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Mittag-Leffler stabilization for an unstable time-fractional anomalous diffusion equation with boundary control matched disturbance

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Summary

This paper addresses the Mittag-Leffler stabilization for an unstable time-fractional anomalous diffusion equation with boundary control subject to the control matched disturbance. The active disturbance rejection control (ADRC) approach is adopted for developing the control law. A state-feedback scheme is designed to estimate the disturbance by constructing two auxiliary systems: One is to separate the disturbance from the original system to a Mittag-Leffler stable system and the other is to estimate the disturbance finally. The proposed control law compensates the disturbance using its estimation and stabilizes system asymptotically. The closed-loop system is shown to be Mittag-Leffler stable and the constructed auxiliary systems in the closed loop are proved to be bounded. This is the first time for ADRC to be applied to a system described by the fractional partial differential system without using the high gain.

KEYWORDS

active disturbance rejection control (ADRC), Mittag-Leffler stabilization, time-fractional anomalous diffusion equation

AMS CLASSIFICATION

37L15, 93D15, 93B52

1 | INTRODUCTION

Stabilization for systems with fractional derivative is one of the fundamental issues in control theory, which has been attracted increasing interests from the control community and many interesting results have been reported in literatures.[1-4] In the work of Li et al,[1] a stabilizing controller for finite dimensional fractional-order system was constructed by utilizing the fractional Lyapunov approach and linear matrix inequality. In the works of He et al,[2,3] asymptotic stability criterions for several classes of nonlinear fractional systems were presented by exploiting two types of fractional Halanay inequalities with bounded/unbounded time-varying delay. In the work of Liang et al,[4] boundary stabilization for time fractional diffusion-wave equation was investigated by numerical simulations, which is the first attempt on stabilization of fractional partial differential equations (PDEs). Recently, boundary feedback stabilization for unstable time fractional reaction diffusion equations was presented in the work of Zhou and Guo,[5] where the backstepping method and Riesz basis method were applied. The same problem was considered in more smooth state space $H^1(0,1)$ in the work of Ge et al,[6] where two types of boundary control conditions with collocated/noncolocated boundary output were investigated. In both papers,[5,6] the diffusion coefficient was supposed to be a constant. Very recently, the spatially-varying diffusion coefficient fractional reaction diffusion was proposed in the work of Chen et al.[7] For controllability and observability aspects for fractional PDEs, the reader can refer to other works.[8-10] However, these results were developed for deterministic fractional system or fractional PDEs without uncertainty.

To cope with internal uncertainty and external disturbance, many control methods have been proposed for fractional-order systems with disturbance, such as integral sliding mode control,[11] adaptive control method,[12,13] and robust control.[14] Over the past few decades, a control technology named active disturbance rejection control (ADRC) was firstly proposed by Han.[15] The key idea of the ADRC is the estimation/cancelation strategy, which is a cost-effective way in dealing with uncertainty and reduces the control energy significantly in practice.[16] The uncertainties dealt with by the ADRC are much more complicated, which can be the coupling of the external disturbances and the system unmodeled dynamics. One of the most striking features of ADRC is that the disturbance is estimated, in real time, through an extended state observer and is canceled in the feedback loop. This has been proved to be an effective tool in stabilizing the one dimensional PDEs,[17-20] multi-dimensional PDEs,[21,22] stochastic differential equations,[23,24] among many others. The generalization of ADRC to the systems described by fractional differential equations can also be found in recent works[25-27] but the ADRC for fractional PDEs has not yet been addressed, which motivates this study.

The time-fractional anomalous diffusion equations can better characterize a subdiffusion process that describes the continuous time random walks phenomenon such as the particles to jump at fixed-time intervals with an incorporating waiting times.[28,29] When there is no disturbance, the stabilization of time-fractional anomalous diffusion equations has been investigated in the works of Zhou and Guo[5] and Chen et al.[7] To the best of authors' knowledge, stabilization of time-fractional anomalous diffusion equations with boundary disturbance has not been addressed so far. In this paper, we adopt the ADRC and the backstepping approach to achieve the Mittag-Leffler stability for a class of factional PDEs with disturbance.

We point out that the fractional ADRC proposed in other works[25-27] is much conservative in stabilizing fractional PDEs. In the work of Li et al,[25] a simple integer-order control scheme for a fractional-order system based on ADRC was developed by treating the fractional-order dynamics as a common disturbance, which is essentially the traditional ADRC.[15] In the works of Li et al[26] and Gao,[27] a fractional-order extended states observer, which generalizes the traditional extended state observer, was proposed and the linear bandwidth-parameterization method was presented to simplify the controller tuning. However, to guarantee the convergence of disturbance estimation error, all of the gain in other works[25-27] has to be high gain. Unfortunately, unlike PDEs,[17] the high-gain extended state observer seems unable to deal with disturbance in fractional PDEs (see Remark 4). In this paper, to overcome the difficulty of high-gain ADRC, we adopt the control design procedure used for stabilization of heat, wave equations in other works,[18,19,22,30] and propose a new fractional disturbance estimator, which is the key step, based on the hidden regularity of fractional PDEs.

We proceed as follows. In Section 2, the problem formulation and some backgrounds are presented. A fractional disturbance estimator for the time fractional-order anomalous diffusion equation (TFADE) with Neumann boundary control matched disturbance is designed in Section 3. Section 4 and Section 5 are attributed to feedback control design, the well-posedness and the stability, respectively. Section 6 discusses briefly the boundary feedback Mittag-Leffler stabilization of the alternative TFADE with the Dirichlet boundary control. To simplify the notation, all obvious domains both for time and spatial variables will be dropped in equations hereafter when there is no confusion.

2 | PROBLEM FORMULATION AND PRELIMINARIES

Consider the following one-dimensional TFADE with Neumann boundary control and boundary disturbance:

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t) + \lambda(x) w(x,t), \\ w_x(0,t) = -q w(0,t), \\ w_x(1,t) = u(t) + d(t), \\ w(x,0) = w_0(x), \end{cases} \tag{1}$$

where $x \in (0,1)$, $t \geq 0$, $w(x,t)$ is the state, $u(t)$ is the control input, $\lambda \in C[0,1]$, $d(t)$ represents an unknown external disturbance, which is only supposed to satisfy $d,\ {}_0^C D_t^\alpha d \in L^\infty(0,\infty)$. ${}_0^C D_t^\alpha w(x,t)$ stands for the Caputo derivative, which is

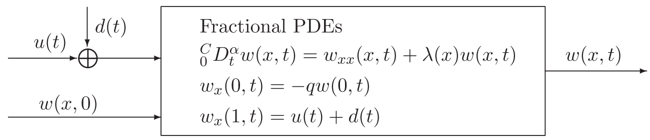

FIGURE 1 Block diagram of the open-loop system (1). PDEs, partial differential equations

a regularized fractional derivative of $w(x,t)$ with respect to time variable t, that is,

$$ {}_0^C D_t^\alpha w(x,t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha} w(x,s)\mathrm{d}s - t^{-\alpha} w(x,0)\right]. $$

It is well known that

$$ \lim_{\alpha \to 1^-} {}_0^C D_t^\alpha w(x,t) = \frac{\partial w(x,t)}{\partial t}. $$

Suppose that $\lambda(x) = \frac{\pi^2}{2}$, $q = 0$, and the initial value is taken as $w_0(x) = \sin(\frac{\pi}{2}(x-1))$. Then, system without control admits a solution $w(x,t) = E_\alpha(\frac{\pi^2}{4}t^\alpha)\sin(\frac{\pi}{2}(x-1))$ satisfying $\|w(\cdot,t)\|_{L^2(0,1)} \to \infty$ as $t \to \infty$, which shows that system (1) is unstable without control and disturbance. When the external disturbance flows in the control end, the stabilization problem for (1) becomes much more complicated. The objective of this paper is to design a state-feedback control law u so that the state of the system depicted in Figure 1 (Mittag-Leffler) converges to zero by rejecting the external disturbance $d(t)$.

Definition 1. The one-parameter Mittag-Leffler function and two-parameter Mittag-Leffler function are defined by

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad \text{and} \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, $$

respectively, where $\alpha > 0, \beta > 0$. In particular, $E_{\alpha,1}(z) = E_\alpha(z)$ and $E_1(z) = E_{1,1}(z) = e^z$.

For more properties of Mittag-Leffler functions, one can refer to the work of Gorenflo et al.[31]

Definition 2 (Mittag-Leffler Stability).
The solution of (1) is said to be Mittag-Leffler stable if

$$ \|w(\cdot,t)\|_{L^2(0,1)} \le \{m(\|w(\cdot,0)\|_{L^2(0,1)})E_\alpha(-\lambda t^\alpha)\}^b, $$

where $\alpha \in (0,1)$, $\lambda > 0$, $b > 0$, $m(0) = 0$, $m(s) \ge 0$, and $m(s)$ is locally Lipschitz in $s \in \mathbb{R}$ with Lipschitz constant m_0.

It is easy to verify that Mittag-Leffler stability implies Lyapunov asymptotic stability, ie, $\lim_{t\to\infty}\|w(\cdot,t)\|_{L^2(0,1)} = 0$. This is because there exists $M > 0$ such that $E_\alpha(-\lambda t^\alpha) \le \frac{M}{1+\lambda t^\alpha}$ for all $t \ge 0$.
Consider the following Cauchy problem in a Banach space H:

$$ \begin{cases} {}_0^C D_t^\alpha X(t) = A_0 X(t), \\ X(0) = x, \end{cases} \tag{2} $$

where A_0 is a closed-linear operator in H.

