The Ellipsoidal Invariant Set of Fractional Order Systems Subject to Actuator Saturation: The Convex Combination Form

Kai Chen, Junguo Lu, and Chuang Li

Abstract—**The domain of attraction of a class of fractional order systems subject to saturating actuators is investigated in this paper. We show the domain of attraction is the convex hull of a set of ellipsoids. In this paper, the Lyapunov direct approach and fractional order inequality are applied to estimating the domain of attraction for fractional order systems subject to actuator saturation. We demonstrate that the convex hull of ellipsoids can be made invariant for saturating actuators if each ellipsoid with a bounded control of the saturating actuators is invariant. The estimation on the contractively invariant ellipsoid and construction of the continuous feedback law are derived in terms of linear matrix inequalities (LMIs). Two numerical examples illustrate the effectiveness of the developed method.**

Index Terms—**Fractional order, saturation, convex hull, invariant set, ellipsoid, domain of attraction.**

I. INTRODUCTION

IN this paper, we focus on the domain of attraction of the original for a class of fractional order systems with saturating actuators. Fractional order systems, which are based on fractional calculus, have attracted much attentions in recent decades. Fractional calculus has a long history over 300 years, and it is a branch of mathematical analysis that deals with the possibility of taking real number or complex number powers of differentiation and integration operators[1]. In recent years, considerable interest in fractional calculus has been stimulated by the applications that this calculus found in numerical analysis and different areas of physics and engineering, possibly including fractal phenomena[2]. Especially in some special areas, such as viscoelastic materials[3] and electro-chemical systems[4], the application of fractional-order models is more adequate and elegant than integer-order models for the investigation of dynamic behavior. The most significant reason is that the fractional differential equations (FDEs) have the ability of revealing inherent memory and inherited character of various materials and processes in real physical world.

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In control engineering field, the fractional order $PI^{\lambda}D^{\mu}$ (FOPID) controller is an active topic for its excellent control performance and the tuning method for FOPID controller has attract more attentions. In the past decades, a number of tuning methods for fractional order controllers are proposed in the literatures, such as [5-7]. However, most of obtained results are derived in the frequency domain, and it is hard to handle nonlinearities, e.g., actuator saturation, of the considered systems.

The stability and stabilization problems of integer order state space model have been widely investigated. In [8], a method for estimating the domain of attraction of the origin for a system under a saturated linear feedback is proposed. And in [9], the stability and stabilization problems of a class of continuous-time and discrete-time Markovian jump linear systems with partly unknown transition probabilities are investigated. The robust stochastic stability problem for discrete-time uncertain singular Markov jump systems with actuator saturation is considered in [10]. As the extension of integer order systems, the stability and stabilization problems of fractional order state space model also have attracted great attractions. Control problems of fractional order systems have achieved tremendous attention in recent years due to the inherent memory advantage of fractional derivatives.

Iterative learning control is one of important robust linear control methods for fractional order linear systems. In [11], Chen investigated the classical Arimoto D-type iterative learning control (ILC) updating law uses the first order derivative (with transfer functions) of tracking error. The convergence of the iterative process for fractional order linear systems was first discussed in time domain in [12], which is a meaningful work, and the fractional order iterative learning control for time-varying systems in convolution form are analyzed. In [13], Li discussed the convergence of the iterative process for fractional order linear time invariant (LTI) system, and proved that the convergence conditions of the fractional order and integer order iterative learning schemes are equivalent for $D = 0$.

Another important issue is the robust stability and stabilization problems of fractional order systems. In [14], the problems of robust stability and stabilization for a class of fractional order linear time invariant systems with convex polytopic uncertainties were considered. Several methods to investigate the stability and stabilization of fractional order linear systems are proposed in [15] and [16] which are based on the conclusion in [14]. In this years, the control synthesis of fractional order system was wide investigated. Reference [17] investigated the robust stability of uncertain parameters FO-LTI interval systems, which have deterministic linear coupling relationship between fractional order and other model parameters. The robust stability for uncertain fractional-order systems of two types of order $\alpha \in (0,1)$ are investigated in [18].

