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# Regional observability for Hadamard-Caputo time fractional distributed parameter systems

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ABSTRACT

In this paper, regional (gradient) exact and approximate observability problems are studied on Hadamard-Caputo time fractional distributed parameter systems. Without any knowledge of the initial vector and its gradient, several equivalent criteria are first provided to achieve the regional observability. Based on these, characterizations for both $\omega-$strategic and gradient $\omega$-strategic zone sensors are developed. Then, by employing the Hilbert Uniqueness Method (HUM), we explicitly reconstruct the initial vector and its gradient respectively. A one-dimension example is finally included to illustrate our results.



## 1. Introduction

During the past two decades, time fractional distributed parameter systems (DPSs) with Caputo or Riemann-Liouville fractional derivatives have been singled out as an outstanding tool to describe the sub-diffusion phenomena [1–4]. This is due to the fact that fractional derivatives are defined as a kind of convolution and are suitable for modeling these dynamics while integer order derivative approaches appear to be less accurate [5]. In recent years, ultra-slow diffusion has attracted increasing attentions. Unfortunately, it's shown in [6,7] that neither Caputo nor Riemann-Liouville fractional derivative could well characterize the dynamics of ultra-slow diffusion processes.

In fact, at the very start, Riemann-Liouville and Caputo fractional derivatives are introduced to bring about particular convenience, especially in analyzing anomalous phenomena. In case these effects are not present, it's more appropriate to adopt a new definition. Therefore, we consider the Hadamard fractional derivatives, in which the power-law kernel functions in Riemann-Liouville and Caputo fractional derivatives are replaced by logarithmic functions of arbitrary exponent [8,9]. In addition, the $t\frac{d}{dt}$ in its definition has shown to be invariant on the half-axis in concerns of dilation. With this definition, new opportunities would be posed to improve the existing results from the theoretical viewpoints. In [10–12], the Mellin transformation for Hadamard fractional integral and derivative was considered and several properties were obtained. Some new results on the initial and boundary value problems of Hadamard differential equations and inclusions were given in [13]. Meanwhile, two survey papers [14,15] proved the existence and uniqueness theorems for the (weak) solutions of several classes of impulse Hadamard fractional differential equations, and provided the analytical solutions presented by the Mittag-Leffler function and its generalization. For more knowledge in this field, we refer the reader to [16–19] and the

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references cited therein. However, even though Hadamard fractional derivative has abundant application prospects in fractional thermoelasticity [20], kinetic theory of gases [21] and physical phenomena in fluctuating environments [22], etc, due to its unique characteristics in describing ultra-slow diffusions, the analysis on the DPSs with Hadamard fractional derivatives is still on its early stage.

Note that observability occupies an important place in the analysis of control systems. By means of characterising sensors, one can identify the trajectory of the considered system, i.e., reconstruct the initial state according to the measurements given by the output function. It's worth remarking that, in case we only concern about the observability on some subregions we are interested in, the systems don't need to be observed on the whole domain. As a result, regional analysis was introduced [23], which raised a new wave in the observability problem over the years. For instance, the regional gradient and boundary observability for integer order DPSs were considered by Zerrik et al. [24,25]. Recently, for time fractional DPSs governed by Riemann-Liouville fractional derivative, the regional gradient observability was firstly developed in [26]. Besides, similar results were developed for the regional enlarged observability of integer order linear parabolic systems [27]. One can find more knowledge about the observability analysis and the design of fractional order observers in [28,29].

Motivated by the above arguments, in this work, we are concerned with the regional (gradient) exact and approximate observability for Hadamard-Caputo time DPSs and explore the characterization for sensors to realize the regional (gradient) strategy. More precisely, the initial state can be also reconstructed according to the output measurements. To this end, the HUM is applied to the reconstruction of both the initial condition and its gradient. It's also worth mentioning that in this paper, we consider a class of Hadamard-Caputo fractional systems with physically interpretable initial conditions similar to those in Caputo fractional systems.

The rest of this paper is organized as follows. In Section 2, the problem under investigation is presented and some needed definitions and lemmas are recalled. Section 3 and 4 are dedicated to our main results on the regional observability and regional gradient observability, respectively. An example is finally given to show the correctness of our results.

## 2. Preliminary results

In this section, we formulate the problem to be considered in this paper, and then recall some basic results.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and consider the following Hadamard-Caputo time fractional distributed parameter system:

$$\begin{cases} {}_a^{HC}D_t^{\alpha} y(x,t) = Ay(x,t) \text{ in } U, \\ y(x,a) = y_0(x) \text{ in } \Omega, \\ y(\xi,t) = 0 \text{ on } \Sigma, \end{cases} \tag{2.1}$$

where $U = \Omega \times [a,b]$, $\Sigma = \partial\Omega \times [a,b]$, $a > 0$ and $0 < \alpha < 1$. $y(x,\cdot) \in AC[a,b] \triangleq \{y(x,\cdot) : [a,b] \to \mathbb{R}$ is absolutely continuous$\}$, $A$ is a bounded linear operator and generates a $C_0-$semigroup $\{T(t)\}$ on Hilbert space $L^2(\Omega)$, while $-A$ a uniformly elliptic operator. The initial vector $y_0 \in L^2(\Omega)$ is unknown. Moreover, ${}_a^{HC}D_t^{\alpha}$ denotes the Hadamard-Caputo fractional derivative to be specified later.

The measurements are given by the following output function:

$$z(t) = Cy(x,t), \tag{2.2}$$

where $C$: $L^2(\Omega \times [a,b]) \to L^2(a,b;\mathbb{R}^m)$ is a bounded operator with dense domain, $m$ donates the number of sensors.

Let $\omega \subseteq \Omega$ and

$$y_0(x) = \begin{cases} y_0^1(x) \text{ in } \omega \text{ to be estimated,} \\ y_0^2(x) \text{ in } \Omega \setminus \omega \text{ undesired.} \end{cases} \tag{2.3}$$

We next consider to reconstruct $y_0^1(x)$ and its gradient in $\omega$.

**Definition 2.1.** [17] The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{R}$, $0 < \alpha < 1$ of a function $f(t)$ is defined by

$${}_a^{H}I_t^{\alpha} f(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}. \tag{2.4}$$

**Definition 2.2.** [17] Let $0 < \alpha < 1$. If $f(t)$ is absolutely continuous on $[a,\ b]$, where $0 < a < b < \infty$. Then the left-sided Hadamard-Caputo fractional derivative is defined by

$${}_a^{HC}D_t^{\alpha} f(t) \triangleq \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{-\alpha} f'(s)ds. \tag{2.5}$$

**Definition 2.3.** The restriction map $p_\omega$ is defined by

$$p_\omega y = \begin{cases} y & \text{in } \omega, \\ 0 & \text{in } \Omega \setminus \omega. \end{cases}$$

And $p_\omega^*$, the adjoint operator of the restriction map $p_\omega$ is defined by

$$p_\omega^* f(x) = \begin{cases} f(x), & x \in \omega, \\ 0, & x \in \Omega \setminus \omega. \end{cases} \tag{2.6}$$

In particular, if $n = 1$, we denote $p_\omega$ as $p_{1,\omega}$, so does $p_{1,\omega}^*$.

