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Finite energy Lyapunov function candidate for fractional order general nonlinear systems

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ABSTRACT

The construction of Lyapunov function candidates and the norm of infinite energy solutions remain among the most elusive unsolved problems of fractional order systems. This paper develops a new framework to construct finite energy Lyapunov function candidates for fractional order general nonlinear systems with randomness, uncertainty, time-delay or memory. Fundamentals of fractional order system and equilibrium are revisited to start up the investigation, where the pseudo and true states of fractional order systems are considered. The process of constructing fractional order Lyapunov function candidate is mainly divided into three steps: Firstly, converting the original system into an equivalent Volterra integro-differential equation, where the weak singularity of fractional order system is included. Secondly, the fractional order Lyapunov function candidate is derived by canceling out the weak singularity that acts as the catalyst, and is absorbed into the fractional finite energy terms. Lastly, the first order derivative of the proposed Lyapunov function candidate is negative definite and bounded by power-law relevant terms. From finite energy aspect, the proposed fractional order Lyapunov function candidate is composed of potential, kinetic and/or Riesz potential energies in terms of physics. The fractional order Lyapunov's theorem and asymptotic stability of equilibrium points are discussed, and some non-L_p stable cases have been shown as fractional order finite energy ones. The impacts of fractional order, region of attraction and initialization state on the stability of equilibrium points are presented as well. Some classical integer order and fractional order results can be deduced from this work. A number of examples are illustrated to substantiate the effectiveness of the proposed unified framework.

1. Introduction

In 1945, H.W. Bode proposed the ideal loop transfer function method [1], which was the first documentary record of fractional order cybernetics. In the following decades [2,3], some effective fractional order control strategies have been gradually integrated into various fields of science and engineering [4–6]. Nevertheless, it is surviving in crevices. The biggest resistance stems from the extremely difficult of proving the stabilization for fractional order non-commensurate linear, linear time vari-

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ant and nonlinear systems. Allow for the crucial roles of stability analysis in design of control system, any novel stabilization method usually is leading a new trend [7–9]. In other words, the fractional order stability analysis is bearing the brunt of urgent needs in fractional order cybernetics and system analysis.

The stability analysis for fractional order nonlinear system is one of the most charming and challenging topics. Early in 1998, the necessary and sufficient condition of BIBO stability for generalized fractional differential systems is discussed in both time and frequency domains [10]. The generalized Gronwall–Bellman lemma [11] provides another useful way for the fractional order stability analysis of equilibrium points, and can be applied to a number of nonlinear control problems. Besides, the Łojasiewicz's inequality is helpful in a wide range of fractional order problems such as the definition of integer order Lyapunov function candidate of fractional order systems [12]. More general results of fractional differential inequalities are summarized in [13]. The fractional order comparison principle provides a round way in stability analysis and stabilization controller design [14]. The indirect Lyapunov method artfully converts the fractional order problem to an integer order one considering the continuous frequency distribution [7,9].

The variational Lyapunov method gives Lyapunov-like functions to connect the solutions of the perturbed and the unperturbed systems in terms of the maximal solution of a comparison problem [15]. The Lyapunov–Krasovskii function approach [16] and the Mittag–Leffler stability [17,18] are two theoretical fundamentals, where the later is an extension of integer order Lyapunov method[19]. The LMI characterization of fractional order system [20–22] formulates a practical way of stability and stabilization for fractional order linear systems. The quadratic Lyapunov function for fractional order system is a big landmark [8,23] that successfully extend the Lyapunov second method to fractional order ones. The stabilization or controller design of a large group of fractional order systems can be practically implemented after the above two works. Lastly, some other relevant works are listed here to extend readability: survey paper [24,25], fractional order stochastic system [26], fractional order switching system [27], applications of fractional calculus [28], and a series of interesting and useful results [29,30].

However, most of the known results are motivated by and extended from integer order references, even if there are essential differences between integer order and fractional order systems such as the intermediate processes [31] and the anomalous phenomena [32]. Besides, the pseudo state instead of true state conditions are assumed. Thus, in the process of fractional order deductions using integer order analogous methods, they are either difficult to achieve desired conclusions, leading to complicated results, or still remaining unsolved such as non-finite time stability [19], non-periodic solutions [33], different definitions of limit circle [34], unpractical Ricatti equation, non-L_p stability [35], extremely difficult to find Lyapunov function candidates with physical meanings, etc.

This paper explores a new path to find Lyapunov function candidate for fractional order general nonlinear systems, where fractional order energy terms are included. The characteristics of this paper can be summarized as:

(1) A number of the above mentioned difficult problems can be solved as the by-products of this work.
(2) Traditional fractional order stability analysis is a case study due to the lack of tools. Nevertheless, this paper, as a general framework, can be widely used in the stability analysis of nonlinear, stochastic, uncertain, time-delay, initialized and distributed fractional order systems. Its extensions to fractional order distributed and mathematical physical systems lead to potential advantages in fractional order biology, physics, electrochemistry and complex networks, etc so that will further draw potential attentions from medicine, science and engineering [36–39].

The innovative contributions of this paper mainly are:

(1) A novel fractional order Lyapunov function candidate for Caputo fractional order general nonlinear systems is presented which is composed of energy terms in physics.
(2) Fractional order finite energy Lyapunov's theorem is presented.
(3) The impacts of fractional order α, region of attraction and initialization state (memory) to the fractional order stability of equilibrium points are discussed.

The rest of this paper is organized as follows. Basic introductions of fractional calculus and fractional order systems are shown in Section 2. Section 3 revisits the equilibrium in fractional order systems. The proposed Lyapunov function candidates of fractional order autonomous and non-autonomous systems are discussed in Sections 4 and 5. In Sections 6 and 7, we provide the analysis of asymptotic stability and boundary problems, respectively. A number of physical examples are illustrated in Section 8. The conclusions and future works are included in Section 9.

2. Preliminaries

2.1. Fractional calculus

Fractional calculus is a generalization of regular calculus, where the integration and differentiation can be extended to non-integer orders including real and complex numbers. Nevertheless, for some historical reasons, to make consistent and to avoid unnecessary confusions, people still use fractional calculus as an alias of non-integer order calculus. This paper focuses on the fractional calculus with real orders, where the two most popular approaches, Riemann–Liouville (RL) and Caputo (C), are introduced as follows.

Definition 1 (Riemann–Liouville Approach). The Riemann–Liouville fractional order integral of order $p \in (0, 1)$ for the continuous function $x(t)$ is expressed as

$$ {}_{t_0}^{RL}I_t^p x(t) = \frac{1}{\Gamma(p)} \int_{t_0}^{t} \frac{x(\tau)}{(t-\tau)^{1-p}} d\tau, \tag{1} $$

where $t \geq t_0$ and $x(t) \equiv 0$ for $t < t_0$. The Riemann–Liouville fractional order derivative of order $q > 0$ is expressed as

$$ {}_{t_0}^{RL}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{x(\tau)}{(t-\tau)^{1-n+q}} d\tau, \tag{2} $$

where $n-1 \leq q < n$, n is an integer and $x(t)$ is unnecessarily smooth such as the Weierstrass function. Besides, given arbitrary q_1, $q_2 > 0$, if $x(t)$ is continuous, then we can obtain

$$ {}_{t_0}^{RL}I_t^{q_1}\, {}_{t_0}^{RL}I_t^{q_2} x(t) = {}_{t_0}^{RL}I_t^{q_1+q_2} x(t), \tag{3} $$

$$ {}_{t_0}^{RL}D_t^{q_1}\, {}_{t_0}^{RL}D_t^{q_2} x(t) = {}_{t_0}^{RL}D_t^{q_1+q_2} x(t) - \sum_{k=1}^{n} \left[{}_{t_0}^{RL}D_t^{q_2-k} x(t) \right]_{t=t_0} \frac{(t-t_0)^{-q_1-k}}{\Gamma(1-q_1-k)}, \tag{4} $$

where $n-1 \leq q_2 < n$ and ${}_{t_0}^{RL}D_t^{q_2-n} = {}_{t_0}^{RL}I_t^{n-q_2}$. Moreover, the negative Riemann–Liouville derivative does not lead to the decrease of function, and the Riemann–Liouville derivative of a constant is not zero, instead a power law function.