In engineering practice, it is important to consider the input saturation. The performance of the closed-loop system may be severely degraded or be unstable when the actuator is under saturation. The actuator saturation for the integer order systems has received great attention in the past decades. However, in the stability analysis of fractional order systems, it is still an open topic. In this paper, we consider the control of fractional order linear systems subject to actuator saturation $D^{\alpha}x(t) = Ax(t) + BSat(u(t))$. To ensure the stabilization of this fractional order linear system, we first concern with the closed-loop stability under a given linear state feedback $u = Fx$. The Lyapunov approach is the mostly fundamental approach to deal with the stability issues of integer order systems. However, it is still an open topic to choose proper Lyapunov function candidates for fractional order systems. Several works have been dedicated to this problem.

For stable fractional order systems, the decay rate of solution is in the sense of Mittag-Leffler function rather than exponential function, which motivates the concept of Mittag-Leffler stability of fractional order systems. Mittag-Leffler stabilities were firstly proposed in [19] and [20] for the commensurate case and incommensurate case, respectively. Even though the generalization of classical Lyapunov direct approach to the fractional order case is proposed, the commonly used quadratic Lyapunov function candidate is not valid for fractional order systems since the fractional derivative of composite function is an infinite series. To the best of our knowledge, stability analysis of fractional order systems based on the Lyapunov direct approach is still an open problem, and only a few works are dedicated to this topic. By the equivalent transformation between the solution of FDEs with that of ODEs, the Lyapunov direct approach was adopted to investigate the stability of fractional order nonlinear systems[21].

Only few works are related to the estimation of domain of attraction for fractional order systems. Such as, in [22], the sector bounded condition of saturation nonlinearity and Gronwall-Bellman inequality were adopted to derive estimation algorithm of attraction in terms of bilinear matrix inequality.

Our contributions of this paper include:

1) Propose a method to obtain the domain of attraction for fractional order systems through a set of ellipsoids.

2) Demonstrate that the convex hull of ellipsoids can be made invariant for fractional order linear systems subject to actuator saturation if each ellipsoid in a set with a bounded control of the saturating actuators is invariant.

By comparing our paper with the previous conclusions, it could be observed that less conservative results were obtained through the proposed method.

The rest of this paper is organized as follows: In Section II, some necessary preliminaries and the problem statement are introduced. The domain of attraction under a given saturated linear feedback is discussed in Section III. To obtain the feedback matrix, the construction of continuous feedback laws are introduced in Section IV. To show the effectiveness of this method, two numerical examples are shown in Section V. Finally, Section VI draws the conclusion.

Notations. $\mathbf{R}^n$ is the set of real n dimensional vectors, and $\mathbf{R}^{n\times m}$ is the set of real $n\times m$ dimensional matrices. $Sat(\cdot)$ stands for the standard saturation function. For a $P \in \mathbf{R}^{n\times n}$, $P > 0$ and $\rho \in (0,\infty)$, an ellipsoid is denoted as $\Psi(P,\rho) := \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px \le \rho\}$ where P is a positive-definite matrix. Especially, we use $\Psi(P)$ to denote $\Psi(P,1)$. In this paper, we are interested in a convex function determined by a set of positive-definite matrices $P_1, P_2, P_3, \dots, P_N \in \mathbf{R}^{n\times n}$ to obtain the maximum estimation on the invariance of the considered fractional order systems.

