The Controllability, Observability, and Stability Analysis of a Class of Composite Systems with Fractional Degree Generalized Frequency Variables

Cuihong Wang, Yafei Zhao, and YangQuan Chen, *Senior Member, IEEE*

***Abstract*—This paper is concerned with fundamental properties of a class of composite systems with fractional degree generalized frequency variables, including controllability, observability and stability. Firstly, some necessary and sufficient conditions are given to guarantee controllability and observability of such composite systems. Then we prove that the stability problem of such composite systems can be reduced to judging whether a fractional degree polynomial is stable. Finally, the stability analysis result is applied in the supervisory control of fractional-order multi-agent systems, and an example is provided to illustrate the effectiveness of the proposed methods.**

***Index Terms*—Composite system, controllability, fractional-order system, observability, stability.**

I. INTRODUCTION

RECENTLY, a class of composite systems called linear time-invariant systems (LTIs) with generalized frequency variables was proposed to provide a unified framework for modeling multi-agent systems [1]−[5]. Specifically, such a composite system can be represented by a composite transfer function $\mathscr{G}(s) = G(\phi(s))$, where $G(s)$ is a proper rational transfer function matrix, and $\phi(s)$ is a scalar function about variable s. The composite transfer function $\mathscr{G}(s)$ can be obtained by simply replacing the Laplace transform s with $\phi(s)$ in $G(s)$, where the function $\phi(s)$ is called the generalized frequency variable. The system represented by $\mathscr{G}(s)$ can also be called a composite system with a generalized frequency variable. Composite systems with generalized frequency variables have been applied to a wide variety of fields such as gene-protein regulatory networks [6], biomolecular communication networks [7], consensus and formation problems [8], as well as tire force distribution for multi-actuator electric vehicles [9], [10]. To our knowledge, the generalized frequency variables mentioned in the above references are all limited to be the integer degree rational functions.

In practice, many physical systems may be better characterized by fractional-order differential equations [11]−[13]. For example, viscoelastic materials [14], biological tissues [15], chaotic dynamic systems [16] and HIV infection [17] can be described by fractional-order dynamic models. Some fundamental properties of fractional-order systems, such as controllability, observability and stability can be found in [18]−[22]. From the view-point of the frequency variable, the LTI fractional-order system with commensurate orders can also be regarded as a composite system with generalized frequency variable $\phi(s) = s^{\alpha}$, $\alpha \in (0, 2)$. The fractional-order system with multiple orders investigated in [23] can be regarded as a composite system with the generalized frequency variable $\phi(s) = s^{\alpha_1} + s^{\alpha_2}$, $\alpha_1, \alpha_2 \in (0, 1)$. In fact, the frequency variable can be generalized to the function with a fractional degree. For example, fractional-order multi-agent systems and distributed-order system investigated in [23]−[25] are all composite systems with fractional degree generalized frequency variable. However, not much attention was paid to fractional-order systems from the view of the generalized frequency variable.

In this paper, we extend the generalized frequency variable to the fractional degree rational function. We will investigate the controllability- (and, by duality, the observability-) properties of a class of composite systems with fractional degree generalized frequency variables, as well as some preliminary stability results. The proposed methods of stability analysis are applied to the supervisory control of fractional-order multi-agent systems and an example is provided to illustrate the effectiveness of the proposed general framework.

Notations: We denote $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ as the set of integer numbers, real numbers and complex numbers, respectively, and $\mathbb{R}_+$ as the set of positive integer numbers, $\mathbb{N}$ as the set of natural numbers and $\mathbb{C}_+$ as the closed right half of complex plane. $A \otimes B$ is Kronecker product of the matrices A and B. For a square matrix A, the set of its eigenvalues is denoted by $\sigma(A)$. For $A \in \mathbb{R}^{m\times n}$, we use $\bar{\sigma}(A)$ to denote the largest singular value of matrix A.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following composite system with fractional

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C. H. Wang and Y. F. Zhao are with the Department of Mathematics and Computer Science, Shanxi Normal University, Linfen 041004, China (e-mail: sxwangcuihong@163.com).