Definition 2 (Caputo Approach). The Caputo fractional order derivative of order $q > 0$ is expressed as

$$ {}_{t_0}^{C}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{x^{(n)}(\tau)}{(t-\tau)^{1-n+q}} d\tau, \tag{5} $$

where n is an integer satisfying $n-1 < q < n$, and $\cdot^{(n)}$ denotes the nth derivative. In addition, the Caputo and Riemann–Liouville derivatives are equivalent under the homogenous initial conditions. Otherwise,

$$ {}_{t_0}^{C}D_t^q x(t) = {}_{t_0}^{RL}D_t^q x(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^{k-q}}{\Gamma(k-q+1)} x^{(k)}(t_0). \tag{6} $$

The unification of integer order operations, other equivalent discussions of Riemann–Liouville and Caputo definitions, physical background of fractional order system, and many other useful conclusions can be found in [40].

Remark 1. It can be easily seen from the above definitions that the properties of fractional order integration and differentiation, such as the Leibniz rule, differentiation of a composite function, differentiation of an integral depending on a parameter, and commutativity of operators are quite different with integer order ones [40]. Besides, in view of the weak singular and multi-valued properties of fractional order operators, the direct extensions of searching Lyapunov function candidates from integer order systems to fractional order systems have encountered enormous difficulties.

2.2. Fractional order systems

Consider the non-commensurate fractional order non-autonomous system

$$ x^{(\alpha)}(t) = {}_{t_0}D_t^{\alpha} x(t) = f(t, x(t)), \tag{7} $$

where $f(t, x(t)) : [t_0, +\infty) \times \mathbb{D} \to \mathbb{R}^n$ is piecewise continuous in t and $x(t)$, $\mathbb{D} \subset \mathbb{R}^n$ is a domain that contains the origin $x(t) = \mathbf{0}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{R}^{+n}$, $x^{(\alpha)} = (x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_n^{(\alpha_n)})^T$ and $\cdot^{(\alpha)}$ or ${}_{t_0}D_t^{\alpha}$ denotes either Riemann–Liouville or Caputo fractional order derivative defined on $[t_0, t]$. Especially, when $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, system (7) becomes a commensurate fractional order one. Moreover, $f(t, x(t)) = f(x(t))$ denotes that (7) is autonomous. In order to construct the Lyapunov function candidate for system (7), we only need the existence and continuity of $x(t)$ on $t \in [t_0, \infty)$ which is a very general condition of $f(t, x(t))$. However, extra restrictions to $f(t, x(t))$ may be introduced in various discussions [41].

Fractional order systems have memory so that the derivatives in (7) should start at $t_0 = -\infty$, theoretically but unpractically. The application of short memory principle [40] can replace ∞ by a large enough L (according to reference [40], L is called the memory length) such that ${}_{-\infty}D_t^{\alpha} \tilde{x}(t) \doteq {}_{-L}D_t^{\alpha} x(t)$ and $x(t)$ is close enough to $\tilde{x}(t)$, where $\tilde{x}(t)$ (true state) and $x(t)$ (pseudo state) denote respectively the solutions of (7) when $t_0 = -\infty$ and $t_0 = -L$, and the larger the L, the closer the $\tilde{x}(t)$ and $x(t)$. Furthermore, the discussion of initial value at t_0 is far from enough due to the fact that, given t_0 and $x(t_0)$, the solution of (7) is not unique if different memories are applied before t_0. This is not contradict to the uniqueness and existence theorem of fractional order system. The reason is non-locality of fractional order operator that falls into the scheme of initialized fractional order system [42]. In this paper, we need to guarantee that the history of the system in different tests remains the same before t_0, i.e., $x(t) = \tilde{x}(t) = \phi(t)$ holds for $-L < t < t_0$, theoretically. In reality, this is implemented by stay still or non-destructive preconditioning on $[-L, t_0)$, and then to determine or to reset the initial state $x(t_0)$ at t_0 [43]. As a result, the pseudo and true states are equivalent on $t \geq t_0$.

3. Revisit of equilibrium in fractional order system

The study of equilibrium plays a central role in various fractional order systems and their applications. In current literatures, the process of searching fractional order equilibrium points is inherited from the integer-order analogy that has been effectively verified every time. Meanwhile, as illustrated in the introduction, many anomalous phenomena are coexisted or emerged from various fractional order problems. Particularly, all approaches of non-integer order derivatives fall into the scheme of fractional calculus such as the Riemann–Liouville and Caputo definitions, thus it is desiderated to systematically reconsider the equilibrium issue in such a diversified and interrelated group of fractional order systems. In this section, we describe what are the fractional order equilibrium points in detail which is to be used in the following discussions of stability and attraction.

3.1. Equilibrium points in fractional order Caputo system

The search of equilibrium points is equivalent to find the equilibrium solution of a system that is either integer-order or fractional order. For the Caputo fractional order system

$$ {}_{t_0}^{C}D_t^{\alpha}x(t)=f(t,x(t)), \tag{8}$$

where $\alpha\in(0,\,1)$. Eq. (8) can be rewritten as

$$ \dot{x}(t)={}_{t_0}^{RL}D_t^{1-\alpha}f(t,x(t)), \tag{9}$$

where $f(t,\,x(t))$ is piecewise continuous [40]. According to the same way of searching integer order equilibrium points, the equilibrium point $\bar{x}$ of (9) satisfies

$$\begin{cases} {}_{t_0}^{RL}D_t^{1-\alpha}f(t,\bar{x})=0, \\ {}_{t_0}^{C}D_t^{\alpha}\bar{x}=f(t,\bar{x}). \end{cases} \tag{10}$$

It follows from the above two equations in (10) that $f(t,\bar{x})=0$. Here comes the following definition.

Definition 3. The equilibrium point $\bar{x}$ of (8) satisfies

$$ f(t,\bar{x})=0, \tag{11}$$

which is the same as integer order case.

Remark 2. It is worth pointing out that the mutual conversion between non-zero and zero equilibrium point can be achieved for fractional order system (8). On the one hand, the non-zero equilibrium point can be translated into the origin. Suppose there exists a non-zero equilibrium point $\bar{x}\neq 0$ satisfying $f(t,\bar{x})=0$. Let $y(t)=x(t)-\bar{x}$, substituting $x(t)=y(t)+\bar{x}$ into (8) yields

$$ {}_{t_0}^{C}D_t^{\alpha}y(t)=f(t,y(t)+\bar{x})=\hat{f}(t,y(t)), \tag{12}$$

where the Caputo derivative of a constant equals zero, and $\bar{y}=0$ is obviously an equilibrium point of (12). On the other hand, it is to be seen that the zero equilibrium point can be a translation of a non-zero solution of the system. To see this point, let $\bar{y}(\tau)$ be a solution of the system

$$ {}_{t_0}^{C}D_\tau^{\alpha}y(\tau)=\tilde{f}(\tau,y(\tau)) \tag{13}$$

defined for all $\tau\geq t_0$. The change of variables $x(\tau)=y(\tau)-\bar{y}(\tau),\ \xi=\tau-t_0$ transforms the above system (13) into

$$ {}_{t_0}^{C}D_\tau^{\alpha}x(\tau)=\tilde{f}(\xi+t_0,x(\xi+t_0)+\bar{y}(\xi+t_0))-{}_{t_0}^{C}D_{\xi+t_0}^{\alpha}\bar{y}(\xi+t_0). \tag{14}$$

Since ${}_{t_0}^{C}D_{\xi+t_0}^{\alpha}\bar{y}(\xi+t_0)=\tilde{f}(\xi+t_0,\bar{y}(\xi+t_0))$, the equilibrium point of the above system at $\xi=0$ is $\bar{x}=0$.

Therefore, without loss of generality, it is assumed in the following main text that the initial time $t_0=0,\ \bar{x}=0$ is an equilibrium point of (8) and $x(0)\in\mathbb{D}$, where $f(t,\bar{x})=0$ and $\mathbb{D}$ is the region of attraction with respect to $\bar{x}=0$ and $t_0=0$.

3.2. Fractional order system without equilibrium

In this subsection, the Riemann–Liouville and initialized fractional order systems are presented as typical fractional order systems without equilibrium.

For the Riemann–Liouville fractional order system

$$ {}_{t_0}^{RL}D_t^{\alpha}x(t)=f(t,x(t)), \tag{15}$$

where $\alpha\in(0,\,1)$. The Riemann–Liouville derivative can be converted into the Caputo form by using

$$ {}_{t_0}^{RL}D_t^{\alpha}x(t)={}_{t_0}^{C}D_t^{\alpha}x(t)+\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}x(t_0), \tag{16}$$

so that (15) becomes

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = f(t,x(t)) - \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}x(t_0). \tag{17} $$

According to Definition 3, any equilibrium point $\bar{x}$ of (17) satisfies

$$ f(t,\bar{x}) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\bar{x}, \tag{18} $$

i.e.

$$ f(t,x(t)) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}x(t) + \hat{f}(t,x(t)), \tag{19} $$

where there exists a $\hat{f}$ satisfying $\hat{f}(t,\bar{x}) = 0$. Then (17) becomes

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = \hat{f}(t,x(t)) + \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[x(t)-x(t_0)], \tag{20} $$

where $\bar{x}$ is an equilibrium point if and only if $\bar{x} = x(t_0)$ and $\hat{f}(t,\bar{x}) = 0$ hold simultaneously. This is too strong and unpractical an assumption. Otherwise, there is no equilibrium point of (15). Particularly, all Riemann–Liouville fractional order autonomous systems have no equilibrium points.

