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Research paper

Fractional derivative modeling for suspended sediment in unsteady flows

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ABSTRACT

This paper makes an attempt to develop a fractional model for describing the distribution of suspended sediment in unsteady flows and study nonlinear dynamic phenomenon of fluids. This model shows the dynamic process of suspended sediment transport. The continuous-time random walk (CTRW) framework provides reasonable physical meaning for fractional order. By solving the equations with different orders and analyzing the results, we get the changing laws of concentration of suspended sediment and find some interesting phenomenon. The above results prove that fractional derivative can well describe the non-local properties of suspended sediment transport, including the non-local properties of time and space. Thus, the fractional derivative model can be serve as a candidate to describe the distribution of suspended sediment in unsteady flows.

1. Introduction

As we all know, fractional calculus has the advantages of the dependence, memory and sufficient preponderance in modeling a given system with viscoelastic substances, compared with integer calculus [1]. Specially, fractional model can capture the occurrence of anomalous events [2,3]. Therefore, fractional modeling has been the focus of research in the past few decades [4]. Many scholars regard fractional differential equations as alternative models to traditional non-linear equations [5,6]. Fractional calculus has been applied to various field, such as biology, hydrology, finance, electronic, control and so on [7-14].

Suspended sediment transport is a typical non-linear behavior. In the traditional study of suspended sediment transport, scholars used to deal with the spread of suspended sediments according to the classical Fick's law [15-20]. Dhamotharan et al. developed a one-dimensional, unsteady numerical model for the prediction of the vertical distribution of suspended sediment concentration. Tomoya et al. established a model to simulate the temporal and spatial distribution of suspended sediment in wave area. Actually, the turbulent diffusion behaviors of suspended sediment are different from the classical Fick's law. However, this part is not involved in the above research. Many previous investigations of sediment suspension have shown that turbulent diffusion has various anomalous diffusion behaviors [21-23]. Nikora et al. experimented on a complex problem of turbulence-sediment interaction in open channel flow and found that both turbulence and sediment

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events are organized in fractal clusters. Berkowitz et al. analyze measured breakthrough curves in a laboratory model and found that it is inconsistent with classical advection diffusion theory. Therefore, sediment transport should be treated with different modeling and analysis methods. Some scholars have done some research on sediment by using fractional derivative models and fractal derivative models [24-28]. Sun et al. used fractional derivative model to describe the overall characteristics of partial bed load transport. Zhang et al. proposed a new simplified fractional order model for bed load sediment transport.

In this paper, there are two highlights. First, few scholars use fractional order to study non-constant suspended sediments in this respect. Secondly, we use the continuous-time random walk (CTRW) framework to provide a basis for this fractional model, so that the appearance of fractional orders is not abrupt. Third, we get the changing laws of concentration of suspended sediment and find some interesting phenomenon by solving the equations and analyzing the results.

The rest of this paper is organized as follows. In Section 2, the relationship between CTRW and fractional derivative model is introduced and the fractional partial differential equation is deduced for suspended sediment transport in unsteady flows. In Section 3, we get the changing laws of concentration of suspended sediment and the dynamic characteristics of suspended sediment transport by analyzing the calculation results under various conditions. In Section 4, we make a conclusion and discuss the advantages and disadvantages.

2. Modeling process

2.1. The continuous-time random walk framework

CTRW theory was first applied to calculate impurity conduction in semiconductors and to analyze the properties of amorphous semiconductors [29-31].

For the connection between CTRW and fractional convection diffusion equations, many scholars have published some interesting works [32-34]. Usually, scholars make a hypothesis that the particle motion is a process of instantaneous point motion. More specifically, the particle starts to randomly jump from the initial position at the initial moment. Usually, the CTRW model is based on the idea that the length of a given jump, as well as the waiting time elapsing between successive jumps is drawn from a probability density function (PDF). The functional shape of the PDF determined the nature of the transport, non-Fickian or Fickian. By adopting different forms of PDF, we can describe different types and different features of anomalous diffusion.

Let $W(x, t)$ be the probability that the particle is at x at time t and $\psi(x, t)$be the density function, which determines the jump length of the particle and the waiting time between the two jumps:

$$\lambda(x)=\int_0^{+\infty}\psi(x,t)dt \tag{1}$$

$$\omega(t)=\int_0^{+\infty}\psi(x,t)dx \tag{2}$$

The initial conditions are: $W_0(x)=W(x,0)=\delta(x)$, $W_0(t)=W(0,t)=\delta(t)$.

