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RESEARCH PAPER

ROBUST STABILITY ANALYSIS OF LTI SYSTEMS WITH FRACTIONAL DEGREE GENERALIZED FREQUENCY VARIABLES

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Dedicated to the memory
of late Professor Wen Chen

Abstract

A novel linear time-invariant (LTI) system model with fractional degree generalized frequency variables (FDGFVs) is proposed in this paper. This model can provide a unified form for many complex systems, including fractional-order systems, distributed-order systems, multi-agent systems and so on. This study mainly investigates the stability and robust stability problems of LTI systems with FDGFVs. By characterizing the relationship between generalized frequency variable and system matrix, a necessary and sufficient stability condition is firstly presented for such systems. Then for LTI systems with uncertain FDGFVs, we present a robust stability method in virtue of zero exclusion principle. Finally, the effectiveness of the method proposed in this paper is demonstrated by analyzing the robust stability of gene regulatory networks.

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1. **Introduction**

The linear time-invariant (LTI) system with a generalized frequency variable (GFV) means that its transfer function has the form $\bar{G}(s) = G(\phi(s))$, where $G(s) = C(sI - A)^{-1}B + D$ is the traditional transfer function of a state-space linear system

$$x(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

and $\phi(s)$ is a scalar function about variable s. $\phi(s)$ is called the GFV, since $\phi(s)$ replaces the traditional frequency variable s in the transfer function $G(s)$. The concept of LTI systems with GFVs was firstly came up by Hara s research group for providing a unifying theoretical framework of analysis and synthesis of multi-agent systems [6, 7]. In recent years, LTI systems with GFVs have appeared in a variety of areas including gene regulatory networks [17, 20], biomolecular communication networks [11], multi-robot formation control problems [6] as well as the control torque distribution of electric vehicles [35].

Due to the wide application of LTI systems with GFVs, many researchers have begun to investigate their fundamental properties including controllability/observability [33], stability [10, 34] and stabilization [9]. However, GFVs mentioned so far are all integer degree rational functions. In fact, GFVs can be generalized to be fractional degree rational functions or even some complex functions. For example, from the view-point of the frequency variable, the LTI fractional-order system [18, 23, 27, 38] can be regarded as an LTI system with GFV, since its transfer function $\bar{G}(s) = C(\phi(s)I - A)^{-1}B + D$, where $\phi(s) = s^{\alpha}$, $\alpha \in (0, 2)$; the multiple-orders fractional system [14, 15, 31] can also be regarded as the LTI system with the GFV $\phi(s) = r_1 s^{\alpha_1} + r_2 s^{\alpha_2} + \cdots + r_n s^{\alpha_n}$, $\alpha_1, \alpha_2, \cdots, \alpha_n \in (0, 1)$; when fractional-order multi-agent systems are concerned, the GFV $\phi(s)$ can be generalized to be fractional degree rational function [37]; the distributed-order system [16] essentially belongs to a class of LTI systems with GFVs, because its transfer function $\bar{G}(s) = C(\frac{s-1}{\ln s}I - A)^{-1}B + D$. However, to the best our knowledge, few literatures have studied fractional-order dynamical systems from the perspective of LTI systems with GFVs.

On the other hand, gene regulatory networks (GRNs) have attracted wide attention in the biological and biomedical science. This network can describe the interaction between DNA, mRNA and protein in genetic expression. Genes on a DNA molecule are firstly transcribed messenger to mRNAs, then mRNAs are translated into proteins, these proteins activate or repress the transcription process. Many pioneering works have witnessed the modeling and analysis of dynamical properties of GRNs. Chen et al. in

[4] provided a mathematical model for homogeneous GRNs with functional differential equations and they analyzed the local stability and bifurcation properties of this model. A sufficient condition was presented in [32] to ensure the stability of GRNs with heterogeneous nominal systems. Wang et al. in [36] modelled GRNs by a differential equation with polytopic uncertainties, they presented a stability condition in terms of linear matrix inequalities (LMIs). The robust stability problem was investigated in [17] for large-scale GRNs with parametric or unstructured uncertainties. Very recently, Hare et al. in [8] further presented robust stability conditions for MIMO GRN systems with three types of perturbations. However, the GRN systems addressed by these paper are all characterized by integer-order differential equations. Ji et al. in [13] built GRN model based on a fractional-order differential equation and experiment results showed that the fractional-order differential equations were more suitable to model genetic regulatory mechanism. In recent years, fractional-order GRNs have aroused great research interest and some criteria on stability analysis have been established by using the fractional Lyapunov method [12, 28]. However, till now, no efforts were made on modeling fractional-order GRNs using LTI systems with fractional degree generalized frequency variables (FDGFVs).

In this paper, we provide an LTI system model with FDGFVs, while the GFVs are generalized to be fractional degree rational functions. The stability and robust stability problems will be investigated for such model. Although the stability and robust stability problems of LTI systems with fractional degree frequency variable have been investigated in many references [1–3, 5, 19, 22, 24–26, 29], the stability analysis methods provided in these references are mostly given for the case that frequency variable $\phi(s) = s^{\alpha}$, $\alpha \in (0, 2)$. For more generalized frequency variables, we will establish stability conditions by analyzing the relationship between GFVs and system matrix. Based on the zero exclusion principle, robust stability conditions are given for LTI systems with uncertain FDGFVs. Finally, the incommensurate fractional-order GRN systems are modelled as an LTI system with FDGFV, the stability and the robust stability problem of this kind of GRN systems are studied by using the GFV method proposed for the first time in this paper.

Notations. We denote $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ as the set of integer numbers, real numbers and complex numbers, respectively. Denote $\mathbb{C}_+$ as the closed right half of complex plane and $\mathbb{C}_- = \mathbb{C} \setminus \mathbb{C}_+$. $j = \sqrt{-1}$ denotes the imaginary unit. For a square matrix A, the set of its eigenvalues is denoted by $\sigma(A)$.

2. **Problem formulation and preliminaries**

Consider the following LTI systems with fractional degree GFVs

$$\bar{G}(s) = C(\phi(s)I_n - A)^{-1}B + D, \tag{2.1}$$

where $A \in \mathbb{R}^{n_p \times n_p}$, $B \in \mathbb{R}^{n_p \times l}$, $C \in \mathbb{R}^{q \times n_p}$ and $D \in \mathbb{R}^{q \times l}$ are constant matrices. $\phi(s)$ is a FDGFV with the following form

$$\phi(s) = \frac{n(s)}{d(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}, \tag{2.2}$$

where $a_i,\ b_j$ $(i = 0, \cdots, n,\ j = 0, \cdots, m)$ are constant coefficients; $\alpha_i,\ \beta_j$ $(i = 0, \cdots, n,\ j = 0, \cdots, m)$ are fractional degrees and we assume that $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0 \geq 0$, and $\beta_m > \beta_{m-1} > \cdots > \beta_0 \geq 0$.