II. Problem Statement and Necessary Preliminaries

A. Preliminaries

Definition 1[23]**.** A quadratic function can be defined as follows:

$$V_c(x) = x^{\mathrm{T}}Px,$$

where $P \in \mathbf{R}^{n\times n}$ is a positive-definite matrix. For a positive number ρ, a level set of $V_c(\cdot)$, denoted by $L_{V_c}(\rho)$, is

$$L_{V_c}(\rho) := \{x \in \mathbf{R}^n; V_c(x) \le \rho\} = \Psi(P,\rho).$$

Definition 2. The α-th ($\alpha > 0$) order fractional integral of a fractional calculus function $f(t)$ is defined as

$$\mathcal{I}^{\alpha}f(t) = D^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau.$$

Definition 3. The α-th ($\alpha > 0$) order fractional derivative of an integrable and differentiable function $f(t)$ is introduced as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{1+\alpha-m}}\mathrm{d}\tau,$$

where m is an integer satisfying $m-1 < \alpha < m$. $f^{(m)}(\cdot)$ is the m-th derivative of function $f(\cdot)$ and $\Gamma(\cdot)$ is the Euler-Gamma function.

Lemma 1[23]**.** For a set of positive-definite matrices $P_1, P_2, P_3, \dots, P_N \in \mathbf{R}^{n\times n}$, Let $P(\gamma) := \sum_{j=1}^{N}\gamma_j P_j$, Then,

$$L_{V_c}(\rho) = co\{\Psi(P_j,\rho), j \in I[1,N]\} = \bigcup_{\gamma\in\Gamma}\Psi(P(\gamma),\rho),$$

where $\gamma \in \mathbf{R}^N$, N is a positive integer number, and $co\{\cdot\}$ stands for the convex hull.

Lemma 2[24]**.** Let $x(t) \in \mathbf{R}^n$ be a vector of differentiable functions. Then, for any time instant $t \ge 0$, the following relationship holds

$$D^{\beta}(x^{\mathrm{T}}(t)Px(t)) \le 2x^{\mathrm{T}}(t)P(D^{\beta}x(t)),$$

where $P \in \mathbf{R}^{n\times n}$ is a constant symmetric positive-definite matrix.

Lemma 3[25]**.** Define $\Phi(F) = \{x \in \mathbf{R}^n : |f_i x| \le 1, i \in [1, m]\}$, and f_i is the i-th row of the matrix F. Let $P \in \mathbf{R}^{n\times n}$ be a positive-definite matrix. Suppose that $\rho > 0$. An ellipsoid $\Omega(P, \rho) = \{x \in \mathbf{R}^n : x^{\mathrm{T}} P x \le \rho\}$ is included in $\Phi(F)$ if and only if,

$$f_i^{\mathrm{T}} P f_i \le \rho.$$

Let Ξ denote the set of $m \times m$ diagonal matrices whose diagonal elements are either 0 or 1. Then, there are 2^m elements in Ξ. Suppose that elements of Ξ are labeled as E_i with $i \in [1, 2^m]$. Then, $E_i^- = I_m - E_i$ is also an element of Ξ, where I_m is the $m \times m$ dimensional identity matrix.

Lemma 4[25]**.** Given $F, H \in \mathbf{R}^{m\times n}$. Suppose that $| h_j x | \le 1$, then we have

$$Sat(Fx) \in co\{E_i F x + E_i^- H x, i \in [1, 2^m]\},$$

where h_j is the j-th row of the matrix H.

Lemma 5[19]**.** Let $x = 0$ be an equilibrium point for the non-autonomous fractional order system $D^\alpha x(t) = f(t, x)$. Assume that there exists a Lyapunov functin $V(t, x)$ and three class-K functions $\beta_i, i = 1, 2, 3$ satisfying

$$\beta_1(\|x\|) \le V(t, x) \le \beta_2(\|x\|),$$
$$D^\beta V(t, x) \le -\beta_3(\|x\|),$$

where $\beta \in (0, 1)$. Then the equilibrium point of system $D^\alpha x(t) = f(t, x)$ is asymptotically Mittag-Leffler stable.

Lemma 6[26]**.** Considering an ellipsoid $\Psi(P)$, if there exists an $H \in \mathbf{R}^{m\times n}$ such that

$$(A + BE_iF + E_i^- H)^{\mathrm{T}} P + P(A + BE_iF + E_i^- H) < 0, \forall i \in I[1, 2^m],$$

and $\Psi(P, \rho) \subset \Phi(F)$, then $\Psi(P, \rho)$ is an invariant set of fractional order system $D^\alpha x(t) = Ax(t) + BSat(Fx)$.