Let $\eta(x, t)$be the probability that the particle just arrives at time t. We get:

$$\eta(x,t)=\int_{-\infty}^{\infty}dx'\int_0^{\infty}\eta(x',t')\psi(x-x',t-t')dt'+\delta(x)\delta(t) \tag{3}$$

$$W(x,t)=\int_0^t\eta(x,t')\Psi(t-t')dt' \tag{4}$$

where $\Psi(t)=1-\int_0^t w(t')dt'$.

Combining Eqs. (3) and (4) and taking the Laplace-Fourier transformation, we get:

$$W(k,u)=\frac{1}{1-\psi(k,u)}\frac{1-w(u)}{u} \tag{5}$$

2.2. The transport of suspended sediment

The suspended sediment transport equation is derived based on turbulent diffusion theory, which contains the most important roles played by advection, turbulent diffusion, and gravitation in the suspension of sediment particles. The physical mechanism of an unsteady sediment suspension distribution is a dynamic of vertical fluxes between downward sediment settling and upward turbulent diffusion. It is available from the theory of turbulent diffusion that suspended sediment transport is an anomalous diffusion behavior. The above conclusions indicate that the probability density function of waiting time and the jump length are likely to be a long-tailed distribution rather than a Poisson distribution or Gaussian distribution. Fig. 1 shows the difference between Long-tailed distribution and Gaussian distribution. Here, we choose $y=\frac{1}{10x}$ as the probability density function of Long-tailed distribution and $y=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ as the probability density function of Gaussian

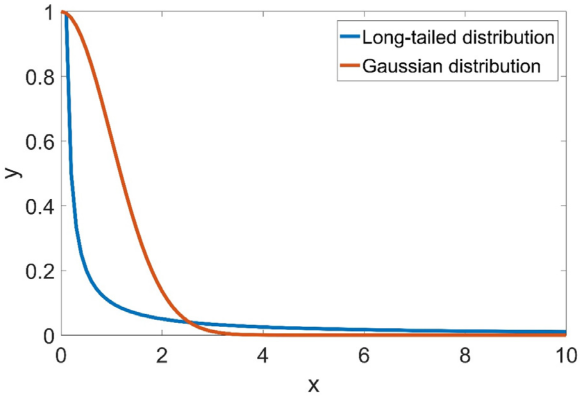
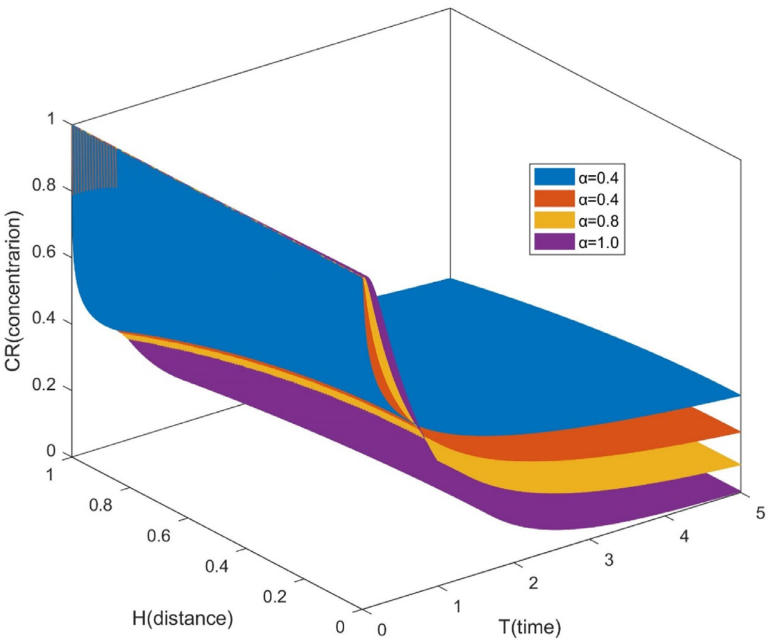


Fig. 1. The curve of long-tailed distribution ($y = \frac{1}{10x}$) and Gaussian distribution ($y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$) with $x \in [0, 10]$.