Y. Q. Chen is with the Mechatronics, Embedded Systems and Automation (MESA) Lab, CA 9530 USA (e-mail: ychen53@ucmerced.edu).

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degree generalized frequency variable $\phi(s)$

$$\mathscr{G}(s) = C_0(\phi(s)I_n - A_0)^{-1}B_0 + D_0 \tag{1}$$

where $A_0 \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times m}$, $C_0 \in \mathbb{R}^{p\times n}$ and $D_0 \in \mathbb{R}^{p\times m}$ are constant matrices. $\phi(s) = 1/h(s)$, $h(s)$ is a fractional degree transfer function with the following form

$$h(s) = \frac{n(s)}{d(s)} = \frac{b_m s^{\beta_m} + b_{m-1}s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}} \tag{2}$$

$a_k (k = 0,\ldots,n)$, $b_k (k = 0,\ldots,m)$ are constants; $\alpha_k (k = 0,\ldots,m)$, $\beta_k (k = 0,\ldots,n)$ are arbitrary real numbers and without loss of generality, we assume that $\alpha_n > \alpha_{n-1} > \ldots > \alpha_0$, and $\beta_m > \beta_{m-1} > \ldots > \beta_0$.

Note that the standard transfer function is expressed as

$$G_0(s) = C_0(sI_n - A_0)^{-1}B_0 + D_0 \tag{3}$$

and $\mathscr{G}(s)$ defined by (1) can be rewritten as

$$\mathscr{G}(s) = G_0(\phi(s)), \quad \phi(s) = \frac{1}{h(s)}.$$

Essentially, $\mathscr{G}(s)$ is the composite function of $G_0(s)$ and $\phi(s)$, that is, $\mathscr{G}(s)$ is generated by just replacing "s" by "$\phi(s)$" in $G_0(s)$. $\phi(s)$ is a fractional degree function. In this paper, we will investigate controllability-, observability- and stability-properties of such composite systems with fractional degree generalized frequency variables.

In particular, the function $h(s)$ is called *commensurate* order if $\alpha_k = \alpha k_1$, $\beta_k = \alpha k_2$ $(0 < \alpha < 1)$, $k_1, k_2 \in \mathbb{Z}$, and has the following form:

$$h(s) = \frac{\sum_{k=0}^{M} b_k (s^\alpha)^k}{\sum_{k=0}^{N} a_k (s^\alpha)^k} = \frac{Q(s^\alpha)}{P(s^\alpha)}. \tag{4}$$

The function $h(s)$ becomes a strictly proper rational function with the complex variable s^α when $N > M$.

Let function $h(s)$, defined by (4), be of commensurate order, the realization of $h(s)$,

$$\begin{aligned} D^\alpha x(t) &= A_h x(t) + b_h u_h(t) \\ y_h(t) &= c_h x(t) \end{aligned} \tag{5}$$

is denoted by $h(s) \sim (D^\alpha,\ A_h,\ b_h,\ c_h, 0)$, where $A_h \in \mathbb{R}^{k\times k}$, $b_h \in \mathbb{R}^{k\times 1}$, $c_h \in \mathbb{R}^{1\times k}$. Then with [26], a realization of $\mathscr{G}(s)$ is as follow:

$$\begin{aligned} D^\alpha \mathscr{X}(t) &= \mathscr{A}\mathscr{X}(t) + \mathscr{B}u(t) \\ y(t) &= \mathscr{C}\mathscr{X}(t) + \mathscr{D}u(t) \end{aligned} \tag{6}$$

where $\mathscr{G}(s) \sim (D^\alpha, \mathscr{A},\ \mathscr{B},\ \mathscr{C},\ \mathscr{D})$ and $\mathscr{X}(t)$ is the nk-dimension state vector,

$$\begin{aligned} \mathscr{A} &= I_n \otimes A_h + A_0 \otimes (b_h c_h) \in \mathbb{R}^{nk\times nk} \\ \mathscr{B} &= B_0 \otimes b_h \in \mathbb{R}^{nk\times m} \\ \mathscr{C} &= C_0 \otimes c_h \in \mathbb{R}^{p\times nk} \\ \mathscr{D} &= D_0 \in \mathbb{R}^{p\times m} \end{aligned} \tag{7}$$