B. Problem Statement

Considering the open-loop fractional order linear system subject to actuator saturation:

$$D^\alpha x(t) = Ax(t) + BSat(u), \tag{1}$$

where the fractional order $0 < \alpha < 1$. $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ are the state and input, respectively. $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ are constant system matrices. $Sat(\cdot) : \mathbf{R}^m \to \mathbf{R}^m$ is the standard saturation function, and $Sat(u) = [Sat(u_1)\ \ Sat(u_2)\ \ \ldots\ \ Sat(u_m)]^{\mathrm{T}}$, where $Sat(u_i) = \mathrm{Sign}(u_i)\min\{1, |u_i|\}$.

The objective of this paper is to obtain an estimation of the domain of attraction for system (1) and obtain a feedback law $u = Sat(Fx)$, under which the closed-loop system,

$$D^\alpha x(t) = Ax(t) + BSat(Fx), \tag{2}$$

is asymptotically stable, where $F \in \mathbf{R}^{m\times n}$ is the feedback matrix. Then the input control is linear with respect to state x in the domain of the state space $\Phi(F)$.

Remark 1. Lyapunov stability method is an important tool for stability analysis of nonlinear systems. There are two methods to explore the stability of nonlinear systems: the Lyapunov indirect approach and the Lyapunov direct approach. In previous work [27], we utilized the Lyapunov indirect approach and Gronwall-Bellman inequality to analyse the decay law of solution. It is an effective way to investigate the global asymptotic stability and controller synthesis of fractional order systems with nonlinearity. However, we are interested in the invariance of considered systems in this study, which means we are more concerned about the local asymptotic stability than the global asymptotic stability. Therefore, the method in [27] is not enough to handle the invariance estimation problem of fractional order systems with actuator saturation.

The Lyapunov direct approach introduce an energy function to analysis the stability of the nonlinear system directly. It is an efficient way to investigate the invariance of nonlinear system. In this paper, we primarily consider Mittag-Leffler stability of the closed-loop systems with the state feedback law using Lyapunov direct approach.

Remark 2. From Definition 1 and [18], one can obtain that the quadratic function $V_c(x) = x^{\mathrm{T}} P x$ is the most popular Lyapunov function candidate to investigate the stability and control method of integer order systems. However, for fractional order systems, it is not suitable to use this function directly. The main reason for this problem is that the quadratic Lyapunov function is not valid since the fractional derivative of composite function is an infinite series. Lemma 2 provides a direct way to adopt this quadratic Lyapunov function for the fractional order systems. Thus the quadratic Lyapunov function can be used to analyze the stability and stabilization of fractional order system by expressing a linear feedback law subject to saturation into a convex hull of a group of auxiliary linear feedback matrices.

The objective of this paper is to obtain a series of feedback functions to reduce the conservatism of the domain of attraction. To that end, we are interested in a function determined by a set of positive-definite matrices $P_1, P_2, P_3, ldots, P_N \in \mathbf{R}^{n\times n}$. Let $Q_j = P_j^{-1}$, $j \in I[1, N]$. For a vector $\gamma \in \mathbf{R}^N$, define

$$Q(\gamma) := \sum_{j=1}^{N} \gamma_j Q_j, \qquad P(\gamma) := Q^{-1}(\gamma),$$

where $\gamma \in \mathbf{R}^N : \sum_{j=1}^{N} \gamma_j = 1$, $\gamma \in \Gamma$. Thus we change the quadratic function as,

$$V_c(x) = x^{\mathrm{T}} P(\gamma) x.$$

From Lemmas 2-4 and Definition 1, the ellipsoid $\Psi(P, \rho) := \{x \in \mathbf{R}^n : x^{\mathrm{T}} P x \le \rho\}$ is said to be contractively invariant if $D^\alpha(x^{\mathrm{T}} P(\gamma) x) < 0$ for all $x \in \Psi(P, \rho) \setminus \{0\}$. Obviously, $\Psi(P, \rho)$ is inside the domain of attraction.