(a)

Fig. 2. Dimensionless numerical simulation results of sediment concentration with different time fractional derivatives $\alpha(\alpha = 0.4, 0.6, 0.8, 1.0)$. (a) 3-D graph of sediment concentration vs. distance and time. (b) The changing law of sediment concentration with the increase of time T when $H=1$. (c) The changing law of sediment concentration with the increase of distance H where $T=1$.

distribution. From Fig. 1, at the beginning, the curve of long-tailed distribution decreases faster than the curve of Gaussian distribution with the increase of x. As x continues to increase, the rate of decline decreases. The anomalous events in diffusion, like the latter half of the long tail distribution, cannot be ignored. Interestingly, this property can be represented by fractional derivatives through CTRW framework.

Considering the case of anomalous diffusion in an external force fields, one gets:

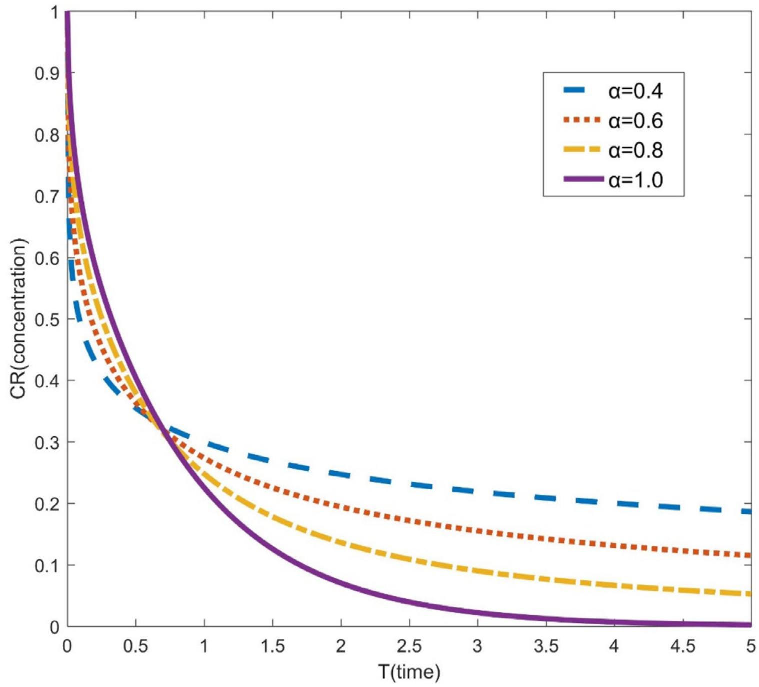
$$\phi(x, t) = \psi(x - \upsilon t, t) \tag{6}$$

Taking the Laplace-Fourier transformation and Taylor expansion, one

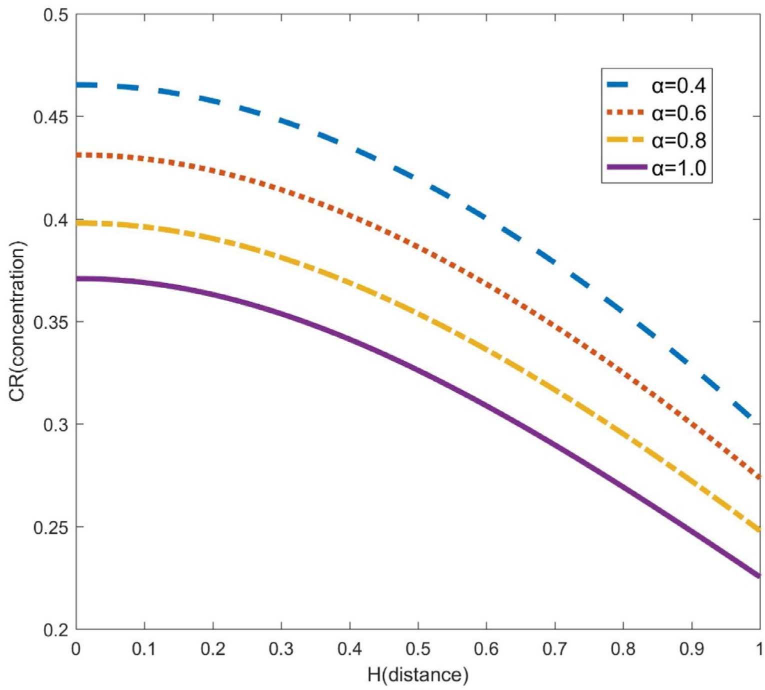
$$\phi(k, u) = \psi(k, u + i\upsilon k) \tag{7}$$