Definition 3. A function $X \in C(\mathbb{R}^+; H)$ is called a strong solution to (2) if $X \in C(\mathbb{R}^+; D(A_0))$, $\int_0^t (X(s) - X(0))/(t-s)^\alpha ds \in C^1(\mathbb{R}^+; H)$ and (2) holds on $\mathbb{R}^+$. The problem (2) is called well-posed if for any $x \in D(A_0)$, there exists a unique strong solution $X(t,x)$ of (2), and $x_n \to 0$ as $n \to \infty$, implies $X(t,x_n) \to 0$ as $n \to \infty$ in H, uniformly on compact intervals.

Lemma 1 (See the work of Bazhlekova[32]).
Suppose that $\alpha \in (0,1)$. Let A_0 be a closed-linear operator densely defined in a Banach space H. If A_0 generates a C_0-semigroup on H, then Cauchy problem (2) admits a unique strong solution $X(t) = S_\alpha(t)x \in C(0,\infty; H)$, where $\{S_\alpha(t)\}_{t\ge 0}$ is a family of continuous linear operator on H and satisfies $S_\alpha(t)A_0 x = A_0 S_\alpha(t)x$ for all $x \in D(A_0)$ and $t \ge 0$.

Consider the following nonhomogeneous Cauchy problem in a Banach space H:

$$\begin{cases} {}_0^C D_t^\alpha X(t) = A_0 X(t) + f(t), \\ X(0) = x, \end{cases} \tag{3}$$

where A_0 is the infinitesimal generator of a C_0-semigroup $e^{A_0 t}$ in H.

Lemma 2. *Let $\alpha \in (0,1)$. Suppose that $\sup_{t\geq 0}\|e^{A_0 t}\| = M < \infty$ for some $M > 0$. If $f \in L^\infty(0,\infty;H)$, then Cauchy problem (3) admits a unique solution $X \in C(0,\infty;H)$.*

Proof. For any fixed $a > 0$, it follows from theorem 3.3 in the work of Zhou and Jiao[33] that (3) admits a unique solution $X \in C(0,a;H)$. Since the above reasoning works for any $a > 0$, (3) admits a global unique solution. □

Lemma 3 (See the work of Aguila-Camacho et al[34]).
Let $x(t) \in \mathbb{R}$ be a continuous and derivable function. Then, for any time instant $t \geq 0$,

$${}_0^C D_t^\alpha x^2(t) \leq 2x(t)\,{}_0^C D_t^\alpha x(t), \quad \forall\, \alpha \in (0,1).$$

If $x(t) \in \mathbb{R}^n$, it holds that for all $\alpha \in (0,1)$ and $t \geq 0$,

$${}_0^C D_t^\alpha (x^\top(t)x(t)) \leq 2x^\top(t)\,{}_0^C D_t^\alpha x(t).$$

3 | DISTURBANCE ESTIMATOR DESIGN

In this section, we design a disturbance estimator to estimate the disturbance. For this purpose, we construct two auxiliary systems, one is to bring the disturbance from original system (1) into a Mittag-Leffler stable system and the other is to estimate the disturbance.

Step 1: The first auxiliary system is designed as follows:

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) &= v_{xx}(x,t) + \lambda(x)w(x,t) - c[v(x,t) - w(x,t)], \\ v_x(0,t) &= -qw(0,t), \quad v_x(1,t) = u(t), \\ v(x,0) &= v_0(x), \end{cases} \tag{4}$$

where the gain c is a positive design parameter, which is used to regulate the convergence speed. Let $\hat{v}(x,t) = v(x,t) - w(x,t)$. Then, it is easy to check that $\hat{v}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \hat{v}(x,t) &= \hat{v}_{xx}(x,t) - c\hat{v}(x,t), \\ \hat{v}_x(0,t) &= 0, \quad \hat{v}_x(1,t) = -d(t), \\ \hat{v}(x,0) &= \hat{v}_0(x) = v_0(x) - w_0(x). \end{cases} \tag{5}$$

Lemma 4. *Suppose that $c > 0$, and d, ${}_0^C D_t^\alpha d \in L^\infty(0,\infty)$. For any initial value $\hat{v}(\cdot,0) \in L^2(0,1)$, there exists a unique solution to (5) such that $\hat{v} \in C(0,\infty;L^2(0,1))$ satisfying $\sup_{t\geq 0}\|\hat{v}(\cdot,t)\|_{L^2(0,1)} < +\infty$. Moreover, If $d \equiv 0$, then $\|\hat{v}(\cdot,t)\|_{L^2(0,1)} \leq M E_\alpha(-\mu t^\alpha)$ with $M, \mu > 0$.*

Proof. Introducing the variable $\tilde{v}(x,t) = \hat{v}(x,t) + x^2 d(t)/2$, we can transform system (5) into an equivalent system

$$\begin{cases} {}_0^C D_t^\alpha \tilde{v}(x,t) &= \tilde{v}_{xx}(x,t) - c\tilde{v}(x,t) + f(x,t), \\ \tilde{v}_x(0,t) &= 0, \quad \tilde{v}_x(1,t) = 0, \\ \tilde{v}(x,0) &= \tilde{v}_0(x), \end{cases} \tag{6}$$

where

$$\begin{cases} f(x,t) = -d(t) + \frac{1}{2}x^2\left[d(t) + {}_0^C D_t^\alpha d(t)\right], \\ \tilde{v}_0(x) = \hat{v}_0(x) - x^2 d(0)/2. \end{cases}$$

Since $d,\ {}_0^C D_t^\alpha d \in L^\infty(0,\infty)$, it has $f \in L^\infty(0,\infty; L^2(0,1))$. Clearly, the well-posedness of (5) is equivalent to that of (6). Define the operator $\mathbb{A} : D(\mathbb{A})(\subset L^2(0,1)) \to L^2(0,1)$ as follows:

$$\begin{cases} \mathbb{A}\phi &= \phi'' - c\phi,\ \forall\ \phi \in D(\mathbb{A}), \\ D(\mathbb{A}) &= \left\{\phi \in H^2(0,1) :\ \phi'(0) = 0,\ \phi'(1) = 0\right\}. \end{cases} \tag{7}$$

It is well known that $\mathbb{A}$ generates an exponentially stable C_0-semigroup $e^{\mathbb{A}t}$ and hence $\sup_{t\geq 0}\|e^{\mathbb{A}t}\| < +\infty$. Since $f \in L^\infty(0,\infty; L^2(0,1))$, it follows from Lemma 2 that (6) has a global solution, ie, (5) is well-posed.

Next, let $V(t) = \frac{1}{2}\int_0^1 \tilde{v}^2(x,t)dx$. Taking Caputo's fractional derivative for $V(t)$ along with the solution of (6), we have, from Lemma 3, that

$$\begin{aligned} {}_0^C D_t^\alpha V(t) &\leq \int_0^1 \tilde{v}(x,t)\,{}_0^C D_t^\alpha \tilde{v}(x,t)\mathrm{d}x = \int_0^1 \tilde{v}(x,t)\left[\tilde{v}_{xx}(x,t) - c\tilde{v}(x,t) + f(x,t)\right]\mathrm{d}x \\ &= -\int_0^1 \tilde{v}_x^2(x,t)\mathrm{d}x - c\int_0^1 \tilde{v}^2(x,t)dx + \int_0^1 \tilde{v}(x,t)f(x,t)\mathrm{d}x \\ &\leq -(c-\varepsilon_0)\int_0^1 \tilde{v}^2(x,t)\mathrm{d}x + \frac{1}{4\varepsilon_0}\int_0^1 f^2(x,t)\mathrm{d}x \\ &= -2(c-\varepsilon_0)V(t) + \frac{1}{4\varepsilon_0}\int_0^1 f^2(x,t)\mathrm{d}x. \end{aligned} \tag{8}$$

In the last step of (8), the inequality $ab \leq \varepsilon_0 a^2 + \frac{4}{\varepsilon_0}b^2$ was used and ε_0 is chosen so that $\varepsilon_0 < c$. From (8), there exists a nonnegative function $P(t)$ satisfying

$${}_0^C D_t^\alpha V(t) + P(t) = -2(c-\varepsilon_0)V(t) + \frac{1}{4\varepsilon_0}\int_0^1 f^2(x,t)\mathrm{d}x,\ t \geq 0. \tag{9}$$

By lemma 2.24 (page 98) in the work of Kilbas et al[35] and taking the Laplace transform on both sides of (9) gives

$$s^\alpha \hat{V}(s) - V(0)s^{\alpha-1} + \hat{P}(s) = -2(c-\varepsilon_0)\hat{V}(s) + F(s),\ t \geq 0, \tag{10}$$

where

$$\hat{V}(s) := \int_0^\infty e^{-st}V(t)\mathrm{d}t,\ \hat{P}(s) := \int_0^\infty e^{-st}P(t)\mathrm{d}t,\ \ F(s) := \frac{1}{4\varepsilon_0}\int_0^\infty e^{-st}\int_0^1 f^2(x,t)\mathrm{d}x\mathrm{d}t$$

are the Laplace transforms of the functions $V(t)$, $P(t)$, and $\frac{1}{4\varepsilon_0}\int_0^1 f^2(x,t)dx$, respectively. By (10), it follows that

$$\hat{V}(s) = \frac{V(0)s^{\alpha-1} - \hat{P}(s) + F(s)}{s^\alpha + 2(c-\varepsilon_0)}. \tag{11}$$

