Received: 24 September 2018 | Revised: 30 March 2019 | Accepted: 7 May 2019
DOI: 10.1002/asjc.2193

BRIEF PAPER

WILEY

Stabilization of uncertain fractional order system with time-varying delay using BMI approach

Bin-Bin He[1] | **Hua-Cheng Zhou[2]** | **Chun-Hai Kou[3]** | **YangQuan Chen[4]**

[1]College of Urban Railway Transportation, Shanghai University of Engineering Science, Shanghai, China
[2]School of Mathematics and Statistics, Central South University, Changsha, China
[3]Department of Applied Mathematics, Donghua University, Shanghai, China
[4]Mechatronics, Embedded Systems and Automation Lab, University of California, Merced, CA, USA

Correspondence
Hua-Cheng Zhou, School of Mathematics and Statistics, Central South University, Changsha, 410075, China.
Email: hczhou@amss.ac.cn

Funding information
Fundamental Research Funds for the Central Universities, Grant/Award Number: CUSF-DH- D-2017083; the National Natural Science Foundation of China, Grant/Award Number: 61803386.

Abstract
This paper considers the systematic design of robust stabilizing state feedback controllers for fractional order nonlinear systems with time-varying delay being possibly unbounded. By using the fractional Halanay inequality and the Caputo fractional derivative of a quadratic function, stabilizability conditions expressed in terms of bilinear matrix inequalities are derived. The controllers can then be obtained by computing the gain matrices. In order to derive the gain matrices, two algorithms are proposed by using the existing computationally linear matrix inequality techniques. Two numerical examples with simulation results are provided to demonstrate the effectiveness of the obtained results.

KEYWORDS
bilinear matrix inequality, fractional Halanay inequality, fractional order system, stability, time-varying delay

1 | INTRODUCTION

Fractional calculus as a hot spot is recently widely used in control theory, viscoelastic systems, engineering, and some interdisciplinary fields although it is considered as a sole mathematical branch with few applications for a long time. Fractional order control theory is first proposed by A. Oustloup in 1991 [1] named as CRONE robust controller. For fractional order control theory, it usually means that the system is a fractional order system or the controller is a fractional order controller. In 1999, when I. Podlubny introduced the $PI^{\lambda}D^{\mu}$ controller [2], fractional order control theory achieves a great continuous development in the past two decades. Since fractional calculus possesses the property of nonlocal and many process with memory and heredity can be well modeled by fractional order systems, it catches more and more researchers' attention and many interesting results of the above issues have been reported in the literature [2–5]. Fractional damping, fractional oscillators and quenching phenomenon characterized by fractional system can also be found in recent works [6–9].

Stability and stabilization are two of the fundamental issues in control theory. Since the seminal paper of D. Matignon in 1996 [10] where the stability condition for linear fractional order system is presented, the problems of stability and stabilization have received a huge interest from the control community. Many interesting results about the stability/stabilization of fractional order systems are obtained, see [11,12,14,18]. In [11], sufficient conditions of the asymptotic stability and output feedback stabilizing controller for fractional order linear systems with positive real uncertainty are presented. In [12], a general stability condition for fractional-order switched non-linear systems is established by using the concepts of Mittag-Leffler increment and average Mittag-Leffler increment. In [13], the stability and L^{∞}-gain analysis problem for continuous-time fractional-order systems with bounded time-varying delays are discussed to prove the asymptotic stability of positive not being sensitive to the magnitude of delays and L^{∞}-gain being independent of the magnitude of delays. In [14], the Riesz basis approach and the semigroup method are respectively exploited to Mittag-Leffler stabilize the unstable time fractional reaction diffusion equation. In [15], the active disturbance rejection control (ADRC) is developed to design a stabilizing controller for time fractional reaction diffusion equation with the general disturbance. The ADRC for wave and beam equations can also be found in [16,17]. In [18], the bounded-input bounded-output stability of two types of distributed-order systems is analyzed. To get the stability conditions, a large number of researchers have developed the different approaches for resolving the issues, such as Laplace transform method [19], linear matrix inequality (LMI) approach [11], Lyapunov method [20], and so on. For the stability of fractional order system with delay, Razumikhin method is explored in [21,22]. In [23], a comparison theorem for a class of fractional-order systems with time delay is proved to investigate the global asymptotic stability conditions of fractional-order Hopfield neural networks with time delay. In [24], the asymptotic stability for fractional nonlinear systems without and with unbounded delays is presented by calculating integer-order derivative of the Lyapunov function, which is essentially to apply the classic Lyapunov direct method not fractional Lyapunov method. In [25] and [26], the authors present the fractional type Halanay inequality with bounded delay and unbounded delay respectively, which provides the effective tools to analyze the stability of fractional order system with time delay. For the mixed delays of neural networks, the reader can refer to [27,28].

LMI is a convex constraint which is widely used in many control problems. For examples, in [29], the stabilizing control law of fractional order nonlinear system is obtained by using the state feedback control and the Lyapunov direct method, and the stabilizability conditions are expressed in terms of LMI. In [30], some simplified LMI stability conditions is developed to globally stabilize the fractional-order linear and nonlinear systems and asymptotic stability of the fractional-order neural networks is ensured. In [31], the fractional-order uncertain systems with the order $0 < \alpha < 1$ and $1 \leq \alpha < 2$ is robustly asymptotically stabilized employing LMI, the method of observer-based control and static output feedback control and the existence condition of a robust stabilizing controller is discussed. More results concerning LMI approach to fractional systems can be found in [32–36] and references therein.

A bilinear matrix inequality (BMI) is usually of the form

$$F(x, y) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j F_{ij} < 0$$

where $x = (x_1, x_2, \ldots, x_n)^{\top} \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_m)^{\top} \in \mathbb{R}^m$, $F_{ij} = F_{ij}^{\top}$ is a symmetric matrix with suitable dimension. It can be seen that the BMI is a LMI in x for fixed y and is a LMI in y for fixed x, so is convex in x or y individual but not jointly. Although BMI is much more difficult to handle computationally than LMI, BMI is still an effective tool for those control problems that cannot be written in terms of an LMI but can be written in terms of BMI. It can describe much wider classes of problems than LMI since the problems described by BMI are not necessarily convex. For the more difference and connections between LMI and BMI, the reader can refer to [37–39].