III. Domain Of Attraction Under a Given Saturated Linear Feedback

In this section, we consider the calculation problems corresponding to the quadratic function and apply it to fractional order linear systems. And then the estimation of the domain of attraction is illustrated.

A. Calculation Problems

From [26] and Remark 2, we know different reference polyhedrons will result in different ellipsoid set. Obviously, a single ellipsoid will result in much conservation. To deal with this situation and reduce the conservatism of the estimation, in this paper, a set of reference polyhedrons is used to produce multiple ellipsoids. Thus, these ellipsoids will combined to a convex hull and it can extend the domain of the invariant set $\Psi(P,\rho)$. One convex hull which is combined by three ellipsoids is shown in Fig. 1. We denote the convex hull of the ellipsoids

$$co\{\Psi(P_j,\rho), j\in I[1,N] := \{\sum_{j=1}^{N}\gamma_j x_j : x_j\in\Psi(P_j,\rho),\gamma\in\Gamma\}.$$

There are various ways to define the composite quadratic function with a set of matrices $P_1, P_2, \ldots, P_N > 0$. For the convenience of analysis, we define this function as follows,

$$V_c(x) := \max_{\gamma\in\Gamma} x^{\mathrm{T}}\left(\sum_{j=1}^{N}\gamma_j P_j\right)x,\quad \sum_{j=1}^{N}\gamma_j = 1.$$

For convenience, only two ellipsoids are considered in this paper. Thus,

$$\begin{aligned} V_c(x) &= \max_{\gamma\in\Gamma} x^{\mathrm{T}}\left(\sum_{j=1}^{N}\gamma_j P_j\right)x \\ &= \max_{\lambda\in[0,1]} x^{\mathrm{T}}\left(\lambda Q_1 + (1-\lambda))Q_2\right)^{-1} x, \end{aligned} \tag{3}$$

where $N = 2$, and λ is a real number. Denote $\Psi(P,\rho)$ as a bounded convex set, it can be easily translated into the following form by Schur complement,

$$\sup_{\gamma\in\Gamma}\delta, \tag{4}$$

$$\text{s.t.}\begin{bmatrix}\delta & x^{\mathrm{T}}\\ x & \sum_{j=1}^{N}\gamma_j Q_j\end{bmatrix}\geq 0,\quad \sum_{j=1}^{N}\gamma_j = 1, \tag{5}$$

$$\Rightarrow \text{s.t.}\begin{bmatrix}\delta & x_i^{\mathrm{T}}\\ x_i & \lambda Q_1 + (1-\lambda)Q_2\end{bmatrix}\geq 0.$$

Fig. 1 shows a two dimensional level set which is the convex hull of three ellipsoids.

Remark 3. In order to calculate the optimal value of $\gamma = \gamma^*(x)$, the LMI problem (5) needs to be solved. However, the calculation is a time-consuming process. Reference [23] provides a simplified way to get $\gamma^*(x)$.

Denote that $\alpha(\lambda,x) = x^{\mathrm{T}}\left(\lambda Q_1 + (1-\lambda))Q_2\right)^{-1} x$. Then, let $x\in\mathbf{R}^n$ and $Q_1, Q_2 > 0$ be given. Assume that $Q_1 - Q_2$ is nonsingular. Let $U\in\mathbf{R}^{n\times n}$ be such that $U^{\mathrm{T}}U = UU^{\mathrm{T}} = I$ and $U^{\mathrm{T}}xx^{\mathrm{T}}U = \mathrm{diag}\{x^{\mathrm{T}}x, 0, \ldots, 0\}$. Let $\hat{Q}_1 = U^{\mathrm{T}}Q_1U, \hat{Q}_2 = U^{\mathrm{T}}Q_2U$ and partition $\hat{Q}_1$ and $\hat{Q}_2$ as

$$\hat{Q}_1 = [\hat{q}_1, \hat{Q}_{12}], \hat{Q}_2 = [\hat{q}_2\quad \hat{Q}_{22}], \hat{q}_1, \hat{q}_2\in\mathbf{R}^{n\times 1}.$$