Taking the inverse Laplace transform on both sides of (11) yields

$$\begin{aligned} V(t) = {} & E_\alpha(-2(c-\varepsilon_0)t^\alpha)V(0) - P(t) * \left[t^{\alpha-1}E_{\alpha,\alpha}(-2(c-\varepsilon_0)t^\alpha)\right] \\ & + \frac{1}{4\varepsilon_0}\int_0^1 f^2(x,t)\mathrm{d}x * \left[t^{\alpha-1}E_{\alpha,\alpha}(-2(c-\varepsilon_0)t^\alpha)\right], \end{aligned} \tag{12}$$

where $*$ denotes the convolution operator. Since $P(t)$, $t^{\alpha-1}$, and $E_{\alpha,\alpha}(-\lambda t^{\alpha})$ are nonnegative functions, it follows from $f \in L^{\infty}(0,\infty;L^2(0,1))$ that

$$
\begin{aligned}
V(t) \leq{}& E_{\alpha}(-2(c-\varepsilon_0)t^{\alpha})V(0) \\
&+ \frac{\|f\|^2_{L^{\infty}(0,\infty;L^2(0,1))}}{4\varepsilon_0}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-2(c-\varepsilon_0)(t-s)^{\alpha}\right)\mathrm{d}s.
\end{aligned}
\tag{13}
$$

To show that $V(t)$ is uniformly bounded for all $t \geq 0$, it suffices to prove that the last term of (13) is bounded. Indeed, it follows from formula 1.10.7 (page 50) in the work of Kilbas et al[35] that

$$
\frac{d}{dt}\left[t^{\alpha}E_{\alpha,\alpha+1}\left(-2(c-\varepsilon_0)t^{\alpha}\right)\right] = t^{\alpha-1}E_{\alpha,\alpha}\left(-2(c-\varepsilon_0)t^{\alpha}\right),
$$

which gives

$$
\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-2(c-\varepsilon_0)s^{\alpha}\right)\mathrm{d}s = t^{\alpha}E_{\alpha,\alpha+1}\left(-2(c-\varepsilon_0)t^{\alpha}\right). \tag{14}
$$

Furthermore, it follows from formula 1.8.28 (page 43) in the work of Kilbas et al[35] that

$$
E_{\alpha,\alpha+1}\left(-2(c-\varepsilon_0)t^{\alpha}\right) = \frac{1}{2(c-\varepsilon_0)t^{\alpha}} + \mathcal{O}\left(\frac{1}{4(c-\varepsilon_0)^2t^{2\alpha}}\right),
$$

which yields

$$
\lim_{t\to\infty} t^{\alpha}E_{\alpha,\alpha+1}\left(-2(c-\varepsilon_0)t^{\alpha}\right) = \frac{1}{2(c-\varepsilon_0)}.
$$

On the other hand, since $\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-2(c-\varepsilon_0)s^{\alpha})ds$ is nondecreasing with respect to t, and so is for $t^{\alpha}E_{\alpha,\alpha+1}(-2(c-\varepsilon_0)t^{\alpha})$ due to (14). Thus,

$$
\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\left(-2(c-\varepsilon_0)s^{\alpha}\right)\mathrm{d}s \leq \frac{1}{2(c-\varepsilon_0)},\ \forall\, t \geq 0. \tag{15}
$$

Therefore, $\sup_{t\geq 0}V(t) < +\infty$. Combining with $x^2d/2 \in L^{\infty}(0,\infty;L^2(0,1))$, we arrive at $\sup_{t\geq 0}\|\hat{v}(\cdot,t)\|_{L^2(0,1)} < +\infty$.

Finally, suppose that $d \equiv 0$. It follows from $\tilde{v}(x,t) = \hat{v}(x,t)$ and (13) that $\|\hat{v}(\cdot,t)\|_{L^2(0,1)} \leq ME_{\alpha}(-\mu t^{\alpha})$ with $M,\mu > 0$. This completes the proof of the lemma. □

Remark 1. When $\alpha = 1$, the assumption on the boundedness of ${}_0^CD_t^{\alpha}d$ can be removed. Actually, when $\alpha = 1$, we can write system (5) as $\dot{\hat{v}}(\cdot,t) = \mathbb{A}\hat{v}(\cdot,t) + \mathbb{B}d(t)$, where the operator $\mathbb{A}$ is given by (7) and $\mathbb{B}$ is given by $\mathbb{B} = \delta_1$, where δ_a is the Dirac pulse at $x = a$. It is well known that $\mathbb{A}$ generates an exponentially stable C_0-semigroup $e^{\mathbb{A}t}$ and $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$.[36] Since $d \in L^{\infty}(0,\infty)$, it follows from lemma 2.1 in the work of Zhou and Weiss[19]or appendix in the work of Zhou[37] that system (5) has a unique bounded solution that is exponentially stable whenever $d \equiv 0$.

Step 2: With system (4), we design the second auxiliary system to estimate the disturbance

$$
\begin{cases}
{}_0^CD_t^{\alpha}z(x,t) &= z_{xx}(x,t) - cz(x,t), \\
z_x(0,t) &= 0, \qquad z(1,t) = w(1,t) - v(1,t), \\
z(x,0) &= z_0(x).
\end{cases}
\tag{16}
$$

where c is a positive design parameter, which is exactly the same as the one in (4). Let $p(x,t) = -z(x,t) - \hat{v}(x,t)$. It is easy to verify that $p(x,t)$ satisfies

$$
\begin{cases}
{}_0^CD_t^{\alpha}p(x,t) &= p_{xx}(x,t) - cp(x,t), \\
p_x(0,t) &= 0, \quad p(1,t) = 0, \\
p(x,0) &= p_0(x),
\end{cases}
\tag{17}
$$

which is a Mittag-Leffler stable system and serves as a target system for the design of disturbance estimator. System (17) can be rewritten as

$$
{}_0^CD_t^{\alpha}p(\cdot,t) = Ap(\cdot,t), \quad p(x,0) = p_0(x),
$$

where the operator $A : D(A)(\subset L^2(0,1)) \to L^2(0,1)$ is given by

$$\begin{cases} [Af](x) = f''(x) - cf(x), \\ D(A) = \left\{ f \in H^2(0,1) \,|\, f'(0) = 0, f(1) = 0 \right\}. \end{cases} \tag{18}$$

Lemma 5 is about the well-posedness and stability of (17), which is proved by Lemma 1.

Lemma 5 (See the work of Zhou and Guo[5]).
Suppose that $c > 0$. For any initial value $p(\cdot,0) \in L^2(0,1)$, there exists a unique solution to (17) such that $p(\cdot,t) = S_\alpha(t)p(\cdot,0) \in C(0,\infty;L^2(0,1))$, which is Mittag-Leffler stable, ie, there exist two constants $M,\mu > 0$ such that $\|p(\cdot,t)\|_{L^2(0,1)} \le ME_\alpha(-\mu t^\alpha)$.

Corollary 1. *Suppose that $c > 0$. For any initial value $p(\cdot,0) \in D(A)$, the solution of (17) satisfies $|p_x(1,t)| \le ME_\alpha(-\mu t^\alpha)$ with some $M,\mu > 0$.*

Proof. Since $p(\cdot,0) \in D(A)$, we have $Ap(\cdot,0) \in \mathcal{H}$. By Lemma 5, $S_\alpha(t)Ap(\cdot,0)$ is Mittag-Leffler stable. Therefore, the system (17) admits a unique solution $p(\cdot,t) = S_\alpha(t)p(\cdot,0) \in C(0,\infty;D(A))$ and there exist two positive constants $M_1,\mu_1 > 0$ such that

$$\|S_\alpha(t)p(\cdot,0)\|_{L^2(0,1)} \le M_1E_\alpha(-\mu_1 t^\alpha)\|p(\cdot,0)\|_{L^2(0,1)}, \tag{19}$$

and

$$\|S_\alpha(t)Ap(\cdot,0)\|_{L^2(0,1)} \le M_1E_\alpha(-\mu_1 t^\alpha)\|Ap(\cdot,0)\|_{L^2(0,1)}. \tag{20}$$

Note that $p(\cdot,t) = S_\alpha(t)p(\cdot,0)$ and $Ap(\cdot,t) = AS_\alpha(t)p(\cdot,0) = S_\alpha(t)Ap(\cdot,0)$ for $p(\cdot,0) \in D(A)$. By (19),

$$\begin{aligned} \|p_{xx}(\cdot,t)\|_{L^2(0,1)} &\le \|Ap(\cdot,t)\|_{L^2(0,1)} + c\|p(\cdot,t)\|_{L^2(0,1)} \\ &\le M_1E_\alpha(-\mu_1 t^\alpha)\left[\|Ap(\cdot,0)\|_{\mathcal{H}} + c\|p(\cdot,0)\|_{\mathcal{H}}\right]. \end{aligned} \tag{21}$$