The function $\phi(s)$ is called commensurate-order if there exist $p_k, q_k \in \mathbb{Z}$ such that $\alpha_k = p_k\alpha$, $\beta_k = q_k\alpha$, $(0 < \alpha < 1)$, that is, $\phi(s)$ has the following form:

$$\phi(s) = \frac{\sum_{k=0}^{m} b_k s^{q_k \alpha}}{\sum_{k=0}^{n} a_k s^{p_k \alpha}}. \tag{2.3}$$

The function $\phi(s)$ becomes a strictly proper rational function when the highest degree of the polynomial in the denominator is greater than that of the polynomial in the numerator.

Now, let us recall the fractional calculus [27]. Without loss of generality, we assume that the lower bound of the fractional integral is 0 throughout this paper. The fractional-order integral is defined as,

$$_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function. The Caputo derivative of function $f(t)$ with order $\alpha \in (0, 1]$ is defined by

$$^C_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} f'(\tau) d\tau, \tag{2.4}$$

where f' is the first-order derivative of function f. We denote D^α instead of $^C_0D_t^\alpha$ for simplicity.

3. **Stability analysis**

In this section, we will investigate the stability of LTI systems with FDGFVs. We say that the LTI system $\bar{G}(s)$ in (2.1) with FDGFV defined by (2.2) is BIBO stable if $\bar{G}(s)$ has no pole in the closed right half complex plane.

Based on $\phi(s)$, we define two domains:

$$_{+} := \{\lambda \quad \mathbb{C} | \quad s \quad \mathbb{C}_{+} \text{ such that } \phi(s) := \frac{n(s)}{d(s)} = \lambda\},$$

and

$$^{c}_{+} := \mathbb{C} \backslash \quad _{+}, \tag{3.1}$$

that is, $^{c}_{+} = \{\lambda \quad \mathbb{C} | \quad s \quad \mathbb{C}_{+}, \ \phi(s) := \frac{n(s)}{d(s)} = \lambda\}$. Then, a stability criterion is stated as:

LEMMA 3.1. *The LTI system $\bar{G}(s)$ defined by (2.1) with FDGFV defined by (2.2) is BIBO stable if and only if $\lambda \quad ^{c}_{+}$, where $\lambda \quad \sigma(A)$ and $^{c}_{+}$ defined in (3.1).*

P r o o f. Base on (2.1), it is easily known that all the poles of $\bar{G}(s)$ satisfy the following characteristic equation

$$|n(s)I \quad d(s)A| = 0. \tag{3.2}$$

Thus all the zeros of (3.2) should lie in the left half plane to ensure that the BIBO stability of system (2.1), which is equivalent to that $\lambda \quad ^{c}_{+}$, where $\lambda \quad \sigma(A)$. □

From Lemma 3.1, we can know that the stability of LTI system $\bar{G}(s)$ is closely related to the GFV $\phi(s)$ and the eigenvalues of matrix A. In the following, some examples are given to illustrate the stability region $^{c}_{+}$ characterized by GFV $\phi(s)$.

EXAMPLE 3.1. Consider LTI system $\bar{G}(s)$ defined by (2.1) with FDGFV $\phi(s) = s^{\alpha}$, $\alpha \quad (0, 2)$. It is obvious that $\bar{G}(s) = C(s^{\alpha}I \quad A)^{-1}B + D$ is the transfer function of commensurate fractional order system:

$$\begin{cases} D^{\alpha}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{cases} \tag{3.3}$$

From references [21, 30], one can know that the stability region $^{c}_{+}$ characterized by $\phi(s) = s^{\alpha}$ is

$$D_{\alpha} := \{z : |\arg(z)| > \frac{\alpha\pi}{2}, z \quad \mathbb{C}\}. \tag{3.4}$$

The domains D_{α} for $1 < \alpha < 2$ and $0 < \alpha < 1$ are depicted in Fig. 1 (a) and Fig. 1 (b), respectively. Thus, fractional-order system (3.3) is stable if and only if for all $\lambda \quad \sigma(A)$, $\lambda \quad D_{\alpha}$.

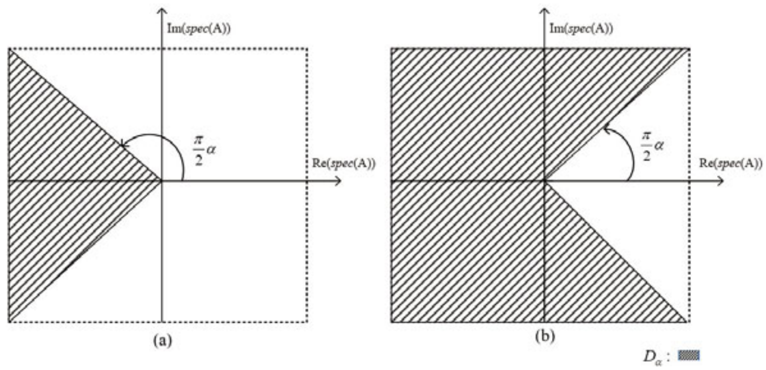


FIGURE 1. Stability domain D_α for: (a) $1 < \alpha < 2$; (b) $0 < \alpha < 1$.

EXAMPLE 3.2. Consider the LTI system $\bar{G}(s)$ defined by (2.1) with GFV $\phi(s) = \sum_{i=1}^{l} r_i s^{\alpha_i}$, $\alpha_i \quad (0,1)$, $(i = 1, 2, \cdots, l)$. From references [14, 15, 31], one can know that $\bar{G}(s)$ is the transfer function of the multiple-order fractional system as follows:

$$\left\{ \begin{array}{rcl} \sum_{i=1}^{l} r_i D^{\alpha_i} x(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t). \end{array} \right. \tag{3.5}$$

The stability region $\;^{c}_{+}$ characterized by $\phi(s) = \sum_{i=1}^{l} r_i s^{\alpha_i}$ is the left side of the curve defined by $l_a \bigcup l_b$ in the complex plane, where l_a and l_b are symmetrical with respect to the real axis, and

$$l_a := \{x + iy |\; x = x(\omega), y = y(\omega), \omega \quad [0, + \quad)\},$$

where

$$\left\{ \begin{array}{rcl} x(\omega) & = & \sum_{i=1}^{l} r_i \omega^{\alpha_i} \cos(\alpha_i \pi/2), \\ y(\omega) & = & \sum_{i=1}^{l} r_i \omega^{\alpha_i} \sin(\alpha_i \pi/2). \end{array} \right.$$

Thus, the multiple orders fractional system (3.5) is stable if and only if for all $\lambda \quad \sigma(A)$, λ lie in the left part of the curve $l_a \bigcup l_b$.