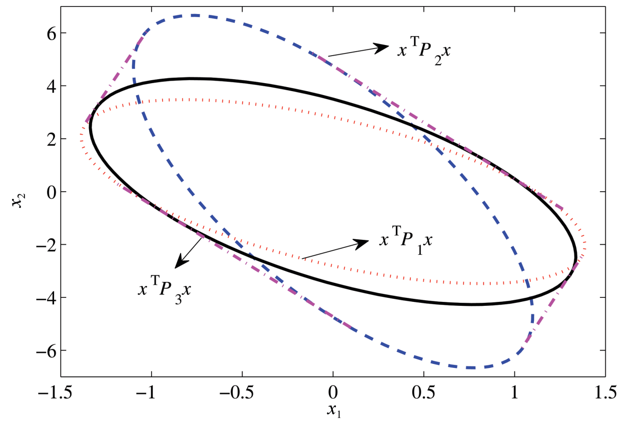

Fig. 1. The invariant set which is the convex hull of three ellipsoids (See the magenta dash-dotted outer curve).

Then $\frac{\partial\alpha}{\partial\lambda} = 0$ at $\lambda\in[0,1]$ if and only if

$$\det\begin{bmatrix}\lambda(\hat{Q}_{12}-\hat{Q}_{22})+\hat{Q}_{22} & \hat{Q}_1-\hat{Q}_2\\ 0_{(n-1)\times(n-1)} & \lambda(\hat{Q}_{12}-\hat{Q}_{22}^{\mathrm{T}})^{\mathrm{T}}+\hat{Q}_{22}^{\mathrm{T}}\end{bmatrix} = 0. \tag{6}$$

Using this method requires less time than solving the LMI problem.

Remark 4. In this section, only two ellipsoids are used in (3). For this case, $Q_1 - Q_2$ is nonsingular and $\gamma^*(x)$ can be easily calculated by the technology in Remark 3. Obviously, using more ellipsoids will make the conservatism less. However, for the case where $N > 2$, $\gamma^*(x)$ could be non unique in some special condition. Such as, the case that one of Q_j might be the combined convex hull of other matrices, which may be considered as degenerated. Although, one can assume that there is no this kind of condition, it is hard to keep the uniqueness and continuity of $\gamma^*(x)$. Thus, for $N > 2$ case, it is difficult to compute the $\gamma^*(x)$ and needs further study.

B. Analysis of Attraction Domain

Assume the set of ellipsoid $\Psi(P_j,\rho_j), j\in I[1,N]$ is given, the fractional order system can be invariant with a corresponding saturated linear feedback Fx. For simplicity and reducing the conservatism of the domain of attraction, a set of invariant ellipsoids $\Psi(P_j,\rho_j), j\in I[1,N]$ was considered with $\rho_j = 1$.

Theorem 1. Suppose that the state feedback law F and an ellipsoid $\Psi(P_j), j\in I[1,N]$ are given. If there exists a matrix $H_j\in\mathbf{R}^{m\times n}$ satisfying

$$\begin{aligned}&(A+B(E_iF+E_i^-H_j))^{\mathrm{T}}P_j + P_j(A+B(E_iF+E_i^-H_j))\\ &\quad\leq 0,\quad \forall i\in I[1,2^m], j\in I[1,N]\end{aligned} \tag{7}$$

and $\Psi(P_j)\subset\Phi(H_j)$, then $co\{\Psi(P_j), j\in I[1,N]\}$ is an invariant set of closed-loop system (2).