In addition, the following two lemmas are necessary to obtain our main results.

**Lemma 2.1.** *[30] Suppose f(t) is continuous, then the unique solution of system*

$$\begin{cases} {}_a^{HC}D_t^\alpha y(x,t) = Ay(x,t) + f(t) \text{ in } U, \\ y(x,a) = y_0(x) \text{ in } \Omega, \\ y(\xi,t) = 0 \text{ on } \Sigma \end{cases} \tag{2.7}$$

*is given by*

$$y(x,t) = S_\alpha\left(\log\frac{t}{a}\right)y_0(x) + \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} K_\alpha\left(\log\frac{t}{s}\right) f(s)\frac{ds}{s}, \tag{2.8}$$

*where $S_\alpha(t) = E_\alpha(At^\alpha)$ and $K_\alpha(t) = E_{\alpha,\alpha}(At^\alpha)$.*

**Proof.** Applying Hadamard fractional integral of order $\alpha$ on both sides of the first equation in (2.7), we have

$$y(x,t) = y_0(x) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}(Ay(x,s) + f(s))\frac{ds}{s}. \tag{2.9}$$

Since $f(t)$ is continuous in $[a, b]$, a positive constant $M$ can be found such that $\|f(t)\| \le M$. This yields that all assumptions in [15, Theorem 3.8] hold. Then, in order to obtain the existence for the solutions of (2.9), we construct the following Picard iterative sequence

$$\begin{cases} \eta_0(x,t) = y_0(x), \\ \eta_i(x,t) = y_0(x) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}(A\eta_{i-1}(x,s) + f(s))\frac{ds}{s}, \quad (x,t) \in U, \ i = 1, 2, \cdots. \end{cases}$$

Similar to Claims $1 \sim 3$ in [15], we get that

(i) $\eta_i(x, t) \in C(U)$, $i = 1, 2, \ldots$;
(ii) $\{\eta_i(x,t)\}_{i=1}^\infty$ converges uniformly to $\eta(x, t)$ for $(x, t) \in U$;
(iii) $\eta(x,t) = \lim_{i\to\infty} \eta_i(x,t)$ is the continuous solution of (2.9).

From the iterative sequence, we get

$$\begin{aligned} \eta_i(x,t) &= y_0(x) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}(A\eta_{i-1}(x,s) + f(s))\frac{ds}{s} \\ &= y_0(x) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} Ay_0(x)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} \\ &\quad + \frac{1}{(\Gamma(\alpha))^2}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} A\int_a^s \left(\log\frac{s}{u}\right)^{\alpha-1}(A\eta_{i-2}(x,u) + f(u))\frac{du}{u}\frac{ds}{s} \\ &= y_0(x) + \frac{A\left(\log\frac{t}{a}\right)^\alpha}{\Gamma(\alpha+1)}y_0(x) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(2\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{2\alpha-1} A(A\eta_{i-2}(x,s) + f(s))\frac{ds}{s} \\ &= \cdots \\ &= \sum_{k=0}^{i} \frac{A^k\left(\log\frac{t}{a}\right)^{k\alpha}}{\Gamma(k\alpha+1)}y_0(x) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\sum_{k=0}^{i}\frac{A^k\left(\log\frac{t}{s}\right)^{k\alpha}}{\Gamma((k+1)\alpha)} f(s)\frac{ds}{s} \\ &\to S_\alpha\left(\log\frac{t}{a}\right)y_0(x) + \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} K_\alpha\left(\log\frac{t}{s}\right) f(s)\frac{ds}{s}, \end{aligned}$$

as $i \to \infty$.

Moreover, suppose that $y_1(x, t)$, $y_2(x, t)$ are two solutions of (2.9), and denote $\tilde{y}(x,t) = y_1(x,t) - y_2(x,t)$. Then we have

$$\|\tilde{y}(x,t)\| \le \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\|A\|\|\tilde{y}(x,s)\|\frac{ds}{s}.$$

According to a generalized Gronwall's Inequality (see [31, Theorem 4]), we get $\tilde{y}(x,t)=0$, and thus, (2.9) exists a unique solution in $U$. It's not difficult to know $y(x, t)$ is a solution of (2.7) if and only if $y(x, t)$ satisfies (2.9). Therefore, (2.7) has a unique solution $y(x,t)=S_\alpha\left(\log\frac{t}{a}\right)y_0(x)+\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}K_\alpha\left(\log\frac{t}{s}\right)f(s)\frac{ds}{s}$. □

**Remark 2.1.** For $A$ being an infinitesimal generator of a $C_0$-semigroup, $S_\alpha(t)$ denotes the so-called $\alpha$-times resolution family of the fractional differential equations. It's not the Mittag-Leffler function as introduced for finite-dimensional system cases, but with similar properties. In particular, if $A$ is a real number or square matrix, then $S_\alpha(t)$ and $K_\alpha(t)$ are known as the Mittag-Leffler function and its generalization. For more arguments and properties of $S_\alpha(t)$, $K_\alpha(t)$, we refer the reader to [32–34].

**Lemma 2.2.** *[35] Let $U$ be a closed, convex subset of a real Hilbert space $H$. Assume $\pi(u,v)$ be a continuous symmetric bilinear form on $H$ satisfying*

$$\pi(u,u)\geq c\|u\|^2,\qquad \forall u\in H,$$

*for some $c>0$. Then there exists a unique element $u\in U$ such that*

$$\pi(u,u)=\inf_{v\in U}\pi(v,v).$$

## 3. Regional observability

By Lemma 2.1, the solution of (2.1) is $y(x,t)=S_\alpha\left(\log\frac{t}{a}\right)y_0(x)$. Then, the output function (2.2) satisfies

$$z(t)=CS_\alpha\left(\log\frac{t}{a}\right)y_0(x). \tag{3.10}$$

Since $C$ is densely defined, both $C^*$ and $(CS_\alpha)^*$, the adjoint operators of $C$ and $CS_\alpha$ exist, respectively, and $(CS_\alpha)^*=S_\alpha^*C^*$. Define $Q(t)=CS_\alpha\left(\log\frac{t}{a}\right)$. Then we have $Q\in\mathcal{L}\left(L^2(\Omega),L^2(a,b;\mathbb{R}^m)\right)$. Hence, $Q^*$: $L^2(a,b;\mathbb{R}^m)\to L^2(\Omega)$, the adjoint operator of $Q$ can be given by

$$Q^*z=\int_a^b S_\alpha^*\left(\log\frac{s}{a}\right)C^*z(s)ds=\int_0^{\log\frac{b}{a}}ae^sS_\alpha^*(s)C^*z(ae^s)ds. \tag{3.11}$$

**Definition 3.1.** [3,25] System (2.1)–(2.2) is said to be regionally exactly observable in $\omega$ and respectively, regionally approximately observable in $\omega$, if $y_0^1(x)\subseteq L^2(\Omega)$ can be uniquely determined by $z(x, t)$ and respectively,

$$Ker\left(Qp_{1,\omega}^*\right)=\{0\}.$$

In fact, the regional exact observation problem is to find an operator $H$: $L^2(a,b;\mathbb{R}^m)\to L^2(\omega)$ such that $Hz=y_0^1$. Here, we introduce $H=p_{1,\omega}Q^*$. Then we conclude the following theorems.

