Controlled Optimization Processes by Systematic Designs: A Dissipativity-Based Approach

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Abstract

Finite-time and fixed-time optimization are crucial in control and decision-making systems that demand fast, guaranteed convergence. Unlike traditional methods that only ensure asymptotic convergence, these approaches are designed to reach the optimal solution within a finite or uniformly bounded time, regardless of initial conditions (in the fixed-time case). This is especially important in real-time, safety-critical, or resource-constrained applications like robotics, autonomous systems, and networked control, where delays or prolonged computation can compromise performance or safety. This poster addresses controlled optimization processes with a focus on convergence properties considering various proposed algorithms with a unified framework. Toward this goal, a new analysis of finite-, fixed-, and prescribed-time convergent optimization algorithms is presented in the perspective of dissipativity theory. This perspective enables the unification of time-constrained optimization algorithms under the framework of dissipativity control theory, and may enable the design of new algorithms that satisfy theses convergence properties.

Preliminaries

Consider the nonautonomous nonlinear system defined as:

$$\dot{x} = f(t,x), \quad x(t_0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function satisfies $f(t,0) = 0$, that defines the origin $x = 0$ as an equilibrium point of the dynamical system. $t_0 \in \mathbb{R}_{\geq 0}$ is the initial time.

Definition 1. [5] The origin of system (1) is considered globally finite-time stable if it is globally asymptotically stable and every solution $x_{(t_0,x_0)}(t)$ of (1) reaches the origin within a finite time, i.e.,

$$x_{(t_0,x_0)}(t) = 0, \quad \forall t \geq t_0 + \mathcal{T}(t_0,x_0) \tag{2}$$

where $\mathcal{T} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a time function that measures the time of convergence to the origin starting from some initial condition (t_0, x_0).

Definition 2. [5, 1] The origin of the system (1) is defined to be fixed time stable if it is globally finite time stable and the time function $\mathcal{T}$ has an upper bound, say, $\tau_{max} > 0$ such that:

$$\mathcal{T}(t_0,x_0) \leq \tau_{max}, \quad \forall x_0 \in \mathbb{R}^n, \forall t_0 \in \mathbb{R}_{\geq 0} \tag{3}$$

Definition 3. The origin of the system (1) is said to be a prescribed/arbitrary time stable if it is fixed time stable, and $\exists \tau_p \in \mathbb{R}_{\geq 0}$, with no dependence on any system parameters or initial conditions and can be predefined or designed in advance. If $\mathcal{T}(t_0,x_0) = \tau_p$, then the origin is strictly prescribed-time convergent, while $\mathcal{T}(t_0,x_0) < \tau_p$ indicates weakly prescribed-time convergence.

Dissipativity theory

Dissipativity theory may be viewed as a counterpart to Lyapunov theory but for systems with inputs. Consider a discrete-time dynamical system satisfying the state-space equation:

$$\zeta_{t_1} = A\zeta_t + Bu_t. \tag{4}$$

In the perspective of classical dissipativity theory, u_t is an external supply that drives the dynamics governed by the state ζ_t.The two key concepts in dissipativity are storage and supply. The storage function $V(\zeta_t)$ can be interpreted as a notion of stored energy in the system. The storage function always satisfies $V(\zeta_t) \geq 0$. The supply rate $\mathcal{S}(\zeta_t; u_t)$. When $\mathcal{S} > 0$, the external force is adding energy to the system. When $\mathcal{S} < 0$, the external force is extracting energy from the system.

Feedback interconnections of gradient algorithms

The concept of time-constrained convergence of optimization algorithms is displayed in a qualitative and quantitative sense. The required Lyapunov condition to satisfy finite- time, fixed- time and prescribes- time convergence is shown.