EXAMPLE 3.3. From reference[16], one can see that the distributed-order LTI system

$$\left\{ \begin{array}{rcl} \int_0^1 D^{\alpha} x(t) d\alpha & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \right. \tag{3.6}$$

with transfer function $G(s) = C(\frac{s-1}{\ln s} I \quad A)^{-1})B + D$ is BIBO stable if and only if all the eigenvalues of A lie on the left of the curve $l_2 := l_c \quad l_d$ in the complex plane, where l_c and l_d are symmetrical with respect to the real axis and

$$l_c := \left\{ x \quad jy | x = \frac{2\pi\omega \ln\omega}{4(\ln\omega)^2 + \pi^2},\ y = \frac{4\omega\ln\omega + 2\pi}{4(\ln\omega)^2 + \pi^2} \right\}$$

with $\omega \quad [0, + \quad)$. The stable boundary curve l_2 of the distributed-order system (3.6) is determinated by setting $s = j\omega$ in the generalized frequency variable $\phi(s) = \frac{s-1}{\ln s}$.

When the fractional-order multi-agent system is concerned , the GFVs $\phi(s)$ can be generalized to be fractional degree rational function as (2.2) [6, 7]. For this case, it is difficult to draw the boundary of curve determined by $\phi(s)$. Thus we give the following stability criterion:

THEOREM 3.1. *Consider LTI system $\bar{G}(s)$ defined by (2.1) with fractional degree GFV $\phi(s) = \frac{n(s)}{d(s)}$ defined by (2.2). Define the fractional degree polynomial $p(\lambda, s)$ for $\lambda \quad \mathbb{C}$*

$$p(\lambda, s) := n(s) \quad \lambda d(s), \tag{3.7}$$

then the following statements are equivalent:

(i) *The LTI system $\bar{G}(s)$ is Hurwitz stable.*
(ii) $\sigma(A) \quad$ ff$(\phi(s)) := \{\lambda \quad \mathbb{C} | p(\lambda, s)$ *is Hurwitz stable*$\}$.

The proof of this theorem can refer to Theorem IV.2 in reference [37].

REMARK 3.1. According to Theorem 3.1, the stable analysis of LTI system $\bar{G}(s)$ can be converted into judging the stability of the polynomial $p(\lambda, s)$.

In fact, it is still a difficult task to determinate the Hurwitz stability of polynomial $p(\lambda, s)$. If $h(s) = \frac{1}{\phi(s)}$ can be easily realized in a state-space form, then a simpler stability criterion is presented in the following theorem.

THEOREM 3.2. *Consider LTI system $\bar{G}(s)$ defined by (2.1) with FDGFV $\phi(s) = \frac{n(s)}{d(s)}$ defined by (2.2), assume that $\phi(s)$ has no zero points in $\mathbb{C}_+$, $h(s) = \frac{1}{\phi(s)}$ has a state realization: $h(s) = C_h(s^\alpha I \quad A_h)^{-1}B_h$, $\alpha \quad (0, 2)$, then LTI system $\bar{G}(s)$ is Hurwitz stable if and only if for each $\lambda \quad \sigma(A)$, all the eigenvalues of $A_h + \lambda B_h C_h$ lie in D_α, where $D_\alpha := \{z : |\arg(z)| > \frac{\alpha\pi}{2}, z \quad \mathbb{C}\}$.*

P r o o f. Let T be a non-singular matrix such that $T^{-1}AT$ is a Jordan matrix and λ_i $(i = 1, 2, \cdots, n)$ be eigenvalues of matrix A. Then

$$\begin{aligned}
&\det(\phi(s)I_n \quad A) = 0\\
&\det(\phi(s)T^{-1}I_nT \quad T^{-1}AT) = 0\\
&\prod_{i=1}^{n}\det(\phi(s))\det(1 \quad \lambda_i h(s)) = 0.
\end{aligned}$$

On the other hand,

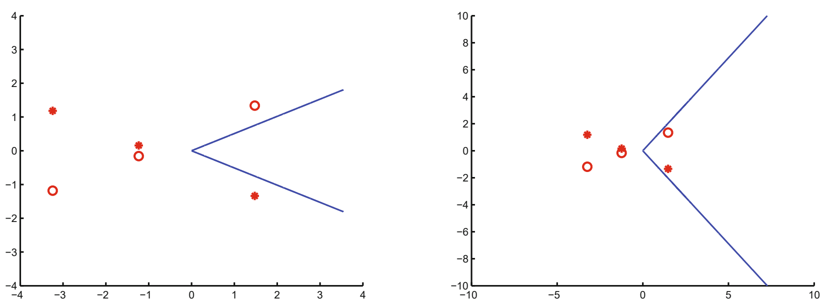
$$\begin{aligned}
&\det(1 \quad \lambda h(s)) = \det(1 \quad \lambda C_h(s^\alpha I \quad A_h)^{-1}B_h) = 0\\
&\det(\begin{bmatrix} s^\alpha I \quad A_h & B_h \\ \lambda C_h & 1 \end{bmatrix}) = 0\\
&\det(s^\alpha I \quad (A_h + \lambda B_h C_h)) = 0.
\end{aligned}$$

Thus, based on the condition that $\phi(s)$ has no zero points in $\mathbb{C}_+$, one can deduce that the LTI system $\bar{G}(s)$ in (2.1) is Hurwitz stable if and only if all the eigenvalues of $A_h + \lambda B_h C_h$ lie in D_α for every $\lambda \quad \sigma(A)$. □

Example 3.4. Consider the LTI system $\bar{G}(s)$ defined by (2.1) with GFV $\phi(s) = 1/h(s)$, $h(s) = \frac{s^\alpha+4}{s^{3\alpha}+3s^{2\alpha}+2s^\alpha+1}$ and matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. A state realization is obtained as $h(s) = C_h(s^\alpha I \quad A_h)^{-1}B_h$ with

$$A_h = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \ B_h = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \ C_h = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}.$$

By using MATLAB, we can check that $\phi(s)$ has no zeros in the closed right half plane and the eigenvalues of A are $\lambda_1 = 1 + 2j$, $\lambda_2 = 1 \quad 2j$. The eigenvalue distributions of matrices $A_h + \lambda_1 B_h C_h$ and $A_h + \lambda_2 B_h C_h$ are labeled by o and * in Fig. 2, respectively. It follows from Theorem 3.2 and Fig. 2(a) that the LTI systems $\bar{G}(s)$ is stable when $\alpha = 0.3$, since all the eigenvalues of matrix $A_h + \lambda_i B_h C_h$, $(i = 1, 2)$, lie in $D_{0.3}$. Otherwise, Fig. 2(b) shows that LTI systems $\bar{G}(s)$ is unstable when $\alpha = 0.6$, since there exist eigenvalues of $A_h + \lambda_i B_h C_h$, $(i = 1, 2)$, that do not lie in $D_{0.6}$.