**Theorem 3.1.** *The following statements are equivalent:*

*(i) System (2.1)–(2.2) is regionally exactly observable in $\omega$;*
*(ii) $Im(H)=L^2(\omega)$;*
*(iii) $Ker\left(p_{1,\omega}\right)+Im(Q^*)=L^2(\Omega)$;*
*(iv) There is a constant $c>0$ such that*

$$\|z\|_{L^2(\omega)}\leq c\|H^*z\|_{L^2(a,b;\mathbb{R}^m)},\quad \forall z\in L^2(\omega). \tag{3.12}$$

**Proof.** The equivalence between $(i)$ and $(ii)$ can be easily obtained from Definition 3.1. In Theorem 1 of [36], choose $X=Z=L^2(\omega)$, $Y=L^2(\Omega)$, $f=I$, the identity operator and $g=H$, we have $(i)\Leftrightarrow(iv)$. So we only need to prove $(ii)\Leftrightarrow(iii)$.

$(ii)\Rightarrow(iii)$: For any $x_1\in Ker(p_{1,\omega})$, $x_2\in Im(Q^*)$, since

$$p_{1,\omega}(x_1+x_2)=p_{1,\omega}x_2\in Im(H)=L^2(\omega),$$

we have $x_1+x_2\in L^2(\Omega)$, that is,

$$Ker(p_{1,\omega})+Im(Q^*)\subseteq L^2(\Omega). \tag{3.13}$$

Next, for any $x\in L^2(\Omega)$, define $\tilde{x}\triangleq p_{1,\omega}x\in L^2(\omega)=Im(H)$, and $y\triangleq x-\tilde{x}$. So, there exists a $z\in L^2(\Omega)$, such that $Hz=\tilde{x}$. Then $\tilde{x}\in Im(Q^*)$ and

$$p_{1,\omega}y=p_{1,\omega}(x-\tilde{x})=p_{1,\omega}x-p_{1,\omega}\tilde{x}=0,$$

that is, $y\in Ker(p_{1,\omega})$. Hence,

$$L^2(\Omega)\subseteq Ker(p_{1,\omega})+Im(Q^*). \tag{3.14}$$

$(iii)\Rightarrow(ii)$: From $(iii)$, we have for any $x\in L^2(\omega)\subseteq L^2(\Omega)$, there exists $x_1\in Ker(p_{1,\omega})$ and $x_2\in Im(Q^*)$, such that $x=x_1+x_2$. Then there is a $z\in L^2(\Omega)$, such that $Q^*z=x_2$. So we have

$$x=p_{1,\omega}x=p_{1,\omega}(x_1+Q^*z)=Hz\in Im(H),$$

that is,

$$L^2(\omega) \subseteq Im(H). \tag{3.15}$$

Next, for any $x \in Im(H)$, together with the definition of $Q^*$, we immediately have that there exists a $z \in L^2(\Omega)$, such that $Hz = x \in L^2(\omega)$, namely

$$Im(H) \subseteq L^2(\omega). \tag{3.16}$$

Combining (3.13)∼(3.16), we know that $(ii) \Leftrightarrow (iii)$ and complete the proof. □

**Theorem 3.2.** *The following statements are equivalent:*

*(i) System (2.1)–(2.2) is regionally approximately observable in* $\omega$*;*
*(ii)* $\overline{Im(H)} = L^2(\omega)$*;*
*(iii)* $Ker(p_{1,\omega}) + \overline{Im(Q^*)} = L^2(\Omega)$*;*
*(iv)* $HH^*$ *is a positive definite operator;*
*(v) Suppose* $y \in L^2(\omega)$ *satisfies that*

$$\langle Hz, y\rangle = 0, \quad \forall z \in L^2(\Omega),$$

*then* $y = 0$*, where* $\langle \cdot, \cdot \rangle$ *denotes the inner product.*

**Proof.** Similar to the proof of Theorem 3.1, we can easily prove $(i) \Leftrightarrow (ii)$ and $(ii) \Rightarrow (iii)$. The equivalence between $(i)$ and $(v)$ can be given according to Proposition 3 in [25] with certain modifications. Thus we omit it.

$(iii) \Rightarrow (ii)$: $\overline{Im(H)} \subseteq L^2(\omega)$ can be easily obtained from the range of $Q^*$.

Next, for any $z \in L^2(\omega)$, we know $p^*_{1,\omega} z \in L^2(\Omega) \subseteq L^2(\omega)$ is also arbitrary. From (iii), there are $x \in Ker(p_{1,\omega})$ and $y \in \overline{Im(Q^*)}$, such that $p^*_{1,\omega} z = x + y$. Then, for any $\varepsilon > 0$, there exists $\tilde{y} \in L^2(a, b; \mathbb{R}^m)$, s.t. $\|\tilde{y} - y\| < \varepsilon$. Hence, we have

$$\left\| H(x + \tilde{y}) - p^*_{1,\omega} z \right\|_{L^2(\omega)} < \varepsilon,$$

that is, $L^2(\omega) \subseteq \overline{Im(H)}$. Therefore,

$$\overline{Im(p_{1,\omega} Q^*)} = L^2(\omega). \tag{3.17}$$

Combining (3.17) with $(ii) \Rightarrow (iii)$, we conclude $(ii) \Leftrightarrow (iii)$.

$(ii) \Leftrightarrow (iv)$: For any $y, z \in L^2(\omega)$, we have

$$\langle HH^* y, z\rangle = \langle y, HH^* z\rangle$$

and $\langle HH^* y, y\rangle = \langle H^* y, H^* y\rangle$. These, together with the equivalence between $\overline{Im(H)} = L^2(\omega)$ and the domain of $HH^*$ is dense in $L^2(\omega)$ lead to the result. □

Next, we discuss the description of the sensors in the case that the studied system is regionally observable and provide the minimum $m$ to determine the initial value.

**Definition 3.2.** [3] The sensor (sensors) is (are) said to be $\omega$-strategic if system (2.1)–(2.2) is regionally approximately observable in $\omega$.