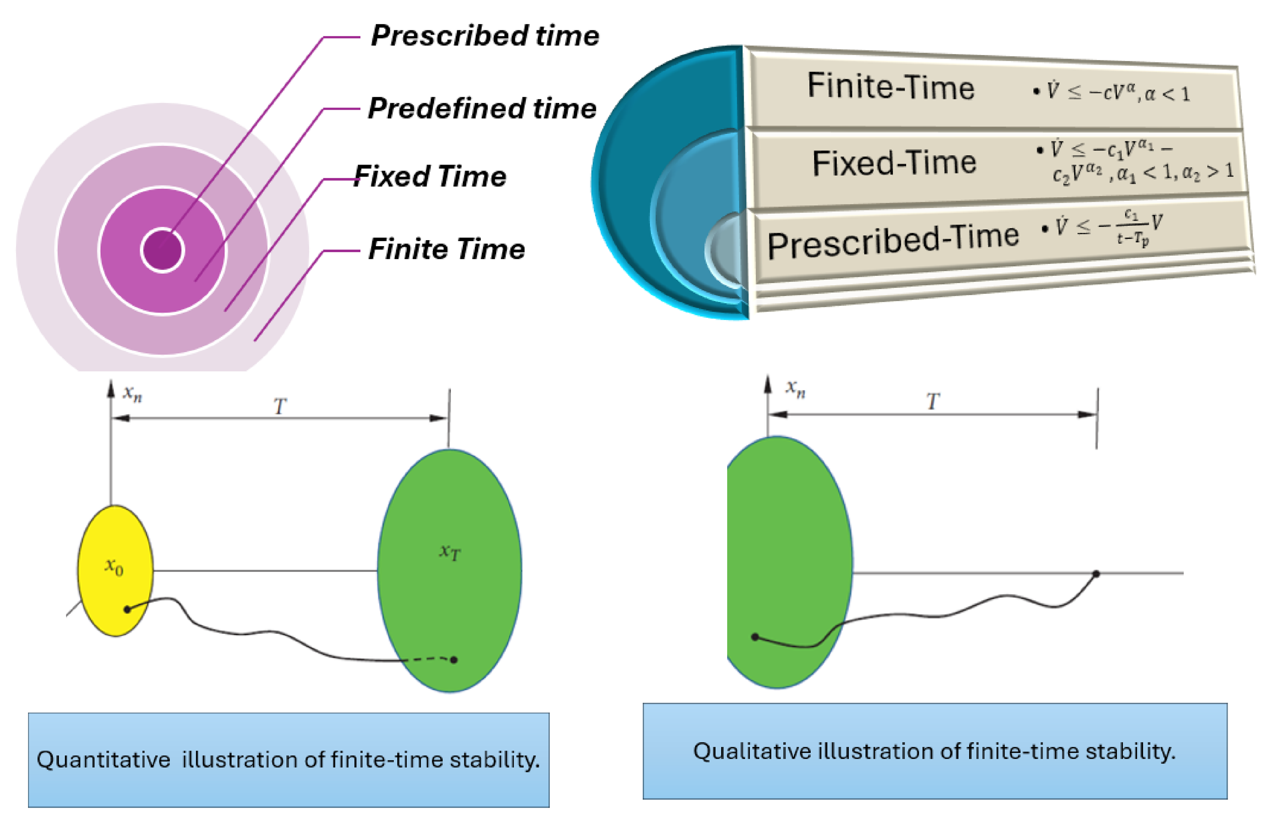

Fig. 1: The concept of Finite/Fixed/Prescribed- Time stability.