or

$$\begin{aligned} \mathscr{A} &= A_h \otimes I_n + (b_h c_h) \otimes A_0 \in \mathbb{R}^{nN\times nN} \\ \mathscr{B} &= b_h \otimes B_0 \in \mathbb{R}^{nk\times m} \\ \mathscr{C} &= c_h \otimes C_0 \in \mathbb{R}^{p\times nk} \\ \mathscr{D} &= D_0 \in \mathbb{R}^{p\times m}. \end{aligned} \tag{8}$$

$D^\alpha f(t)$ is the Caputo fractional derivative of function $f(t)$ defined by [11]

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \tag{9}$$

where m is an integer satisfying $m - 1 < \alpha \le m$, $m \in \mathbb{N}$, and $\Gamma(\cdot)$ is the Gamma function.

Remark 1: In order to distinguish the state realization of the integer-order system represented by $G(s) \sim (A, B, C, D)$, we denote the state realization of the fractional-order system as $G(s) \sim (D^\alpha, A, B, C, D)$, where $G(s)$ is the transfer function and α is derivative order of state variable.

III. Controllability and Observability Analysis

In this section, we will investigate the controllability and observability of the state realization system of $\mathscr{G}(s)$ for the case where $h(s)$ is a commensurate order transfer function. The observability results can be deduced by the duality method.

Lemma 1: If system (6) is controllable (observable) then system (5) is controllable (observable).

Proof: Suppose, by contradiction, system (5) is not controllable. From [19], we know that $s^\alpha I_k - A_h$ and b_h are not left coprime, which implies that $s^\alpha I_{nk} - I_n \otimes A_h$ and $I_n \otimes b_h$ are not left coprime,i.e., there exist a scalar $s \in \mathbb{C}$ and a nonzero vector $v \in \mathbb{R}^{nk}$ such that

$$v^T[s^\alpha I_{nk} - A_h \otimes I_n \mid b_h \otimes I_n] = [\mathbf{0} \mid \mathbf{0}]. \tag{10}$$

However, it is easy to check that for scalar s and nonzero vector v

$$v^T[s^\alpha I_{nk} - \mathscr{A} \mid \mathscr{B}] = v^T[\tilde{A} \mid b_h \otimes B_0] = [\mathbf{0} \mid \mathbf{0}].$$

where $\tilde{A} = s^\alpha I_{nk} - A_h \otimes I_n - (b_h c_h) \otimes A_0$. This implies that (6) is not controllable.

The proof for observability is similar and omitted here. ■

Proposition 1: If $\text{rank}(B_0) = n$, then system (6) is controllable (observable) if and only if system (5) is controllable (observable).

Proof: (Sufficiency) Suppose, by contradiction, that system (6) is not controllable. Then there exist a scalar $s \in \mathbb{C}$ and a nonzero vector $v \in \mathbb{R}^{nk}$ such that

$$v^T[s^\alpha I_{nk} - \mathscr{A} \mid \mathscr{B}] = v^T[\tilde{A} \mid b_h \otimes B_0] = [\mathbf{0} \mid \mathbf{0}]. \tag{11}$$

where $\tilde{A} = s^\alpha I_{nk} - A_h \otimes I_n - (b_h c_h) \otimes A_0$. Since B_0 is of full row rank, then $v^T b_h \otimes B_0 = 0$ implies $v^T b_h \otimes I_n = 0$. By easy computation, we can deduce that equality (11) implies that

$$v^T[s^\alpha I_{nk} - A_h \otimes I_n \mid b_h \otimes I_n] = [\mathbf{0} \mid \mathbf{0}]. \tag{12}$$

This means that $(D^\alpha, A_h \otimes I_n, b_h \otimes I_n)$ is not controllable, hence (D^α, A_h, b_h) is not controllable.

(Necessity) The necessity has been proved in Lemma 1.