In practice engineering application, the uncertainties are widely exist in real control systems, like interval uncertainty [40,41], unknown external disturbances [32] and positive real uncertainty [11]. If the uncertainties are not well-handled, some unexpected performances, like unstable and uncontrollable, may occur. Therefore, in order to avoid this phenomenon happens, it is worthy studying the control design of the robust controller to copy with the uncertainties .

Two basic control problems to be solved of this paper are: First, for a class of given fractional order system with uncertainties, how to verify whether the system is stabilizable or not? Second, for a stabilizable system, how to construct a stabilizing control law? With these problems in mind, in this paper, we present a BMI method for fractional order system with time-varying structured uncertainties and time delay being possibly unbounded.

The main contributions of this work are summarized as follows: (i) By using the fractional type Halanay inequality and an important fractional derivative inequality of quadratic function, we derive a new stabilization criteria in terms of BMIs, which gives the design of stabilizing state feedback controllers. These controllers can be solved by using the existing computationally effective algorithms. (ii) The proposed BMI-based criteria are quite general since many factors, such as time-varying delay that can either be bounded or be unbounded, time-varying uncertainties, nonlinear Lipschitz functions, are considered. (iii) The derived results are easily checkable and two numerical examples are presented to confirm this.

The rest of the paper is organized as follows: Section 2 presents some basic definitions and lemmas that will be used in the follows. Section 1 is the main result of the paper. Two examples are given in Section and Section 5 is a conclusion about this paper..

2 | PRELIMINARY AND PROBLEM STATEMENT

There are some definitions of fractional derivatives, among many others, we know that the fractional derivative in Caputo sense than in other fractional derivatives is more applicable to real-world problems and well understood in physical situations due to its consistency with the derivative of constants and the initial state of the integer order. In this paper, we adopt the Caputo fractional derivative. Some important definitions and useful lemmas are given in order.

Definition 1 ([2], Page 79). The Caputo's fractional derivative of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\theta)}{(t-\theta)^{\alpha-m+1}} d\theta, \ t > 0 $$

with $m = \min\{k \in \mathbb{N} : k > \alpha > 0\}$, where $f^{(m)}(t)$ is the m-order derivative of $f(t)$, and $\Gamma(\cdot)$ is the Gamma function defined as $\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau} d\tau$.

In particular, when $0 < \alpha < 1$, we have

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\theta)}{(t-\theta)^{\alpha}} d\theta. $$

Definition 2 ([2], Page 16). The one-parameter Mittag-Leffler function and two-parameter Mittag-Leffler function are defined by

$$ E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + 1)} $$

and

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} $$

respectively, where $\alpha > 0, \beta > 0$.

Especially, from the above definition, we have that $E_{\alpha,1}(z) = E_\alpha(z)$ and $E_1(z) = E_{1,1}(z) = e^z$.

As we all know, the well-known Leibniz chain rule is invalid for fractional order derivative. Luckily, the following lemma make it possible to estimate the Caputo fractional derivative of quadratic function, which plays a vital role in applying fractional Lyapunov method and fractional Halanay inequality to derive the stability conditions for fractional system with or without time delay.

Lemma 1 ([42]). *Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable functions. Then, for any $t \geq 0$, the following relationship holds*

$$ {}_0^C D_t^\alpha (x^\top(t) P x(t)) \leq 2x^\top(t) P {}_0^C D_t^\alpha x(t), \forall \alpha \in (0,1), $$

where $P \in \mathbb{R}^{n\times n}$ is a constant, square, symmetric and positive define matrix.

Lemma 2 ([26]). *Let $\alpha \in (0,1)$ and $V : [-h, +\infty) \to \mathbb{R}^+$ be bounded on $[-h, 0]$ and continuous on $[0, +\infty)$. Suppose that $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies $\tau(t) \leq t + h$ for some fixed $h > 0$, $t - \tau(t) \to +\infty$ as $t \to +\infty$. For some positive constants $\lambda > \mu > 0$, the following inequality holds: for all $t \geq 0$,*

$$ {}_0^C D_t^\alpha V(t) \leq -\lambda V(t) + \mu \sup_{-\tau(t)\leq\sigma\leq 0} V(t+\sigma). \quad (1) $$

Then $\lim_{t\to+\infty} V(t) = 0.$

Lemma 3 ([43,44], Page 38, Lemma 2.8). *(Schur complement lemma)Let the partitioned matrix*

$$ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{pmatrix} \quad (2) $$

be symmetric. Then

$$ \begin{aligned} A < 0 &\Longleftrightarrow A_{11} < 0, A_{22} - A_{12}^\top A_{11}^{-1} A_{12} < 0 \\ &\Longleftrightarrow A_{22} < 0, A_{11} - A_{12} A_{22}^{-1} A_{12}^\top < 0. \end{aligned} \quad (3) $$

Lemma 4 ([44], Page 30, Lemma 2.3). *Let $X \in \mathbb{R}^{m\times n}, Y \in \mathbb{R}^{n\times m}, F \in \mathbb{R}^{n\times n}$ with $F^\top F \leq I$. Then for arbitrary scalar $\delta > 0$, arbitrary nonzero vectors x and y, there holds*

$$ 2x^\top XFYy \leq \delta x^\top XX^\top x + \delta^{-1} y^\top Y^\top Y y. $$

Consider the following fractional order system with time-varying structured uncertainties and time delay:

$$ \begin{cases} {}_0^C D_t^\alpha x(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]u(t) \\ \qquad\qquad +[A_1 + \Delta A_1(t)]x(t-\tau(t)) \\ \qquad\qquad +f(x(t)) + g(x(t-\tau(t))), \\ x(\theta) = \phi(\theta), \theta \in [-h, 0] \end{cases} \quad (4) $$

where $0 < \alpha < 1$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are two continuous functions and there exist two constants $L_1, L_2 > 0$ such that for all $x, y \in \mathbb{R}^n$:

$$\|f(x) - f(y)\|_{\mathbb{R}^n} \leq L_1 \|x - y\|_{\mathbb{R}^n}, \tag{5}$$

and

$$\|g(x) - g(y)\|_{\mathbb{R}^n} \leq L_2 \|x - y\|_{\mathbb{R}^n}. \tag{6}$$

$\Delta A(t), \Delta A_1(t), \Delta B(t)$ represent time-varying structured uncertainties which are assumed to be of form $\Delta A(t) = G_a F_a(t) H_a$, $\Delta A_1(t) = G_{a_1} F_{a_1}(t) H_{a_1}$, $\Delta B(t) = G_b F_b(t) H_b$ where $A, A_1, B, G_a, G_{a_1}, G_b, H_a, H_{a_1}, H_b$ are known real constant matrices of appropriate dimensions and $F_a(t), F_{a_1}(t), F_b(t)$ are unknown real time-varying matrices satisfying $F_a^\top(t)F_a(t) \leq I$, $F_{a_1}^\top(t)F_{a_1}(t) \leq I$ and $F_b^\top(t)F_b(t) \leq I$. $\tau(t) \in C(\mathbb{R})$ represents the time delay with $\tau(t) \geq -h$ for all $t \geq 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$. It is worth to note that the condition that $t - \tau(t) \to \infty$ as $t \to \infty$ implies that the time delay $\tau(t)$ can be unbounded, for example, $\tau(t) = t/2 + \sin^2(t)$.