Since by interpolation inequality (page 75) in the work of Adams and Fournier,[38]

$$\|p_x(\cdot,t)\|_{L^2(0,1)} \le C_1\left[\|p(\cdot,t)\|_{L^2(0,1)} + \|p_{xx}(\cdot,t)\|_{L^2(0,1)}\right]$$

with some $C_1 \ge 1$ independent of p, it follows from the Sobolev trace theorem, (19), and (21) that there is $C_2 > 0$ such that

$$|p_x(1,t)| \le C_2\left[\|p(\cdot,t)\|_{L^2(0,1)} + \|p_x(x,t)\|_{L^2(0,1)} + \|p_{xx}(x,t)\|_{L^2(0,1)}\right] \le ME_\alpha(-\mu t^\alpha)$$

with $M = 2(C_1+1)C_2M_1[\|Ap(\cdot,0)\|_{L^2(0,1)} + (c+1)\|p(\cdot,0)\|_{L^2(0,1)}]$ and $\mu = u_1$. ☐

Finally, putting all these systems (4) and (16) together, we obtain a disturbance estimator for system (1) as follows:

$$\begin{cases} {}_0^CD_t^\alpha v(x,t) &= v_{xx}(x,t) + \lambda(x)w(x,t) - c[v(x,t) - w(x,t)], \\ v_x(0,t) &= -qw(0,t), \quad v_x(1,t) = u(t), \\ {}_0^CD_t^\alpha z(x,t) &= z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) &= 0, \quad z(1,t) = w(1,t) - v(1,t). \end{cases} \tag{22}$$

which estimates the total disturbance $d(t) \approx z_x(1,t)$ because of $p_x(1,t) = z_x(1,t) - d(t)$ and Corollary 1.

Remark 2. We emphasize that the disturbance estimator (22) does not use the high gain, which is a remarkable feature different from traditional fractional extended state observer (which can estimate the disturbance) presented in the works of Li et al[26] and Gao.[27] To explain this clearly, inspired by the works of Guo and Jin[17] and Guo and Zhou,[21] where the test function was used to make the boundary control and boundary disturbance to be included into an ODE, we define two new variables $Y(t)$ and $Z(t)$ as follows:

$$Y(t) = \int_0^1 h(x)w(x,t)\mathrm{d}x, \quad Z(t) = \int_0^1 [h(x)\lambda(x) + h''(x)]w(x,t)\mathrm{d}x, \tag{23}$$

where $h(x)$ is any test function $h \in C^2[0,1]$ with $h(0) = h'(0) = h'(1) = 0$, and $h(1) = 1$. Obviously, we can take $h(x) = x^2(3-2x)$. A simple computation shows $Y(t)$ and $Z(t)$ are governed by

$$ {}_0^C D_t^\alpha Y(t) = u(t) + d(t) + Z(t). $$

From this equation, by the works of Li et al[26] and Gao,[27] the fractional extended state observer is designed by

$$ \begin{cases} {}_0^C D_t^\alpha \hat{Y}(t) = u(t) + \hat{d}(t) + Z(t) - \beta_1[\hat{Y}(t) - Y(t)], \\ {}_0^C D_t^\alpha \hat{d}(t) = -\beta_2[\hat{Y}(t) - Y(t)], \end{cases} \tag{24} $$

where $\beta_1 = 2\omega_o$ and $\beta_2 = \omega_o^2$ with ω_o being the linear-bandwidth-parameterization.[26] By lemma 2 in the work of Gao,[27] it follows that

$$ \lim_{t\to\infty} \sup |\hat{Y}(t) - Y(t)| \leq \frac{M}{\omega_o^2}, \quad \lim_{t\to\infty} \sup |\hat{d}(t) - d(t)| \leq \frac{2M}{\omega_o}, \tag{25} $$

where $M = \sup_{t\geq 0} |{}_0^C D_t^\alpha d(t)|$. It is clearly seen from (25) that the large ω_o improves convergence of fractional-order extended states observer and decreases the disturbance estimation error $|\hat{d}(t) - d(t)|$. However, the larger ω_o would amplify the sensor noise. In contrast to present (22), (24) uses high gain, which is not good for applications.

Remark 3. In Corollary 1, it is shown that when the initial state of system is smooth, the $z_x(1,t)$ can be regarded as an estimate of the disturbance $d(t)$ because $|p_x(1,t)| = |z_x(1,t) - d(t)|$ converges to zero, Mittag-Leffler asymptotically. It is worth emphasizing that this estimation is not in traditional sense since we do not have estimation error for nonsmooth initial state. However, this approximation turns out to be enough for stabilization for any initial state in the state space. In what follows, we will use this estimated value of $d(t)$ to design a disturbance-estimator-based stabilizing controller.

4 | FEEDBACK CONTROL DESIGN

In this section, based on the disturbance estimator (22), we design a stabilizing control for system (1). For this purpose, we introduce an invertible transformation $w \to \hat{w}$ [5]:

$$ \hat{w}(x,t) = [(I + \mathbb{P})w](x,t) = w(x,t) - \int_0^x k(x,y)w(y,t)\mathrm{d}y, \tag{26} $$

where the kernel function $k(x,y)$ is the solution of the following PDE:

$$ \begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = (\lambda(y) + c)k(x,y), \\ k_y(x,0) + qk(x,0) = 0, \\ k(x,x) = -q - \frac{1}{2}\int_0^x (\lambda(y) + c)\mathrm{d}y. \end{cases} \tag{27} $$

The inverse of transform (26) is given by

$$ w(x,t) = [(I + \mathbb{P})^{-1}\hat{w}](x,t) = \hat{w}(x,t) + \int_0^x l(x,y)\hat{w}(y,t)\mathrm{d}y, \tag{28} $$

where the kernel function $l(x,y)$ is the solution of the following PDE:

$$ \begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = -(\lambda(x) + c)l(x,y), \\ l_y(x,0) + ql(x,0) = 0, \\ l(x,x) = -q - \frac{1}{2}\int_0^x (\lambda(y) + c)\mathrm{d}y. \end{cases} \tag{29} $$

By theorem 2.2 in the work of Smyshlyaev and Krstic,[39] the PDE (29) has a unique solution $l \in C^2(\bar{\mathcal{T}})$. By theorem 2.1 in the work of Smyshlyaev and Krstic,[39] the PDE (27) has a unique solution $k \in C^2(\bar{\mathcal{T}})$. Under the transformation (26) and its inverse transformation (28), system (1) is equivalent to

$$\begin{cases} {}_0^C D_t^\alpha \hat{w}(x,t) &= \hat{w}_{xx}(x,t) - c\hat{w}(x,t), \\ \hat{w}_x(0,t) &= 0, \\ \hat{w}_x(1,t) &= u(t) + d(t) - k(1,1)w(1,t) - \int_0^1 k_x(1,y)w(y,t)\mathrm{d}y. \end{cases} \tag{30}$$

If the disturbance $d(t)$ vanishes, the stabilizing control law is chosen in the work of Zhou and Guo[5] as

$$u(t) = k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)\mathrm{d}y. \tag{31}$$

However, when the disturbance d is nonzero, the control law (31) cannot stabilize system (1). For example, suppose that the disturbance $d(t)$ is a constant $d(t) = \sqrt{c}\sinh(\sqrt{c})$. Then, system (1) admits a nonzero solution

$$w(x,t) = \cosh\left(\sqrt{c}x\right) + \int_0^x l(x,y)\cosh\left(\sqrt{c}y\right)\mathrm{d}y, \quad \forall\, t \geq 0.$$

Now, since we have estimated the disturbance $d(t)$, which is just $z_x(1,t)$ presented in Section 3, it is natural to propose the following disturbance-estimator-based feedback controller:

$$u(t) = -z_x(1,t) + k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)\mathrm{d}y. \tag{32}$$

It is seen that the "$-z_x(1,t)$" term in (32) is used to cancel (compensate) the disturbance, and the remaining term is to stabilize system (30) without $d(t)$ suggested by (31). The closed loop of (30) corresponding to controller (32) becomes

$$\begin{cases} {}_0^C D_t^\alpha \hat{w}(x,t) &= \hat{w}_{xx}(x,t) - c\hat{w}(x,t), \\ \hat{w}_x(0,t) &= 0, \quad \hat{w}_x(1,t) = p_x(1,t). \end{cases} \tag{33}$$

Lemma 6. *Suppose that $c > 0$ and the signal $p_x(1,t)$ is generated by system (17). For any initial value $\hat{w}(x,0) \in L^2(0,1)$, system (33) admits a unique solution $\hat{w} \in C(0,\infty;\mathcal{X})$, which is Mittag-Leffler stable, ie, there exist two constants $M, \mu > 0$ such that $\|\hat{w}(\cdot,t)\|_{L^2(0,1)} \leq ME_\alpha(-\mu t^\alpha)$ for all $t \geq 0$.*

Proof. We first introduce a new variable $\tilde{w}(x,t) = \hat{w}(x,t) - p(x,t)$. Noting that $p(x,t)$ is governed by system (17), it is easy to verify that $\tilde{w}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \tilde{w}(x,t) &= \tilde{w}_{xx}(x,t) - c\tilde{w}(x,t), \\ \tilde{w}_x(0,t) &= 0, \quad \tilde{w}_x(1,t) = 0. \end{cases} \tag{34}$$