The proof for observability is parallel to that for controllability and omitted here. ■

Proposition 2: If $\text{rank}(B_0) < n$ (resp., $\text{rank}(C_0) < n$), then system (6) is controllable (resp., observability) if and only if (A_0, B_0) is controllable (resp., observable) and system (5) is both controllable and observable .

Proof: We assume system (5) is controllable and hence we have the following controllable form:

$$A_h = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{k-1} \end{bmatrix}, \ b_h = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$c_h = \begin{bmatrix} b_0 & b_1 & \cdots & b_{k-1} \end{bmatrix}.$$

Based on this canonical form of controllability, the transfer function of system (5) can be deduced to be

$$h(s) := \frac{n(s^\alpha)}{d(s^\alpha)} = \frac{b_{k-1}s^{\alpha(k-1)} + \cdots + b_1 s^\alpha + b_0}{s^{\alpha k} + a_{k-1}s^{\alpha(k-1)} + \cdots + a_1 s^\alpha + a_0}.$$

From (7), we can obtain

$$\mathscr{A} = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ -\bar{A}_1 & -\bar{A}_2 & -\bar{A}_3 & \cdots & -\bar{A}_k \end{bmatrix}$$

$$\mathscr{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 & B_0^T \end{bmatrix}^T$$

where $\bar{A}_i \triangleq a_{i-1}I_n - b_{i-1}A_0$ $(i = 1, \ldots, k)$. Hence, we have

$$\text{rank}[s^\alpha I_{nk} - \mathscr{A} \mid \mathscr{B}] = n(k-1) + \text{rank}[\tilde{D} \mid B_0] \quad (13)$$

where $\tilde{D} = d(s^\alpha)I_{nk} - n(s^\alpha)A_0$. This means that the controllability of system (6) is equivalent to the condition given by

$$\text{rank } [d(s^\alpha)I_{nk} - n(s^\alpha)A_0 \mid B_0] = n, \ \forall s \in \mathbb{C}. \quad (14)$$

(Necessity) Suppose system (5) is not observable. Then there exists an $s_0 \in \mathbb{C}$ that satisfies $d(s_0^\alpha) = n(s_0^\alpha) = 0$. Since $\text{rank}(B_0) < n$, it follows that

$$\text{rank } [d(s_0^\alpha)I_{nk} - n(s_0^\alpha)A_0 \mid B_0] = \text{rank}(B_0) < n \quad (15)$$

which violates the condition (14).

Next, suppose(A_0, B_0) is not controllable, i.e., there exists a $\lambda \in \mathbb{C}$ such that

$$\text{rank}[\lambda I - A_0, B_0] < n. \quad (16)$$

Note that we can always find an $s \in \mathbb{C}$ such that $\lambda = d(s^\alpha)/n(s^\alpha)$, which violates condition (14). This completes the proof of necessity.

(Sufficiency) Suppose (A_0, B_0) is controllable and system (5) is observable, i.e., there exists no common factor $s \in \mathbb{C}$ such that $d(s^\alpha) = n(s^\alpha) = 0$. Now, for any $s \in \mathbb{C}$ satisfying $n(s^\alpha) = 0$, we have

$$\text{rank } [\tilde{D} \mid B_0] = \text{rank}[d(s^\alpha)I_n \mid B_0] = n. \quad (17)$$

Alternatively for any $s \in \mathbb{C}$ satisfying $n(s^\alpha) \neq 0$, we have

$$\text{rank } [\tilde{D} \mid B_0] = \text{rank}[\frac{d(s^\alpha)}{n(s^\alpha)}I - A_0 \mid B_0] = n \quad (18)$$

where $\tilde{D} = d(s^\alpha)I_{nk} - n(s^\alpha)A_0$. Hence system (6) is controllable. This completes the proof of sufficiency. ■

Meanwhile, we can obtain the following theorem.

Theorem 1: Consider the composite system $\mathscr{G}(s)$ given by (1), where $h(s)$ is strictly proper fractional degree transfer function defined by (2). Then the realization $\mathscr{G}(s) \sim (D^\alpha, \mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D})$ given by (6) is controllable (observable) if and only if the realization $G_0(s) \sim (A_0, B_0, C_0, D_0)$ is controllable (observable) and $h(s) \sim (D^\alpha, A_h, b_h, c_h, 0)$ is both controllable and observable.