The objective of this paper is to design a state feedback controller

$$u(t) = Kx(t) + K_1 x(t - \tau(t))$$

such that the following closed-loop system

$$\begin{cases} {}_0^C D_t^\alpha x(t) = [A + G_a F_a(t) H_a + BK \\ \qquad +G_b F_b(t) H_b K] x(t) + f(x(t)) \\ \qquad +[A_1 + G_{a_1} F_{a_1}(t) H_{a_1} + BK_1 \\ \qquad +G_b F_b(t) H_b K_1] x(t - \tau(t)) \\ \qquad +g(x(t - \tau(t))), \\ x(\theta) = \phi(\theta), \theta \in [-h, 0] \end{cases} \tag{7}$$

is asymptotically stable while rejecting the time-varying structured uncertainties.

3 | MAIN RESULTS

Theorem 1. *The closed-loop system 7 is asymptotically stable if there exist a symmetric positive definite matrix P, matrices K, K_1 with appropriate dimensions, six positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$ and two constants $\lambda > \mu > 0$, such that BMI (8) holds,*

$$\begin{pmatrix} -\varepsilon_3 I & 0 & 0 & 0 & H_{a_1}P & 0 & 0 & 0 \\ 0 & -\varepsilon_4 I & 0 & 0 & H_b K_1 P & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon_6 I & 0 & L_2 P & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{11} & A_1 P + BK_1 P & PH_a^\top & PK^\top H_b^\top & L_1 P \\ PH_{a_1}^\top & PK_1^\top H_b^\top & L_2 P & PA_1^\top + PK_1^\top B^\top & -\mu P & 0 & 0 & 0 \\ 0 & 0 & 0 & H_a P & 0 & -\varepsilon_1 I & 0 & 0 \\ 0 & 0 & 0 & H_b KP & 0 & 0 & -\varepsilon_2 I & 0 \\ 0 & 0 & 0 & L_1 P & 0 & 0 & 0 & -\varepsilon_5 I \end{pmatrix} \leq 0, \tag{8}$$

where

$$\begin{aligned} M_{11} = {} & AP + PA^\top + BKP + PK^\top B^\top \\ & + \varepsilon_1 G_a G_a^\top + (\varepsilon_2 + \varepsilon_4) G_b G_b^\top \\ & + \varepsilon_3 G_{a_1} G_{a_1}^\top + (\varepsilon_5 + \varepsilon_6) I + \lambda P, \end{aligned} \tag{9}$$

Moreover, the state feedback control law is designed by letting

$$u(t) = Kx(t) + K_1 x(t - \tau(t)). \tag{10}$$

Proof. Since P is a symmetric positive definite matrix, so does P^{-1}. We construct a Lyapunov function candidate as $V(t) = x^\top(t) P^{-1} x(t)$.

Finding Caputo's derivative of $V(t)$ with respect to t along the solution to (7) and using Lemma 1 yield

$$\begin{aligned} {}_0^C D_t^\alpha (x^\top(t) P^{-1} x(t)) & \leq 2x^\top(t) P^{-1} {}_0^C D_t^\alpha x(t) \\ = {} & 2x^\top(t) P^{-1} \big[(A + G_a F_a(t) H_a + BK \\ & +G_b F_b(t) H_b K) x(t) + f(x(t)) \\ & + \left(A_1 + G_{a_1} F_{a_1}(t) H_{a_1} + BK_1 + G_b F_b(t) H_b K_1 \right) \\ & \times\ x(t - \tau(t)) + g(x(t - \tau(t))) \big] \\ = {} & x^\top(t) \left(P^{-1}A + A^\top P^{-1} + P^{-1}BK + K^\top B^\top P^{-1} \right) \\ & \times x(t) + 2x^\top(t) P^{-1} G_a F_a(t) H_a x(t) \\ & + 2x^\top(t) P^{-1} G_b F_b(t) H_b K x(t) + 2x^\top(t) P^{-1} f(x(t)) \\ & + x^\top(t) \left(P^{-1} A_1 + P^{-1} B K_1 \right) x(t - \tau(t)) \\ & + x^\top(t - \tau(t)) \left(A_1^\top P^{-1} + K_1^\top B^\top P^{-1} \right) x(t) \\ & + 2x^\top(t) P^{-1} G_{a_1} F_{a_1}(t) H_{a_1} x(t - \tau(t)) \\ & + 2x^\top(t) P^{-1} G_b F_b(t) H_b K_1 x(t - \tau(t)) \\ & + 2x^\top(t) P^{-1} g(x(t - \tau(t))). \end{aligned}$$

In order to further estimate ${}_0^C D_t^\alpha V(t)$ and to deal with uncertainties terms $\Delta A(t) = G_a F_a(t) H_a$, $\Delta A_1(t) = G_{a_1} F_{a_1}(t) H_{a_1}$, $\Delta B(t) = G_b F_b(t) H_b$, noting the properties of F_a, F_{a_1}, F_b, $f(x)$, $g(x)$, we apply Lemma 4 to give the following matrix inequalities:

$$\begin{aligned} & 2x^\top(t) P^{-1} G_a F_a(t) H_a x(t) \\ & \leq \varepsilon_1 x^\top(t) P^{-1} G_a G_a^\top P^{-1} x(t) \\ & \quad +\varepsilon_1^{-1} x^\top(t) H_a^\top H_a x(t), \end{aligned} \tag{11}$$