It follows from lemma 3.1 in the work of Zhou and Guo[5] that system (34) admits a unique solution $\tilde{w} \in C(0,\infty;L^2(0,1))$ and there exist $M_1, \mu_1 > 0$ such that $\|\tilde{w}(\cdot,t)\|_{L^2(0,1)} \leq M_1E_\alpha(-\mu_1 t^\alpha)$ for all $t \geq 0$. On the other hand, by Lemma 5, we obtain the existence of solution of system (33). By Lemma 5 again, $\|p(\cdot,t)\|_{L^2(0,1)} \leq M_2E_\alpha(-\mu_2 t^\alpha)$ with some $M_2, \mu_2 > 0$. Therefore,

$$\|\hat{w}(\cdot,t)\|_{L^2(0,1)} \leq \|\tilde{w}(\cdot,t)\|_{L^2(0,1)} + \|p(\cdot,t)\|_{L^2(0,1)} \leq ME_\alpha(-\mu t^\alpha), \forall\, t \geq 0,$$

with $M = M_1 + M_2$, $\mu = \min\{\mu_1, \mu_2\}$. This ends the proof. □

5 | WELL-POSEDNESS AND MITTAG-LEFFLER STABILITY OF CLOSED-LOOP SYSTEM

We go back to the closed-loop system (1) under the feedback (32)

$$\begin{cases} {}_0^CD_t^\alpha w(x,t) &= w_{xx}(x,t)+\lambda(x)w(x,t), \\ w_x(0,t) &= -qw(0,t), \\ w_x(1,t) &= -z_x(1,t)+k(1,1)w(1,t)+\int_0^1 k_x(1,y)w(y,t)\mathrm{d}y+d(t), \\ {}_0^CD_t^\alpha v(x,t) &= v_{xx}(x,t)+\lambda(x)w(x,t)-c[v(x,t)-w(x,t)], \\ v_x(0,t) &= -qw(0,t), \\ v_x(1,t) &= -z_x(1,t)+k(1,1)w(1,t)+\int_0^1 k_x(1,y)w(y,t)\mathrm{d}y, \\ {}_0^CD_t^\alpha z(x,t) &= z_{xx}(x,t)-cz(x,t), \\ z_x(0,t) &= 0, \quad z(1,t)=v(1,t)-w(1,t). \end{cases} \tag{35}$$

We consider system (35) in the state space $\mathcal{H}=(L^2(0,1))^3$.

Theorem 1. *Let $k(x,y)$ be the solution of (27). Suppose that $c>0$, and $d,\ {}_0^CD_t^\alpha d \in L^\infty(0,\infty)$. For any initial value $(w(\cdot,0),v(\cdot,0),z(\cdot,0))\in\mathcal{H}$, there exists a unique solution to (35) such that $(w,v,z)\in C(0,\infty;\mathcal{H})$ satisfying $\|w(\cdot,t)\|_{L^2(0,1)}\le ME_\alpha(-\mu t^\alpha)$, with some $M,\mu>0$, and $\sup_{t\ge0}\|(v(\cdot,t),z(\cdot,t))\|_{[L^2(0,1)]^2}<+\infty$. If we assume further that $d(t)\equiv0$, then there exist two constants $M',\mu'>0$ such that $\|(v(\cdot,t),z(\cdot,t))\|_{\mathbb{H}^2}\le M'E_\alpha(-\mu't^\alpha),\ t\ge0$.*

Proof. Using the variables $\hat{v}(x,t)=v(x,t)-w(x,t)$, $p(x,t)=z(x,t)-\hat{v}(x,t)$ and the transformation (26), we can write the equivalent system of (35) as follows:

$$\begin{cases} {}_0^CD_t^\alpha \hat{v}(x,t) &= \hat{v}_{xx}(x,t)-c\hat{v}(x,t), \\ \hat{v}_x(0,t) &= 0, \quad \hat{v}_x(1,t)=-d(t), \\ {}_0^CD_t^\alpha p(x,t) &= p_{xx}(x,t)-cp(x,t), \\ p_x(0,t) &= 0, \quad p(1,t)=0, \\ {}_0^CD_t^\alpha \hat{w}(x,t) &= \hat{w}_{xx}(x,t)-c\hat{w}(x,t), \\ \hat{w}_x(0,t) &= 0, \quad \hat{w}_x(1,t)=p_x(1,t). \end{cases} \tag{36}$$

The existence of solution of system (36) follows from Lemmas (4), (5), and (6). By Lemmas (5) and (6) again, there exist $M_1,\mu_1>0$ such that $\|(p(\cdot,t),\hat{w}(\cdot,t))\|_{[L^2(0,1)]^2}\le M_1E_\alpha(-\mu_1t^\alpha)$. Noting the invertible transformation

$$\begin{pmatrix} w(x,t)\\ v(x,t)\\ z(x,t)\end{pmatrix} = \begin{pmatrix} 0 & 0 & (I+\mathbb{P})^{-1}\\ I & 0 & (I+\mathbb{P})^{-1}\\ -I & -I & 0\end{pmatrix}\begin{pmatrix}\hat{v}(x,t)\\ p(x,t)\\ \hat{w}(x,t)\end{pmatrix}, \tag{37}$$

where $I+\mathbb{P}$ is defined by (28), we have that $(w,v,z)\in C(0,\infty;\mathcal{H})$ is well defined and satisfies $\|w(\cdot,t)\|_{L^2(0,1)}\le \|(I+\mathbb{P})^{-1}\|M_1E_\alpha(-\mu_1t)$ and $\sup_{t\ge0}\|(v(\cdot,t),z(\cdot,t))\|_{[L^2(0,1)]^2}<+\infty$.

Next, suppose that $d(t)\equiv0$. By Lemma 4 again, we have $\|\hat{v}(\cdot,t)\|_{L^2(0,1)}\le M_2E_\alpha(-\mu_2t^\alpha)$ with some $M_2,\mu_2>0$, which, together with (37), implies that $\|(v(\cdot,t),z(\cdot,t))\|_{\mathbb{H}^2}\le M'E_\alpha(-\mu't^\alpha),\ t\ge0$ for some $M',\mu'>0$. This completes the proof. □

Remark 4. To achieve Mittag-Leffler stability for the closed-loop system in Theorem 1, the new disturbance estimator is used, which is high-gain free. In addition, the traditional fractional extended state observer proposed in the works of Li et al[26] and Gao,[27] where the high-gain observer was used seems not to be able to suppress the disturbance, like $d,\ {}_0^CD_t^\alpha d\in L^\infty(0,\infty)$, in fractional PDEs. Actually, from Remark 2, the $\hat{d}(t)$ is an estimate of $d(t)$. By ADRC strategy, the control law should be designed by

$$u(t) = -\widehat{d}(t) + k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)\mathrm{d}y. \tag{38}$$

With this control law, the closed loop becomes

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t) + \lambda(x)w(x,t), \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = -\widehat{d}(t) + k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)\mathrm{d}y + d(t), \\ {}_0^C D_t^\alpha \widehat{Y}(t) = Z(t) - \beta_1[\widehat{Y}(t) - Y(t)] + k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)\mathrm{d}y, \\ {}_0^C D_t^\alpha \widehat{d}(t) = -\beta_2[\widehat{Y}(t) - Y(t)], \end{cases} \tag{39}$$

where $Y(t)$ and $Z(t)$ are given by (23), $\beta_1 = 2\omega_o$ and $\beta_2 = \omega_o^2$. Using transformation (28) and the error variables $\widetilde{Y}(t) = \widehat{Y}(t) - Y(t)$, $\widetilde{d}(t) = \widehat{d}(t) - d(t)$, system (39) is equivalent to

$$\begin{cases} {}_0^C D_t^\alpha \widehat{w}(x,t) = \widehat{w}_{xx}(x,t) - c\widehat{w}(x,t), \\ \widehat{w}_x(0,t) = 0, \quad \widehat{w}_x(1,t) = -\widetilde{d}(t), \\ {}_0^C D_t^\alpha \widetilde{Y}(t) = \widetilde{d}(t) - \beta_1 \widetilde{Y}(t), \\ {}_0^C D_t^\alpha \widetilde{d}(t) = -\beta_2 \widetilde{Y}(t) - {}_0^C D_t^\alpha d(t). \end{cases} \tag{40}$$

If $\alpha = 1$, by linear system theory,[36] with the similar estimation way in the works of Guo and Jin[17] or Zhou and Guo,[20] one can show that

$$\lim_{t\to\infty,\omega_o\to\infty} \sup \|\widehat{w}(\cdot,t)\|_{L^2(0,1)} = 0.$$

However, when $\alpha \in (0,1)$, the admissibility theory for fractional system is not available. By (25), we have

$$\lim_{t\to\infty} \sup |\widetilde{d}(t)| \le \frac{2M}{\omega_o}, \quad \lim_{t\to\infty} \sup \left| {}_0^C D_t^\alpha \widetilde{d}(t) \right| \le 2M,$$

where $M = \sup_{t\ge 0} |{}_0^C D_t^\alpha d(t)|$. Compared (40) with (5), one can only prove, by Lemma 4, that $\sup_{t\ge 0} \|\widehat{w}(\cdot,t)\|_{L^2(0,1)} < +\infty$.

