satisfying the L^2 Mittag–Leffler stability estimate (13) in $L^2(0,1)$ or the H^1 Mittag–Leffler stability estimate (14) in $H^1(0,1)$, respectively.

4.1.2 Neumann boundary conditions:

Consider the following observer for the system (26) with the available measurement being at the opposite end of the actuation and the Neumann boundary conditions

$$\begin{cases} {}_0^C D_t^\alpha \hat{z}(x,t) = \hat{z}_{xx}(x,t) + a(x)\hat{z}(x,t) + \rho(x)\left[z(0,t) - \hat{z}(0,t)\right], \\ \hat{z}_x(0,t) = c\left[z(0,t) - \hat{z}(0,t)\right], \\ \hat{z}_x(b,t) = u(t). \end{cases} \tag{41}$$

Here $\rho(x)$ and c are output injection functions ($c \in \mathbf{R}$ is a constant) to be designed. Then the observer error $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \tilde{z}(x,t) = \tilde{z}_{xx}(x,t) + a(x)\tilde{z}(x,t) - \rho(x)\tilde{z}(0,t), \\ \tilde{z}_x(0,t) = -c\tilde{z}(0,t), \quad \tilde{z}_x(b,t) = 0 \end{cases} \tag{42}$$

and the system (42) can be transformed into

$$\begin{cases} {}_0^C D_t^\alpha \tilde{\omega}(x,t) = \tilde{\omega}_{xx}(x,t) - \lambda\tilde{\omega}(x,t) + \left[k_y(x,0) - \rho(x)\right]\tilde{\omega}(0,t), \\ \tilde{\omega}_x(0,t) = \left[k(0,0) - c\right]\tilde{\omega}(0,t), \\ \tilde{\omega}_x(b,t) = 0 \end{cases} \tag{43}$$

by using the following invertible coordinate transformation

$$\tilde{\omega}(x,t) = \tilde{z}(x,t) + \int_0^x k(x,y)\tilde{z}(y,t)\,\mathrm{d}y \tag{44}$$

and the Neumann boundary feedback controller

$$\tilde{z}_x(b,t) = -k(b,b)\tilde{z}(b,t) - \int_0^b k_x(b,y)\tilde{z}(y,t)\,\mathrm{d}y, \quad t \in (0,\infty), \tag{45}$$

where $\lambda > 0$ is a constant, $\tilde{\omega}_0(x) = \tilde{z}_0(x) + \int_0^x k(x,y)\tilde{z}_0(y)\,\mathrm{d}y$ and $k(\cdot,\cdot)$ satisfies (30).

Corresponding to the system (29), we obtain that the observer gains should be chosen as

$$\rho(x) = k_y(x,0), \quad c = k(0,0). \tag{46}$$

Moreover, it follows from Lemma 4 that if $a \in C^1[0,b]$, the function $k(x,y)$ exists uniquely in $0 \le y \le x \le b$ and we conclude that:

Theorem 4: Let $k(x,y)$ be the solution of (30) and if $\rho(x), c$ be given by (46). Suppose that all conditions in Theorem 2 hold. Then for $\tilde{z}_0 \in L^2(0,b)$ or $\tilde{z}_0 \in H^1(0,b)$, the system (42) admits a unique solution $\tilde{z}(x,t)$ satisfying the L^2 Mittag–Leffler stability estimate (33) in $L^2(0,1)$ or the H^1 Mittag–Leffler stability estimate (34) in $H^1(0,1)$, respectively.

4.2 Observer design for collocated sensor/actuator pairs

4.2.1 Dirichlet boundary conditions:

Consider the following observer for the system (1) with the available measurement being at the same end of the actuation and the Dirichlet boundary conditions under the assumption that the observer error at $x = b$ is zero

$$\begin{cases} {}_0^C D_t^\alpha \hat{z}(x,t) = \hat{z}_{xx}(x,t) + a(x)\hat{z}(x,t) + \rho(x)\left[z(b,t) - \hat{z}(b,t)\right], \\ \hat{z}(0,t) = 0, \quad \hat{z}(b,t) = u(t), \end{cases} \tag{47}$$

where $\rho(x)$ is a output injection function to be designed. The difference with the anti-collocated case is that the gain $\rho(x)$ is introduced in the other boundary condition.