$$\begin{aligned}
&2x^\top(t)P^{-1}G_bF_b(t)H_bKx(t)\\
&\le \varepsilon_2 x^\top(t)P^{-1}G_bG_b^\top P^{-1}x(t)\\
&\quad+\varepsilon_2^{-1}x^\top(t)K^\top H_b^\top H_bKx(t),
\end{aligned}\tag{12}$$

$$\begin{aligned}
&2x^\top(t)P^{-1}G_{a_1}F_{a_1}(t)H_{a_1}x(t-\tau(t))\\
&\le \varepsilon_3 x^\top(t)P^{-1}G_{a_1}G_{a_1}^\top P^{-1}x(t)\\
&\quad+\varepsilon_3^{-1}x^\top(t-\tau(t))H_{a_1}^\top H_{a_1}x(t-\tau(t)),
\end{aligned}\tag{13}$$

$$\begin{aligned}
&2x^\top(t)P^{-1}G_bF_b(t)H_bK_1x(t-\tau(t))\\
&\le \varepsilon_4 x^\top(t)P^{-1}G_bG_b^\top P^{-1}x(t)\\
&\quad+\varepsilon_4^{-1}x^\top(t-\tau(t))K_1^\top H_b^\top H_bK_1x(t-\tau(t)),
\end{aligned}\tag{14}$$

$$\begin{aligned}
&2x^\top(t)P^{-1}f(x(t))\\
&\le \varepsilon_5 x^\top(t)P^{-1}P^{-1}x(t)+\varepsilon_5^{-1}f^\top(x(t))f(x(t))\\
&\le \varepsilon_5 x^\top(t)P^{-1}P^{-1}x(t)+\varepsilon_5^{-1}L_1^2x^\top(t)x(t),
\end{aligned}\tag{15}$$

and

$$\begin{aligned}
&2x^\top(t)P^{-1}g(x(t-\tau(t)))\\
&\le \varepsilon_6 x^\top(t)P^{-1}P^{-1}x(t)\\
&\quad+\varepsilon_6^{-1}g^\top(x(t-\tau(t)))g(x(t-\tau(t)))\\
&\le \varepsilon_6 x^\top(t)P^{-1}P^{-1}x(t)\\
&\quad+\varepsilon_6^{-1}L_2^2x^\top(t-\tau(t))x(t-\tau(t)).
\end{aligned}\tag{16}$$

Combining (11)-(16), we can obtain

$$\begin{aligned}
&{}_0^CD_t^\alpha V(t)\le x^\top(t)\overline{\Omega}x(t)+x^\top(t-\tau(t))\overline{\Phi}x(t-\tau(t))\\
&+x^\top(t)\overline{\Psi}x(t-\tau)+x^\top(t-\tau(t))\overline{\Psi}^\top x(t)
\end{aligned}$$

where

$$\begin{aligned}
\overline{\Omega}=\;&P^{-1}A+A^\top P^{-1}+P^{-1}BK+K^\top B^\top P^{-1}\\
&+\varepsilon_1P^{-1}G_aG_a^\top P^{-1}+(\varepsilon_2+\varepsilon_4)P^{-1}G_bG_b^\top P^{-1}\\
&+\varepsilon_3P^{-1}G_{a_1}G_{a_1}^\top P^{-1}+(\varepsilon_5+\varepsilon_6)P^{-1}P^{-1}\\
&+\varepsilon_1^{-1}H_a^\top H_a+\varepsilon_2^{-1}K^\top H_b^\top H_bK+\varepsilon_5^{-1}L_1^2I,
\end{aligned}$$

$$\overline{\Phi}=\varepsilon_3^{-1}H_{a_1}^\top H_{a_1}+\varepsilon_4^{-1}K_1^\top H_b^\top H_bK_1+\varepsilon_6^{-1}L_2^2I,$$

and

$$\overline{\Psi}=P^{-1}A_1+P^{-1}BK_1.$$

Hence we have

$$\begin{aligned}
&{}_0^CD_t^\alpha V(t)+\lambda V(t)-\mu\sup_{-\tau(t)\le\sigma\le0}V(t+\sigma)\\
&\le x^\top(t)\overline{\Omega}x(t)+x^\top(t-\tau(t))\overline{\Phi}x(t-\tau(t))\\
&\quad+x^\top(t)\overline{\Psi}x(t-\tau(t))+x^\top(t-\tau(t))\overline{\Psi}^\top x(t)\\
&\quad+\lambda x^\top(t)P^{-1}x(t)\\
&\quad-\mu\sup_{-\tau(t)\le\sigma\le0}x^\top(t+\sigma)P^{-1}x(t+\sigma)\\
&\le x^\top(t)(\overline{\Omega}+\lambda P^{-1})x(t)\\
&\quad+x^\top(t-\tau(t))(\overline{\Phi}-\mu P^{-1})x(t-\tau(t))\\
&\quad+x^\top(t)\overline{\Psi}x(t-\tau(t))+x^\top(t-\tau(t))\overline{\Psi}^\top x(t)\\
&=\begin{pmatrix}x^\top(t) & x^\top(t-\tau(t))\end{pmatrix}\begin{pmatrix}\overline{\Omega}+\lambda P^{-1} & \overline{\Psi}\\ \overline{\Psi}^\top & \overline{\Phi}-\mu P^{-1}\end{pmatrix}\\
&\times\begin{pmatrix}x(t)\\ x(t-\tau(t))\end{pmatrix}\le 0
\end{aligned}$$

provided that

$$\begin{pmatrix}\overline{\Omega}+\lambda P^{-1} & \overline{\Psi}\\ \overline{\Psi}^\top & \overline{\Phi}-\mu P^{-1}\end{pmatrix}\le 0,\tag{17}$$

holds. Now, using $\mathrm{diag}(P,P)$, it is easy to verify that LMI (17) is equivalent to

$$\begin{pmatrix}P\overline{\Omega}P+\lambda P & P\overline{\Psi}P\\ P\overline{\Psi}^\top P & P\overline{\Phi}P-\mu P\end{pmatrix}\le 0,$$

that is

$$\begin{pmatrix}\Omega & \Psi\\ \Psi^\top & \Phi\end{pmatrix}\le 0,\tag{18}$$

where

$$\begin{aligned}
\Omega=\;&AP+PA^\top+BKP+PK^\top B^\top+\varepsilon_1G_aG_a^\top\\
&+(\varepsilon_2+\varepsilon_4)G_bG_b^\top+\varepsilon_3G_{a_1}G_{a_1}^\top+(\varepsilon_5+\varepsilon_6)I\\
&+\varepsilon_1^{-1}PH_a^\top H_aP+\varepsilon_2^{-1}PK^\top H_b^\top H_bKP\\
&+\varepsilon_5^{-1}L_1^2PP+\lambda P,\\
\Phi=\;&\varepsilon_3^{-1}PH_{a_1}^\top H_{a_1}P+\varepsilon_4^{-1}PK_1^\top H_b^\top H_bK_1P\\
&+\varepsilon_6^{-1}L_2^2PP-\mu P,
\end{aligned}$$

and

$$\Psi=A_1P+BK_1P.$$

It follows from Lemma 3 that BMI (18) is equivalent to BMI (8). In the light of Lemma 2, we know that the closed-loop system (7) is asymptotically stable. □