6 | STABILIZATION FOR SYSTEM WITH DIRICHLET BOUNDARY CONTROL

In the previous sections, we have designed a stabilizing control law for system (1). In this section, we consider the stabilizing control law design for the TFADE with Dirichlet boundary control and boundary disturbance, described by the following PDE:

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) &= w_{xx}(x,t) + \lambda(x)w(x,t), \\ w_x(0,t) &= -qw(0,t), \\ w(1,t) &= u(t) + d(t), \\ w(x,0) &= w_0(x), \end{cases} \tag{41}$$

where $w(x,t)$ is the state, $u(t)$ is the control input, and $d(t)$ is the unknown external disturbance. We design the following auxiliary system to separate disturbance from original system (41) to a Mittag-Leffler stable system.

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) &= v_{xx}(x,t) + \lambda(x)w(x,t) - c[v(x,t) - w(x,t)], \\ v_x(0,t) &= -qw(0,t), \quad v(1,t) = u(t), \\ v(x,0) &= v_0(x), \end{cases} \tag{42}$$

where the gain $c > 0$ is a design parameter. Let $\hat{v}(x,t) = v(x,t) - w(x,t)$. Then, it is easy to check that $\hat{v}(x,t)$ is governed by

$$\begin{cases} {}_0^C D_t^\alpha \hat{v}(x,t) & = \hat{v}_{xx}(x,t) - c\hat{v}(x,t), \\ \hat{v}_x(0,t) & = 0, \quad \hat{v}(1,t) = -d(t), \\ \hat{v}(x,0) & = \hat{v}_0(x) = v_0(x) - w_0(x). \end{cases} \tag{43}$$

Lemma 7. *Suppose that $c > 0$, and d, ${}_0^C D_t^\alpha d \in L^\infty(0,\infty)$. For any initial value $\hat{v}(\cdot,0) \in L^2(0,1)$, there exists a unique solution to (43) such that $\hat{v} \in C(0,\infty;L^2(0,1))$ satisfying $\sup_{t\geq 0}\|\hat{v}(\cdot,t)\|_{L^2(0,1)} < +\infty$. Moreover, if $d \equiv 0$, then $\|\hat{v}(\cdot,t)\|_{L^2(0,1)} \leq ME_\alpha(-\mu t^\alpha)$ with $M,\mu > 0$.*

Proof. Introducing the variable $\tilde{v}(x,t) = \hat{v}(x,t) + x^2 d(t)$, we can transform system (5) into an equivalent system

$$\begin{cases} {}_0^C D_t^\alpha \tilde{v}(x,t) & = \tilde{v}_{xx}(x,t) - c\tilde{v}(x,t) + f(x,t), \\ \tilde{v}_x(0,t) & = 0, \quad \tilde{v}(1,t) = 0, \\ \tilde{v}(x,0) & = \hat{v}_0(x) - x^2 d(0), \end{cases} \tag{44}$$

where $f(x,t) = -d(t) + x^2[d(t) + {}_0^C D_t^\alpha d(t)]$. Since d, ${}_0^C D_t^\alpha d \in L^\infty(0,\infty)$, it has $f \in L^\infty(0,\infty;L^2(0,1))$. The remaining proof is similar to Lemma 4, which is omitted here. □

Next, we propose the following system to estimate the disturbance.

$$\begin{cases} {}_0^C D_t^\alpha z(x,t) & = z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) & = 0, \quad z_x(1,t) = w_x(1,t) - v_x(1,t), \\ z(x,0) & = z_0(x). \end{cases} \tag{45}$$

Indeed, let $p(x,t) = -z(x,t) - \hat{v}(x,t)$. It is easy to verify that $p(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha p(x,t) & = p_{xx}(x,t) - cp(x,t), \\ p_x(0,t) & = 0, \quad p_x(1,t) = 0, \\ p(x,0) & = p_0(x), \end{cases} \tag{46}$$

which can be rewritten as

$${}_0^C D_t^\alpha p(\cdot,t) = \mathcal{A}p(\cdot,t), \quad p(x,0) = p_0(x),$$

with the operator $\mathcal{A} : D(\mathcal{A})(\subset L^2(0,1)) \to L^2(0,1)$ given by

$$\begin{cases} [\mathcal{A}f](x) = \varepsilon f''(x) - cf(x), \\ D(\mathcal{A}) = \{f \in H^2(0,1) | f'(0) = 0, f'(1) = 0\}. \end{cases} \tag{47}$$

Lemma 8 (See the work of Zhou and Guo[5]).
Suppose that $c > 0$. For any initial value $p(\cdot,0) \in L^2(0,1)$, there exists a unique solution to (46) such that $p \in C(0,\infty;L^2(0,1))$, which is Mittag-Leffler stable, ie, there exist two constants $M,\mu > 0$ such that $\|p(\cdot,t)\|_{L^2(0,1)} \leq ME_\alpha(-\mu t^\alpha)$.

Similar to Corollary 1, $p(1,t)$ has the following estimation.

Corollary 2. *Suppose that $c > 0$. For any initial value $p(\cdot,0) \in D(\mathcal{A})$, the solution of (46) satisfies $|p(1,t)| \leq ME_\alpha(-\mu t^\alpha)$ with some $M,\mu > 0$.*

Under the invertible transformation (26), system (41) is converted into

$$\begin{cases} {}_0^C D_t^\alpha \widehat{w}(x,t) & = \widehat{w}_{xx}(x,t) - c\widehat{w}(x,t), \\ \widehat{w}_x(0,t) & = 0, \\ \widehat{w}(1,t) & = u(t) + d(t) - \int_0^1 k(1,y)w(y,t)\mathrm{d}y. \end{cases} \tag{48}$$

If the disturbance $d(t)$ vanishes, the stabilizing control law[5] is chosen as

$$u(t) = \int_0^1 k(1,y)w(y,t)\mathrm{d}y. \tag{49}$$

When the disturbance $d(t)$ is nonzero, the control law (49) cannot stabilize system (41). A counter-example can be taken as $d(t) = \cosh(\sqrt{c})$. In this case, system (41) admits a nonzero solution

$$w(x,t) = \cosh\left(\sqrt{c}x\right) + \int_0^x l(x,y)\cosh\left(\sqrt{c}y\right)\mathrm{d}y, \ \forall\, t \geq 0.$$

Now, since we have estimated $d(t)$ and its approximated value is $z(1,t)$, it is natural to propose the following disturbance-estimator-based feedback controller:

$$u(t) = -z(1,t) + \int_0^1 k(1,y)w(y,t)\mathrm{d}y. \tag{50}$$

It is seen that the "$-z(1,t)$" term in (50) is used to cancel (compensate) the disturbance, and the remaining term is to stabilize system (41) whenever $d(t)$ vanishes. The closed loop of (41) corresponding to controller (50) becomes

$$\begin{cases} {}_0^C D_t^\alpha \widehat{w}(x,t) & = \widehat{w}_{xx}(x,t) - c\widehat{w}(x,t), \\ \widehat{w}_x(0,t) & = 0, \widehat{w}(1,t) = p(1,t). \end{cases} \tag{51}$$

Lemma 9. *Suppose that $c > 0$ and the signal $p(1,t)$ is generated by system (46). For any initial value $\widehat{w}(x,0) \in L^2(0,1)$, system (51) admits a unique solution $\widehat{w} \in C(0,\infty;\mathcal{X})$, which is Mittag-Leffler stable, ie, there exist two constants $M, \mu > 0$ such that $\|\widehat{w}(\cdot,t)\|_{L^2(0,1)} \leq ME_\alpha(-\mu t^\alpha)$ for all $t \geq 0$.*

Proof. Similar to the proof of Lemma 6, we introduce a new variable $\widetilde{w}(x,t) = \widehat{w}(x,t) - p(x,t)$. Noting that $p(x,t)$ is a solution of (46), it is easy to check that $\widetilde{w}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{w}(x,t) & = \widetilde{w}_{xx}(x,t) - c\widetilde{w}(x,t), \\ \widetilde{w}_x(0,t) & = 0, \quad \widetilde{w}(1,t) = 0. \end{cases} \tag{52}$$

It follows from lemma 3.1 in the work of Zhou and Guo[5] that system (52) admits a unique solution $\widetilde{w} \in C(0,\infty;L^2(0,1))$ and there exist $M_1, \mu_1 > 0$ such that $\|\widetilde{w}(\cdot,t)\|_{L^2(0,1)} \leq M_1E_\alpha(-\mu_1 t^\alpha)$ for all $t \geq 0$. On the other hand, by Lemma 5, we obtain the existence of solution of system (51). By Lemma 5 again, $\|p(\cdot,t)\|_{L^2(0,1)} \leq M_2E_\alpha(-\mu_2 t^\alpha)$ with some $M_2, \mu_2 > 0$. Therefore,

$$\|\widehat{w}(\cdot,t)\|_{L^2(0,1)} \leq \|\widetilde{w}(\cdot,t)\|_{L^2(0,1)} + \|p(\cdot,t)\|_{L^2(0,1)} \leq ME_\alpha(-\mu t^\alpha), \text{for all } \ t \geq 0$$

with $M = M_1 + M_2$, $\mu = \min\{\mu_1, \mu_2\}$. This ends the proof. □

We go back to the closed-loop system (41) under the feedback (50)

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) &= w_{xx}(x,t) + \lambda(x)w(x,t), \\ w_x(0,t) &= -qw(0,t), \\ w(1,t) &= -z(1,t) + \int_0^1 k(1,y)w(y,t)\mathrm{d}y + d(t), \\ {}_0^C D_t^\alpha v(x,t) &= v_{xx}(x,t) + \lambda(x)w(x,t) - c[v(x,t) - w(x,t)], \\ v_x(0,t) &= -qw(0,t), \\ v(1,t) &= -z(1,t) + \int_0^1 k(1,y)w(y,t)\mathrm{d}y, \\ {}_0^C D_t^\alpha z(x,t) &= z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) &= 0, \quad z_x(1,t) = w_x(1,t) - v_x(1,t). \end{cases} \tag{53}$$

We consider system (35) in the state space $\mathcal{H} = (L^2(0,1))^3$.