From Theorem 1, it follows that the zeros and poles cancellation do not occur for the composite system $\mathscr{G}(s)$. This property may not hold for parallel, cascade and feedback connection of two fractional-order systems.

IV. Stability Analysis

The nk-dimensional state vector $\mathscr{X}(t)$ of the state-space realization of the composite transfer function $\mathscr{G}(s)$ becomes quite large as n and N increase. Thus we do not directly investigate the stability properties of $\mathscr{G}(s)$ by its realization, but we study its stability by its structure information, that is, $\mathscr{G}(s)$ is compounded by $G_0(s)$ and $h(s)$. In addition, it is sometimes not a very easy task to realize the composite transfer function $\mathscr{G}(s)$, especially when $h(s)$ is of non-commensurate order.

The composite system $\mathscr{G}(s)$ defined by (1) is BIBO stable if there exists a real number $M > 0$ such that $\|\mathscr{G}(s)\|_\infty \leq M < \infty$, where $\|\mathscr{G}(s)\|_\infty := ess\sup_{s\in\mathbb{C}_+} \bar{\sigma}(\mathscr{G}(s))$.

Before analyzing the stability of composite system (1), we will give the following lemma.

Lemma 2 [27]: For a commensurate order fractional degree polynominal defined as

$$p(s) = p_n s^{n\alpha} + p_{n-1}s^{(n-1)\alpha} + \cdots + p_1 s^\alpha + p_0, \ p_0 \neq 0 \quad (19)$$

where $p_0, p_1, \ldots, p_n \in \mathbb{C}$, $\alpha \in \mathbb{R}_+$, is stable if and only if $0 < \alpha < 2$ and all zeros of its companion polynomial defined as

$$p_c(\omega) = p_n\omega^n + p_{n-1}\omega^{n-1} + \cdots + p_1\omega + p_0, \ p_0 \neq 0 \quad (20)$$

satisfy the condition $|\arg(\omega_i)| > \frac{\pi}{2}\alpha$.

In the following section, we will provide a necessarily and sufficiently BIBO stable condition for the composite system $\mathscr{G}(s)$ defined in (1).

Theorem 2: Consider the composite system $\mathscr{G}(s)$ in (1) with generalized frequency variable $\phi(s) = 1/h(s)$, where $h(s) = n(s)/d(s)$, $n(s)$ and $d(s)$ are fractional degree polynomials. Define the fractional degree polynomial $p(\lambda, s)$ for $\lambda \in \mathbb{C}$ by

$$p(\lambda, s) := d(s) - \lambda n(s) \quad (21)$$

then the following statements are equivalent.

1) Composite system $\mathscr{G}(s)$ is BIBO stable.

2) $\sigma(A_0) \subset \Lambda(h(s)) := \{\lambda \in \mathbb{C} | p(\lambda, s) \text{ is stable}\}$.

Proof: Note that $h(s)$ defined in (2) is the fractional degree transfer function and s^α $(\alpha \in \mathbb{R}_+)$ defines a multi-valued function of the complex variable s whose domain can be viewed as a Riemann surface with a number of Riemann sheets which are finite when α is a rational number, or are infinite when α is an irrational number. For this multi-valued function, only the first Riemann sheet has physical significance.

1) $\Leftrightarrow$ 2) The composite system $\mathscr{G}(s)$ is BIBO stable if and only if $\mathscr{G}(s)$ has no poles in the closed right-half plane of

the principal Riemann sheet [18], which is equivalent to that $\det(\phi(s)I - A_0)) \neq 0$ for all $s \in \mathbb{C}_+$, i.e., $\phi(s) \neq \lambda$ for all $s \in \mathbb{C}_+$ and $\lambda \in \sigma(A_0)$. This condition is equivalent to the property that $p(\lambda, s)$ is stable for all $\lambda \in \sigma(A_0)$.