It then follows that the observer error $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \tilde{z}(x,t) = \tilde{z}_{xx}(x,t) + a(x)\tilde{z}(x,t) - \rho(x)\tilde{z}(b,t), \\ \tilde{z}(0,t) = 0, \quad \tilde{z}(b,t) = 0. \end{cases} \tag{48}$$

Try to find the transformation

$$\tilde{z}(x,t) = \tilde{\omega}(x,t) - \int_x^b k(x,y)\tilde{\omega}(y,t)\,\mathrm{d}y \tag{49}$$

that convert (48) into the following target system

$$\begin{cases} {}_0^C D_t^{\alpha} \tilde{\omega}(x,t) = \tilde{\omega}_{xx}(x,t) - \lambda \tilde{\omega}(x,t), \\ \tilde{\omega}(0,t) = \tilde{\omega}(b,t) = 0 \end{cases}. \tag{50}$$

By substituting (49) into (48) and utilise (50), we see that $k(x,y)$ must satisfy

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = (a(y) + \lambda)k(x,y), & 0 \le y \le x \le b, \\ k(x,b) = 0, & 0 \le x \le b, \\ -k_x(x,x) - k_y(x,x) - \dfrac{\mathrm{d}}{\mathrm{d}x} k(x,x) = a(x) + \lambda, & 0 \le x \le b \end{cases} \tag{51}$$

that yield

$$\begin{cases} {}_0^C D_t^{\alpha} \tilde{\omega}(x,t) = \tilde{\omega}_{xx}(x,t) - \lambda \tilde{\omega}(x,t) + (k_y(x,b) - \rho(x))\tilde{\omega}(x,b), \\ \tilde{\omega}(0,t) = \tilde{\omega}(b,t) = 0, \end{cases} \tag{52}$$

where $\lambda > 0$ is a positive constant. Then the observer gains should be chosen as

$$\rho(x) = k_y(x,b). \tag{53}$$

Moreover, similar to the argument in the proof of Lemma 2.2 in [7], we see that if $a \in C^1[0,b]$, the function $k(x,y)$ exists uniquely in $0 \le y \le x \le b$. Choose the observer gains as (53), we see that

Theorem 5: Let $k(x,y)$ be the solution of (30) and $\rho(x)$ be given by (53). Suppose that all conditions in Theorem 1 hold. Then for any $\tilde{z}_0 \in L^2(0,b)$ or $\tilde{z}_0 \in H^1(0,b)$, the system (48) admits a unique solution $\tilde{z}(x,t)$ satisfying the L^2 Mittag–Leffler stability estimate (13) in $L^2(0,1)$ or the H^1 Mittag–Leffler stability estimate (14) in $H^1(0,1)$, respectively.

4.2.2 Neumann boundary conditions: Consider the following observer for the system (26) with the available measurement of our system being at the same end with actuation and the Neumann boundary conditions

$$\begin{cases} {}_0^C D_t^{\alpha} \hat{z}(x,t) = \hat{z}_{xx}(x,t) + a(x)\hat{z}(x,t) + \rho(x)\left[z(b,t) - \hat{z}(b,t)\right], \\ \hat{z}_x(0,t) = 0, \\ \hat{z}_x(b,t) = c\left[z(b,t) - \hat{z}(b,t)\right] + u(t). \end{cases} \tag{54}$$

Here $\rho(x)$ and c are output injection functions ($c \in \mathbf{R}$ is a constant) to be designed. The difference with the collocated case is that the gain c is introduced in the other boundary condition. Then it follows that the observer error $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^{\alpha} \tilde{z}(x,t) = \tilde{z}_{xx}(x,t) + a(x)\tilde{z}(x,t) - \rho(x)\tilde{z}(b,t), \\ \tilde{z}_x(0,t) = 0, \quad \tilde{z}_x(b,t) = -c\tilde{z}(b,t). \end{cases} \tag{55}$$