For the initialized fractional order nonlinear system

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = f(t,x(t)) + h(t), \tag{21} $$

where $h(t) = -\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t_0}(t-\tau)^{-\alpha}\dot{x}(\tau)d\tau$ and $x(t)$ is called the true state on $(-\infty, t]$. Obviously, any of the equilibrium points of (21) $\bar{x}$ satisfies $f(t,\bar{x}) = 0$ and $\bar{x}$ is constant on $(-\infty, t]$. The region of attraction is defined by two parts, $x(t_0) \in \mathbb{D}$ and the initialization state $x(t) = \bar{x}$ for $t < 0$. Otherwise, $x(t)$ is called the pseudo state if $x(t)$ doesn't equal to $\bar{x}$ on $t < 0$. Besides, if $x(t)$ is not constant on $(-\infty, t_0]$, there is no equilibrium point of (21). In real applications, as mentioned in the end of Section 2.2 that the initialization state is assumed as zero or is properly preconditioned so that the pseudo and true states are consistent with each other [43].

4. Lyapunov function candidate of Caputo fractional order autonomous system

Consider a class of nonlinear Caputo fractional order autonomous system

$$ {}_{0}^{C}D_t^{\alpha}x(t) = f(x(t)), \tag{22} $$

where $\alpha \in (0,\ 1)$ and the initial condition is $x(0) \in \mathbb{D}$. It can be seen from Section 3 that the equilibrium point $\bar{x} = 0$ of (22) satisfies $f(0) = 0$, remarkablely. Then, applying the Laplace transform to (22) yields

$$ sX(s) - x(0) = \gamma F(s) + \frac{s-\gamma s^{\alpha}}{s^{\alpha}}F(s), \tag{23} $$

where $F(s) = \mathcal{L}\{f(x(t))\}$ and γ is a to be determined parameter. Define the following auxiliary functions

$$ \begin{cases} g(t) = \mathcal{L}^{-1}\left\{\frac{s-\gamma s^{\alpha}}{s^{\alpha}}\right\}, \\ \tilde{g}(t) = \mathcal{L}^{-1}\left\{\frac{s-\gamma s^{\alpha}}{s^{\alpha+1}}\right\}, \end{cases} \tag{24} $$

where $g(t)$ and $\tilde{g}(t)$ have several attractive properties *Property 1:*

$$ g(t) = \frac{d}{dt}\tilde{g}(t) + \tilde{g}(0)\delta(t) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + [\tilde{g}(0)-\gamma]\delta(t). \tag{25} $$

Property 2:

$$ \int_0^{\infty} g(\tau)d\tau = \tilde{g}(\infty) - \tilde{g}(0) + \tilde{g}(0) = \tilde{g}(\infty) = -\gamma. \tag{26} $$

Property 3:

$$ \int_0^{\infty} |g(\tau)|d\tau = -\int_0^{\infty} g(\tau)d\tau = \gamma, \tag{27} $$

where $\gamma = \tilde{g}(0)$ implies that $\gamma = \frac{1}{2}\,{}_0I_t^{\alpha}\delta(t)\big|_{t=0}$.

According to the properties of unilateral Laplace transform, (22) can be rewritten as

$$ \dot{x}(t) = \gamma f(x(t)) + g(t) * f(x(t)), \tag{28} $$

where $*$ denotes the convolution on $[0,\ t]$. The above system (28) is similar to Volterra form that motivates us the following theorem.

Theorem 1. *For the Caputo fractional order autonomous system (22), if* $f(0)=0$ *and there exists a* $t_1 \geq 0$ *such that*

$$V(t,x) = -\int_0^x f(\tau)d\tau + \frac{1}{2}{}_0I_t^\alpha\left[f^T(x(t))f(x(t))\right] > 0 \tag{29}$$

holds for all $t \geq t_1 \geq 0$*. Then V(t, x) is a fractional order Lyapunov function candidate of (22).*

Proof. It can be seen that (22) is equivalent to (28), where $\gamma = \tilde{g}(0)$ is assumed. Let us define the following Lyapunov function candidate

$$V(t,x) = -\int_0^x f(\tau)d\tau + \tfrac{1}{2}\int_0^t \int_t^{+\infty} |g(\theta-\tau)|d\theta f^T(x(\tau))f(x(\tau))d\tau. \tag{30}$$

Considering the first derivative of equation (30), we have

$$\begin{aligned}\dot{V}(t,x) = &-\int_0^t g(t-\tau)f^T(x(\tau))d\tau f(x(t)) + \frac{1}{2}\int_t^{+\infty}|g(\theta-t)|d\theta f^T(x(t))f(x(t)) - \gamma f^T(x(t))f(x(t)) \\ &-\frac{1}{2}\int_0^t |g(t-\tau)|f^T(x(\tau))f(x(\tau))d\tau \leq \left[-\gamma + \frac{1}{2}\int_t^{+\infty}|g(\theta-t)|d\theta\right]f^T(x(t))f(x(t)) \\ &+\frac{1}{2}\int_0^t |g(t-\tau)|f^T(x(\tau))f(x(\tau))d\tau + \frac{1}{2}\int_0^t |g(t-\tau)|d\tau f^T(x(t))f(x(t)) \\ &-\frac{1}{2}\int_0^t |g(t-\tau)|f^T(x(\tau))f(x(\tau))d\tau = \frac{1}{2}\int_t^{+\infty}|g(\theta-t)|d\theta f^T(x(t))f(x(t)) \\ &+\left[\frac{1}{2}\int_0^t |g(t-\tau)|d\tau - \gamma\right]f^T(x(t))f(x(t)) = -\frac{t^{\alpha-1}}{2\Gamma(\alpha)}f^T(x(t))f(x(t)) \leq 0,\end{aligned}$$

where $\gamma = \frac{1}{2}{}_0I_t^\alpha \delta(t)\big|_{t=0}$ and the Young's inequality are applied. Furthermore, $\int_0^\infty |g(\tau)|d\tau = \gamma$, and the equality holds if $t=+\infty$ or $f(\cdot)=0$. It is worth noting that for different g and γ, (30) represents a large group of fractional order Lyapunov function candidates such as for higher order fractional order systems and distributed order or distributed parameter systems [44].

In addition, the integrand $\int_t^{+\infty}|g(\theta-\tau)|d\theta$ in the above $V(t, x)$ can be expressed as

$$\begin{aligned}\int_t^{+\infty}|g(\theta-\tau)|d\theta &= \int_t^{+\infty}\left(2\gamma - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\bigg|_{t\to 0}\right)\delta(\theta-\tau)d(\theta-\tau) - \int_t^{+\infty}\frac{(\theta-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d(\theta-\tau) \\ &= \int_{t-\tau}^{+\infty}\left[\left(2\gamma - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\bigg|_{t\to 0}\right)\delta(\xi) - \frac{\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]d\xi = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)},\end{aligned} \tag{31}$$

where $\tau \in (0, t)$, $|g(\theta-\tau)| = -g(\theta-\tau)$ and the above equation holds if and only if $\alpha \in (0, 1)$. It follows that the fractional order Lyapunov function candidate (30) becomes

$$V(t,x) = -\int_0^x f(\tau)d\tau + \frac{1}{2}\int_0^t\int_t^{+\infty}|g(\theta-\tau)|d\theta f^T(x(\tau))f(x(\tau))d\tau = -\int_0^x f(\tau)d\tau + \frac{1}{2}{}_0I_t^\alpha\left[f^T(x(t))f(x(t))\right], \tag{32}$$

which is equivalent with (30). The proof is completed. □

Remark 3. It was mentioned that γ played like the catalyst that disappeared in the final $V(t, x)$ and $\dot{V}(t,x)$. Moreover, we used only the integral of $g(t)$ that is proper. Last but not least, in (31), when $\alpha = 1$,

$$\gamma = \frac{t^{1-1}}{2\Gamma(1)}\bigg|_{t\to 0} = \frac{1}{2}.$$

Substituting $\alpha = 1$ and $\gamma = \frac{1}{2}$ back into (24), (28) and (30) yields

$$\begin{aligned}&\dot{x}(t) = f(x(t)), \\ &V(t,x) = -\int_0^x f(\tau)d\tau, \\ &\dot{V}(t,x) = -f^T(x(t))f(x(t)) \leq -\frac{t^{\alpha-1}}{2\Gamma(\alpha)}f^T(x(t))f(x(t))\bigg|_{\alpha=1} = -\frac{1}{2}f^T(x(t))f(x(t)).\end{aligned}$$

Remark 4. With proposed fractional order Lyapunov function candidate (29), the positive definite of $V(t, x)$ can be guaranteed if $x^T(t)f(x(t)) < 0$ holds for all $t \geq t_1 \geq 0$, which is a sufficient condition. This condition also guarantees the stability of zero equilibrium for integer order system $\dot{x}(t) = f(x(t))$.