Remark 2: As stated in Theorem 2, the stability of $\mathscr{G}(s)$ in (1) can be reduced to judging whether the fractional degree polynomial $p(\lambda, s)$ is stable for all $\lambda \in \sigma(A_0)$. The stability of $p(\lambda, s)$ means that $p(\lambda, s)$ has no zeros in the closed right-half plane of the first Riemann sheet. If $p(\lambda, s)$ is commensurate order, we can use the result in Lemma 2, that is, the stability of $p(\lambda, s)$ can be transferred into judging the stability of its companion polynomial $p_c(\lambda, s)$. See the following example:

Example 1: Consider the composite system (1) with fractional degree generalized frequency variable $\phi(s) = \frac{1}{h(s)}$, where

$$h(s) = \frac{100(s^{0.25}+2)(\frac{19}{10}s^{0.5} - \frac{1}{500000}s^{0.25} + \frac{21}{10})}{(s^{0.25}-1)^2(s^{0.25}+1)(s^{0.25}+100)}.$$

We set system matrix

$$A_0 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

By Theorem 2, we determine the stability of $p(\lambda, s)$ in order to analyze the stability of the composite system (1), where $p(\lambda, s)$ is given as

$$\begin{aligned} p(\lambda, s) = {} & (s^{0.25}-1)^2(s^{0.25}+1)(s^{0.25}+100) \\ & - \lambda 100(s^{0.25}+2)(\frac{19}{10}s^{0.5} - \frac{1}{500000}s^{0.25} + \frac{21}{10}) \end{aligned}$$

where $\lambda \in \sigma(A_0)$. Let $\omega = s^{0.25}$, we obtain the companion polynomial of $p(\lambda, s)$ to be

$$\begin{aligned} p_c(\lambda, \omega) = {} & (\omega-1)^2(\omega+1)(\omega+100) \\ & - \lambda 100(\omega+2)(\frac{19}{10}\omega^2 - \frac{1}{500000}\omega + \frac{21}{10}). \end{aligned}$$

It is easy to see that the two eigenvalues of A_0 are $\lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2}j$ and using [8], one can determine that the zeros $\omega_i,\ (i = 1, 2, \ldots, 4)$ of $p_c(\lambda, \omega)$ are all in the open left-half plane which implies that $|\arg(\omega_i)| > \frac{\pi}{8}$. It follows from Lemma 2 that $p(\lambda_i, s)$ $(i = 1, 2)$ is stable, which means that the composite system (1) is BIBO stable.

The theory of composite system can be applied into the supervisory control of fractional-order multi-agent systems. For example, we consider an n SISO autonomous fractional-order multi-agent system, where the input and output behavior of each agent is represented by $h(s)$, which is a strictly proper scalar transfer function. The input and output behavior of the n agents can be described as $H(s) = h(s)I_n$. Then we design a supervisory controller with the transfer function $G_0(s) = C_0(sI_n - A_0)^{-1}B_0 + D_0$ such that the overall interconnected system is stable. The logical scheme describing the fractional-order plant and the supervisory controller connection is given in Fig. 1.

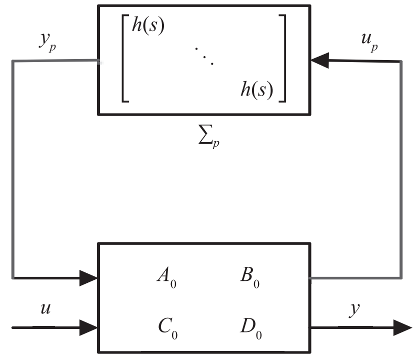

Fig. 1. The scheme of overall interconnected system.

We denote u_p and y_p as the input and output of the plant, and u and y as the input and output of the overall interconnected system. Therefore the transfer function of the overall interconnected system from input u to output y is

$$\mathscr{H}(s) = C_0(\frac{1}{h(s)}I_n - A_0)^{-1}B_0 + D_0. \tag{22}$$

From (22), we can determine that $\mathscr{H}(s)$ belongs to the composite system defined in (1). It follows from Theorem IV.2 that if we want to design a supervisory controller such that the overall interconnected system is stable, the eigenvalues of matrix A_0 should lie in a particular region specified by the generalized frequency variable $\phi(s) = \frac{1}{h(s)}$. The following example is provided to illustrate the design method of supervisory controller.