Corollary 1. *The closed-loop system (7) is asymptotically stable if there exist a symmetric positive definite matrix P, matrices Y, Y_1 with appropriate dimensions and six positive constants $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6$ and two constants $\lambda>\mu>0$, such that the following BMI (19) holds,*

$$\begin{pmatrix}
-\varepsilon_3I & 0 & 0 & 0 & H_aP & 0 & 0 & 0\\
0 & -\varepsilon_4I & 0 & 0 & H_bY_1 & 0 & 0 & 0\\
0 & 0 & -\varepsilon_6I & 0 & L_2P & 0 & 0 & 0\\
0 & 0 & 0 & N_{11} & A_1P+BY_1 & PH_a^\top & Y^\top H_b^\top & L_1P\\
PH_a^\top & Y_1^\top H_b^\top & L_2P & PA_1^\top+Y_1^\top B^\top & -\mu P & 0 & 0 & 0\\
0 & 0 & 0 & H_aP & 0 & -\varepsilon_1I & 0 & 0\\
0 & 0 & 0 & H_bY & 0 & 0 & -\varepsilon_2I & 0\\
0 & 0 & 0 & L_1P & 0 & 0 & 0 & -\varepsilon_5I
\end{pmatrix}\le 0,\tag{19}$$

where

$$\begin{aligned}
N_{11}=\;&AP+PA^\top+BY+Y^\top B^\top\\
&+\varepsilon_1G_aG_a^\top+(\varepsilon_2+\varepsilon_4)G_bG_b^\top\\
&+\varepsilon_3G_{a_1}G_{a_1}^\top+(\varepsilon_5+\varepsilon_6)I+\lambda P,
\end{aligned}$$

Moreover, the state back control law is designed by letting

$$u(t)=YP^{-1}x(t)+Y_1P^{-1}x(t-\tau(t)).$$

Proof. In the proof of Theorem 1, letting $K=YP^{-1}$ and $K_1=Y_1P^{-1}$, (19) can be immediately derived. □

Theorem 1 and Corollary 1 solve two basic control problems raised in Introduction Section.

Remark 1. In [29], the authors used the Lyapunov direct method [20] to prove the Mittag-Leffler stability of fractional order system with time-varying structured uncertainties. Here, we use the fractional type

Halanay inequality [26] to obtain the asymptotic stability result of fractional order system with time-varying structured uncertainties and time delay. Our result can be regarded as a generalization/continuation of [29].

Remark 2. A robust stability criteria for integer order delay system is presented in [45], where the time-varying delay satisfies $\dot{\tau}(t) \leq \mu < 1$. Here, using fractional type Halanay inequality, we only need the delay satisfies $t - \tau(t) \to \infty$ as $t \to \infty$ and we don't need the condition $\dot{\tau}(t) \leq \mu < 1$. Moreover, the time-varying delay is allowed to be not only bounded such as $\tau(t) = \sin^2(t^2)$ but also unbounded such as $\tau(t) = t/3 + \cos^2(t^3)$.

Matrix inequality (8) is a bilinear matrix inequality with respect to variables P, K, K_1 and $\lambda, \mu, \varepsilon_1$-$\varepsilon_6$. It is clear that if P is fixed, finding K, K_1, λ, μ and ε_1-ε_6 is a LMI feasibility problem, which is realizable with existing algorithms. Thus, we denote BMI (8) as $\Lambda(P, K, K_1, \lambda, \mu, \varepsilon_1\text{-}\varepsilon_6)$, and propose the following algorithm:

Algorithm 3.1

• Given: $k := 0$, arbitrary symmetric positive definite matrix $P^{(0)} \in \mathbb{R}^{n\times n}$.

• Repeat:

Step 1. Set k:=k+1.

Step 2. Solve the minimum eigenvalue of $\Lambda(P^{(k-1)}, K, K_1, \lambda, \mu, \varepsilon_1\text{-}\varepsilon_6)$ with solution $K, K_1, \lambda, \mu, \varepsilon_1$-$\varepsilon_6$, denote the solution as $(K^{(k)}, K_1^{(k)}, \lambda^{(k)}, \mu^{(k)}, \varepsilon_1^{(k)}\text{-}\varepsilon_6^{(k)})$.

Step 3. Solve the minimum eigenvalue of $\Lambda(P, K^{(k)}, K_1^{(k)}, \lambda^{(k)}, \mu^{(k)}, \varepsilon_1^{(k)}\text{-}\varepsilon_6^{(k)})$ with solution P, denote the solution as $P^{(k)}$.

• Until:

$\mathrm{eig}\{\Lambda(P^{(k-1)}, K^{(k)}, K_1^{(k)}, \lambda^{(k)}, \mu^{(k)}, \varepsilon_1^{(k)}\text{-}\varepsilon_6^{(k)})\} < 0$ or

$\mathrm{eig}\{\Lambda(P^{(k)}, K^{(k)}, K_1^{(k)}, \lambda^{(k)}, \mu^{(k)}, \varepsilon_1^{(k)}\text{-}\varepsilon_6^{(k)})\} < 0$.

Remark 3. Algorithm 3.1 includes iterative process, which may be an endless loop whenever the closed-loop system (7) is unstable. To avoid this case, the maximum number of iterations should be given.

Note that (19) contains variables λP and $-\mu P$, which leads (19) to be nonlinear with respect to λ, μ and P. However, we can use searching algorithm to determinate all the variables. That is, for some λ and μ with $\lambda > \mu > 0$, find the feasible solution of LMI (19) (when λ and μ are given, BMI (19) becomes LMI). Therefore, we can propose the following algorithm to find the feasible solution of BMI (19).