We choose [21,22]:

$$\lambda(k) = e^{-\sigma\left[V_1 k - qD_1(ik)^{\mu} - (1-q)D_1(-ik)^{\mu}\right]} \sim 1 - \sigma[V_1 ik - qD_1(ik)^{\mu} - (1-q)D_1(-ik)^{\mu}] \tag{8}$$



(b)



(c)

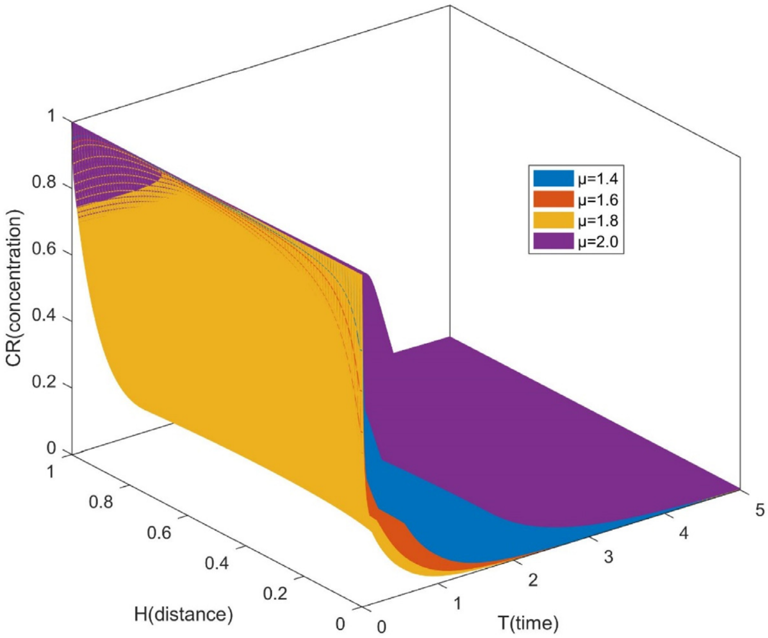
Fig. 2. Continued

$$w(u) = e^{-(\tau u)^{\alpha}} \sim 1 - (\tau u)^{\alpha} \tag{9}$$

where V_1 is the drift coefficient, D_1 is the diffusion one and the parameter $0 \le q \le 1$ called "skewness" represents the proportion of high-velocity "jets" in the direction of flow [35,36].

Thus, we can get:

$$\phi(k,u) = \lambda(k)w(u) \sim 1 - (\tau u)^{\alpha} - \sigma[V_1 ik - qD_1(ik)^{\mu} - (1-q)D_1(-ik)^{\mu}] \tag{10}$$



(a)

Fig. 3. Dimensionless numerical simulation results of sediment concentration with different time fractional derivatives $\mu(\mu = 1.4, 1.6, 1.8, 2.0)$. (a) 3-D graph of sediment concentration vs. distance and time. (b) The changing law of sediment concentration with the increase of time T when $H = 0.5$. (c) The changing law of sediment concentration with the increase of distance H where $T = 1$.

Combining Eqs. (5), (9) and (10), we get:

$$W(k,u) = \frac{1}{u + Mu^{1-\alpha}\left[V_1 ik - qD_1(ik)^{\mu} - (1-q)D_1(-ik)^{\mu}\right]} \tag{11}$$

where $M = \frac{\sigma}{\tau^{\alpha}}$.

Taking the Laplace-Fourier inverse transformation to obtain the following fractional advection diffusion equation, we get:

$$ {}_0^cD_t^{\alpha} W(x,t) = -V\frac{\partial W(x,t)}{\partial x} + qD\frac{\partial^{\mu} W(x,t)}{\partial x^{\mu}} + (1-q)D\frac{\partial^{\mu} W(x,t)}{\partial(-x)^{\mu}} \tag{12}$$

where $V = MV_1$,$D = MD_1$,${}_0^cD_t^{\alpha}$ denotes the caputo derivative and $\frac{\partial^{\alpha}}{\partial x^{\alpha}}$ denotes the Riemann-Liouville derivative. For suspended sediment, we chose q= 1.Thus, Eq. (12) reduces to the following space-time FADE:

$$ {}_0^cD_t^{\alpha} W(x,t) = -V\frac{\partial W(x,t)}{\partial x} + D\frac{\partial^{\mu} W(x,t)}{\partial x^{\mu}} \tag{13}$$

The choice of q is mainly based on the observation of the anomalously rapid transport of contaminants in heterogeneous systems.