(A) (B)

FIGURE 2. The eigenvalue distributions of $A_h + \lambda B_h C_h$

4. Robust stability analysis

In this section, we study robust stability analysis of LTI systems with uncertain FDGFVs, and assume that the parameters (a_i, b_j) and (α_i, β_j) in $\phi(s)$ defined by (2.2) are unknown, but these parameters belong to the sets:

$$\mathrm{P}_{ab} := \{(a_i, b_j) |\ \underline{a} \le a_i \le \bar{a},\ \underline{b} \le b_j \le \bar{b}, \},$$

and

$$\mathrm{P}_{\alpha\beta} := \{(a_i, b_j) |\ 0 < \underline{\alpha} \le \alpha_i \le \bar{\alpha},\ 0 < \underline{\beta} \le \beta_j \le \bar{\beta}\},$$

where $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$.

The LTI system $\bar{G}(s)$ defined by (2.1) with FDGFV $\phi(s)$ defined by (2.2) is called to be robustly stable if it is BIBO stable for all parameters $(a_i, b_j) \in \mathrm{P}_{ab}$ and $(\alpha_i, \beta_j) \in \mathrm{P}_{\alpha\beta}$ $(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$.

According to Theorem 3.1, the LTI system $\bar{G}(s)$ defined by (2.1) with GFV $\phi(s)$ defined by (2.1) is robustly stable if and only if $p(\lambda, s) = n(s) - \lambda d(s)$ is Hurwitz stable for every $\lambda \in \sigma(A)$ and for all $(a_i, b_j) \in \mathrm{P}_{ab}$ and $(\alpha_i, \beta_j) \in \mathrm{P}_{\alpha\beta}$ $(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$. Thus, the robust stability problem of LTI systems with FDGFVs can be converted into analyzing the robust stability of interval uncertain polynomials $p(\lambda, s) = n(s) - \lambda d(s)$. The Zero Exclusion Principle provided in reference [39] has given a method to determinate the robust stability of a fractional degree polynomial with interval uncertainties. Based on the Zero Exclusion Principle, we easily have the following proposition:

PROPOSITION 4.1. *Given matrix A, $\lambda \in \sigma(A)$, consider the fractional degree polynomial $p(s, \lambda) = n(s) - \lambda d(s)$, where $n(s)$, $d(s)$ are defined in*

(2.2) and parameters $(a_i, b_j) \quad \mathrm{P}_{ab}$ *and* $(\alpha_i, \beta_j) \quad \mathrm{P}_{\alpha\beta}$*. Then* $p(s,\lambda)$ *is robustly stable if and only if* $p(s,\lambda)$ *contains at least one Hurwitz stable polynomial and* $0 \ / \ p(j\omega,\lambda)$ *for* $\ \omega \quad [0,+ \quad)$.

5. An application to fractional-order GRN systems

Consider a fractional-order GRN system described by the following equation

$$\left\{ \begin{array}{lcl} D^{\alpha_1} r_i & = & \quad a_i r_i + d_i u_i \\ D^{\alpha_2} p_i & = & c_i r_i \quad b_i p_i \end{array} \right. , \qquad y_i = [0\ 1] \left[\begin{array}{c} r_i \\ p_i \end{array} \right], \tag{5.1}$$

where r_i and p_i are the concentrations of mRNA and protein associated with the i-th gene $(i = 1, 2, \cdots, n)$, respectively. Parameters $a_i > 0$ and $b_i > 0$ are the degradation rates of the mRNA, and protein, respectively. $c_i > 0$ and $d_i > 0$ are the translation and transcription rates, respectively. $\alpha_1, \alpha_2 \quad (0, 1)$ are fractional-orders, and in this paper we admit α_1 and α_2 to be incommensurate order.

The GRN has a cyclic feedback control, the input variable $u_i(t)$ is modeled as

$$u_i(t) = \left\{ \begin{array}{ll} \xi_i p_n(t), & \text{for } i = 1 \\ \xi_i p_{i-1}(t), & \text{for } i = 2, 3, \cdots, n, \end{array} \right.$$

$u_i(t) > 0$ activates the transcription of a gene, otherwise $u_i(t) < 0$ represses the transcription.

The transfer function from u_i to y_i is

$$g_i(s) = \frac{c_i d_i}{s^{\alpha_1+\alpha_2} + b_i s^{\alpha_1} + a_i s^{\alpha_2} + a_i b_i}. \tag{5.2}$$

The overall GRN system with cyclic activation-repression interconnections is modelled as the feedback control system depicted in Fig. 3, where $u(t) := [u_1(t), u_2(t), \cdots, u_n(t)]^T = Ky(t)$ with

$$K := \left[\begin{array}{ccccc} 0 & 0 & 0 & \cdots & \xi_1 \\ \xi_2 & 0 & 0 & \cdots & 0 \\ 0 & \xi_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_n & 0 \end{array} \right]. \tag{5.3}$$

To facilitate the stability analysis described below, we move the gain $c_i d_i$ into the corresponding feedback gain ξ_i in K. Thus, the overall system can be equivalently transformed into the feedback system shown in Fig. 3 by replacing $g_i(s)$ by $\bar{g}_s$ and K by $\bar{K}$, where

$$\bar{g}_i(s) = \frac{1}{T_{a_i b_i} s^{\alpha_1+\alpha_2} + T_{a_i} s^{\alpha_1} + T_{b_i} s^{\alpha_2} + 1}, \tag{5.4}$$

and

$$\bar{K} := \begin{bmatrix} 0 & 0 & 0 & \cdots & \xi_1 R_1^2 \\ \xi_2 R_2^2 & 0 & 0 & \cdots & 0 \\ 0 & \xi_3 R_3^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_n R_n^2 & 0 \end{bmatrix}, \tag{5.5}$$

where $T_{a_i b_i} = \frac{1}{a_i b_i}$, $T_{a_i} = \frac{1}{a_i}$, $T_{b_i} = \frac{1}{b_i}$, $R_i = \frac{\sqrt{c_i d_i}}{\sqrt{a_i b_i}}$.