Algorithm 3.2

Step 1. For some constants λ, μ with $\lambda > \mu > 0$, substitute them into BMI (19), this makes BMI (19) be a LMI.

Step 2. Solve LMI (19) to obtain $P, Y, Y_1, \varepsilon_1 - \varepsilon_6$.

Remark 4. If the LMI (19) in Step 2 of Algorithm 3.2 is infeasible, it means that the feasible solution to BMI (19) can not be obtained via the above Algorithm 3.2. In such a case, some other algorithms should be pursued, such as interior point method [46] and local minima method [47]. Compared with Algorithm 3.1 that uses the iterative process, Algorithm 3.2 can be easily implemented by utilizing the existing LMI optimization techniques [43].

4 | NUMERICAL EXAMPLES

Example 1. Consider the robust stability of the fractional order system shown in (4) with the following coefficients matrix $A = \begin{pmatrix} -8 & 20 \\ -50 & -20 \end{pmatrix}$, $B = \begin{pmatrix} -10 \\ 10 \end{pmatrix}$, $A_1 = \begin{pmatrix} -5 & -2 \\ -3 & -6 \end{pmatrix}$, $G_a = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$, $G_{a_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $G_b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $F_a = F_{a_1} = F_b = \frac{1}{t+1}$, $H_a = \begin{pmatrix} -1 & 1 \end{pmatrix}$, $H_{a_1} = \begin{pmatrix} 0 & 0.5 \end{pmatrix}$, $H_b = \begin{pmatrix} -0.05 \end{pmatrix}$, and $f(x(t)) = \begin{pmatrix} \sin x_1^2(t) \\ \sin x_2^2(t) \end{pmatrix}$, $g(x(t - \tau(t))) = \begin{pmatrix} \sin x_1^2(t-\tau(t)) \\ \sin x_2^2(t-\tau(t)) \end{pmatrix}$.

With the help of MATLAB Toolbox, we found that the LMI (19) is feasible and one of possible solution is obtained as follows: $\varepsilon_1 = 11.4403$, $\varepsilon_2 = 4.6994$, $\varepsilon_3 = 2.8845$, $\varepsilon_4 = 4.6830$, $\varepsilon_5 = 4.8894$, $\varepsilon_6 = 18.9786$, and $\lambda = 2$, $\mu = 1$, $P = \begin{pmatrix} 2.4 & 0 \\ 0 & 0.25 \end{pmatrix}$, $Y = \begin{pmatrix} 12 & -5 \end{pmatrix}$, $Y_1 = \begin{pmatrix} 0.6 & 0.1 \end{pmatrix}$.By computing the feedback control (10), the state feedback control law is given by

$$u(t) = \begin{pmatrix} 5 & -20 \end{pmatrix} x(t) + \begin{pmatrix} 0.25 & 0.4 \end{pmatrix} x(t - \tau(t)).$$

In view of Corollary 1, the system under the above feedback control law is asymptotically stable. The simulation of the system with $\alpha = 0.7$($\alpha = 0.7$ has no speciality. It can be any value in $(0, 1)$) with bounded delay $\tau(t) = \sin^2(3t)$ and unbounded delay $\tau(t) = t/4 + t\sin^2(10t)/4 + 3$ can be seen in Figure 1 and Figure 2 respectively.

Example 2. Consider the robust stability of fractional order system (4) with parameters $A = \begin{pmatrix} -10 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 \\ 0 \\ -10 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -2 & 0 \\ 1 & 0 & -5 \end{pmatrix}$, $G_a = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, $H_a = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $G_{a_1} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $H_{a_1} = \begin{pmatrix} 0 & -2 & 0 \end{pmatrix}$,

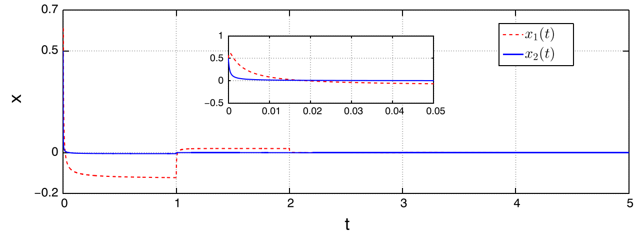

FIGURE 1 Time responses of $x_1(t), x_2(t)$ for the closed-loop system with bounded delay $\tau(t) = \sin^2(3t)$ [Color figure can be viewed at wileyonlinelibrary.com]

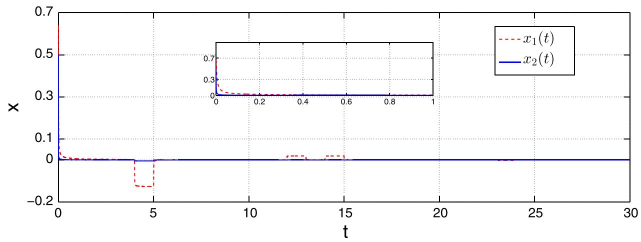

FIGURE 2 Time responses of $x_1(t), x_2(t)$ for the closed-loop system with unbounded delay $\tau(t) = t/4 + t\sin^2(10t)/4 + 3$ [Color figure can be viewed at wileyonlinelibrary.com]

$$G_b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},\ H_b = \begin{pmatrix} 1 \end{pmatrix},\ F_a(t) = F_{a_1}(t) = F_b(t) = \sin t,\ f(x(t)) = \begin{pmatrix} \sin x_1^2(t) \\ 0 \\ \sin x_3^2(t) \end{pmatrix} \text{ and } g(x(t-\tau(t))) = \begin{pmatrix} 0 \\ \sin x_2^2(t-\tau(t)) \\ 0 \end{pmatrix}.$$

The purpose of designing a feedback control law is to stabilize the given system. When there is no extra control, that is, $u(t) = 0$, the system is unstable, this can be seen in Figure 3, where $\alpha = 0.7$, $x_1(0) = x_2(0) = 2$ and the delay $\tau(t) = 1$.