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots$ are the eigenvalues of $-A$ with the corresponding multiplicities $r_1, r_2, \ldots, r_k, \ldots$, satisfying $0 < \lambda_1 < \ldots < \lambda_k < \ldots$, and $\lim_{k\to\infty} \lambda_k = \infty$. The orthonormal eigenfunctions $\alpha_{kj}(x)$, $j = 1, 2, \ldots, r_k$ corresponding to $\lambda_k$, for $k = 1, 2, \ldots$ form an orthonormal basis of $L^2(\Omega)$. So we know, for any $y_0(x) \in L^2(\Omega)$,

$$S_\alpha(t) y_0(x) = \sum_{k=1}^{\infty} \sum_{j=1}^{r_k} E_\alpha\left(-\lambda_j t\right) \langle y_0(x), \alpha_{kj}(x)\rangle \alpha_{kj}(x).$$

In [37], El Jai et al. put forward that a sensor can be characterized by $(P, d)$, where $P \subseteq \Omega$ stands for the location of the sensor and $d$ is its corresponding spatial distribution. Thus, we consider system (2.1) with $m$ zone sensors $z(t) = (z_1(t), z_2(t), \ldots, z_m(t))^\top$, where $z_i(t) = \int_{P_i} d_i(x) y(x, t) dx$, $i = 1, 2, \ldots, m$.

**Theorem 3.3.** *Denote* $d^i_{kj}(x) = \langle \chi_{P_i} d_i(x), \alpha_{kj}(x)\rangle$ *and define*

$$D_k = \begin{bmatrix} d^1_{k1}(x) & \cdots & d^1_{kr_k}(x) \\ \vdots & \cdots & \vdots \\ d^m_{k1}(x) & \cdots & d^m_{kr_k}(x) \end{bmatrix}.$$

*Then the sensors* $(P_i, d_i(x))$, $i = 1, 2, \ldots, m$ *are* $\omega$*-strategic if and only if*

$$m \geq r \triangleq \sup\{r_k\} \;\; and \;\; rank\, D_k = r_k, \quad for \;\; k = 1, 2, \ldots. \tag{3.18}$$

**Proof.** According to Definition 3.1 and 3.2, $(P_i, d_i(x))$, $i = 1, 2, \ldots, m$ are $\omega$-strategic if and only if

$$Ker(H^*) = Ker\left(CS_\alpha\left(\log\frac{t}{a}\right)p^*_{1,\omega}\right) = \{0\}. \tag{3.19}$$

For any $y \in L^2(\Omega)$, according to

$$H^*y(x) = C\sum_{k=1}^{\infty}\sum_{j=1}^{r_k} E_\alpha\left(-\lambda_j\left(\log\frac{t}{a}\right)^\alpha\right)\langle p^*_{1,\omega}y(x),\ \alpha_{kj}(x)\rangle\alpha_{kj}(x), \tag{3.20}$$

we have the equivalence between (3.19) and

$$\sum_{k=1}^{\infty}\sum_{j=1}^{r_k} E_\alpha\left(-\lambda_j\left(\log\frac{t}{a}\right)^\alpha\right)D_k y_k = 0 \quad\Rightarrow\quad y = 0, \tag{3.21}$$

where $y_k = \left(y_{k1}(x), y_{k2}(x), \ldots, y_{kr_k}(x)\right)^T$ and $y_{kj}(x) = \left\langle p^*_{1,\omega}y(x),\ \alpha_{kj}(x)\right\rangle$, $j = 1, 2, \ldots, r_k$. Now, we only need to prove (3.18)⇔(3.21).

(3.21)⇒(3.18): If $m < sup\{r_k\}$, that is, there exists an integer number $\widetilde{k}$, such that $m < r_{\widetilde{k}}$. Then there is a nonzero $\widetilde{y} \in L^2(\omega)$, s.t. $D_{\widetilde{k}}y_{\widetilde{k}} = 0$, i.e.,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{r_k} E_\alpha\left(-\lambda_j\left(\log\frac{t}{a}\right)^\alpha\right)D_{\widetilde{k}}y_{\widetilde{k}} = 0,$$

which leads to a contradiction.

If $m \geq sup\{r_k\}$, there is an integer number $\widetilde{k}$, such that $rank\, D_{\widetilde{k}} < r_{\widetilde{k}}$. From the knowledge of linear algebra, we can easily know that there is a nonzero $\widetilde{y} \in L^2(\omega)$, s.t. $D_{\widetilde{k}}y_{\widetilde{k}} = 0$, which contradicts to (3.21).

(3.18)⇒(3.21): If there exists a nonzero $\widetilde{y} \in L^2(\omega)$, such that for some $\widetilde{k}$, $y_{\widetilde{k}} \neq 0$ and

$$\sum_{k=1}^{\infty}\sum_{j=1}^{r_k} E_\alpha\left(-\lambda_j\left(\log\frac{t}{a}\right)^\alpha\right)D_{\widetilde{k}}y_{\widetilde{k}} = 0.$$

Since $E_\alpha\left(-\lambda_j\left(\log\frac{t}{a}\right)^\alpha\right) > 0$, for all $t \in [a, b]$, we immediately have $D_{\widetilde{k}}y_{\widetilde{k}} = 0$. Then $rank\, D_{\widetilde{k}} < r_{\widetilde{k}}$, which leads to a contradiction. And the proof is completed. □

**Remark 3.1.** If for every $k$, $\lambda_k$ is a single value, that is, $r_k = 1$, then Theorem 3.3 shows that the initial condition $y_0^1(x)$ of system (2.1)–(2.2) can be reconstructed by one sensor; and if there is a $\widetilde{k}$, such that the multiplicity of $\lambda_{\widetilde{k}}$ is infinite, then the number of sensors should be infinite.

The last part in this section is to provide a method to reconstruct $y_0^1(x)$ on $\omega$. The HUM, which was first introduced by Lions [35,38], is the main approach to be applied here. Moreover, the residual initial condition in $\Omega\backslash\omega$ doesn't need to be taken into account.

Define $G = Im(H)$. For any $g \in G^*$, where $G^*$ is the dual space of $G$, consider the following system

$$\begin{cases} {}_a^{HC}D_t^\alpha \varphi(x,t) = A\varphi(x,t) \text{ in } U, \\ \varphi(x,a) = p^*_\omega g(x) \text{ in } \Omega, \\ \varphi(\xi,t) = 0 \text{ on } \Sigma. \end{cases} \tag{3.22}$$

Define

$$\| g \|^2_{G^*} \triangleq \int_a^b \left\| \frac{1}{t}C\varphi\left(x, \frac{ab}{t}\right)\right\|^2 dt, \tag{3.23}$$

then we have the following lemma.