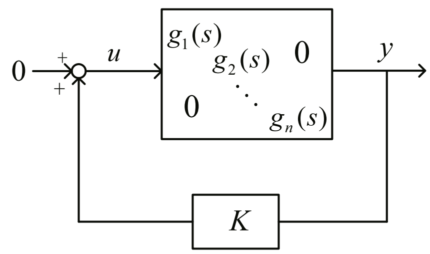


FIGURE 3. The scheme representation of the GRN systems with cyclic interconnection

If $a_1 = a_2 = \cdots = a_n$, $b_1 = b_2 = \cdots = b_n$, $c_1 = c_2 = \cdots = c_n$ and $d_1 = d_2 = \cdots = d_n$, then $\bar{g}_1(s) = \bar{g}_2(s) = \cdots = \bar{g}_n(s) = \bar{g}(s)$, the GRN system belongs to a class of homogenous multi-agent systems. The dynamics of the GRN systems can be represented by

$$H(s) = (\varphi(s)I \quad \bar{K})^{-1}, \quad \varphi(s) = \frac{1}{\bar{g}(s)}, \tag{5.6}$$

where $\bar{g}(s)$ is defined in (5.4), $\bar{K}$ is defined in (5.5).

To this end, we will investigate the stability of the GRN system base on the GFV method.

THEOREM 5.1. *The homogenous GRN system with transfer function defined in (5.6) is BIBO stable if and only if all the eigenvalues of $\bar{K}$ lie on the left part of curve $l_a \bigcup l_b$ in the complex plane, where l_a and l_b are symmetrical with respect to the real axis, and*

$$l_a := \{x + yj | x = x(\omega), y = y(\omega), \ \omega \quad [0, + \quad)\}, \tag{5.7}$$

where

$$x(\omega) = T_{ab}\omega^{\alpha_1+\alpha_2}\cos(\frac{\pi}{2}(\alpha_1+\alpha_2)) + T_a\omega^{\alpha_1}\cos(\frac{\pi}{2}\alpha_1) + T_b\omega^{\alpha_2}\cos(\frac{\pi}{2}\alpha_2) + 1,$$

$$y(\omega) = T_{ab}\omega^{\alpha_1+\alpha_2}\sin(\frac{\pi}{2}(\alpha_1+\alpha_2)) + T_a\omega^{\alpha_1}\sin(\frac{\pi}{2}\alpha_1) + T_b\omega^{\alpha_2}\sin(\frac{\pi}{2}\alpha_2).$$

P r o o f. From Lemma 3.1, we can know that the homogenous GRN system with transfer function $H(s)$ defined in (5.6) is Hurwitz stable if and only if for all $\lambda \quad \sigma(\bar{K})$, $\lambda = \varphi(s)$ when $s \quad \mathbb{C}_+$. It is natural to determine the boundary of $\varphi(s)$ when s lies on the imaginary axis. Then, for $s = j\omega$, $0 \quad \omega < + \quad$, we have

$$\varphi(j\omega) = x(\omega) + jy(\omega), \tag{5.8}$$

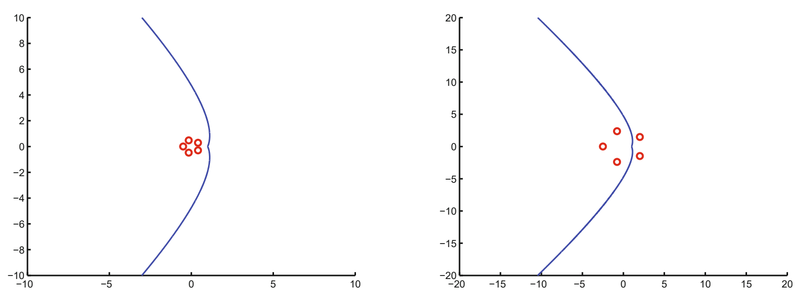
for $s = j\omega, \qquad < \omega < 0$, we have

$$\varphi(j\omega) = x(\omega) \quad jy(\omega), \tag{5.9}$$

where $x(\omega)$ and $y(\omega)$ are defined in (5.7). Therefore the stability region of system (5.6) is the left part of curve $l_a \bigcup l_b$ in the complex plane. □

Example 5.1. Assume that the number of gene is $n = 5$ and the parameters (a, b, c, d) and (α_1, α_2) are given as $a = 3$, $b = 2$, $c = 2$, $d = 1.5$ and $\alpha_1 = \frac{\sqrt{2}}{2}$, $\alpha_2 = \frac{\sqrt{3}}{2}$, respectively. The nominal dynamics of each gene is given as

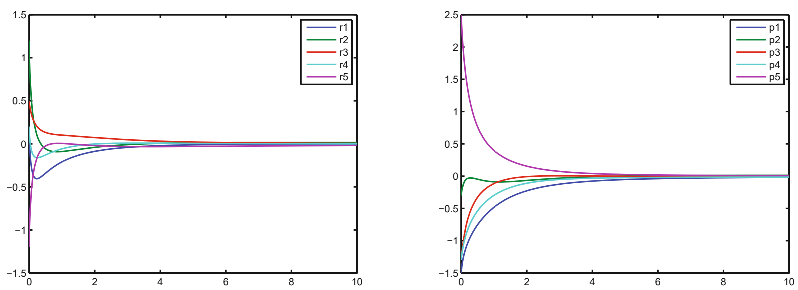
$$\bar{g}(s) = \frac{1}{\frac{1}{6}s^{\frac{\sqrt{2}+\sqrt{3}}{2}} + \frac{1}{2}s^{\frac{\sqrt{2}}{2}} + \frac{1}{2}s^{\frac{\sqrt{3}}{2}} + 1}, \quad R^2 = \frac{1}{2}. \tag{5.10}$$



(A) (B)

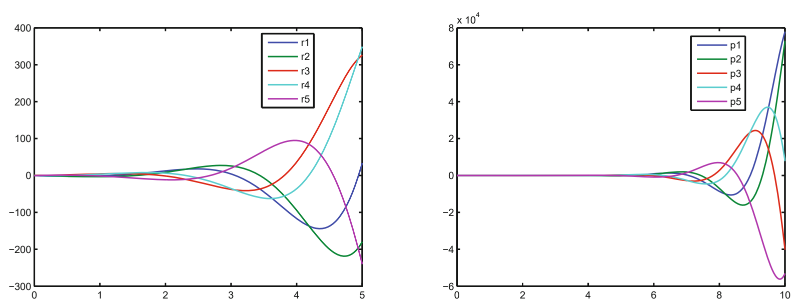
Figure 4. The domain $\;^c_+$ and the position of eigenvalues of $\bar{K}$.