Remark 5. Large $\dot{V}(t,x)$ around $t=0$ is a typical feature of fractional order systems. For example, the power law function with negative exponent, and the first order derivative of Mittag–Leffler function both equal infinity at the origin [45]. Nevertheless, $\dot{V}(t,x)$ can also be bounded in closed domain if certain conditions are added that will be illustrated in the subsequent discussions.

Remark 6. From the perspective of energies, it is obvious that the above fractional order Lyapunov function candidate

$$V(t,x) = -\int_0^x f(\tau)d\tau + \frac{1}{2}{}_0I_t^\alpha\left[f^T(x(t))f(x(t))\right]$$

can be divided into two parts: $-\int_0^x f(\tau)d\tau$ indicates potential term and $\frac{1}{2}{}_0I_t^\alpha\left[f^T(x(t))f(x(t))\right]$ represents fractional order kinetic term. Therefore, if $x^T(t)f(x(t)) > 0$ and the zero equilibrium of (22) is stable, the potential term $-\int_0^x f(\tau)d\tau$ which stored at the initial moment decreases and is transferred into other forms of energy such as the fractional order kinetic term $\frac{1}{2}{}_0I_t^\alpha\left[f^T(x(t))f(x(t))\right]$ and/or dissipations.

5. Lyapunov function candidate of Caputo fractional order non-autonomous system

A real world system may slightly change due to the environmental changes, aging and some other factors that can be introduced in the vary of parameters of equation based models, where the zero equilibrium point is assumed to be existed. In this section, a novel method of constructing Lyapunov function candidates for Caputo fractional order non-autonomous system is investigated.

Theorem 2. *For the Caputo fractional order non-autonomous system*

$${}_0^CD_t^\alpha x(t) = f(t,x(t)), \tag{33}$$

if $f(t,0)=0$ and there exists a $t_1 \geq 0$ such that

$$\begin{aligned} V(t,x) &= -\int_0^x m(\tau)d\tau + \frac{1}{2}\left\{{}_0I_t^\alpha\left[f^T(x(t))f(x(t))\right] + {}_tI_{+\infty}^\alpha\left[n^T(x(t))n(x(t))\right]\right\} \\ &= -\int_0^x m(\tau)d\tau + \frac{1}{2}{}_0I_t^\alpha\left[f^T(x(t))f(x(t)) - n^T(x(t))n(x(t))\right] + H_\alpha\left[n^T(x(t))n(x(t))\right] > 0 \end{aligned} \tag{34}$$

holds for all $t \geq t_1 \geq 0$, then V(t, x) is a fractional order Lyapunov function candidate of (33), where $m(\cdot)$, $n(\cdot)$ and $H_\alpha(\cdot)$ are defined in the following proof.

Proof. Consider the following Lyapunov function associated with system (33):

$$\begin{aligned} V(t,x) = &\frac{1}{2}\int_0^t\int_t^{+\infty}|g(\theta-\tau)|d\theta f^T(\tau,x(\tau))f(\tau,x(\tau))d\tau + \frac{1}{2}\int_t^{+\infty}\int_t^{+\infty}|g(|\theta-\tau|)|d\theta n^T(x(\tau))n(x(\tau))d\tau \\ &-\int_0^x m(\tau)d\tau > 0, \end{aligned} \tag{35}$$

where $m^T(x)m(x) - n^T(x)n(x) \leq f^T(\tau,x(\tau))f(\tau,x(\tau)) \leq f^T(t,x)m(x)$. Similar to the proof of Theorem 1 and taking the time derivative of Eq. (35), it follows that:

$$\begin{aligned} \dot{V}(t,x) = &-\gamma f^T(t,x(t))m(x(t)) - \int_0^t g(t-\tau)f^T(\tau,x(\tau))d\tau m(x(t)) + \frac{1}{2}\int_t^{+\infty}|g(\theta-t)|d\theta f^T(t,x(t))f(t,x(t)) \\ &-\frac{1}{2}\int_0^t|g(t-\tau)|f^T(\tau,x(\tau))f(\tau,x(\tau))d\tau - \frac{1}{2}\int_t^{+\infty}|g(|\theta-t|)|d\theta n^T(x(t))n(x(t)) \\ &-\frac{1}{2}\int_t^{+\infty}|g(|\tau-t|)|n^T(x(\tau))n(x(\tau))d\tau \leq -\frac{\gamma}{2}f^T(t,x(t))f(t,x(t)) - \frac{\gamma}{2}n^T(x(t))n(x(t)) \\ &+\frac{1}{2}\int_0^t|g(t-\tau)|d\tau m^T(x(t))m(x(t)) - \frac{1}{2}\int_t^{+\infty}|g(\tau-t)|n^T(x(\tau))n(x(\tau))d\tau \\ &\leq -\frac{\gamma}{2}\left[f^T(t,x(t))f(t,x(t)) + n^T(x(t))n(x(t)) - m^T(x(t))m(x(t))\right] \\ &-\frac{1}{2}\int_t^{+\infty}|g(\tau-t)|n^T(x(\tau))n(x(\tau))d\tau \leq -\frac{1}{2}\int_t^{+\infty}|g(\tau-t)|n^T(x(\tau))n(x(\tau))d\tau < 0. \end{aligned}$$

In this situation, suppose $\gamma = {}_0I_t^\alpha\delta(t)\big|_{t=0}$, the integrand $\int_t^{+\infty}|g(|\theta-\tau|)|d\theta$ in the above V(t, x) can be expressed as

$$\begin{aligned} \int_t^{+\infty}|g(|\theta-\tau|)|d\theta &= -\int_t^{+\infty}\frac{|\theta-\tau|^{\alpha-2}}{\Gamma(\alpha-1)}d\theta = -\int_t^\tau\frac{(\tau-\theta)^{\alpha-2}}{\Gamma(\alpha-1)}d\theta - \int_\tau^{+\infty}\frac{(\theta-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\theta \\ &= \frac{(\tau-t)^{\alpha-1}}{\Gamma(\alpha)} - 2\gamma + 2\gamma = \frac{(\tau-t)^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned} \tag{36}$$

Substituting (36) into (35) yields (34), where

$$H_\alpha\left[n^T(x(t))n(x(t))\right] = \frac{1}{2}\int_0^{+\infty} \frac{|t-\tau|^{\alpha-1}}{\Gamma(\alpha)} n^T(x(\tau))n(x(\tau))d\tau \tag{37}$$

denotes the Riesz potential term of $n^T(x(t))n(x(t))$ on $t \in [0, +\infty)$. The proof is completed. □

Remark 7. Through the above analysis, we can conclude that if $V(t, x) > 0$ and $f^T(t, x)m(x) \geq m^T(x)m(x)$, then the condition

$$m^T(x)m(x) - n^T(x)n(x) \leq f^T(t, x(t))f(t, x(t)) \leq f^T(t, x)m(x) \tag{38}$$

in (35) can be replaced by

$$m^T(x)m(x) - n^T(x)n(x) \leq f^T(t, x(t))f(t, x(t)) \leq m^T(x)m(x)$$

which is a sufficient condition of (38).

Remark 8. It should be noted that

$$V(t, x) = -\int_0^x m(\tau)d\tau + \frac{1}{2}\int_0^t \int_t^{+\infty} |g(\theta - \tau)|d\theta n^T(x(\tau))n(x(\tau))d\tau$$

is a fractional order Lyapunov function candidate if $f(t, x) = f(x)$ and $m(x) = n(x) = f(x)$. In other words, using the above strategy, the second double integral terms in (35) is existed for fractional order non-autonomous systems.

Remark 9. From the above analysis, when $\alpha = 1$, it follows that $g(t) = 0$ and $V(t, x) = -\int_0^x m(\tau)d\tau$, where $f^T(t, x)f(t, x) \leq f^T(t, x)m(x)$ and $\dot{V}(t, x) = -f^T(t, x)f(t, x)$.