**Lemma 3.1.** *If system (2.1)–(2.2) is regionally approximately observable in $\omega$, then (3.23) defines a norm on $G^*$.*

**Proof.** We can easily prove that (3.23) is a semi-norm on $G^*$. According to Lemma 2.1, the solution of (3.22) is given by

$$\varphi(x,t) = S_\alpha\left(\log\frac{t}{a}\right)p^*_{1,\omega}g(x). \tag{3.24}$$

Since system (2.1)–(2.2) is regionally approximately observable in $\omega$, from Definition 3.2, we have

$$Ker(H^*) = Ker\left(CS_\alpha\left(\log\frac{t}{a}\right)p^*_{1,\omega}\right) = \{0\}. \tag{3.25}$$

This, together with $\| g \|_{G^*} = 0 \Leftrightarrow CS_\alpha(\log\frac{b}{t})p^*_{1,\omega}g = 0$, leads to

$$\| g \|_{G^*} = 0 \quad \Rightarrow \quad g = 0. \tag{3.26}$$

Therefore, (3.23) defines a norm on $G^*$. □

Now, consider about the following system

$$\begin{cases} {}_a^{HC}D_t^\alpha \Psi(x,t) = A^*\Psi(x,t) + C^*\nu(x,t) \text{ in } U, \\ \Psi(x,a) = 0 \text{ in } \Omega, \\ \Psi(\xi,t) = 0 \text{ on } \Sigma, \end{cases} \tag{3.27}$$

which can be regarded as the dual system of (2.1), where

$$\nu(x,t) = \frac{1}{t}C\varphi(x,\frac{ab}{t}) = \frac{1}{t}CS_\alpha\left(\log\frac{b}{t}\right)p^*_{1,\omega}g(x).$$

For any $g \in G^*$, define

$$Fg \triangleq p_{1,\omega}\,{}_a^H I_t^{1-\alpha}\Psi(x,b). \tag{3.28}$$

Then, according to the HUM, system (2.1)–(2.2) is regionally observable if and only if (3.28) has a solution in $G^*$.

**Theorem 3.4.** *If system (2.1)–(2.2) is regionally approximately observable in $\omega$, then (3.28) has a unique solution $g \in G^*$. Furthermore, the initial vector can be reconstructed by*

$$y_0^1 = g.$$

The following lemma plays a key role to prove Theorem 3.4.

**Lemma 3.2.** *For any $t \in [a, b]$ and $\alpha \in (0, 1)$, the solution $\Psi(x, t)$ of system (3.27) has the property*

$$ {}_a^H I_t^{1-\alpha}\Psi(x,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\int_a^t E_\alpha\left(-\lambda_j\left(\log\frac{t}{\tau}\right)^\alpha\right)\langle\nu(x,\tau),\,\alpha_{kj}(x)\rangle\frac{d\tau}{\tau}\cdot\alpha_{kj}(x). \tag{3.29}$$

**Proof.** According to Lemma 2.1, the solution of (3.27) can be expressed by

$$\begin{aligned}\Psi(x,t) &= \int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}K_\alpha^*\left(\log\frac{t}{s}\right)C^*\nu(x,s)\frac{ds}{s} \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_j\left(\log\frac{t}{s}\right)^\alpha\right)\langle C^*\nu(x,s),\,\alpha_{kj}(x)\rangle\frac{ds}{s}\cdot\alpha_{kj}(x).\end{aligned} \tag{3.30}$$

From (2.4) and (3.30), we have

$$\begin{aligned}{}_a^H I_t^{1-\alpha}\Psi(x,t) &= \frac{1}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\Psi(x,s)\frac{ds}{s} \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\int_a^t\int_\tau^t\sum_{n=0}^{\infty}\frac{(-\lambda_j)^n(\log\frac{t}{s})^{-\alpha}(\log\frac{s}{\tau})^{n\alpha+\alpha-1}}{\Gamma(n\alpha+\alpha)\Gamma(1-\alpha)}\frac{ds}{s}\cdot\langle\nu(x,\tau),\,\alpha_{kj}(x)\rangle\frac{d\tau}{\tau}\cdot\alpha_{kj}(x) \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\int_a^t\sum_{n=0}^{\infty}\frac{(-\lambda_j)^n h(t,\tau)}{\Gamma(n\alpha+\alpha)B(n\alpha+\alpha,1-\alpha)}\cdot\langle\nu(x,\tau),\,\alpha_{kj}(x)\rangle\frac{d\tau}{\tau}\cdot\alpha_{kj}(x),\end{aligned} \tag{3.31}$$

where $B(n\alpha+\alpha, 1-\alpha)$ is the Beta function defined by

$$B(n\alpha+\alpha,1-\alpha) = \int_0^1 x^{n\alpha+\alpha-1}(1-x)^{-\alpha}dx \tag{3.32}$$

and

$$\begin{aligned}h(t,\tau) &= \int_\tau^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{\tau}\right)^{n\alpha+\alpha-1}\frac{ds}{s} \\ &= \left(\log\frac{t}{\tau}\right)^{n\alpha}\int_0^1 u^{n\alpha+\alpha-1}(1-u)^{-\alpha}du \\ &= \left(\log\frac{t}{\tau}\right)^{n\alpha}B(n\alpha+\alpha,1-\alpha).\end{aligned} \tag{3.33}$$

The second equality in (3.33) is provided by variable transformation $u = \frac{\log s - \log\tau}{\log t - \log\tau}$.

Substituting (3.32) and (3.33) into (3.31), we get

$$ {}_{a}^{H}I_{t}^{1-\alpha}\Psi(x,t)=\sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\int_a^t E_\alpha\left(-\lambda_k\left(\log\frac{t}{\tau}\right)^{\alpha}\right)\cdot\langle\nu(x,\tau),\,\alpha_{kj}(x)\rangle\frac{d\tau}{\tau}\cdot\alpha_{kj}(x). \tag{3.34}$$

The proof is completed. □

Now, we prove Theorem 3.4.

**Proof of** Theorem 3.4. Since system (2.1)–(2.2) is regionally approximately observable in $\omega$, from Lemma 3.1, we know $\|\cdot\|_{G^*}$ defines a norm on $G^*$.

According Lemma 3.2, $\forall g\in G^*$, we have

$$\begin{aligned}\langle Fg,g\rangle&=\left\langle p_{1,\omega}\,{}_{a}^{H}I_{t}^{1-\alpha}\Psi(x,b),\,g\right\rangle\\&=\left\langle p_{1,\omega}\sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\int_a^t E_\alpha\left(-\lambda_k\left(\log\frac{t}{\tau}\right)^{\alpha}\right)\cdot\langle\nu(x,\tau),\,\alpha_{kj}(x)\rangle\frac{d\tau}{\tau}\cdot\alpha_{kj}(x),\,g(x)\right\rangle\\&=\left\langle\int_a^b\frac{1}{\tau^2}S_\alpha^*\left(\log\frac{b}{\tau}\right)C^*CS_\alpha^*\left(\log\frac{b}{\tau}\right)p_{1,\omega}^*g(x)d\tau,\,p_{1,\omega}^*g(x)\right\rangle\\&=\int_a^b\left\|\frac{1}{t}C\varphi\left(x,\frac{ab}{t}\right)\right\|^2dt=\|g\|_{G^*}^2.\end{aligned}\tag{3.35}$$

Applying Lemma 2.2, (3.28) has a unique solution $g(x)=y_0^1(x)$ and the initial vector $y(x)$ in $\omega$ is also estimated.