With the help of MATLAB, we found that the BMI (19) are satisfied with $\varepsilon_1 = 4.24$, $\varepsilon_2 = 3.95$, $\varepsilon_3 = 10.04$, $\varepsilon_4 = 3.74$, $\varepsilon_5 = 4.66$, $\varepsilon_6 = 10.31$, and $\lambda = 1$, $\mu = 0.5$, $P = \begin{pmatrix} 2 & 0 & 0.5 \\ 0 & 2 & 0 \\ 0.5 & 0 & 0.4 \end{pmatrix}$, $Y = \begin{pmatrix} -0.6 & -0.7 & 1.5 \end{pmatrix}$, $Y_1 = \begin{pmatrix} -0.08 & 0.01 & -0.12 \end{pmatrix}$.Then under the state feedback control law

$$\begin{aligned} u(t) = {} & \begin{pmatrix} -1.8 & -0.35 & 6 \end{pmatrix} x(t) \\ & + \begin{pmatrix} 0.05 & 0.005 & -0.36 \end{pmatrix} x(t-\tau(t)), \end{aligned}$$

system (4) is asymptotically stable.

The simulation of system (4) with $\alpha = 0.5$ with constant delay $\tau(t) = 1$ and unbounded delay $\tau(t) = t/4 + t\sin^2(10t)/4 + 3$ can be seen in Figure 4 and Figure 5,

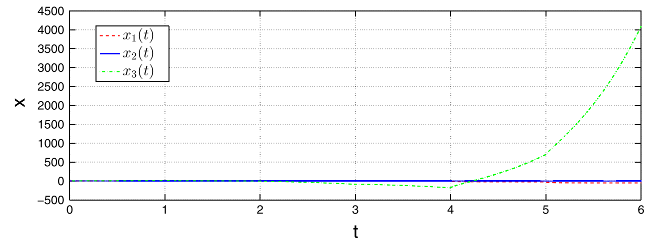

FIGURE 3 Time responses of $x_1(t), x_2(t), x_3(t)$ for fractional order system with bounded delay $\tau(t) = 1$ and without controller [Color figure can be viewed at wileyonlinelibrary.com]

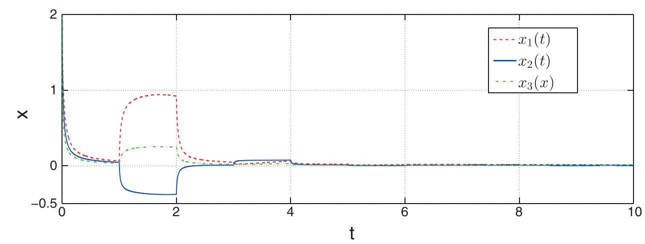

FIGURE 4 Time responses of $x_1(t), x_2(t), x_3(t)$ for the closed-loop system with bounded delay $\tau(t) = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

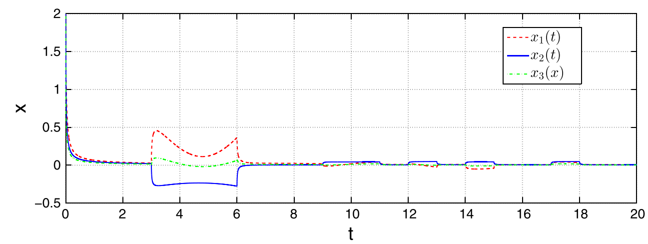

FIGURE 5 Time responses of $x_1(t), x_2(t), x_3(t)$ for the closed-loop system with bounded delay $\tau(t) = t/4 + t\sin^2(10t)/4 + 3$ [Color figure can be viewed at wileyonlinelibrary.com]

respectively. Moreover, compared Figure 4 with Figure 5, we can see that the state of the closed-loop system with bounded delay converges to zero fast than the one of system with unbounded delay.

5 | CONCLUDING REMARKS

In this paper, we present BMIs method for the stabilization of fractional order differential system with time-varying delay, where the fractional Halanay inequality is used. With the help of MATLAB Toolbox, either matrix P or λ, μ are fixed, the feedback control law can be easily constructed by two proposed algorithms. How to construct an output feedback controller to stabilize the fractional order nonlinear systems with time-varying delay is interesting and is what we will do in future.

ACKNOWLEDGMENTS

This work is partially supported by the Fundamental Research Funds for the Central Universities(No. CUSF-DH-D-2017083) and the National Natural Science Foundation of China (No.61803386).

ORCID

Bin-Bin He https://orcid.org/0000-0002-8860-6263
Hua-Cheng Zhou https://orcid.org/0000-0001-6856-2358
YangQuan Chen https://orcid.org/0000-0002-7422-5988