Remark 10. Comparing Theorems 1 and 2, the Riesz potential term is shown in the non-autonomous case that is an acausal one. The introduce of short memory principle [40] helps the fast calculation of this term, i.e.,

$$\begin{aligned} H_\alpha\left[n^T(x(\tau))n(x(\tau))\right] &= \frac{1}{2}\int_0^{+\infty} \frac{|t-\tau|^{\alpha-1}}{\Gamma(\alpha)} n^T(x(\tau))n(x(\tau))d\tau \\ &\doteq \frac{1}{2}\int_{\tilde{t}_0}^{t+L} \frac{|t-\tau|^{\alpha-1}}{\Gamma(\alpha)} n^T(x(\tau))n(x(\tau))d\tau, \end{aligned}$$

where $\tilde{t}_0 = \max\{0, t-L\}$, and the memory length L is a large enough constant depending on α and norm of $n(x(t))$. The fast computations of Riesz potential term, fractional integral and derivative, and accurate time domain responses are cutting edge topics that are all welcomed in here [46].

6. Fractional order Lyapunov's theorem and asymptotic stability

Using the idea of integer order derivative of a fractional order Lyapunov function candidate [12], the above conclusions can be introduced to the discussions of fractional order asymptotic stabilities, and a number of useful results are derived as follows.

From the perspective of energies, the above discussed Lyapunov function $V(t, x)$ is relevant to three terms which is shown in Table 1. Thereinto, the kinetic term or power is a fractional order one, the Riesz potential relevant term

$$H_\alpha\left[n^T(x(t))n(x(t))\right] - {}_tI_{+\infty}^\alpha\left[n^T(x(t))n(x(t))\right]$$

is defined on $[t, \infty)$ so that $|t-\tau| = \tau - t$ and the potential term is bounded for any finite $x(t)$. It is reasonable to assume all the above three functions are bounded due to $\dot{V}(t, x(t)) < 0$.

In the first place, for kinetic term or power, the boundedness of $\frac{1}{2}{}_0I_t^\alpha\left[f^T(t, x(t))f(t, x(t))\right]$ implies only $\lim_{t\to\infty} f^T(t, x(t))f(t, x(t)) = 0$, that is to say, ${}_0^CD_t^\alpha x(t) = f(t, x(t)) \to 0$ as $t \to \infty$. For example, given $\alpha < \beta < \alpha + 1$ and let $x(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, then we have

$${}_0^CD_t^\alpha x(t) = {}_0^CD_t^\alpha \frac{t^{\beta-1}}{\Gamma(\beta)} = \frac{t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} = f(t, x(t)) \to 0,$$

Table 1
Composition of Lyapunov function candidate.

Function	Energy representation
$\frac{1}{2}{}_0I_t^\alpha\left[f^T(t, x(t))f(t, x(t))\right]$	kinetic term or power
$H_\alpha[n^T(x(t))n(x(t))]$	Riesz potential term
$-\int_0^x m(\tau)d\tau$	potential term

as $t \to \infty$. Nevertheless,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{t^{\beta-1}}{\Gamma(\beta)} = \begin{cases} 0, & \alpha < \beta < 1, \\ 1, & \beta = 1, \\ \infty, & 1 < \beta < \alpha + 1. \end{cases} \tag{39}$$

In other words, the boundedness of the above fractional order kinetic term is not directly leading to stability, although it successfully extends the classical L_2 norm to the fractional order one.

In the next place, for Riesz potential relevant term, the boundedness of $\frac{1}{2} {}_tI^{\alpha}_{+\infty}\left[n^T(x(t))n(x(t))\right]$ implies that $\int_0^\infty \frac{\tau^{\alpha-1}}{\Gamma(\alpha)} n^T(x(\tau))n(x(\tau))d\tau$ is finite, i.e., $n(x(t))$ is bounded on $t \in [0, \infty)$ and is fractional order L_2 convergence. Besides the Riesz potential relevant term ${}_tI^{\alpha}_{+\infty}\left[n^T(x(t))n(x(t))\right] \to 0$ as $t \to +\infty$ due to the boundedness of $n(x(t))$. Nevertheless, the stability of zero equilibrium can still not be guaranteed such as $n(x(t)) = \pm\sin(x(t))$. Note that the Riesz potential term is not shown in autonomous cases. Here comes to mind that the potential term is the key to the fractional order stability of equilibrium points.

Finally, the form of potential term $-\int_0^x f(\tau)d\tau$ or $-\int_0^x m(\tau)d\tau$ is exactly the same with the integer order Lyapunov function candidate from variable gradient method, although they are derived from different ways. A necessary condition is that the potential term is bounded as the zero equilibrium point is stable.

Recall the system requirement below (7) and in view of the above discussions, the existence of a positive definite $V(t, x)$ can not directly imply the conclusion of asymptotic stability under such a weak restriction even if $V^{(\beta)}(t, x) \le 0$, where $\beta \in (0, 1)$. A few counterexamples can be illustrated such as certain spiky solutions. Thus two theorems and one remark are summarized below to link the proposed $V(t, x)$ and the asymptotic stability for fractional order systems.

Theorem 3 (Fractional order Lyapunov's theorem)**.** *Let $x = 0$ be an equilibrium point for (22) or (33), and $\mathbb{D} \subset \mathbb{R}$ be the region of attraction containing $x = 0$, where $f(t, x)$ is locally Lipschitz to $x(t)$. Let the proposed $V(t,x) : [0, +\infty) \times \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function such that $V(t,x)|_{x=0} = 0$, $V(t, x) > 0$ in $\mathbb{D}\backslash\{0\}$ and $\dot{V}(t,x) \le 0$ in $\mathbb{D}$. Then, $x = 0$ is stable. Moreover, if $\dot{V}(t,x) < 0$ in $\mathbb{D}$, then $x = 0$ is asymptotically stable.*

Proof. The proof is the same with the proof of Theorem 4.1 in [47], where the uniqueness and existence of solution is guaranteed by the locally Lipschitz condition [40], and the Lyapunov surface equals zero if $V(t, x) > 0$ and $\dot{V}(t,x) < 0$.

The above theorem holds for both autonomous and nonautonomous fractional order system. Besides, the previously mentioned fractional order L_2 convergence is an efficient way to quantize the energy of fractional order systems that are not L_p convergent. Moreover, to find an upper bound of $\|x(t)\|$, and to prove the uniform (asymptotic) stability for fractional order systems, the following theorem is introduced. □

Theorem 4. *For the fractional order system (22) or (33), where $f(x(t))$ or $f(t, x(t))$ is Lipschtiz continuity. Suppose the proposed Lyapunov function candidate $V(t, x)$ satisfies*

$$\alpha_1 \|x\|^a \le V(t,x) \le \alpha_2 \|x\|^{ab}, \tag{40}$$

$$\dot{V}(t,x) \le -\alpha_3 \|x\|^{ab}, \tag{41}$$

where a, b, α_1, α_2 and α_3 are positive constants. It can be proved that

$$\|x(t)\| \le \left[\frac{V(t_0, x(t_0))}{\alpha_1} E_{1-\alpha}\left(-\frac{\alpha_3}{\alpha_2} t^{1-\alpha}\right)\right]^{1/a}. \tag{42}$$

Proof. The proof is a combination of Theorems 2 and 8 in [48] and Example 7.3 in [18]. Besides, the constants α_1, α_2 and α_3 can be replaced by class-κ functions, and some more results regarding the integer order derivative of a fractional order Lyapunov function candidate can be found in [18]. Furthermore, $\dot{V}(t,x)$ is bounded by the Mittag–Leffler function that is finite at the origin, and this is a special case of Remark 5. □

Remark 11. The locally Lipschitz condition in Theorem 3 guarantees that $x(t)$ and $f(t, x(t))$ are smooth at almost everywhere. This condition can be weakened to any uniqueness and existence condition of fractional order system. Besides, if the *bi*-Lipschitz condition is held, i.e., $d_{x(t)}(x(t), 0) \le K d_f(f(t, x(t)), f(t, 0))$, where $d(\cdot, \cdot)$ denotes any operator of distance in a certain domain and $K > 0$, then the asymptotic stability can be arrived due to $f(t, x(t)) \to 0$.

7. Bounded and boundary properties

This section focuses on the impacts of α, initial value and initialization function (memory) to the stability of equilibrium points in fractional order systems.