We are looking for the transformation

$$\tilde{z}(x,t) = \tilde{\omega}(x,t) - \int_x^b k(x,y)\tilde{\omega}(y,t)\,\mathrm{d}y, \tag{56}$$

that maps the system (55) into the Mittag–Leffler stability target system

$$\begin{cases} {}_0^C D_t^{\alpha} \tilde{\omega}(x,t) = \tilde{\omega}_{xx}(x,t) - \lambda \tilde{\omega}(x,t), \\ \tilde{\omega}_x(0,t) = \tilde{\omega}_x(b,t) = 0. \end{cases} \tag{57}$$

By substituting (56) into (55) and utilise (57), we see that $k(x,y)$ must satisfy

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = (a(y) + \lambda)k(x,y), & 0 \le y \le x \le b, \\ k(x,b) = 0, & 0 \le x \le b, \\ -k_x(x,x) - k_y(x,x) - \dfrac{\mathrm{d}}{\mathrm{d}x} k(x,x) = a(x) + \lambda, & 0 \le x \le b \end{cases} \tag{58}$$

that yield

$$\begin{cases} {}_0^C D_t^{\alpha} \tilde{\omega}(x,t) = \tilde{\omega}_{xx}(x,t) - \lambda \tilde{\omega}(x,t) + \left[-k_y(x,b) - \rho(x)\right]\tilde{\omega}(b,t), \\ \tilde{\omega}_x(0,t) = 0, \\ \tilde{\omega}_x(b,t) = \left[-k(b,b) - c\right]\tilde{\omega}(b,t), \end{cases} \tag{59}$$

where $\lambda > 0$ is a positive constant. Then the observer gains should be chosen as

$$\rho(x) = k_y(x,b), \quad c = -k(b,b). \tag{60}$$

Similarly, if $a \in C^1[0,b]$, we get that the function $k(x,y)$ exists uniquely in $0 \le y \le x \le b$. Choose the observer gains as (60), we obtain that