Thus $T_a = \frac{1}{3}$, $T_b = \frac{1}{2}$, $T_{ab} = \frac{1}{6}$. The values of ξ_i are given as $\xi_i = 1 (i = 2, 4)$, $\xi_j = \quad 1 (j = 1, 3, 5)$. One can compute the eigenvalues of $\bar{K}$ are $\lambda_i = \frac{1}{2}e^{\frac{j(2i-1)\pi}{5}}$, $(i = 1, 2, \cdots, 5)$, which are labeled by o in Fig. 4(a). This figure demonstrates that all the eigenvalues of $\bar{K}$ lie in the stability region $\;^c_+$ characterized by GFV $\phi(s) = \frac{1}{\bar{g}(s)}$. Thus it follows from Theorem 5.1 that the homogenous GRN system with cyclic gain matrix $\bar{K}$ is stable. Given the initial conditions $r_1(0) = 0.2$, $p_1(0) = \quad 1.5$, $r_2(0) = 1.2$, $p_2(0) = \quad 0.3$, $r_3(0) = 0.5$, $p_3(0) = \quad 1.3$, $r_4(0) = 0.2$, $p_4(0) = \quad 1.3$, $r_5(0) = \quad 1.2$, $p_5(0) = 2.5$, the state responses of mRNAs r_i and proteins p_i, $(i = 1, 2, \cdots, 5)$ are depicted in Fig. 5(a) and Fig. 5(b), respectively. From these figures, we can deduce that the GRN system is stable, which is coincided with theoretical analysis results.



(A) (B)

FIGURE 5. The stable state responses of mRNAs and proteins.



(A) (B)

FIGURE 6. The unstable state responses of mRNAs and proteins.

Otherwise, let we update $c = 3$ and $d = 5$ and the other parameters remain the same, then we can computer $R^2 = \frac{2}{5}$ and the eigenvalues of $\bar{K}$ are $\lambda_i = \frac{2}{5}e^{\frac{j(2i-1)\pi}{5}}$, $(i = 1, 2, \cdots, 5)$, which are labeled by o in Fig. 4(b). This figure demonstrate that matrix $\bar{K}$ has two eigenvalues that do not lie in the stability region $\ ^{c}_{+}$. Thus it follows from Theorem 5.1

that the homogenous GRN system with cyclic gain matrix $\bar{K}$ is unstable. Meanwhile, given above initial conditions, the state responses of mRNAs r_i and proteins p_i, $(i = 1, 2, \cdots, 5)$ are depicted in Fig. 6(a) and Fig. 6(b), respectively. From these figures, we can deduce that the GRN system is unstable, which verifies the theoretical analysis results.

In the following, we will investigate the robust stability of homogenous GRN system with transfer function

$$\bar{H}(s) = (\phi(s)I \quad K)^{-1}, \;\; \phi(s) = \frac{1}{g(s)}, \tag{5.11}$$

where

$$g(s) = \frac{cd}{s^{\alpha_1+\alpha_2} + bs^{\alpha_1} + as^{\alpha_2} + ab}. \tag{5.12}$$

and K is defined in (5.5). We assume that parameters (a, b, c, d) and (α_1, α_2) are unknown, but they belong to the following set:

$$\begin{aligned} \mathrm{P}_{abcd} := \{(a, b, c, d)|\ 0 \quad \underline{a} \quad a_i \quad \bar{a}, 0 \quad \underline{b} \quad b_i \quad \bar{b}, \\ 0 \quad \underline{c} \quad c_i \quad \bar{c}, 0 \quad \underline{d} \quad d_i \quad \bar{d}\}, \end{aligned}$$

and

$$\mathrm{P}_{\alpha_1\alpha_2} := \{(\alpha_1, \alpha_2)|\ 0 \quad \underline{\alpha}_1 \quad \alpha_1 \quad \bar{\alpha}_1 \quad 1, 0 \quad \underline{\alpha}_2 \quad \alpha_2 \quad \bar{\alpha}_2 \quad 1\}.$$

Based on Theorem 3.1, we can get the following propositions:

PROPOSITION 5.1. *The homogenous GRN system with transfer function $\bar{H}(s)$ defined in (5.11) is robustly stable if and only if for all $\lambda \quad \sigma(K)$, the fractional degree polynomial*

$$p(\lambda, s) = s^{\alpha_1+\alpha_2} + bs^{\alpha_1} + as^{\alpha_2} + ab \quad \lambda cd \tag{5.13}$$

is robustly stable for all $(a, b, c, d) \quad \mathrm{P}_{abcd}$ and $(\alpha_1, \alpha_2) \quad \mathrm{P}_{\alpha_1\alpha_2}$.

P r o o f. Let $n(s) = s^{\alpha_1+\alpha_2} + bs^{\alpha_1} + as^{\alpha_2} + ab$, $d(s) = cd$ in (3.7), Theorem 3.1 implies that this proposition holds. □

PROPOSITION 5.2. *The fractional degree polynomial*

$$p(\lambda, s) = s^{\alpha_1+\alpha_2} + bs^{\alpha_1} + as^{\alpha_2} + ab \quad \lambda cd$$

is robustly stable for all $(a, b, c, d) \quad \mathrm{P}_{abcd}$ and $(\alpha_1, \alpha_2) \quad \mathrm{P}_{\alpha_1\alpha_2}$, if and only if $p(\lambda, s)$ contains at least one Hurwitz stable polynomial and

$$\;_1(j\omega) = \;_2(\lambda, j\omega),$$

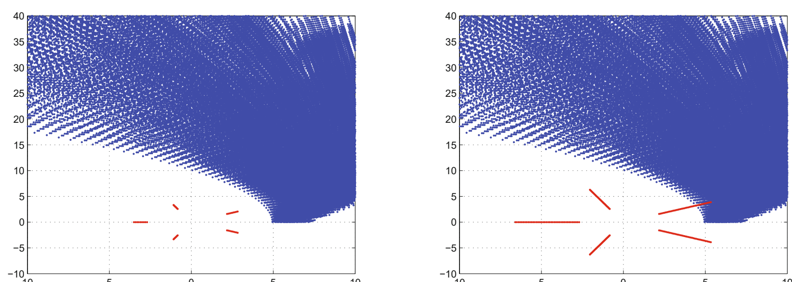
for all $\omega \quad [0, + \quad)$, *where*

$$
\begin{aligned}
&{}_1(j\omega) = (j\omega)^{\alpha_1+\alpha_2} + b(j\omega)^{\alpha_1} + a(j\omega)^{\alpha_2} + ab,\\
&{}_2(\lambda, j\omega) = \lambda cd.
\end{aligned}
$$

P r o o f. It follows from Proposition 4.1 that $p(\lambda, s)$ is robustly stable for all $(a, b, c, d) \quad \mathrm{P}_{abcd}$ and $(\alpha_1, \alpha_2) \quad \mathrm{P}_{\alpha_1,\alpha_2}$ if and only if $p(\lambda, s)$ contains at least one Hurwitz stable polynomial and $0 \ / \ p(\lambda, j\omega)$ for all $\omega \quad [0, + \quad)$, which is equivalent to that $\quad {}_1(j\omega) = \quad {}_2(\lambda, j\omega)$ for all $\omega \quad [0, + \quad)$. □

Example 5.2. Assume that the number of genes is $n = 5$, and the parameters (a, b, c, d) satisfy $\underline{a} = 2.75 \quad a \quad \bar{a} = 3.25$, $\underline{b} = 1.80 \quad b \quad \bar{b} = 2.20$, $\underline{c} = 1.90 \quad c \quad \bar{c} = 2.20$, $\underline{d} = 1.42 \quad d \quad \bar{d} = 1.63$. The fractional degree parameters (α_1, α_2) satisfy $\underline{\alpha}_1 = 0.5 \quad \alpha_1 \quad \bar{\alpha}_1 = 1$, $\underline{\alpha}_2 = 0.5 \quad \alpha_2 \quad \bar{\alpha}_2 = 1$. The control gains are given as $\xi_i = 1$ $(i = 2, 4)$, $\xi_j = \quad 1$ $(j = 1, 3, 5)$, thus one can calculate the eigenvalues of K as $\lambda_i = e^{\frac{j(2i-1)\pi}{5}}$, $(i = 1, 2, \cdots, 5)$.