Example 2: Consider n SISO fractional order agents with $h(s) = \frac{1}{s^{\sqrt{5}-2}+0.8s^{\sqrt{3}-1}-0.5}$. The corresponding $p(\lambda, s)$ is given as

$$p(\lambda, s) = s^{\sqrt{5}-2} + 0.8s^{\sqrt{3}-1} - 0.5 - \lambda$$

which is of non-commensurate order. One can not determine the stability of $p(\lambda, s)$ by using Lemma 2, however we can determine that the stability of $p(\lambda, s)$ is equivalent to $d(s) = s^{\sqrt{5}-2} + 0.8s^{\sqrt{3}-1} - 0.5 \neq \lambda$, $\forall s \in \mathbb{C}_+$. It is necessary to derive the boundary of $d(s)$ when $s \in \mathbb{C}_+$. Since $d(s)$ is the analytic function for $s \in \mathbb{C}_+$, the boundary of $d(s)$ is determined by setting $s = j\omega$ [23]. Then, for $s = j\omega$ $(-\infty < \omega < 0)$, we have

$$d(j\omega) = x(\omega) - jy(\omega)$$

where

$$\begin{aligned} x(\omega) &= \omega^{\sqrt{5}-2}\cos(\frac{\sqrt{5}-2}{2}\pi) \\ &\quad + 0.8\omega^{\sqrt{3}-1}\cos(\frac{\sqrt{3}-1}{2}\pi) - 0.5, \\ y(\omega) &= \omega^{\sqrt{5}-2}\sin(\frac{\sqrt{5}-2}{2}\pi) \\ &\quad + 0.8\omega^{\sqrt{3}-1}\sin(\frac{\sqrt{3}-1}{2}\pi) \end{aligned}$$

while for $s = j\omega$ $(0 \leq \omega < \infty)$, we have

$$d(j\omega) = x(\omega) + jy(\omega).$$

Therefore, if we want to design the supervisory controller $u_p = A_0 y_p$ such that the overall system is stable, we just set all the eigenvalues of A_0 to lie on the left of curve $l := l_1 \cup l_2$, where l_1 and l_2 are symmetrical with respect to the real axis, and $l_1 := \{x + jy \mid x = x(\omega), y = y(\omega), \omega \in [0\ \infty)\}$. The stable boundary of the overall interconnected system (22) is plotted on Fig. 2 by referring to the program code provided in [23].

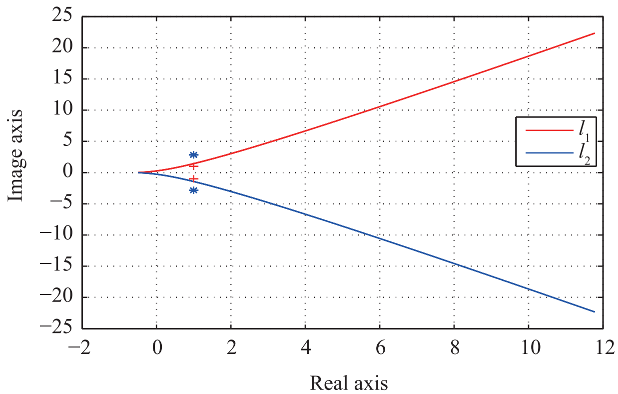

Fig. 2. The stable boundary of the overall system.

For example, we set

$$A_0 = \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix},\ B_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ C_0 = \begin{bmatrix} 2 & 1 \end{bmatrix},\ D_0 = 0.$$

The eigenvalues of A_0 are $\lambda_1 = 1 + 2.8284j$, $\lambda_2 = 1 - 2.8284j$, which are labeled by symbol " $*$ " in Fig. 2 and are all in the left-half part of curve l, so the multi-agent system (22) is BIBO stable. The stable impulse response for $\mathscr{H}(s)$ in (22) with null initiations is shown in Fig. 3 by referring to the program code provided in [23].