First of all, in physics, the term $f(x(t))$ or $f(t, x(t))$ always includes α to make consistence of physical meanings, i.e., $f(\cdot,\cdot,\cdot) = f(t, x(t), \alpha)$ so that the sufficient condition $x^T f(x) < 0$ may only hold for a subset of $\alpha \in (0, 1)$ such as $x^{(\alpha)}(t) = f(t, x(t), \alpha) = \left(2^\alpha - \sqrt{2}\right)x(t)$, where α is not known exactly and the equilibrium point $x = 0$ is asymptotically stable if and only if $\alpha \in \left(0, \frac{1}{2}\right)$. However, in reality, the relationships of α and sign of $x^T f(x) < 0$ are usually too complex to be analyzed, particularly for nonlinear cases. For uncertain α, the order sensitivity method can be applied [45]. The structure of nonlinearity of fractional order system requires fractional order Akaike information criterion, neural network, deep learning or any other methods of structural identification.

In the second place, the search of region of attraction for fractional order system is more complex than integer-order cases considering the physical meaning of system, history of state, order sensitivity and the continuous dependence of the fractional order solutions [40]. For simplicity, if zero initialization state is assumed, i.e., $x(t) \equiv 0$ for $t < 0$, the region of attraction should satisfy certain physical rules and can be verified around the equilibrium points.

Lastly, given a fractional order system satisfying the uniqueness and existence conditions, the future of state $x(t)$, where $t > 0$, is determined by the initial condition instead of initial value, which covers all information before and around initial instant such as

$$\begin{cases} x(0), \\ \lim\limits_{t\to 0} {}_0D_t^{\alpha-1}x(t) = \lim\limits_{t\to 0} \int_0^t \frac{(t-\tau)^{-\alpha}x(\tau)}{\Gamma(1-\alpha)}d\tau, \\ x(t) = \phi(t), t < 0. \end{cases} \tag{43}$$

For the above initialization state (memory) $\phi(t)$, its impact to the current state is called initialization response [42]. The pseudo state is named after neglecting the initialization response. Nevertheless, it is unnecessary to consider $\phi(t)$ from the Big Bang, and the short memory principle [40] or preconditioning can guarantee the close enough of pseudo and true states. It should be noted that preconditioning and keep static are practical ways to restrict the influence of initialization response so that pseudo and true states are equivalent [43]. Out of the above discussions, some other tools, such as comparison principle and various of numerical methods are helpful to the stability analysis of this work.

8. Illustrative examples

In this section, five examples are given to illustrate the preceding theoretical results.

Example 1: Non-existence of finite-time stability

This example presents the non-existence of finite-time stability for fractional order non-autonomous system. Given the nonlinear fractional order system

$${}_{t_0}^{C}D_t^{\alpha}x(t) = f(t, x(t)), \tag{44}$$

where $\alpha \in (0, 1)$ and $\bar{x}$ is an arbitrary equilibrium point with region of attraction $\mathbb{D}$. If $x(t)$ reaches $\bar{x}$ at a finite time instant $t_1 > t_0$, and stays at $\bar{x}$ for all $t \geq t_1$, applying ${}_{t_0}I_t^{\alpha}$ to (44) yields

$${}_{t_0}I_t^{\alpha} f(t, x(t)) = \psi(t) + {}_{t_1}I_t^{\alpha} f(t, \bar{x}) = x(t) - x(t_0) = \bar{x} - x_0, \tag{45}$$

where $t \geq t_1$, $x(t_0) = x_0$ belongs to the region of attraction $\mathbb{D}$, and

$$\psi(t) = \int_{t_0}^{t_1} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau))d\tau. \tag{46}$$

It follows from $f(t, \bar{x}) = 0$ that

$$\psi(t) - \bar{x} + x_0 = \int_{t_0}^{t_1} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau))d\tau + \int_{t_0}^{t_1} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{x_0 - \bar{x}}{t_1 - t_0} \Gamma(\alpha)(t-\tau)^{1-\alpha} \right) d\tau = 0, \tag{47}$$

where $\forall t \geq t_1$. The above equation holds if and only if [49]

$$f(\tau, x(\tau)) + \frac{x_0 - \bar{x}}{t_1 - t_0} \Gamma(\alpha)(t-\tau)^{1-\alpha} \equiv 0, \tag{48}$$

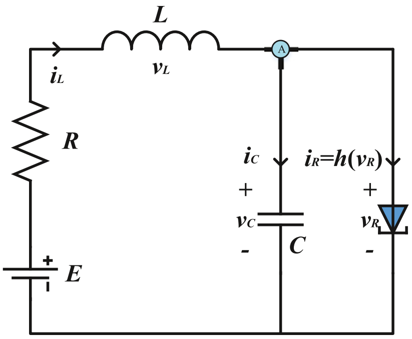
which implies that

$$\begin{cases} x_0 = \bar{x}, \\ f(\tau, x(\tau)) = 0, \end{cases} \tag{49}$$

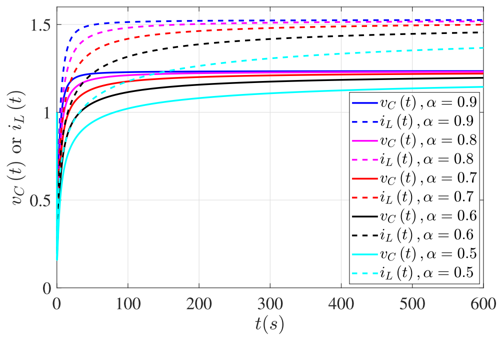
where $\tau \in [t_0, t_1]$. In other words, (47) holds if and only if

$$x(t_0) = \bar{x}, \tag{50}$$

which is contradict to $t_1 > t_0$ and implies the non-finite-time stability of nonlinear fractional order system (44) [19]. It should be noted that extra conditions are required to guarantee the finite time stability of fractional order systems such as the time varying homogeneous and nonhomogeneous delayed discussions in [50].



(a) The fractional order tunnel diode circuit.



(b) The trajectory of voltage and current for different α.

Fig. 1. System setups and solutions of Example 2.

Example 2: Fractional order tunnel diode circuit

In order to illustrate the preceding theoretical results, we perform simulations for realistic fractional order tunnel diode circuit in this subsection.

Consider the typical nonlinear fractional order system-tunnel diode circuit [47,51] as an example, in which the state is multi-variables. The electrical circuit schematic diagram is shown in Fig. 1(a). It should be noticed that both capacitor and inductor in this circuit have fractional order characteristics and corresponding tunnel diode has nonlinear characteristics. According to Kirchoff's laws, the state-space model of this system can be written in the following form:

$$\begin{cases} {}_0^C D_t^\alpha v_C(t) = \frac{1}{C} i_L(t) - \frac{1}{C} h(v_R(t)), \\ {}_0^C D_t^\alpha i_L(t) = \frac{E}{L} - \frac{1}{L} v_C(t) - \frac{R}{L} i_L(t), \end{cases} \tag{51}$$

where $E = 1.2V$ is the terminal voltage of the battery, $R = 1.5k\Omega = 1.5 \times 10^3\Omega$ is the resistance value, $C = 2pF = 2 \times 10^{-12}F$ is the capacitance, $L = 5\mu H = 5 \times 10^{-6}H$ is the inductance. Without loss of generality, suppose $h(v_R(t)) = h(v_C(t)) = v_C^2(t)$ represents the nonlinear function of tunnel diode. Then, by using the definition of equilibrium point in fractional order system, two equilibrium points $\bar{v}_{C1} = \left(\sqrt{5} - 1\right)V$, $\bar{i}_{L1} = \left(\sqrt{5} - 1\right)^2 A$ and $\bar{v}_{C2} = -\left(\sqrt{5} + 1\right)V$, $\bar{i}_{L2} = \left(\sqrt{5} + 1\right)^2 A$ can be obtained. There is no clear physical meaning for equilibrium point $\left(\bar{v}_{C2}, \bar{i}_{L2}\right)^T$, thus, we choose $\left(\bar{v}_{C1}, \bar{i}_{L1}\right)^T$ as an unique equilibrium point for system (51) and the initial state set as $[v_C(0), i_L(0)]^T = [1V, 0.5A]^T$. By calculation, the straightforward inequality is derived:

$$\frac{1}{C} v_C(0)\left[i_L(0) - v_C^2(0)\right] + \frac{1}{L} i_L(0)[E - v_C(0) - Ri_L(0)] = -0.305 < 0.$$

It can be validated that the conditions in Theorem 1 hold.

Fig. 1 (b) shows the simulation results for nonlinear fractional order tunnel diode circuit (51). The voltage trajectories of capacitor are plotted by different colored solid line as well as the current trajectories of inductor are plotted by various dotted line. It can be observed from this figure that the trajectories of the voltage and current will be asymptotically convergent to equilibrium point as time goes to infinity. In addition, the relationship between convergence speed and fractional order can be established. We can obviously see that the convergence speed of voltage and current can be dramatically increased with the increasing of the α.