Theorem 2. *Suppose that $c_0, c_1 > 0$, and $d \in L^\infty(0,\infty)$ (or $d \in L^2(0,\infty)$). For any initial value $(w_0, v_0, z_0, \hat{w}_0) \in \mathcal{H}$, there exists a unique solution to (35) such that $(w,v,z) \in C(0,\infty;\mathcal{H})$ satisfying $\|w(\cdot,t)\|_{L^2(0,1)} \le ME_\alpha(-\mu t^\alpha)$, with some $M, \mu > 0$, and $\sup_{t\ge0}\|(v(\cdot,t), z(\cdot,t))\|_{[L^2(0,1)]^2} < +\infty$. If we assume further that $d \equiv 0$, then there exist two constants $M', \mu' > 0$ such that $\|(v(\cdot,t), z(\cdot,t))\|_{[L^2(0,1)]^2} \le M'E_\alpha(-\mu' t^\alpha)$, $t \ge 0$.*

Proof. Using the variables $\hat{v}(x,t)$ and $p(x,t)$ given by (43) and (46), respectively, and the invertible transformation (26), we can write (53) to its equivalent form

$$\begin{cases} {}_0^C D_t^\alpha \hat{v}(x,t) &= \hat{v}_{xx}(x,t) - c\hat{v}(x,t), \\ \hat{v}_x(0,t) &= 0, \quad \hat{v}_x(1,t) = -d(t), \\ {}_0^C D_t^\alpha p(x,t) &= p_{xx}(x,t) - cp(x,t), \\ p_x(0,t) &= 0, \quad p_x(1,t) = 0, \\ {}_0^C D_t^\alpha \hat{w}(x,t) &= \hat{w}_{xx}(x,t) - c\hat{w}(x,t), \\ \hat{w}_x(0,t) &= 0, \quad \hat{w}(1,t) = p(1,t). \end{cases} \tag{54}$$

The existence of solution of system (36) follows from Lemmas 7, 8, and 9. By Lemmas 8 and 9 again, there exist $M_1, \mu_1 > 0$ such that $\|(p(\cdot,t), \hat{w}(\cdot,t))\|_{(L^2(0,1))^2} \le M_1E_\alpha(-\mu_1 t^\alpha)$. Noting the invertible transformation

$$\begin{pmatrix} w(x,t) \\ v(x,t) \\ z(x,t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & (I+\mathbb{P})^{-1} \\ I & 0 & (I+\mathbb{P})^{-1} \\ -I & -I & 0 \end{pmatrix} \begin{pmatrix} \hat{v}(x,t) \\ p(x,t) \\ \hat{w}(x,t) \end{pmatrix}, \tag{55}$$

where $I + \mathbb{P}$ is defined by (28), we have that $(w,v,z) \in C(0,\infty;\mathcal{H})$ is well defined and satisfies $\|w(\cdot,t)\|_{L^2(0,1)} \le \|(I+\mathbb{P})^{-1}\|M_1E_\alpha(-\mu_1 t^\alpha)$ and $\sup_{t\ge0}\|(v(\cdot,t), z(\cdot,t))\|_{(L^2(0,1))^2} < +\infty$.

Next, suppose that $d(t) \equiv 0$. By Lemma 7 again, we have $\|\hat{v}(\cdot,t)\|_{L^2(0,1)} \le M_2E_\alpha(-\mu_2 t^\alpha)$ with some $M_2, \mu_2 > 0$, which, together with (37), implies $\|(v(\cdot,t), z(\cdot,t))\|_{\mathbb{H}^2} \le M'E_\alpha(-\mu' t^\alpha)$, $t \ge 0$ for some $M', \mu' > 0$. This completes the proof. □

7 | NUMERICAL SIMULATION

In this section, we present some numerical simulations for illustration of system (35). For numerical computations, the parameters are taken as $q = 0, c = 4$, and the disturbance is taken as $d(t) = \sin(2t) + \cos(\pi t)$. The fractional derivative is taken as $\alpha = 0.6$ and $\alpha = 0.9$. The numerical algorithm is based on the combination of the $L1$ scheme[40] in time and the second-order centered difference scheme[41,42] in space. We take the spacial step $dx = 0.001$ and the time step $dt = 0.01$.

With these parameters, the $k(x,y)$ and $l(x,y)$ can be computed as

$$k(x,y) = -4x\frac{I_1\left(\sqrt{4(x^2-y^2)}\right)}{\sqrt{4(x^2-y^2)}}, \quad l(x,y) = -4y\frac{J_1\left(\sqrt{4(x^2-y^2)}\right)}{\sqrt{4(x^2-y^2)}},$$

where $I_1(x)$ and $J_1(x)$ are a first-order modified Bessel function and a first-order Bessel function given by

$$I_1(x) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^{2n+1}n!(n+1)!}, \quad J_1(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}n!(n+1)!},$$

respectively. The initial values for system (35) are taken as

$$w(x,0) = x^2,\ v(x,0) = \frac{2}{\pi}\cos\frac{\pi}{2}x + x^2,\ z(x,0) = 4 - 4\cos 2\pi x + \frac{2}{\pi}\cos\frac{\pi}{2}x.$$

Figures 2, 3, and 4 show the solution of system (35). The convergence of $w(x,t)$ is very fast in Figure 2. The boundedness of $(v(x,t), z(x,t))$ is satisfactory in Figures 3 and 4. Figure 5 shows the norm $\|w(\cdot,t)\|_{L^2(0,1)^2}$ and the value of $w(x,t)$ at $x = 0.5$. It is clearly seen from these figures that the convergence speed for system with $\alpha = 0.6$ is slower than that of system with $\alpha = 0.9$. Figure 6 shows the disturbance $d(t)$ and its estimation $z_x(1,t)$. It is seen that the disturbances for both $\alpha = 0.6$ and $\alpha = 0.9$ are estimated as the time t is sufficiently large. The estimation of $d(t)$ for $\alpha = 0.9$ is faster then that of $d(t)$ for $\alpha = 0.6$.

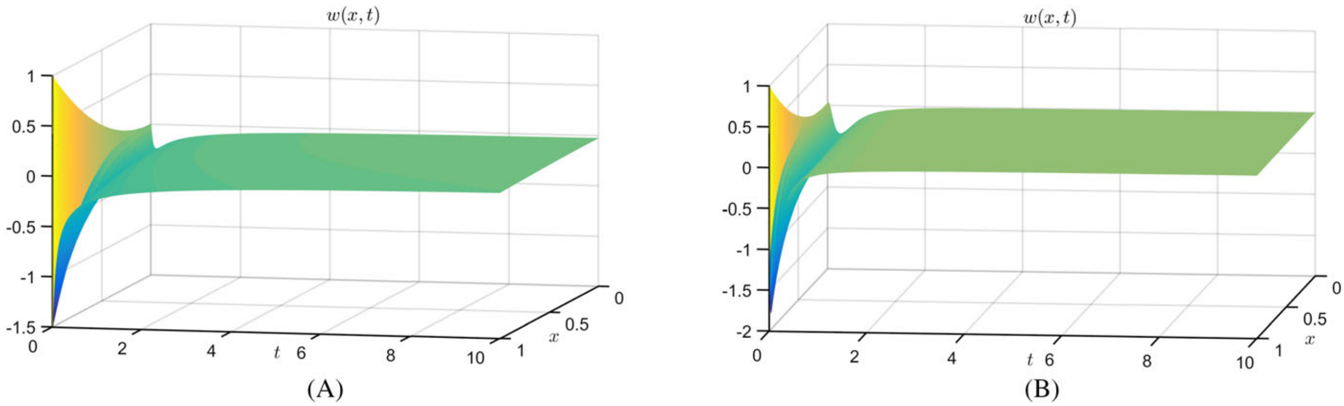

FIGURE 2 The displacement of $w(x,t)$ for system (35) with $\alpha = 0.6$ and $\alpha = 0.9$, respectively. A, The state $w(x,t)$ for system (35) with $\alpha = 0.6$; B, The state $w(x,t)$ for system (35) with $\alpha = 0.9$ [Colour figure can be viewed at wileyonlinelibrary.com]