For suspended sediment transport in unsteady flow, Eq. (13) becomes:

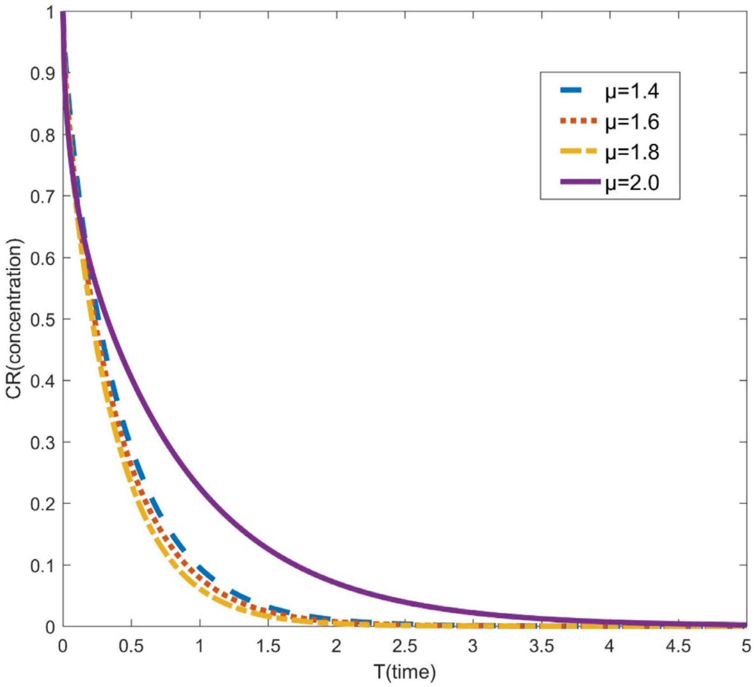
$$ {}_0^cD_t^{\alpha} S(y,t) = w\frac{\partial S(y,t)}{\partial y} + \varepsilon_{sy}\frac{\partial^{\mu} S(y,t)}{\partial y^{\mu}} \tag{14}$$

where $S(y, t)$ is the concentration of sediment, y is the vertical distance from the riverbed, w is the sedimentation velocity of the sediment, ε_{sy} is the vertical component of the turbulent diffusion coefficient of the sediment,α is the index of the time fractional derivative and μ is the index of the space fractional derivative.

In the following contents, we will simplify its initial conditions and boundary conditions [37]. The initial condition is:

$$S(y,0) = C \tag{15}$$

where C is a constant.



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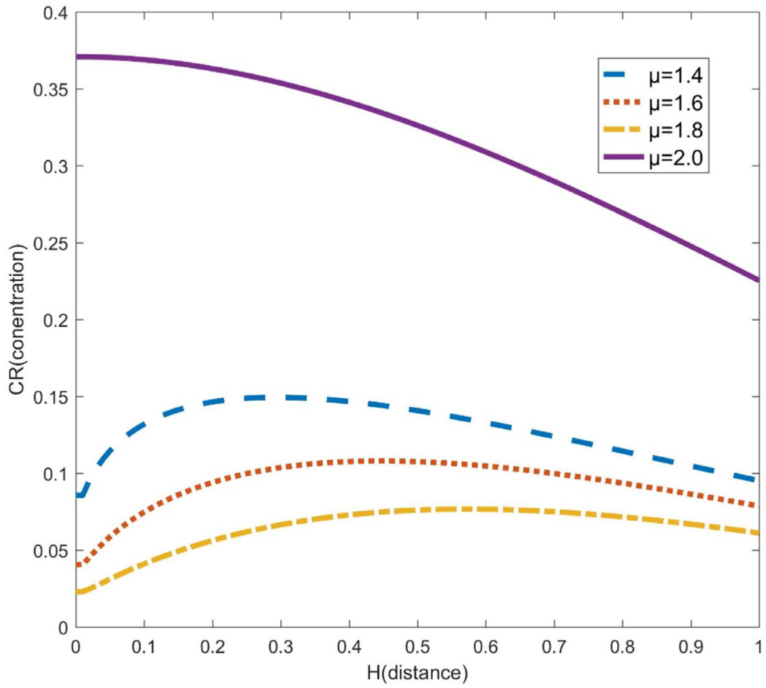