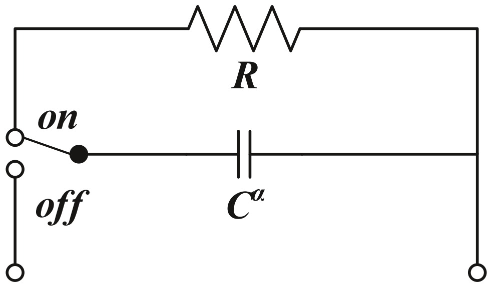
Example 3: Fractional order RC circuit

In this example, we test the theoretical results by using Resistor-Capacitance (RC) circuit. Given a fractional order interval system:

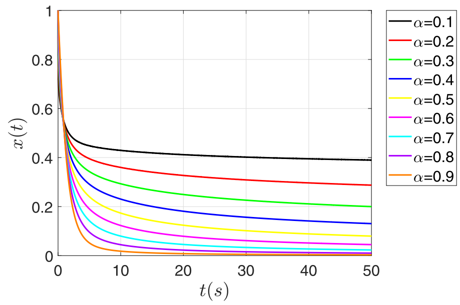
$${}_0^C D_t^\alpha x(t) + \frac{1}{R(t)C^\alpha(t)} x(t) = 0, \tag{52}$$

where the positive variables $R(t)$ and $C(t)$ are bounded and time varying, and the fractional order $\alpha \in (0, 1)$ is assumed to be an unknown constant. This equation describes typically the discharging process of RC circuit (Fig. 2(a)), or the relaxation process of Kelvin-Voigt material, where $x(t)$ denotes voltage or stress, respectively. Suppose the fractional order time constant satisfying $a_{\min} \le \frac{1}{R(t)C^\alpha(t)} \le a_{\max}$, there exist

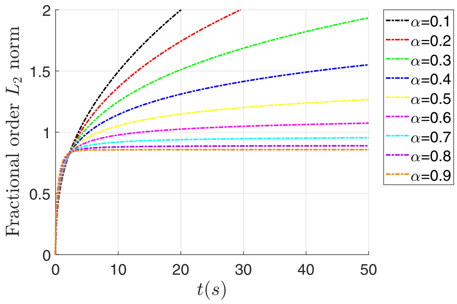
$$\begin{cases} m(x(t)) \ = -a_{\max} x(t), \\ n^2(x(t)) = \left(a_{\max}^2 - a_{\min}^2\right) x^2(t), \end{cases} \tag{53}$$



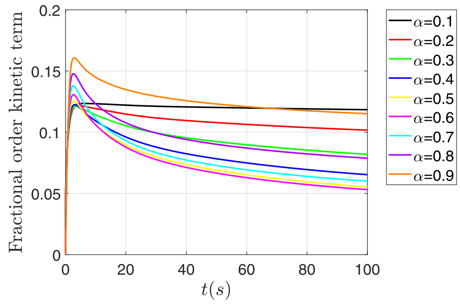
(a) The fractional order RC circuit.



(b) The trajectory of $x(t)$ for $\alpha \in \{0.1, 0.2, \cdots, 0.9\}$ and $x(0) = 1$.



(c) The L_2 norm of $x(t)$ for $\alpha \in \{0.1, 0.2, \cdots, 0.9\}$.



(d) The fractional order kinetic term of $x(t)$ for $\alpha \in \{0.1, 0.2, \cdots, 0.9\}$. The infinity L_p norm can be quantitively described by the fractional order kinetic term.

Fig. 2. System setups and solutions of Example 3.

so that the fractional order Lyapuonv function candidate can be defined as (34), and the zero equilibrium point is asymptotically stable, where the positivity of $V(t, x(t))$ is guaranteed by $\int_0^x m(\tau)d\tau < 0$. In addition, when $\alpha = 1$, the Lyapunov function candidate becomes $V(t, x) = a_{\max} \int_0^x \tau d\tau$, where $\forall a_{\max} > 0$. Then, a number of numerical solutions are shown in Figs. 2(b)–(d), where $\alpha \in (0,\ 1)$ and $\frac{1}{R(t)C^{\alpha}(t)} = 1 - e^{-t}$. In Fig. 2(b), the $x(t)$ reduces faster for larger α, in which all the fractional order decreasing curves are heavily tailed comparing to the integer order exponential decay. The heavily tailed characteristic breaks the possibility of L_p stability such as the L_2 norm of $x(t)$ is shown in Fig. 2(c). For $\alpha \in (0,\ 1]$, $\|x(t)\|_p$ is bounded if and only if $\alpha = 1$. Moreover, it indicates another fact that fractional order control systems are usually high dissipative ones. Nevertheless, the fractional order kinetic helps to quantify the energy of fractional order system (Fig. 2(d)).

Example 4: Fractional order nonlinear time-delay system

Consider the following fractional-order nonlinear time-delay system:

$$ {}_0^C D_t^{\alpha} x(t) = -a(t)x(t) - b(t)x^2(t - d(t)), \tag{54}$$

where $\alpha \in (0, 1)$, $a(t) > 0$, $0 < b(t) < 1$ are bounded variables, $d(t)$ is a time-varying delay satisfying $\lim_{t\to\infty}(t - d(t)) = \infty$ and $d(t_2) - d(t_1) \le t_2 - t_1$ where $t_2 > t_1$, $x(t) \equiv 0$ for $t < 0$, $\left|x^2(t) - x^2(t - d(t))\right| \le \rho_1$ and $\left|x^4(t) - x^4(t - d(t))\right| \le \rho_2$. It follows from $f^2(x(t)) \le m(x(t))f(x(t))$ that $m(x(t))$ can be chosen as

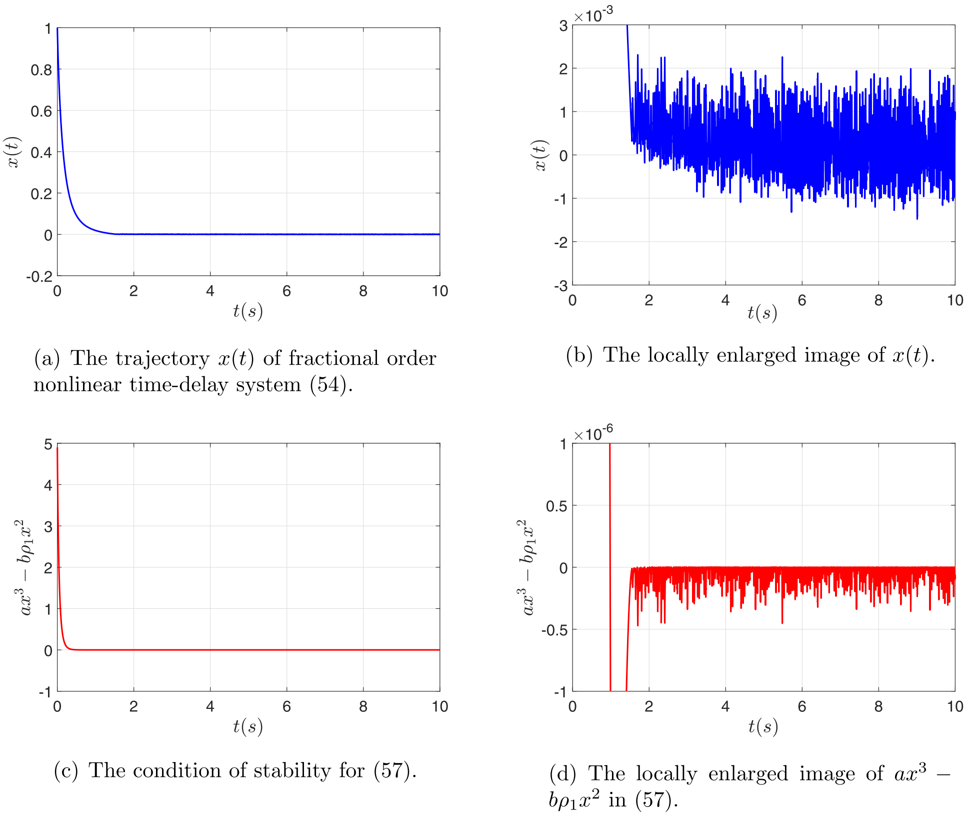
$$ m(x(t)) = -2ax(t) - \lambda_2 x^2(t), \tag{55}$$

where

$$ \lambda_2 \mathrm{sgn}(x(t)) \ge \max\left\{ b,\ \frac{b^2\rho_2}{ax^3(t) - b\rho_1 x^2(t)} \right\}, \tag{56}$$

$a = \max\{a(t)\}$, $b = \max\{b(t)\}$ and $n(x(t))$ is an arbitrary one satisfying $m^2(x(t)) - n^2(x(t)) \le f^2(t, x(t))$. The zero equilibrium of (54) is asymptotically stable if

$$ ax^3(t) - b\rho_1 x^2(t) > 0, \tag{57}$$



(a) The trajectory $x(t)$ of fractional order nonlinear time-delay system (54).