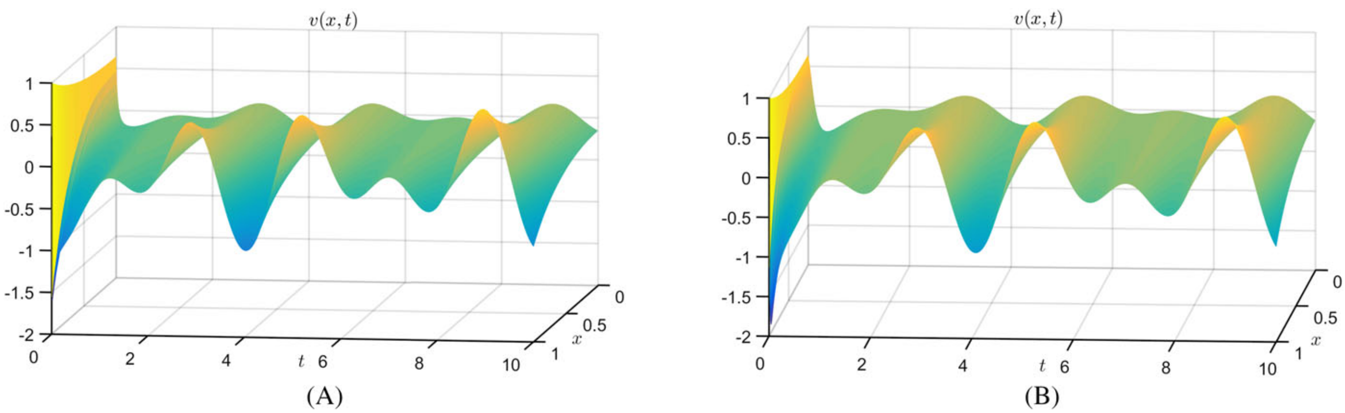

FIGURE 3 The displacement of $v(x,t)$ for system (35) with $\alpha = 0.6$ and $\alpha = 0.9$, respectively. A, The state $v(x,t)$ for system (35) with $\alpha = 0.6$; B, The state $v(x,t)$ for system (35) with $\alpha = 0.9$ [Colour figure can be viewed at wileyonlinelibrary.com]

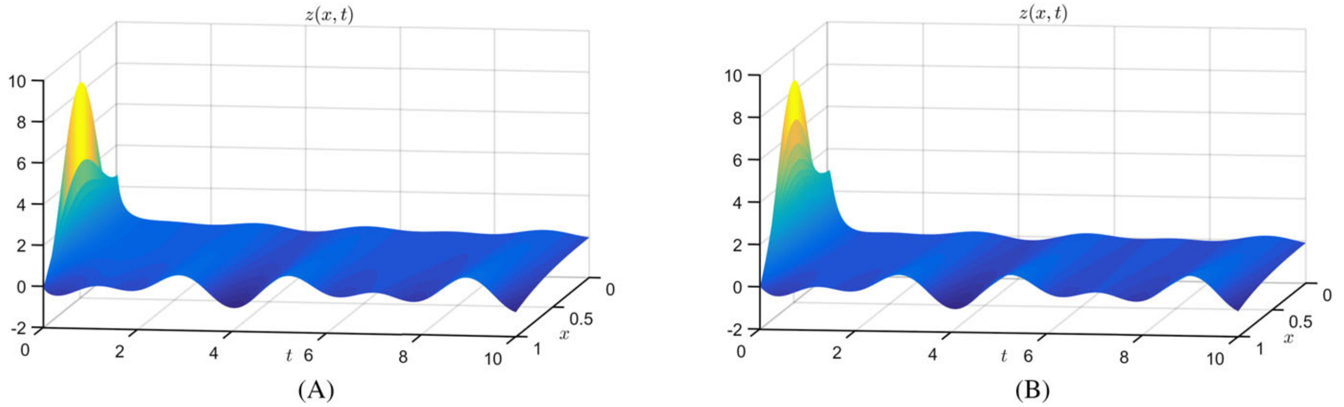

FIGURE 4 The displacement of $z(x, t)$ for system (35) with $\alpha = 0.6$ and $\alpha = 0.9$, respectively. A, The state $z(x, t)$ for system (35) with $\alpha = 0.6$; B, The state $z(x, t)$ for system (35) with $\alpha = 0.9$ [Colour figure can be viewed at wileyonlinelibrary.com]

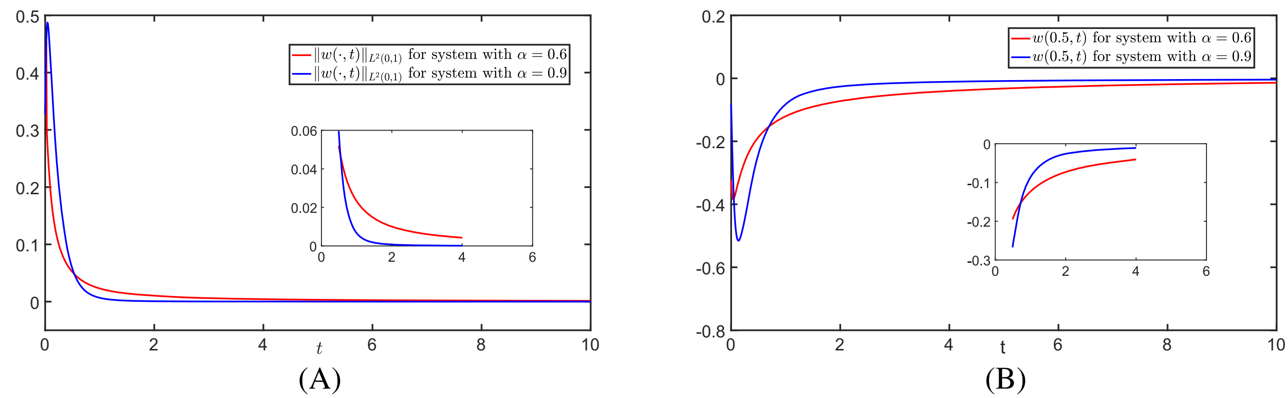

FIGURE 5 $\|w(\cdot, t)\|_{L^2(0,1)}$ and $w(0.5, t)$ for system (35) with $\alpha = 0.6$ and $\alpha = 0.9$, respectively. A, $\|w(\cdot, t)\|_{L^2(0,1)}$ for system (35); B, $w(0.5, t)$ for system (35) [Colour figure can be viewed at wileyonlinelibrary.com]

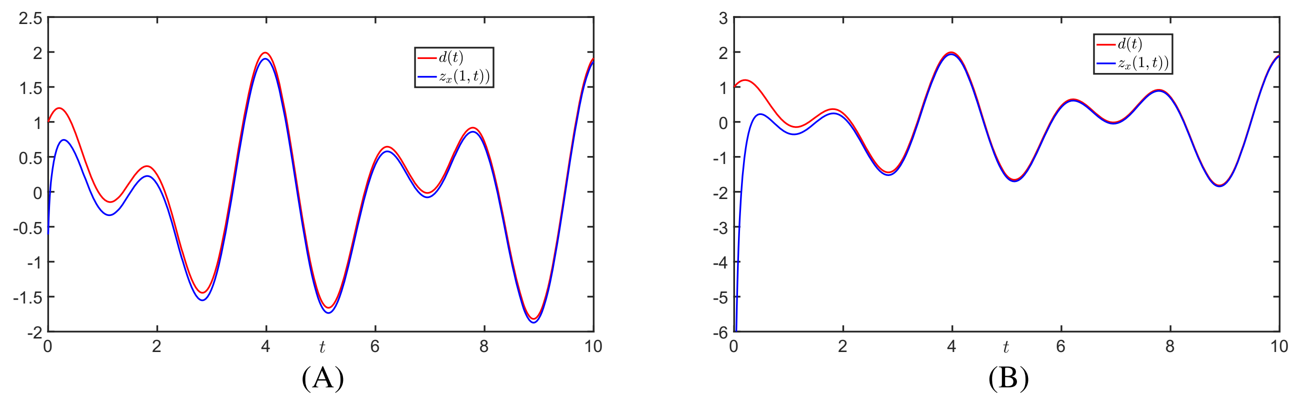

FIGURE 6 The disturbance $d(t)$ and its estimation $z_x(1, t)$ for system (35) with $\alpha = 0.6$ and $\alpha = 0.9$, respectively. A, The disturbance $d(t)$ and its estimation $z_x(1, t)$ for system (35) with $\alpha = 0.6$; B, The disturbance $d(t)$ and its estimation $z_x(1, t)$ for system (35) with $\alpha = 0.9$ [Colour figure can be viewed at wileyonlinelibrary.com]

8 | CONCLUDING REMARKS

In this paper, we study the ADRC approach to boundary feedback Mittag-Leffler stabilization for a one-dimensional time-fractional anomalous diffusion equation with the disturbance suffered from the control matched boundary. By constructing two auxiliary systems, the disturbance can be estimated and thus can be canceled in the feedback loop. By the estimation of the disturbance, we are able to design a stabilizing feedback control for system with unknown input. The results obtained here could provide some insights into the qualitative analysis of the design of fractional PDEs with boundary disturbance. The approach is potentially usable for treating other fractional PDEs with boundary control matched disturbance, such as the following model

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t) + \lambda(x) w(x,t), \ x \in (0,1), t \geq 0, \\ w(0,t) = 0, w(1,t) = u(t) + d(t), \ t \geq 0, \\ w(x,0) = w_0(x), \ 0 \leq x \leq 1, \end{cases} \tag{56}$$

where $w(x,t)$ is the state, $u(t)$ is the control input, and $d(t)$ is the unknown external disturbance, as well as the spatially-varying diffusion coefficient for fractional reaction diffusion model in the work of Chen et al,[7] which are of great interest and will be considered in our future works. Finally, the feedback in this paper is about the full state feedback, and a more interesting future work is on output-feedback stabilization for fractional PDEs, based certainly on the state-feedback stabilization results developed in this paper.

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