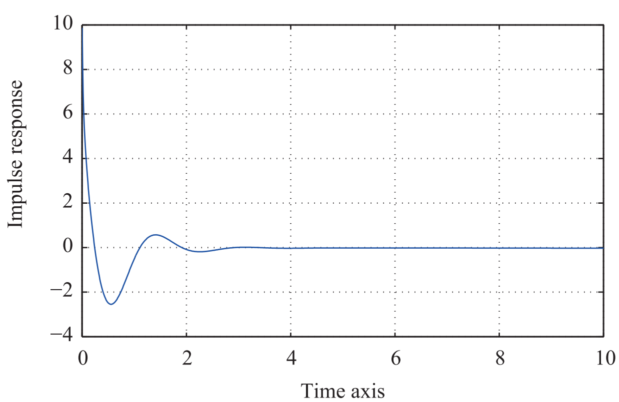

Fig. 3. The stable impulse response of the overall interconnected system.

Otherwise, if we set $A_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, the eigenvalues of A_0 are $\lambda_1 = 1 + j$, $\lambda_2 = 1 - j$, which are labeled by symbol " $+$ " in Fig. 2 and are all in the right-half part of curve l, so the multi-agent system (22) is not BIBO stable. The unstable impulse response for $\mathscr{H}(s)$ in (22) with null initiations is shown in Fig. 4.

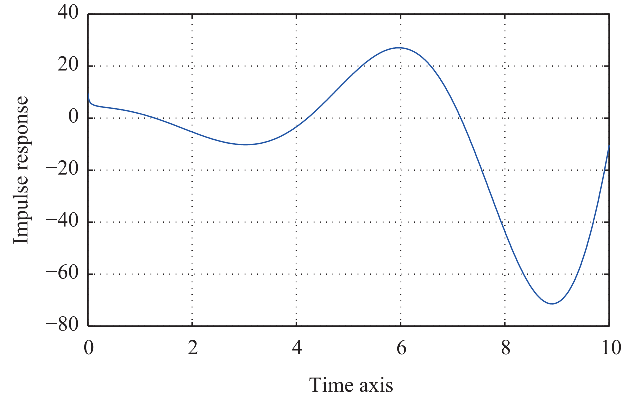

Fig. 4. The unstable impulse response of the overall interconnected system.

V. Conclusions

In this paper, we have considered some fundamental properties of a class of composite systems with fractional degree frequency variables, described by $\mathscr{G}(s) = C_0(\phi(s)I - A_0)^{-1}B_0 + D_0$. In particular, we have showed the relationship of the controllability and observability between the state realization systems about $\mathscr{G}(s)$ and $h(s) = \frac{1}{\phi(s)}$. Then we reduced the stability condition of such composite system to judging whether a fractional degree polynomial is stable. Finally the stability analysis method for such composite systems is applied to the supervisory control of fractional-order multi-agent systems and an example is provided to illustrate the effectiveness of the proposed method.

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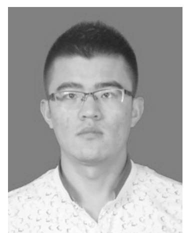
Cuihong Wang received the M. Sc. and Ph. D. degrees from Southwest Jiaotong University, China, in 2005 and 2008, respectively. She is currently an Associate Professor at Shanxi Normal University, China. Her research interests include singular systems, time delay systems, fractional system, stability analysis, model reduction, and robust control.



Yafei Zhao Master student at Shanxi Normal University, China. His current research interests include fractional order systems, model reduction, and robust control.

YangQuan Chen received the Ph. D. degree in advanced control and instrumentation from Nanyang Technological University, Singapore, in 1998. Dr. Chen was at the Faculty of Electrical and Computer Engineering, Utah State University before he joined the School of Engineering, University of California, Merced in 2012 where he teaches "Mechatronics" for juniors and "Fractional order mechanics" for graduates. His current research interests include mechatronics for sustainability, cognitive process control and hybrid lighting control, multi-UAV based cooperative multi-spectral "personal remote sensing" and applications, applied fractional calculus in controls, signal processing and energy informatics; distributed measurement and distributed control of distributed parameter systems using mobile actuator and sensor networks.