(c)

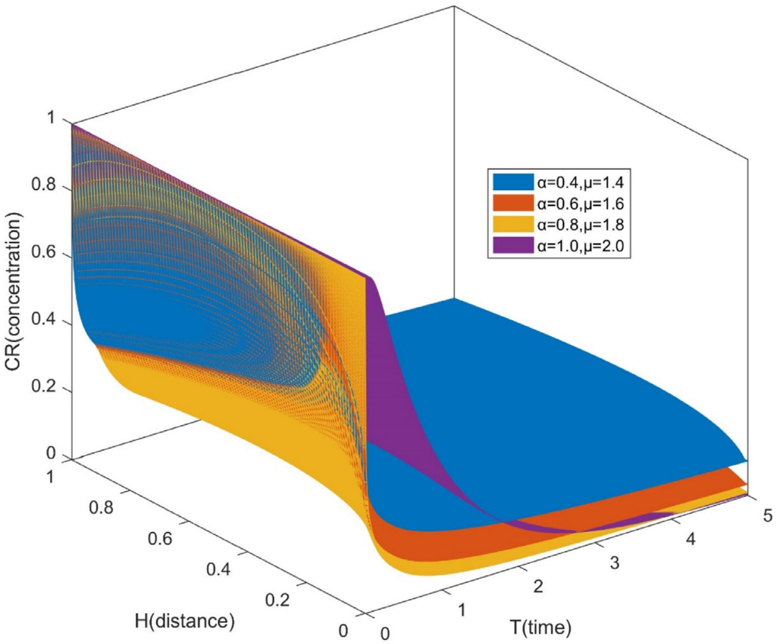
Fig. 3. Continued

Bottom boundary condition of the river bed (there is no resuspension) is:

$$\left.\frac{\partial S}{\partial y}\right|_{y=0} = 0 \tag{16}$$

Free water boundary condition (there is no net transfer of sediment) is:

$$\left. ws + \varepsilon_{sy}\frac{\partial S}{\partial y}\right|_{y=h} = 0 \tag{17}$$



(a)

Fig. 4. Dimensionless numerical simulation results of sediment concentration with different time fractional derivatives α and space fractional derivatives μ(α=0.4, $\mu = 1.4$; α=0.6, $\mu = 1.6$; α=0.8, $\mu = 1.8$; α= 1.0, $\mu = 2.0$). (a) 3-D graph of sediment concentration vs. distance and time. (b) The changing law of sediment concentration with the increase of time T when H= 1. (c) The changing law of sediment concentration with the increase of distance H where T= 1.

3. Numerical calculation

We simplify the equation and make the equation dimensionless to explore the features of the equation. $w$and ε_{sy} are set to be a constant, respectively. We take the water depth ratio $H = y/h$, the concentration ratio$CR = S/S_0$, and the time ratio $T = w * t/h$, where h is the vertical distance from the free water surface to the riverbed, S_0 is the concentration at the bottom of the riverbed. Then the following Dimensionless result is:

$$ {}_0^cD_T^\alpha CR = \frac{\partial CR}{\partial H} + D\frac{\partial^\mu CR}{\partial H^\mu}\left(D = \frac{\varepsilon_{sy}}{wh}\right) \tag{18} $$

In order to solve the above equation, we use the finite difference method to discretize the equation. We take the time step τ=0.01 and the space step $h = 0.01$.The discrete format is as follows:

$$ {}_0^cD_T^\alpha CR(H_i, T_{k+1}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k} b_j^{(\alpha)}\Big(u\big(H_i, T_{k-j+1}\big) - u\big(H_i, T_{k-j}\big)\Big) + O(\tau) \tag{19} $$