REFERENCES

1. A. Oustaloup and B. Mathieu, *La commande CRONE*, Hermes, Paris, 1991.
2. I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1998.
3. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Amsterdam, 1993.
4. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.
5. F. Mainardi, *Fractional calculus and wave in linear viscoelasticity: An introduction to mathematical models*, World Scientific, Singapore, 2010.
6. Y. Xu, Y. G. Li, and D. Liu, *A method to stochastic dynamical systems with strong nonlinearity and fractional damping*, Nonlinear Dynam. **83** (2016), no. 4, 2311–2321.
7. Y. Xu, Y. G. Li, and D. Liu, *"Response of fractional oscillators with viscoelastic term under random excitation"*, J. Comput. Nonlin. Dyn. **9** (2014), no. 3, 031015.
8. Y. F Xu and Z. Zheng, *Quenching phenomenon of a time-fractional diffusion equation with singular source term*, Math. Methods Appl. Sci. **40** (2017), no. 16, 5750–5759.
9. Y. F Xu, *Quenching phenomenon in a fractional diffusion equation and its numerical simulation*, Int. J. Comput. Math. **95** (2018), no. 1, 98–113.
10. D. Matignon, *Stability results on fractional differential equations to control processing*, IMACS/IEEE-SMC Multiconference, Symposium on Control, Optimization and Supervision(CESA), 1996, pp. 963–968.
11. Y. Ma, J. Lu, and W. Chen, *Robust stability and stabilization of fractional order linear systems with positive real uncertainty*, ISA Trans. **53** (2014), no. 2, 199–209.
12. H. Yang and B. Jiang, *Stability of fractional-order switched non-linear systems*, IET Contr. Theory Appl. **10** (2016), no. 8, 965–970.
13. J. Shen and J. Lam, *Stability and performance analysis for positive fractional-order systems with time-varying delays*, IEEE Trans. Autom. Control. **61** (2016), no. 9, 2676–2681.
14. H. C. Zhou and B. Z. Guo, *Boundary feedback stabilization for an unstable time fractional reaction diffusion equation*, SIAM J. Control Optim. **56** (2018), no. 1, 75–101.
15. H. C. Zhou and Others, *Mittag-Leffler stabilization for an unstable time fractional anomalous diffusion equation with boundary control matched disturbance*, Internat. J. Robust Nonlinear Control. https://doi.org/10.1002/rnc.4632
16. H. C. Zhou and G. Weiss, *Output feedback exponential stabilization for one-dimensional unstable wave equations with boundary control matched disturbance*, SIAM J. Control Optim. **56** (2018), no. 6, 4098–4129.
17. H. C. Zhou and W. Guo, *Performance output tracking and disturbance rejection for an Euler-Bernoulli beam equation with unmatched boundary disturbance*, J. Math. Anal. Appl. **470** (2019), no. 2, 1222–1237.
18. Z. Jiao, Y. Q. Chen, and Y. Zhong, *Stability analysis of linear time-invariant distributed-order systems*, Asian J. Control. **15** (2013), no. 3, 640–647.
19. W. H. Deng, C. P. Li, and J. H. Lü, *Stability analysis of linear fractional differential system with multiple time delays*, Nonlinear Dynam. **48** (2007), no. 4, 409–416.
20. Y. Li, Y. Q. Chen, and I. Podlubny, *Mittag-Leffler stability of fractional order nonlinear dynamic systems*, Automatica. **45** (2009), no. 8, 1965–1969.
21. B. Chen and J. Chen, *Razumikhin-type stability theorems for functional fractional-order differential systems and applications*, Appl. Math. Comput. **254** (2015), 63–69.
22. V. N. Phat, M. V. Thuan, and T. N. Tuan, *New criteria for guaranteed cost control of nonlinear fractional-order delay systems: a Razumikhin approach*, Vietnam J. Math. (2018). https://doi.org/10.1007/s10013-018-0323-x
23. H. Wang and Others, *Global stability analysis of fractional-order Hopfield neural networks with time delay*, Neurocomputing. **154** (2015), 15–23.
24. S. Liu and Others, *Asymptotical stability of Riemann-Liouville fractional nonlinear systems*, Nonlinear Dynam. **86** (2016), no. 1, 65–71.
25. B. B. He and Others, *Asymptotical stability of fractional order systems with time delay via an integral inequality*, IET Control Theory Appl. **12** (2018), no. 12, 1748–1754.
26. B. B. He and Others, *New integral inequalities and asymptotic stability of fractional order systems with unbounded time delay*, Nonlinear Dynam. **94** (2018), no. 2, 1523–1534.
27. J. N. Li and S. L. Li, *Mean-square exponential stability for stochastic discrete-time recurrent neural networks with mixed time delays*, Neurocomputing **151** (2015), 790–797.
28. J. N. Li and Others, *Exponential synchronization of discrete-time mixed delay neural networks with actuator constraints and stochastic missing data*, Neurocomputing **207** (2016), 700–707.
29. M. V. Thuan and D. C. Huong, *New results on stabilization of fractional-order nonlinear systems via an LMI Approach*, Asian J. Control **20** (2018), no. 5, 1–10.
30. S. Zhang, Y. Yu, and J. Yu, *LMI conditions for global stability of fractional-order neural networks*, IEEE Trans. Neural Netw. Learn. Syst. **28** (2017), no. 10, 2423–2433.
31. Y. H. Lan and Y. Zhou, *LMI-based rebust control of fractional-order uncertain linear systems*, Comput. Math. Appl. **62** (2011), no. 3, 1460–1471.
32. C. Yin and Others, *Robust stability analysis of fractional-order uncertain singular nonlinear system with external disturbance*, Appl. Math. Comput. **269** (2015), 351–362.

33. S. Dadras and H. R. Momeni, *Fractional-order dynamic output feedback sliding mode control design for robust stabilization of uncertain fractional-order nonlinear systems*, Asian J. Control **16** (2014), 489–497.
34. Y. Wei and Others, *The output feedback control synthesis for a class of singular fractional order systems*, ISA Trans. **69** (2017), 1–9.
35. M. V. Thuan, D. C. Huong, and D. T. Hong, *New results on robust finite-time passivity for fractional-order neural networks with uncertainties*, Neural Process Lett. (2018). https://doi.org/10.1007/s11063-018-9902-9
36. D. C. Huong and M. V. Thuan, *Design of unknown input reduced-order observers for a class of nonlinear fractional-order time-delay systems*, Int J Adapt Control Signal Process. **32** (2018), no. 3, 412–423.
37. J. G. VanAntwerp and R. D. Braatz, *A tutorial on linear and bilinear matrix inequalities*, Journal of Process Control **10** (2000), 363–385.
38. K. C. Goh, *Robust control synthesis via bilinear matrix inequalities*, A Bell & Howell Information Company, Ann Arbor, 1995.
39. X. Song and Z. Wang, *Dynamic output feedback control for fractional-order systems*, Asian J. Control **15** (2013), no. 3, 834–848.
40. C. Li and J. Wang, *Robust stability and stabilization of fractional order interval systems with coupling relationships: The* $0 < \alpha < 1$ *case*, J. Franklin Inst. **349** (2012), no. 7, 2406–2419.
41. S. Zheng, *Robust stability of fractional order system with general interval uncertainties*, Systems Control Lett. **99** (2017), 1–8.
42. N. Aguila-Camacho, M. Duarte-Mermoud, and J. A. Gallegos, *Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems*, Commun. Nonlinear Sci. Numer. Simul. **22** (2015), no. 1-3, 650–659.
43. S. Boyd and Others, *Linear Matrix Inequalities in System and Control Theory*, The Society for Industrial and Applied Mathematics, Philadelphia, 1994.
44. G. R. Duan and H. H. Yu, *LMIs in Control Systems, Analysis, Design and Applications*, CRC Press, Boca Raton, 2013.
45. M. Wu and Others, *Delay-dependent criteria for robust stability of time-varying delay systems*, Automatica **40** (2004), no. 8, 1435–1439.
46. K. C. Goh and Others, *Biaffine matrix inequality properties and computational methods*, Proc. Amer. Control Conf., Baltimore, 1994, pp. 850–855.
47. K. C. Goh and Others, μ/K_m- *synthesis via bilinear matrix inequalities*, Proceeding of the 33rd Conference on Decision and Control, Lake Buena Vista, 1994, pp. 2032–2037.

How to cite this article: He B-B, Zhou H-C, Kou C-H, Chen Y. Stabilization of uncertain fractional order system with time-varying delay using BMI approach. *Asian J Control*. 2021;23: 582–590. https://doi.org/10.1002/asjc.2193