From Example 5.1, we can know that the polynomial $p(\lambda, s)$ defined in (5.13) is stable when the parameters (a, b, c, d) and (α_1, α_2) are given as $a = 3$, $b = 2$, $c = 2$, $d = 1.5$ and $\alpha_1 = \frac{\sqrt{2}}{2}$, $\alpha_2 = \frac{\sqrt{3}}{2}$, respectively. Thus $p(\lambda, s)$ defined in (5.13) contains at least one Hurwitz stable polynomial. $\quad {}_1(j\omega)$ depicted by blue lines and $\quad {}_2(\lambda, j\omega)$ depicted by red lines are shown in Fig. 7(a), respectively. From this figure, we can see that $\quad {}_1(j\omega) = \quad {}_2(\lambda, j\omega)$ for all $\omega \quad [0, + \quad)$. It follows from Propositions 5.1 and 5.2 that the GRN system is robustly stable.



(A) (B)

Figure 7. The unstable state responses of mRNAs and proteins.

Next, assume $\underline{d} = 1.42 \quad d \quad \bar{d} = 3$ and other parameters are kept the same, $\quad {}_1(j\omega)$ depicted by blue lines and $\quad {}_2(j\omega)$ depicted by red lines are shown in Fig. 7(b), respectively. This figure demonstrates that the red lines are contained in the blue lines, that is, there are some $\omega \quad [0, \quad)$ such that $_1(j\omega) = \quad _2(\lambda, j\omega)$. Thus Propositions 5.1 and 5.2 imply that the GRN system is not robustly stable.

6. **Conclusions**

In this paper, we generalized the GFVs to fractional degree rational functions. The stability and robust stability problems were investigated for LTI systems with fractional degree GFVs. We provided necessary and sufficient conditions to ensure the stability and robust stability of LTI systems with fractional degree GFVs. Finally, the effectiveness of methods proposed in this paper is verified by analyzing the stability and robust stability of incommensurate fractional-order gene regulatory network systems. The tool of stability analysis adopted in this paper is mainly discriminating the roots distributions of fractional degree polynomial. However, it is still a hard and complex task especially for the case that fractional degree polynomial is with incommensurate order. Therefore, the future research is to find a simpler tool for checking the stability and robust stability of LTI systems with FDGFVs.

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References

[1] B.B. Alagoz, Hurwitz stability analysis of fractional order LTI systems according to principal characteristic equations. *ISA Trans.* **70** (2017), 7–15; DOI: 10.1016/j.isatra.2017.06.005.

[2] C. Bonnet, J. Partington, Coprime factorizations and stability of fractional differential systems. *Syst. Control Lett.* **41**, No 3 (2000), 167–174; DOI: 10.1016/S0167-6911(00)00050-5.

[3] C. Bonnet, J. Partington, Analysis of fractional delay systems of retarded and neutral type. *Automatica* **38**, No 7 (2002), 1133–1138; DOI: 10.1016/S0005-1098(01)00306-5.

[4] L. Chen, K. Aihara, Stability of genetic regulatory networks with time delay. *IEEE Trans. Circuits-I.* **49**, No 5 (2002), 602–608; DOI: 10.1109/TCSI.2002.1001949.

[5] Z. Gao, Analytical criterion on stabilization of fractional-order plants with interval uncertainties using fractional-order $PD\mu$ controllers with a filter. *ISA Trans.* **83** (2018), 25–34; DOI: 10.1016/j.isatra.2018.09.004.

[6] S. Hara, T. Hayakawa, and H. Sugata, Stability analysis of linear systems with generalized frequency variables and its applications to formation control. *46th IEEE Conf. on Decision and Control*, Dec 12–14, (2007), New Orleans, LA, USA, 1459–1466; DOI: 10.1109/CDC.2007.4435000.

[7] S. Hara, T. Hayakawa, and H. Sugata, LTI systems with generalized frequency variables: A unified framework for homogeneous multi-agent dynamical systems. *SICE J. Control, Measurement, and System Integration* **2**, No 5 (2009), 299–306; DOI: 10.9746/jcmsi.2.299.

[8] S. Hara, T. Iwasaki, Y. Hori, Robust stability analysis for LTI systems with generalized frequency variables and its application to gene regulatory networks. *Automatica* **105** (2019), 96–106; DOI: 10.1016/j.automatica.2019.03.019.

[9] S. Hara, M. Kanno, and H. Tanaka, Cooperative gain output feedback stabilization for multi-agent dynamical systems. *48th IEEE Conf. on Decision and Control held jointly with the 28th Chinese Control Conf.*, Dec 15-18, (2009), Shanghai, China, 877–882; DOI: 10.1109/CDC.2009.5399889.

[10] S. Hara, H. Tanaka, and T. Iwasaki, Stability analysis of systems with generalized frequency variables. *IEEE Trans. Automat. Contr.* **59**, No 2 (2014), 313–326; DOI: 10.1109/TAC.2013.2281482.

[11] Y. Hori, H. Miyazako, S. Kumagai, and S. Hara, Coordinated spatial pattern formation in biomolecular communication networks. *IEEE Trans. Molecular, Biological and Multi-Scale Communications* **1**, No 2 (2015), 111–121; DOI: 10.1109/TMBMC.2015.2500567.

[12] C. Huang, J. Cao, M. Xiao, Hybrid control on bifurcation for a delayed fractional gene regulatory network. *Chaos Soliton. Fract.* **87** (2016), 19–29; DOI: 10.1016/j.chaos.2016.02.036.

[13] R. Ji, L. Ding, X. Yan and M. Xin, Modelling gene regulatory network by fractional order differential equations. *IEEE 5th Internat. Conf. on Bio-Inspired Computing: Theories and Applications (BIC-TA)*, Sept 23-26, (2010), Changsha, China, 431–434; DOI: 10.1109/BICTA.2010.5645163.