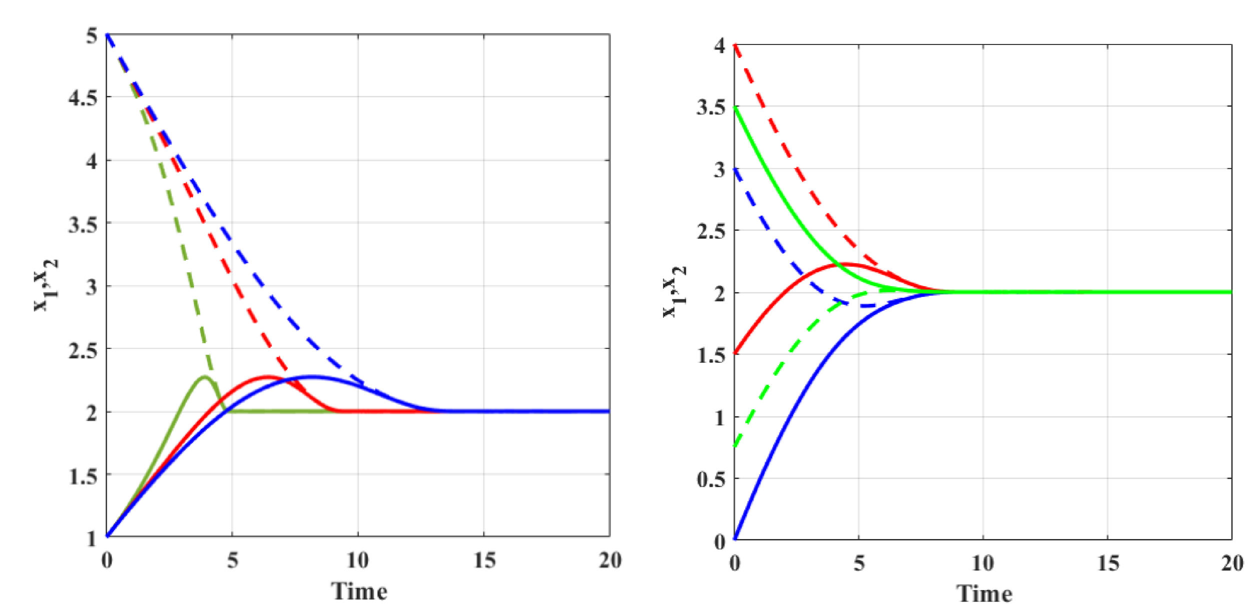

Fig. 2: Example on prescribed-time convergent of gradient flow with time varying gradient feedback for the Trid function (x_1 solid, x_2 dashed: Left: for different prescribed times $T_p = 5, 10$ and 15, Right: For different initial conditions with $T_p = 10$ [2].

The dissipativity can be applied to the analysis of iterative optimization algorithms. The benefit of using dissipativity for algorithm analysis is that it provides a principled and modular framework where algorithms and oracles can be interchanged and analyzed [4, 3, 7].