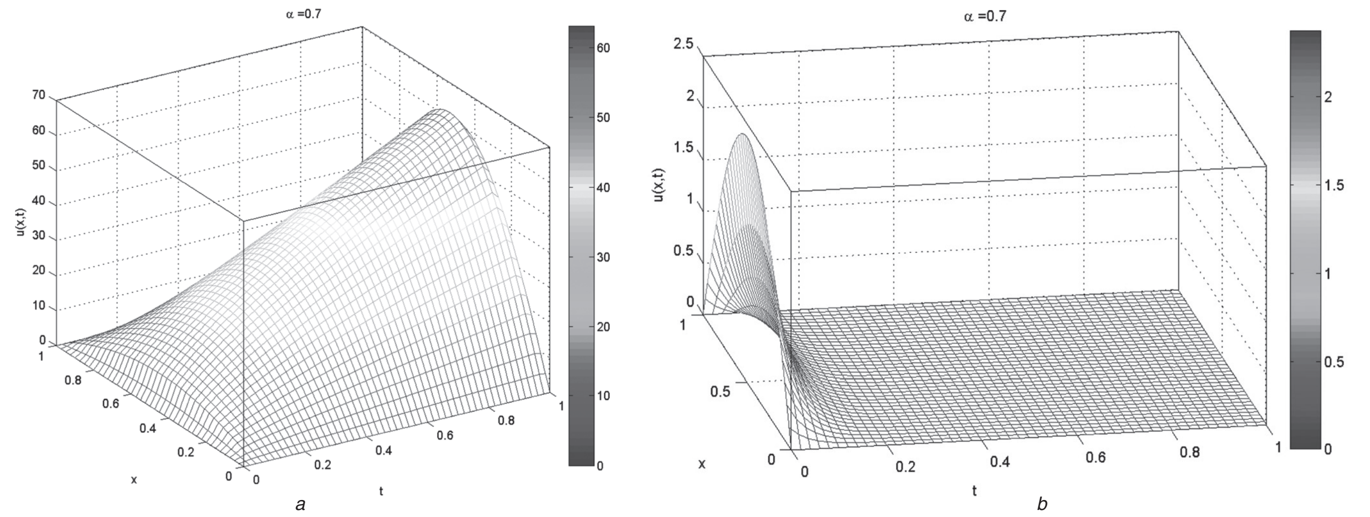

Fig. 1 *Solution of the system (1) when $\alpha = 0.7$*

a Without control
b Approximation of controlled system

Theorem 6: Let $k(x,y)$ be the solution of (51) and if $\rho(x), c$ be given by (60). Suppose that all conditions in Theorem 2 are satisfied. Then for any $\tilde{z}_0 \in L^2(0,b)$ or $\tilde{z}_0 \in H^1(0,b)$, the system (55) has a unique solution $\tilde{z}(x,t)$ satisfying the L^2 Mittag–Leffler stability estimate (33) in $L^2(0,1)$ or the H^1 Mittag–Leffler stability estimate (34) in $H^1(0,1)$, respectively.

5 Numerical simulation

The aim of this section is to carry out a simulate example to test the effectiveness of our theoretical results. For simplicity, here we consider the Dirichlet boundary condition cases.

In system (1), we take $a(x) = 10$, $b = 1$. Let the initial data be

$$z_0(x) = 9x(1-x), \quad x \in (0,1). \tag{61}$$

Fig. 1*a* shows that the system (1) is unstable without control [i.e. $z(1,t) = u(t) \equiv 0$] when $\alpha = 0.7$.

Next, we let $\lambda = 1$ in the kernel function. According to [38] (see also [33]), it follows that

$$k(x,y) = -11y\frac{I_1\left(\sqrt{11(x^2-y^2)}\right)}{\sqrt{11(x^2-y^2)}}, \tag{62}$$

where I_1 is the modified Bessel functions of order one. By using the coordinate transformation (5), Fig. 1*b* shows that system (1) converges smoothly. The results show that our method yields satisfactory performance in dealing with the unstable heat process described by the time fractional-order anomalous diffusion system.

6 Conclusion

In this paper, we present explicitly the closed-loop solutions to the boundary feedback stabilisation problem for the time fractional-order anomalous diffusion system. We hope that the results obtained here could provide some insights into the qualitative analysis of the design of fractional order controller and observer. It should be pointed out that the described method is not limited if the existence of the explicit solution of system of kernel function $k(x,y)$ can be solved. For example, consider the following more general time fractional-order anomalous diffusion system

$${}_0^C D_t^{\alpha} z(x,t) = z_{xx}(x,t) + a(x,t)z(x,t) \quad \text{in } (0,b)\times(0,\infty) \tag{63}$$

with the boundary conditions $z(0,t) = 0$, $z(b,t) = u(t)$, $t \geq 0$, where the function a depends on t. By using the coordinate transformation

$$\omega(x,t) = z(x,t) + \int_0^x k(x,y)z(y,t)\,\mathrm{d}y, \tag{64}$$

the boundary stabilisation problem is then converted into a problem of solving the following linear hyperbolic PDE

$$\begin{cases} k_{xx}(x,y,t) - k_{yy}(x,y,t) - k_t(x,y,t) = (a(y,t)+\lambda)k(x,y,t), \\ \quad 0 \leq y \leq x \leq b, \\ k(x,0,t) = 0, \quad 0 \leq x \leq b, \\ k_x(x,x,t) + k_y(x,x,t) + \dfrac{\mathrm{d}}{\mathrm{d}x}(k(x,x,t)) = a(x,t) + \lambda, \quad 0 \leq x \leq b, \end{cases} \tag{65}$$

where λ is a constant. Then all results here are still holding provided that the problem (65) admits at least a solution. The approach here can also be modified to obtain a controller that minimises a reasonable cost functional that puts penalty on both state and control [39].

On the other hand, studied the results here can also be extended to complex fractional-order distributed parameter systems and various open questions are still under consideration. For instance, the problem of boundary feedback stabilisation of spatial fractional-order distributed parameter systems, time-space fractional-order distributed parameter systems as well as the sensor configurations are of great interest.

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