where $b_j^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}$

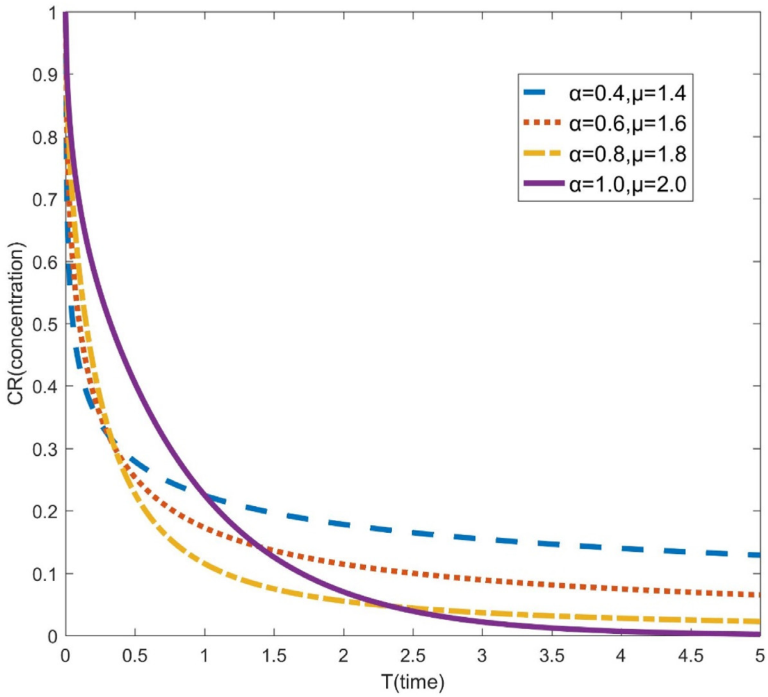
$$ \frac{\partial^\mu CR(H_i, T_{k+1})}{\partial H^\mu} = h^{-\mu}\sum_{j=0}^{i} w_j^{(\mu)} u(H_i - (j-1)h, T_{k+1}) + O(h) \tag{20} $$

where $w_j^{(\mu)} = (-1)^j \frac{\mu(\mu-1)\cdots(\mu-j+1)}{j!}$

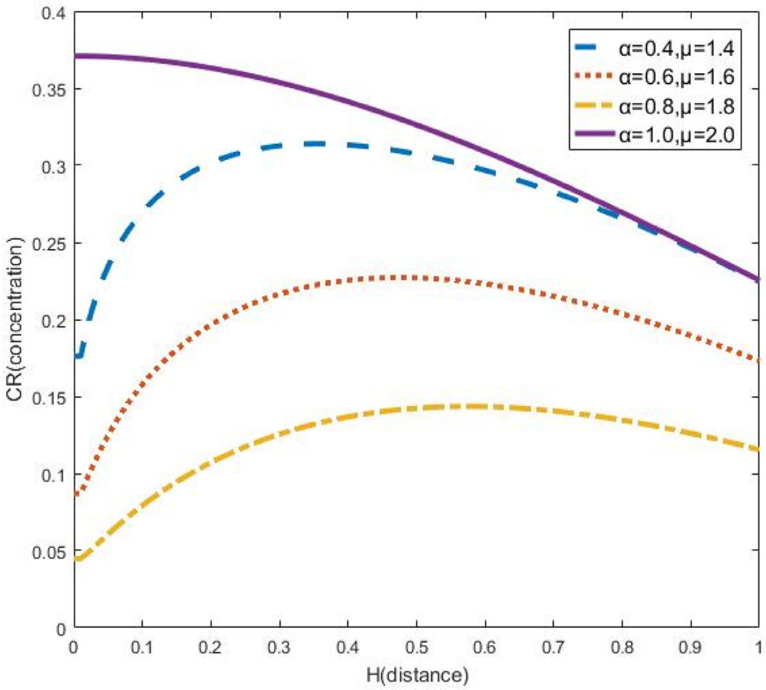
Then we solve the time fractional equations, the spatial fractional equations, and the space-time fractional equations separately to explore the characteristics under different conditions.

3.1. The time fractional derivative advection-diffusion model

We want to study the law of sediment concentration with the changing of time fractional derivatives α.Here, we solve the Eq. (18) with $D = 1$, $\mu = 2$ and get Fig. 2. Fig. 2 shows dimensionless numerical simulation results of sediment concentration with different time fractional derivatives α. From Fig. 2(a), the 3-D graph shows the trends of sediment concentration clearly with the changing of the distance and the time. More specifically, the sediment concentration decreases with the increase of the time T and the distance H, respectively. From Fig. 2(b), the time curve of sediment concentration decreases from steep to



(a)



(b)

Fig. 4. Continued

slow with the increase of the time T. On the other hand, the distribution of long-tailed becomes more and more obviously with the decrease of time fractional derivative α. Note that, the rate of change of sediment concentration when $0 < \alpha < 1$ is less than the rate in the normal ($\alpha = 1$). From Fig. 2(c), the downward trend of the curves of sediment concentration are almost the same with the decrease of time fractional derivative α. The changing law of sediment concentration can be used to describe the memory effect (time-nonlocality) in the anomalous diffusion of sediment transport which means the previous sediment deposition will have an impact on the current sediment transport.