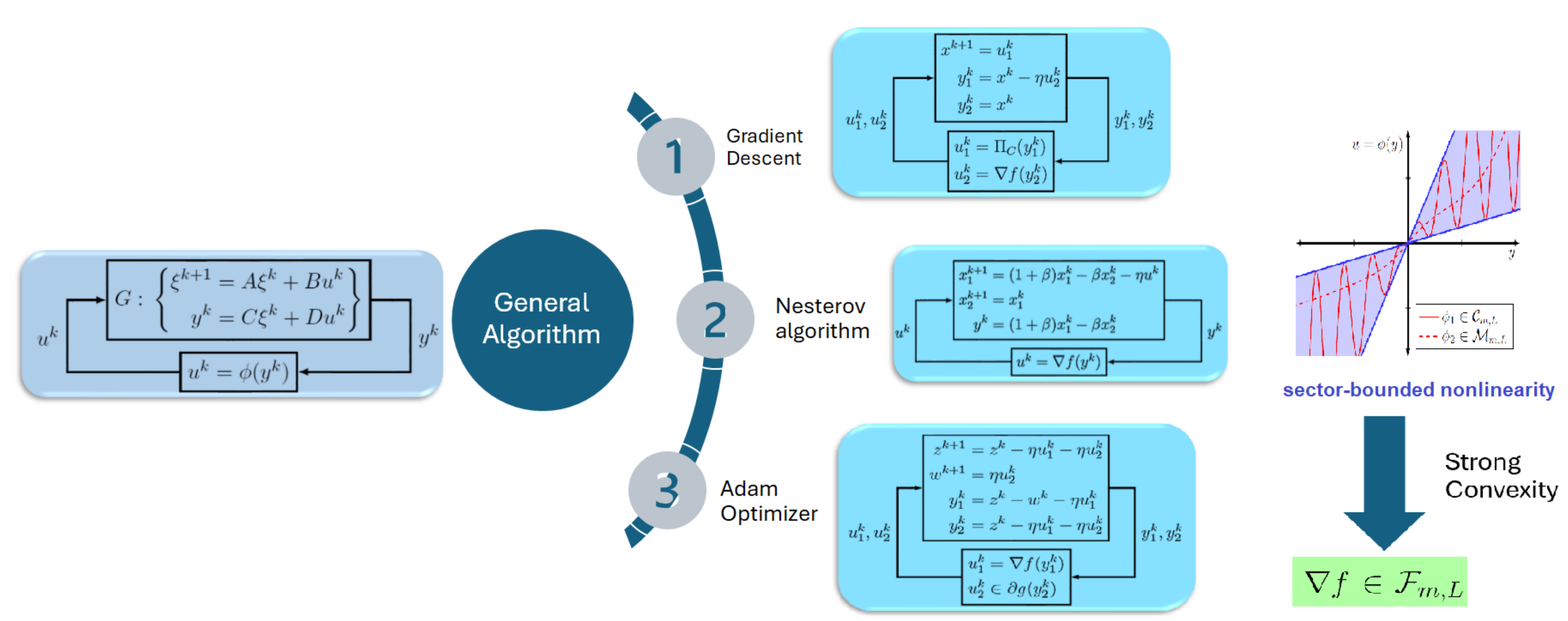

Fig. 3: Equivalent feedback interconnections for general gradient algorithms [4, 6].

Dissipativity-based optimization design with controllable convergence performance

Definition 4. The Optimization algorithm, say Σ is said to be *Finite-time* convergent, if it is dissipative with the supply rate

$$\mathcal{S}(u(t), \dot{y}(t), z(t)) = u^\top(t)\dot{y}(t) - \beta\mathcal{V}^k(t,z(t)) - W(z(t)),$$

$\forall\, t \geq t_0$, for $\beta > 0$ and $0 < k < 1$. where $W(\cdot)$ is a positive definite function, and $\mathcal{V}$ is a storage function.

Definition 5. The Optimization algorithm Σ is said to be *Fixed-time* convergent, if it is dissipative with the supply rate

$$\mathcal{S}(u(t), \dot{y}(t).z(t)) = u^\top(t)\dot{y}(t) - \beta_1\mathcal{V}^{k_1}(t,z(t)) - \beta_2\mathcal{V}^{k_2}(t,z(t)) - W(z(t)),$$

$\forall\, t \geq t_0$ for $\beta_1, \beta_2 > 0$, $0 < k_1 < 1$ and $k_2 > 1$. where $W(\cdot)$ is a positive definite function, and $\mathcal{V}$ is a storage function

Definition 6. The Optimization algorithm Σ is said to be *Prescribed-time* convergent, if it is dissipative with the supply rate

$$\mathcal{S}(u(t), \dot{y}(t).z(t)) = u^\top(t)\dot{y}(t) - k\frac{\alpha(\mathcal{V}(t,z(t)))}{T_p - t} - W(z(t)),$$

$\forall\, t \geq t_0$ for $\beta_1, \beta_2 > 0$, $0 < k_1 < 1$ and $k_2 > 1$. where $W(\cdot)$ is a positive definite function,$\alpha(\cdot)$ is a class $\mathcal{K}$ function, and $\mathcal{V}$ is a storage function

Remarks

This study aims to define and analyze a passivity-based controller design technique for discrete time and continuous time gradient flows. The approach establishes finite-time and fixed-time convergence of the gradient algorithm trajectories [1] to its minimum point in the perspective of dissipativity theory, which can be applied to the analysis of iterative optimization algorithms. As a future work, a new form of a dissipation function, selected a priori, can be introduced to design a feedback control rule to achieve such convergence properties. The proposed framework may also extended to study time- constrained convergence in distributed optimization, where the algorithms can be modeled as shown in the figure below.

References

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