## 4. Regional gradient observability

First, we introduce some new definitions necessary in this section and need to modify parts of definitions and assumptions due to the gradient operator.

Similar to (3.10), the output function is given by

$$z(x,t)=CS_\alpha\left(\log\frac{t}{a}\right)y_0(x)\triangleq\widetilde{Q}(t)y_0(x),\tag{4.36}$$

where $\widetilde{Q}:H_0^1(\Omega)\to L^2(a,b;\mathbb{R}^m)$. Moreover, the adjoint operator of $\widetilde{Q}$ has the same representation as that of $Q$, shown in (3.11).

**Definition 4.1.** [39] $\nabla:H_0^1(\Omega)\to(L^2(\Omega))^n$ is the gradient operator defined by

$$\nabla y\triangleq\left(\frac{\partial y}{\partial x_1},\frac{\partial y}{\partial x_2},\cdots,\frac{\partial y}{\partial x_n}\right).$$

And $\nabla^*:\left(L^2(\Omega)\right)^n\to H^{-1}(\Omega),\ y\mapsto h$, the adjoint operator of $\nabla$, is given by the unique solution of

$$\begin{cases}\Delta h=-div\,y & in\ \Omega,\\ h=0 & on\ \partial\Omega.\end{cases}$$

**Definition 4.2.** [3,25] System (2.1)–(2.2) is said to be regionally gradient exactly observable in $\omega$ and respectively, regionally gradient approximately observable in $\omega$, if the gradient of $y_0^1(x)$ can be uniquely determined by the observation $z(x, t)$ and respectively,

$$Ker\left(\widetilde{Q}\nabla^*p_\omega^*\right)=\{0\}.$$

In addition, we assume $y_0(x)\in H_0^1(\Omega)$. Then, the regional gradient observability problem aims to reconstruct $\nabla y_0^1$, the gradient of the initial vector in $\omega$. Here,we define $\widetilde{H}:L^2(a,b;\mathbb{R}^m)\to(L^2(\omega))^n,\ z\mapsto p_\omega\nabla\widetilde{Q}^*z$ such that $\widetilde{H}z=\nabla y_0^1$. Then we have the following theorems for the regional gradient observability.

**Theorem 4.1.** *The following statements are equivalent:*

*(i) System (2.1)–(2.2) is regionally gradient exactly observable in $\omega$;*

*(ii) $Im(\widetilde{H})=(L^2(\omega))^n$;*

*(iii) $Ker(p_\omega)+Im(\nabla\widetilde{Q}^*)=(L^2(\Omega))^n$;*

*(iv) There is a constant $c>0$ such that, for any $z\in(L^2(\omega))^n$,*

$$\|z\|_{(L^2(\omega))^n}\le c\|\widetilde{H}^*z\|_{L^2(a,b;\mathbb{R}^m)}.\tag{4.37}$$

**Theorem 4.2.** *The following statements are equivalent:*

*(i) System (2.1)–(2.2) is regionally gradient approximately observable in $\omega$;*

*(ii)* $\overline{Im(\widetilde{H})} = \left(L^2(\omega)\right)^n$;
*(iii)* $Ker(p_\omega) + \overline{Im(\nabla \widetilde{Q}^*)} = \left(L^2(\Omega)\right)^n$;
*(iv)* $\widetilde{H}\widetilde{H}^*$ *is a positive definite operator;*
*(v) Suppose* $y \in (L^2(\omega))^n$*. If for any* $z \in (L^2(\Omega))^n$,

$$\left\langle \widetilde{H}z, y \right\rangle = 0,$$

*then* $y = 0$.

The proofs of Theorems 4.1 and 4.2 can be given similarly to those of Theorems 3.1 and 3.2, respectively, with some modifications due to the gradient operator $\nabla$, its adjoint operator $\nabla^*$ and different Hilbert spaces. Hence, we omit it.

**Remark 4.1.** For both regional observability and regional gradient observability, there are certain systems which are not (gradient) observable on $\Omega$, but regionally (gradient) observable on $\omega \subsetneqq \Omega$. An example will be given to illustrate this situation later in Section 5.

Then, we aim to link the characteristic of sensors with the regional gradient approximate observability of the system under consideration. Suppose that the measurements of the system are given by $m$ zone sensors $(P_i, d_i(x))$, $i = 1, \dots, m$, that is

$$z_i(t) = \langle y(x,t), d_i(x) \rangle_{L^2(P_i)},$$

for $i = 1, \dots, m$.

**Definition 4.3.** The sensor (sensors) is (are) said to be gradient $\omega$-strategic if system (2.1)–(2.2) is regionally gradient approximately observable in $\omega$.

**Theorem 4.3.** *Define*

$$D_k^l \triangleq \frac{\partial}{\partial x_l} \begin{bmatrix} d_{k1}^1(x)\alpha_{k1}(x) & \cdots & d_{kr_k}^1(x)\alpha_{kr_k}(x) \\ \vdots & \cdots & \vdots \\ d_{k1}^m(x)\alpha_{k1}(x) & \cdots & d_{kr_k}^m(x)\alpha_{kr_k}(x) \end{bmatrix},$$

*where* $d_{kj}^i(x) \triangleq \langle \chi_{P_i} d_i(x),\ \alpha_{kj}(x) \rangle$, $l = 1, 2, \dots, n$*. Then* $(P_i, d_i(x))$, $i = 1, 2, \dots, m$ *are gradient* $\omega$*-strategic if and only if for any* $y \in (L^2(\omega))^n$,

$$\sum_{k=1}^{\infty} \sum_{j=1}^{r_k} E_\alpha(-\lambda_k t^\alpha) \sum_{l=1}^{n} D_k^l \Phi_{kl} = 0 \ \Rightarrow \ y = 0, \tag{4.38}$$

*where* $\Phi_{kl} \triangleq \left(y_{k1l}, \dots, y_{kr_k l}\right)^\top$, $y_{kjl} \triangleq \left\langle p_{1,\omega}^* y_l,\ \alpha_{kj}(x) \right\rangle$ *and* $y = (y_1, y_2, \dots, y_n)^\top$, $l = 1, 2, \dots, n$*. When* $n = 1$, *(4.38) is equivalent to*

$$m \geq r \triangleq \sup\{r_k\} \text{ and } \operatorname{rank} D_k^1 = r_k,$$

*for* $k = 1,\ 2, \dots$.