3.2. The space fractional derivative advection-diffusion model

We want to study the law of sediment concentration with the changing of space fractional derivatives μ. Here, we solve the Eq. (18) with $D=1$, $\alpha=1$ and get Fig. 3. Fig. 3 shows dimensionless numerical simulation results of sediment concentration with different space fractional derivatives μ. Fig. 3(a) is similar to Fig. 2(a). From Fig. 3(b), time curve of the sediment concentration is steeper when $1<\mu<2$ than normal (μ=2).The changing law conforms to the phenomenon of super diffusion. From Fig. 3(c), space curve is more gentle when $1<\mu<2$ than normal (μ=2) with the increase of distance H. The changing law conforms to the phenomenon of super diffusion and shows the space-nonlocality of sediment. However, the sediment concentration rises and then falls with the increase of the distance H. In this respect, it is not in line with the actual situation. Problem may arise on the boundary condition and need further analysis.

3.3. The time–space fractional derivative advection-diffusion model

We want to study the law of sediment concentration with the changing of time fractional derivatives α and space fractional derivatives μ. Here, we solve the Eq. (18) with $D=1$ and get Fig. 4. Fig. 4(a) is similar to Figs. 2(a) and 3(a). Fig. 4 shows dimensionless numerical simulation results of sediment concentration with different time fractional derivatives α and space fractional derivativesμ. From Fig. 4(b),the sediment concentration decreases faster when $0<\alpha<1$, $1<\mu<2$ than normal (α=1, μ=2) in the early time which represents the super diffusion. As the time increasing, the memory effect becomes more and more obviously which represents the sub-diffusion. From Fig. 4(c), space curve is more gentle when $0<\alpha<1$, $1<\mu<2$ than normal (α=1, μ=2) with the increase of distance H which shows the space-nonlocality of sediment. Therefore, for a time–space fractional derivative advection-diffusion model, whether the diffusion is super-diffusion or sub-diffusion depends on the actual situation.

4. Conclusion and discussion

In this paper, a fractional model based on CTRW framework is established, and suspended sediment transport is studied by numerical simulation. By comparing different order and different parameter values, the influence of each parameter on the equation motion characteristics is explored. Three major conclusions are drawn from the above study.

(1) Compared with the traditional advection diffusion, changing the time derivative into fractional order can effectively describe the phenomenon of sub-diffusion and show the non-local property of sediment in time direction. By changing the different order, the sub-diffusion speed under different conditions can be well described.
(2) By changing the spatial derivative to fractional order, super-diffusion will occur in the early stage, especially in Figs. 3(b) and 4(b). This can be used to capture the phenomenon of fast displacement of suspended sediment in unsteady flow.
(3) In CTRW framework, we choose power law distribution for the probability density functions of waiting time and jump distance. From the results of calculation, the non-local characteristics of suspended sediment movement, a complex anomalous diffusion phenomenon, can be well captured. It is proved that it is feasible to use power law function as probability density function for suspended sediment transport.

In the classical diffusion equation, the derivative order is an integer. From the mathematical point of view, the essence of the integer derivative is the local linear approximation of the function through the concept of the limit. Thus, the integer derivative has local properties. For an ideal problem with many assumptions, the classical partial differential equations are good at characterizing the problem. However, for many real problems, especially complex systems or non-linear situations, a certain state is determined not only by the last moment or the next moment, but also by the nature of the previous period or the whole domain. At this time, the original classical diffusion equation will no longer be applicable. But for the fractional derivatives, the definition is competent because of non-locality. Thus, fractional derivatives are advantageous to describe a physical problem. Another advantage of the fractional derivatives is that it has less parameters and the physical meaning is very clear. Just like the suspended sediments discussed above, it has four parameters including two fractional derivatives. It's much easier compared with other nonlinear models.

Declaration of Competing Interest

No conflict of interest.

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