[14] Z. Jiao and Y.Q. Chen, Stability analysis of fractional-order systems with double noncommensurate orders for matrix case. *Fract. Calc. Appl. Anal.* **14**, No 3 (2011), 436–453; DOI: 10.2478/s13540-011-0027-3; https://www.degruyter.com/view/j/fca.2011.14.issue-3/

issue-files/fca.2011.14.issue-3.xml.

[15] Z. Jiao and Y.Q. Chen, Stability of fractional-order linear time-invariant systems with multiple noncommensurate orders. *Comput. Math. Appl.* **64**, No 10 (2012), 3053–3058; DOI: 10.1016/j.camwa.2011.10.014.

[16] Z. Jiao, Y. Chen, and I. Podlubny, *Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives.* Springer, London (2013).

[17] T.H. Kim, Y. Hori, and S. Hara, Robust stability analysis of gene-protein regulatory networks with cyclic activation-repression interconnections. *Syst. Control Lett.* **60**, No 6 (2011), 373–382; DOI: 10.1016/j.sysconle.2011.03.003.

[18] C. Li, Q. Yi, Finite difference method for two-dimensional nonlinear time-fractional subdiffusion equation. *Fract. Calc. Appl. Anal.* **21**, No 4 (2018), 1046–1472; DOI: 10.1515/fca-2018-0057; https://www.degruyter.com/view/j/fca.2018.21.issue-4/issue-files/fca.2018.21.issue-4.xml.

[19] S. Liang, S.G. Wang, Y. Wang, Routh-type table test for zero distribution of polynomials with commensurate fractional and integer degrees. *J. Franklin. Inst.* **354** No 1 (2017), 83–104; DOI: 10.1016/j.jfranklin.2016.08.019.

[20] G. Ling, Z. Guan, R. Liao, and X. Cheng, Stability and bifurcation analysis of cyclic genetic regulatory networks with mixed time delays. *SIAM J. Appl. Dyn. Syst.* **14**, No 1 (2015), 202–220; DOI: 10.1137/140954131.

[21] D. Matignon, Stability properties for generalized fractional differential systems. In: *ESAIM: Proceedings and Surveys* **5** (1998), 145–158; DOI: 10.1051/proc:1998004.

[22] R. Mohsenipour, J.M. Fathi, Robust stability analysis of fractional-order interval systems with multiple time delays. *Int. J. Robust Nonlin. Control* **29** No 6 (2019), 1823–1839; DOI: 10.1002/rnc.4461.

[23] C.A. Monje, Y.Q. Chen, B.M. Vinagre, D. Xue and V. Feliu-Batlle, *Fractional-order Systems and Controls: Fundamentals and Applications.* Springer Science & Business Media, London (2010).

[24] I. Petras, Stability of fractional order systems with rational orders: A survey. *Fract. Calc. Appl. Anal.* **12**, No 3 (2009), 269–298.

[25] I. Petras, L. Dorcak, The frequency method for stability investigation of fractional control systems. *Journal of SACTA* **2**, No. 1–2 (1999), 75–85.

[26] I. Petras, Y. Q. Chen and B. M. Vinagre, *Robust Stability Test for Interval Fractional Order Linear Systems, Unsolved Problems in Mathematics and Control Systems*. Princeton University Press, USA, (2004).

[27] I. Podlubny, *Fractional Di erential Equations: an Introduction to Fractional Derivatives, Fractional Di erential Equations, Methods of Their Solution and Some of Their Applications*. **198**, Academic Press, San Diego (1999).

[28] F. Ren, F. Cao, J. Cao, Mittag-Le er stability and generalized Mittag-Le er stability of fractional-order gene regulatory networks. *Neurocomputing* **160** (2015), 185–90; DOI: 10.1016/j.neucom.2015.02.049.

[29] J. Sabatier, C. Farges, J.C. Trigeassou, A stability test for non-commensurate fractional order systems. *Syst. Control Lett.* **62**, No 9 (2013), 739–746; DOI: 10.1016/j.sysconle.2013.04.008.

[30] J. Sabatier, M. Moze, and C. Farges, LMI stability conditions for fractional order systems. *Comput. Math. Appl.* **59**, No 5 (2010), 1594–1609; DOI: 10.1016/j.camwa.2009.08.003.

[31] H. Taghavian and M.S. Tavazoei, Robust stability analysis of uncertain multiorder fractional systems: Young and Jensen inequalities approach. *Int. J. Robust Nonlin. Control* **28**, No 4 (2017), 1127–1144; DOI: 10.1002/rnc.3919.

[32] C.D. Thron, The secant condition for instability in biochemical feedback control–I. The role of cooperativity and saturability. *Bull. Math. Biol.* **53**, No 3 (1991), 383–401; DOI: 10.1007/BF02460724.

[33] M.E. Valcher and P. Misra, On the controllability and stabilizability of non-homogeneous multi-agent dynamical systems. *Syst. Control Lett.* **61**, No 7 (2012), 780–787; DOI: 10.1016/j.sysconle.2012.04.006.

[34] M.E. Valcher and P. Misra, On the stabilizability and consensus of positive homogeneous multi-agent dynamical systems. *IEEE Trans. Automat. Contr.* **59**, No 7 (2014), 1936–1941; DOI: 10.1109/TAC.2013.2294621.

[35] Y. Wang, H. Fujimoto, and S. Hara, Torque distribution-based range extension control system for longitudinal motion of electric vehicles by LTI modeling with generalized frequency variable. *IEEE/ASME Trans. Mechatron.* **21**, No 1 (2016), 443–452; DOI: 10.1109/TMECH.2015.2444651.

[36] Z. Wang, H. Gao, J. Cao and X. Liu, On delayed genetic regulatory networks with polytopic uncertainties: robust stability analysis. *IEEE Trans. Nanobiosci.* **7**, No 2 (2008), 154–163; DOI: 10.1109/TNB.2008.2000746.

[37] C. Wang, Y. Zhao and Y.Q. Chen, The controllability, observability, and stability analysis of a class of composite systems with fractional

degree generalized frequency variables. *IEEE/CAA J. Automat. Sin.* **6**, No 3 (2019), 859–864; DOI: 10.1109/JAS.2019.1911501

[38] Y. Wei, Y.Q. Chen, S. Cheng and Y. Wang, A note on short memory principle of fractional calculus. *Frac. Calc. Appl. Anal.* **20**, No 6 (2017), 1382–1404; DOI: 10.1515/fca-2017-0073; https://www.degruyter.com/view/j/fca.2017.20.issue-6/issue-files/fca.2017.20.issue-6.xml.

[39] S. Zheng, Robust stability of fractional order system with general interval uncertainties. *Syst. Control Lett.* **99** (2017), 1–8, DOI: 10.1016/j.sysconle.2016.11.001.

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