**Proof.** According to Definition 4.3 and Theorem 4.2, we know $(P_i,\ d_i(x))$, $i = 1, 2, \dots, m$ are gradient $\omega$-strategic if and only if the following condition holds.

If $y \in (L^2(\omega))^n$ satisfies that for any $z \in (L^2(\Omega))^n$, $\left\langle \widetilde{H}z, y \right\rangle = 0$, then $y = 0$.

From the definition of the adjoint operator and the characteristics of the sensors, we have

$$C^* z(t) = \sum_{i=1}^{m} \chi_{P_i} d_i(x) z_i(t). \tag{4.39}$$

These, together with (3.11), lead to

$$\begin{aligned} \left\langle \widetilde{H}z, y \right\rangle &= \left\langle \nabla \int_0^{\log \frac{b}{a}} ae^s S_\alpha^*(s) \sum_{i=1}^{m} \chi_{P_i} d_i(x) z_i(ae^s) ds,\ p_\omega^* y \right\rangle \\ &= \left\langle \nabla \left( \int_0^{\log \frac{b}{a}} ae^s \sum_{k=1}^{\infty} \sum_{j=1}^{r_k} \sum_{i=1}^{m} E_\alpha(-\lambda_k s^\alpha) \left\langle \chi_{P_i} d_i(x) z_i(ae^s),\ \alpha_{kj}(x) \right\rangle ds \cdot \alpha_{kj}(x) \right),\ p_\omega^* y \right\rangle \\ &= \sum_{l=1}^{n} \left\langle \sum_{k=1}^{\infty} \sum_{j=1}^{r_k} \sum_{i=1}^{m} \int_0^{\log \frac{b}{a}} ae^s E_\alpha(-\lambda_k s^\alpha) z_i(ae^s) ds \cdot \frac{\partial}{\partial x_l} \left( d_{kj}^i(x) \alpha_{kj}(x) \right),\ p_{1,\omega}^* y_l \right\rangle = 0. \end{aligned} \tag{4.40}$$

According to Lemma 1 of Chapter 5 in [40], the arbitrariness of $z$ leads to the equivalence between (4.40) and

$$\sum_{k=1}^{\infty}\sum_{j=1}^{r_k} E_\alpha(-\lambda_k t^\alpha)\sum_{l=1}^{n} D_k^l \Phi_{kl} = 0$$

and the proof of the first part is completed.

When $n = 1$, then (4.38) reduces to

$$\sum_{k=1}^{\infty}\sum_{j=1}^{r_k} E_\alpha(-\lambda_k t^\alpha) D_k^1 \Phi_{k1} = 0 \;\Rightarrow\; y = 0. \tag{4.41}$$

Since for any $t\in[a, b]$, $E_\alpha(-\lambda_k t^\alpha) > 0$, the equivalence between (4.41) and

$$m \geq \sup\{r_k\},\;\; rank\, D_k^1 = r_k, \quad k = 1, 2, \ldots,$$

can be obtained similar to that between (3.18) and (3.21) in Theorem 3.3. □

Finally, we provide an approach to reconstruct the gradient of the initial vector in $\omega$. Without taking the residual initial gradient in $\Omega\backslash\omega$ into consideration, define $V \triangleq \{f \in (L^2(\Omega))^n | f = 0 \; in \; \Omega\backslash\omega\} \bigcap \{\nabla f | f \in H_0^1(\Omega)\}$. Therefore, for any $h\in V$, there exists $\widetilde{h} \in H_0^1(\Omega)$, such that $\widetilde{h} = \nabla^* p_\omega^* h$. Then, consider the system

$$\begin{cases} {}_a^{HC}D_t^\alpha \varphi(x,t) = A\varphi(x,t) \text{ in } U, \\ \varphi(x,a) = \widetilde{h}(x) \text{ in } \Omega, \\ \varphi(\xi,t) = 0 \text{ on } \Sigma, \end{cases} \tag{4.42}$$

which admits a unique solution $\varphi \in C(\Omega \times [a,b]) \bigcap L^2\big(a, b; H_0^1(\Omega)\big)$ presented by

$$\varphi(x,t) = S_\alpha\left(\log\frac{t}{a}\right)\widetilde{h}(x) = S_\alpha\left(\log\frac{t}{a}\right)\nabla^* p_\omega^* h(x). \tag{4.43}$$

Then we consider the semi-norm on $V$

$$\| h \|_V^2 \triangleq \int_a^b \left\| \frac{1}{t} C S_\alpha\left(\log\frac{b}{t}\right)\nabla^* p_\omega^* h(x)\right\|^2 dt \tag{4.44}$$

and have the following lemma.

**Lemma 4.1.** *If system (2.1)–(2.2) is regionally gradient approximately observable in $\omega$, then (4.44) defines a norm on $V$.*

Similar to Lemma 3.1, we can easily conclude the result with certain revisions due to the gradient operator and omit the proof. Changing the map $F$ in (3.28) to

$$Fh = p_\omega \nabla\psi(x,b), \tag{4.45}$$

where $\psi(x, t)$ is the solution of (3.27). What needs to be explained is that $\nu(x, t)$ in (3.27) is obtained from the solution of (4.42), other than the solution of (3.22). Similarly, it can be shown that

$$\langle Fh, h\rangle_{(L^2(\Omega))^n} = \|h\|_V^2, \tag{4.46}$$

that is, $F$ is coercive from $V$ to $V^*$. Hence, we conclude the following theorem to estimate $\nabla y_0^1(x)$ according to Lemma 2.2. Since the proof of the following theorem is very similar to that of Theorem 3.4, we omit it.

**Theorem 4.4.** *If system (2.1)–(2.2) is regionally gradient approximately observable in $\omega$, then (4.45) has a unique solution $h\in V$. Furthermore, the gradient of initial vector can be reconstructed by*

$$\nabla y_0^1(x) = h(x).$$

## 5. An example

Consider the following one-dimensional diffusion system with one zone sensor:

$$\begin{cases} {}_2^{HC}D_t^{0.5} y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t) \text{ in } [-1,1]\times[2,4], \\ y(x,\,2) = y_0(x),\;\; x\in[-1,1], \\ y(-1,t) = y(1,t) = 0,\;\; t\in[2,4], \\ z(t) = Cy(x,t) = \int_P y(x,t)dx,\;\; t\in[2,4], \end{cases} \tag{5.47}$$

where $P = [p_1, p_2] \subseteq [-1, 1]$.