(b) The locally enlarged image of $x(t)$.

(c) The condition of stability for (57).

(d) The locally enlarged image of $ax^3 - b\rho_1 x^2$ in (57).

Fig. 3. The numerical solutions for ${}_0^C D_t^{0.9} x(t) = -5x(t) - 0.1x^2(t - d(t))$, where $d(t)$ in [0.009, 0.011] is a bounded random time-delay.

which is a sufficient condition. This condition can be implemented by designing a state observer so that $b(t)x^2(t - d(t))$ can be cancelled out to a great extent. In the following numerical results, a small $b(t) = 0.1$ and a relative larger $a(t) = 5$ are applied. Besides, it is assumed that $\alpha = 0.9$, $x(0) = 1$, $\rho_1 = 1$ and the delay $d(t)$ is randomly changing in [0.009, 0.011]. The state $x(t)$ is shown in Fig. 3(a) and the condition (57) is verified in Fig. 3(c), respectively. The locally enlarged images of $x(t)$ and $ax^3(t) - b\rho_1 x^2(t)$ are shown in Figs. 3(b) and (d), in which the condition (57) is broken near $t = 1$ and the monotone decreasing $x(t)$ starts to oscillate shortly after that moment.

In this example, the proposed $V(t, x(t))$ in Theorem 2 is a fractional order Lyapunonv function candidate if the sufficient condition (57) is held. Moreover, the violation of (57) doesn't mean to result nonasymptotic stability, but is closely related to the performance of system (54). In (57), the parameter b is critical to the stability of zero equilibrium point for (54), where worse performance can be observed for larger b (Figs. 4(b)–(d)). When $b = 0$, it is asymptotically stable (Fig. 4(a)) which is exactly the same with Example 3 (Fig. 3).

Example 5: Initialized fractional order system

This example shows the consistence of stabilities for fractional order system and initialized fractional order system with constant initialization states.

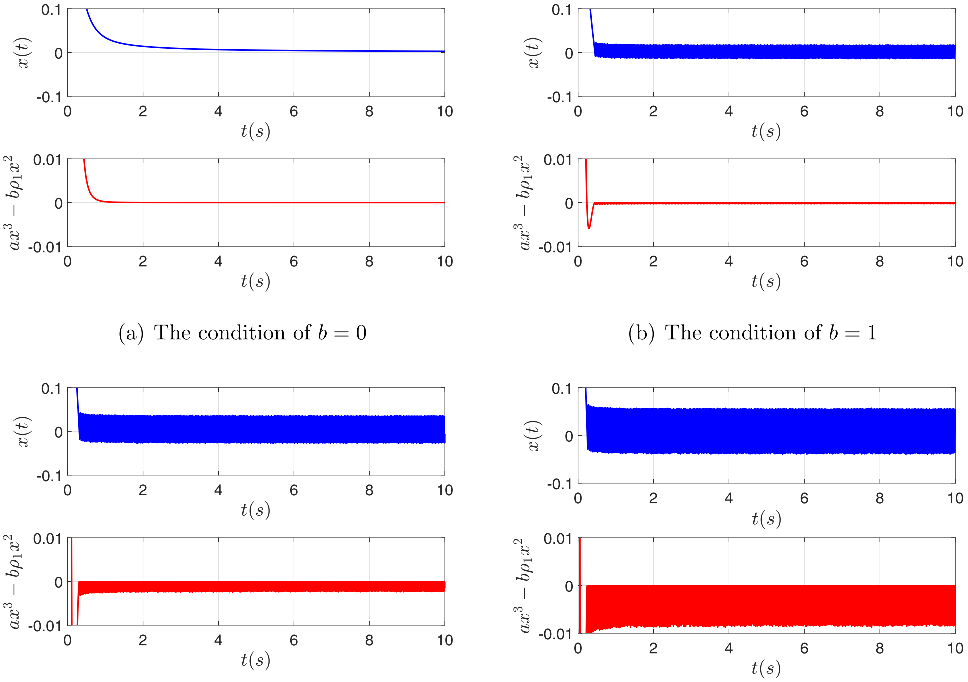
The initialized fractional order system is shown as

$$ {}_0^C D_t^{\alpha} x(t) = -x(t) - x^3(t) - \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{0} \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} d\tau, \tag{58} $$

where $\alpha = 0.6$ and $x(t)$ is constant for any $t \le 0$. It can be seen that (58) is the same as

$$ {}_0^C D_t^{\alpha} x(t) = -x(t) - x^3(t) $$

due to its constant initialization state. There are three equilibrium points $\bar{x}_1 = 0$, $\bar{x}_2 = 1$ and $\bar{x}_3 = -1$. The initialization state $x(t) = \phi(t), t < 0$, is set to be five different constants, i.e. $\phi(t) = 1$, $\phi(t) = 0.5$, $\phi(t) = 0$, $\phi(t) = -0.5$ and $\phi(t) = -1$, respectively, and $x(0) = x(0^-)$. It can be easily verified that $-x^2(1 - x^2) \le 0$ for $x(0) \le 1$ due to the non-existence of finite-time solution. Thus the equilibrium points are stable.



(a) The condition of $b = 0$ (b) The condition of $b = 1$

(c) The condition of $b = 2$ (d) The condition of $b = 3$

Fig. 4. Numerical solutions for ${}_0^C D_t^{0.9} x(t) = -5x(t) - bx^2(t - d(t))$, where $b \in \{0, 1, 2, 3\}$ and $d(t)$ in [0.009, 0.011] is a bounded random time-delay.

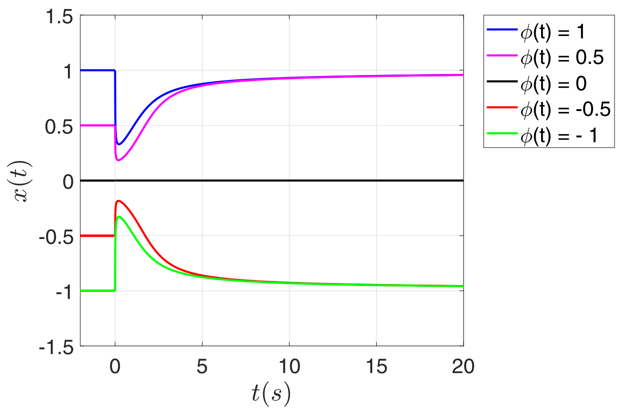


Fig. 5. The trajectories of fractional order nonlinear system with different initialization states.

Nevertheless, in reality, instead of $t_0 = -\infty$, we can only assume $x(t)$ exists on $[-L, 0]$ and $x(t) \equiv 0$ for $t < -L$, where the memory length L is a given constant. It follows that (58) becomes

$$ {}_0^C D_t^{\alpha} x(t) = -x(t) - x^3(t) - \frac{x(0)}{\Gamma(1-\alpha)(t+L)^{\alpha}}. \quad (59) $$

Let $L = 2$, the numerical solutions of (59) for different initialization states are plotted in Fig. 5. For positive initialization states and $x(0)$, the trajectory of $x(t)$ will be asymptotically convergent to $\bar{x}_2 = 1$ as time goes to infinity. For the negative initialization states and $x(0)$, the trajectory of $x(t)$ is asymptotically convergent to $\bar{x}_3 = -1$. It should be noted that, when $x(0) = 1$ or $x(0) = -1$ the solutions of (59) do not stay at $\bar{x}_2$ or $\bar{x}_3$, respectively. This is the impact of non-zero initialization state, and (59) is thus a fractional order system without equilibrium points except for $x(t) \equiv 0$ as $t \le 0$.

9. Conclusions and future works

This paper has developed a novel framework to construct Lyapunov function candidates for fractional order general nonlinear systems. The proposed Lyapunov function has potential, kinetic and/or Riesz potential energy terms that can quantitively evaluate the infinite energy solutions of fractional order systems. The first order derivative of the proposed Lyapunov function has been proved as negative definite, and the relevant fractional order Lyapunov's theorem has been discussed. The discussions on asymptotic stability and a number of boundary problems have been shown to extend the availability of the proposed method. Five examples with different engineering applications have been illustrated to verify the proposed methods. Particularly, it has be shown from the examples that the stability of equilibirum for fractional order systems with nonlinearity, randomness, uncertainty, time-delay, memory and so on can be generally discussed under the framework of this paper.

The equivalence of Lyapunov stability for fractional order system and initialized fractional order system with time-varying initialization state will be further studied.

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