Here, $A = \frac{\partial^2}{\partial x^2}$, the one-dimension Laplace operator, whose eigenvalue $\lambda_k$ and the corresponding eigenfunction $\alpha_k(x)$ are known as $\lambda_k = -k^2\pi^2$, and $\alpha_k(x) = \sqrt{2}\sin(k\pi x)$, for $k = 1, 2, \dots$ and $x \in [-1, 1]$. It can be obtained that for any $y_0(x) \in L^2([-1, 1])$,

$$\begin{aligned} Q(t)y_0(x) &= CS_\alpha\left(\log\frac{t}{a}\right)y_0(x) \\ &= 2\sum_{k=1}^{\infty} E_{0.5}\left(-k^2\pi^2\left(\log\frac{t}{2}\right)^{0.5}\right)\langle y_0(x), \sin(k\pi x)\rangle \int_{p_1}^{p_2}\sin(k\pi x)dx \end{aligned}$$

and

$$\tilde{Q}(t)\nabla^* y_0(x) = 2\sum_{k=1}^{\infty} E_{0.5}\left(-k^2\pi^2\left(\log\frac{t}{2}\right)^{0.5}\right)\langle \nabla^* y_0(x), \sin(k\pi x)\rangle \int_{p_1}^{p_2}\sin(k\pi x)dx.$$

First, we consider the approximate observability and gradient approximate observability on the whole region, that is, $p_1 = -1,\ p_2 = 1$. Then, we have

$$Q(t)y_0(x) \equiv 0 \ \ and \ \ \tilde{Q}(t)\nabla^* y_0(x) \equiv 0, \quad \forall y_0 \in L^2([-1, 1]), \tag{5.48}$$

which implies that $Ker(Q) \neq \{0\}$ and $Ker(\tilde{Q}\nabla^*) \neq \{0\}$. From Definition 3.1 and 4.2, we conclude that system (5.47) is neither approximate observable nor gradient approximate observable on the whole domain.

Next, we show that system (5.47) is both regionally approximately observable and regionally gradient approximately observable on a subregion $\omega \subsetneqq [-1, 1]$.

Let $p_1 = 0,\ p_2 = 1$. Choose $y_0(x) = \sin m\pi x$, where $m$ is an integer number. Then we have

$$Q(t)y_0(x) = \sum_{k \neq 2q} E_{0.5}\left(-k^2\pi^2\left(\log\frac{t}{2}\right)^{-0.5}\right)M_{km} \neq 0,$$

where $q = 1, 2, \dots$ and $M_{km} = \frac{1}{k\pi}\left(\frac{\sin(m+k)\pi}{(k+m)\pi} - \frac{\sin(m-k)\pi}{(m-k)\pi}\right)$. Hence, $Ker(Qp^*_{1,[0,1]}) = \{0\}$ and $y_0(x)$ is regionally approximately observable on [0, 1].

Choose $y_0(x) = \cos m\pi x$. Then

$$\tilde{Q}(t)\nabla^* y_0(x) = m\pi \sum_{k \neq 2q} E_{0.5}\left(-k^2\pi^2\left(\log\frac{t}{2}\right)^{-0.5}\right)M_{km} \neq 0.$$

Hence, $\cos m\pi x$ is regionally gradient approximately observable on [0, 1].

Since the eigenvalues of $\frac{\partial^2}{\partial x^2}$ are all single, that is, $r_k = 1$ for all $k$, then $r = 1$ and $rank D_k^1 = 1$. Hence, from Theorems 3.3 and 4.3, we see that the zone sensor is both [0,1]-strategic and gradient [0,1]-strategic. The above results coincide with Remark 3.1 and explain Remark 4.1.

Finally, we reconstruct the initial condition and its gradient in [0,1]. On one hand, when (5.47) is regionally approximately observable in [0,1], it follows from Lemma 3.1 that for $g \in G^*$, (3.23) defines a norm on $G^*$, where $\varphi(x, t)$ solves

$$\begin{cases} {}^{HC}_{\ 2}D_t^{0.5}\varphi(x,t) = \dfrac{\partial^2}{\partial x^2}\varphi(x,t) \text{ in } [-1, 1] \times [2, 4], \\ \varphi(x, 2) = p^*_{1,[0,1]}g(x) \text{ in } [-1, 1], \\ \varphi(-1, t) = \varphi(1, t) = 0 \text{ on } [2, 4]. \end{cases} \tag{5.49}$$

Then the regionally observable problem is equivalent to solve

$$Fg = p_{1,[0,1]}{}^{H}_{2}I_t^{0.5}\Psi(x, 4), \tag{5.50}$$

where $\Psi(x, t)$ is the solution of

$$\begin{cases} {}^{HC}_{\ 2}D_t^{0.5}\Psi(x,t) = \dfrac{\partial^2}{\partial x^2}\Psi(x,t) + \dfrac{1}{t}C^*C\varphi\left(x, \dfrac{8}{t}\right) \text{ in } [-1, 1] \times [2, 4], \\ \Psi(x, 2) = 0 \text{ in } [-1, 1], \\ \Psi(-1, t) = \Psi(1, t) = 0 \text{ on } [2, 4]. \end{cases} \tag{5.51}$$

From Theorem 3.4, we conclude that $F$ is an isomorphism. Thus, (5.50) exists a unique solution $g \in G^*$ and $y_0^1(x) = g(x)$ provides the initial condition in [0,1].

On the other hand, according to Lemma 4.1, (4.44) defines a norm on $V$ provided by the regional gradient approximate observability of system (5.47) on [0,1]. Consider the system

$$\begin{cases} {}_{2}^{HC}D_t^{0.5}\Psi(x,t) = \dfrac{\partial^2}{\partial x^2}\Psi(x,t) + \dfrac{1}{t}C^*CS_{0.5}\left(\log\dfrac{4}{t}\right)\nabla^* p^*_{1,[0,1]}h(x) \text{ in } [-1,1]\times[2,4], \\ \Psi(x,2) = 0 \text{ in } [-1,1], \\ \Psi(-1,t) = \Psi(1,t) = 0 \text{ on } [2,4], \end{cases} \tag{5.52}$$

It follows from Theorem 4.4 that $F$: $h \mapsto p_{1,[0,1]}\nabla\psi(x,4)$ has a unique solution in $V$, which is the initial gradient $\frac{d}{dx}y_0^1(x)$ in [0,1].

## 6. Conclusion

In this paper, we have explored the regional observability for time fractional DPSs involving an Hadamard-Caputo time fractional derivative. The regional (gradient) exact and approximate observability results are guaranteed by several necessary and sufficient conditions. The minimum number of sensors are also derived to achieve the regional (gradient) approximately observability. By utilizing the HUM, the initial vector and its gradient are both reconstructed. Finally, an example shows the applications of our results in practical models. In further works, investigating the numerical solutions for the Hadamard-Caputo time fractional distributed parameter systems is of great interest. Since the Hadamard-Caputo fractional derivative is a particular case of the Caputo-Katugampola fractional derivative with $\rho \to 0^+$, the numerical methods discussed in [41] can be extended and applied to the investigation on the numerical solutions of a Hadamard-Caputo time fractional distributed parameter system, such as (5.47).

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