Proof. From Lemma 5 we know, the following inequality needs to be proved to ensure the asymptotic Mittag-Leffler stability of the fractional order linear system (2) corresponding the quadratic function (1),

$$D^{\alpha}V(t,x) = D^{\alpha}(x^{\mathrm{T}}P_jx) < 0, \forall x\in\Psi(P_j,\rho)\setminus\{0\}, \tag{8}$$

applying fractional inequality in Lemma 2 and Lemma 4 to inequality (8), gives

$$\begin{aligned} D^{\alpha}(x^{\mathrm{T}}P_jx) &\leq 2x^{\mathrm{T}}P_j(D^{\alpha}(x)) \\ &= 2x^{\mathrm{T}}P_j(Ax + BSat(Fx)) \\ &= 2x^{\mathrm{T}}A^{\mathrm{T}}P_jx + 2x^{\mathrm{T}}P_jB\sum_{i=1}^{2^m}\eta_i(E_iFx + E_i^-H_jx), \end{aligned}$$

where $0 \leq \eta_i \leq 1$ and $\sum_{i=1}^{2^m}\eta_i = 1$.

From Theorem 1, one can obtain that $D^{\alpha}(x^{\mathrm{T}}P_jx) < 0$ for all $x \in \Psi(P_j,\rho)\backslash\{0\}$. And it is easy to notice that $Sat(Fx)$ is a convex hull of $E_iFx + E_i^-H_jx$ for all $i \in [1, 2^m], j \in I[1, N]$ in (9). Thus, the Mittag-Leffler stability of the fractional order linear closed-loop system $D^{\alpha}x(t) = Ax(t) + BSat(Fx)$ is ensured.

The condition of $\Psi(P_j) \subset \Phi(H_j)$ is equivalent to $\rho h_{ji}P_j^{-1}h_{ji}^{\mathrm{T}} \leq 1$, then it can be written as follows by Schur complement,

$$\begin{bmatrix} 1 & h_{ji}(\frac{P_j}{\rho})^{-1} \\ (\frac{P_j}{\rho})^{-1}h_{ji}^{\mathrm{T}} & (\frac{P_j}{\rho})^{-1} \end{bmatrix} \geq 0, \tag{9}$$

for all $i \in [1, m]$.

Denote $Q_j = (\frac{P_j}{\rho})^{-1}$. Let $G_j = H_j(\frac{P_j}{\rho})^{-1}$ and the i-th row of matrix G_j be g_{ji}, i.e., $g_{ji} = h_{ji}(\frac{P_j}{\rho})^{-1}$. Hence, the condition $\Psi(P_j, \rho) \subset \Phi(H_j), j \in I[1, N]$ can be formulated as,

$$\begin{bmatrix} 1 & g_{ji} \\ g_{ji}^{\mathrm{T}} & Q_j \end{bmatrix} \geq 0, \text{ for } i \in [1, m], j \in I[1, N], \tag{10}$$

where $Q_j \in \mathbf{R}^{n\times n}$, $G_j \in \mathbf{R}^{m\times n}$. Due to the fact that $x_0 \in co\{\Psi(P_j, j \in I[1, N])\}$, there exists $x_j \in \Psi(P_j)$ and $\gamma_j \geq 0, j \in I[1, N]$, such that $\gamma_1 + \gamma_2 + \cdots + \gamma_N = 1$ and $x_0 = \gamma_1x_1 + \gamma_2x_2 + \cdots + \gamma_Nx_N$. Denote $Q = \gamma_1Q_1 + \gamma_2Q_2 + \cdots + \gamma_NQ_N$ and $P = Q^{-1}$. Then from Lemma 1, one can obtain that $\Psi(P) \subset co\{\Psi(P_j), j \in I[1, N]\}$.

Let $G = \gamma_1G_1 + \gamma_2G_2 + \cdots + \gamma_NG_N$ and g_i be the ith row of G, combining inequality (7) and inequality (10), gives

$$\begin{aligned} &Q(A+BE_iF)^{\mathrm{T}} + (A + BE_iF)Q \\ &\quad + G^{\mathrm{T}}E_i^-B^{\mathrm{T}} + BE_i^-G \leq 0, \forall i \in [1, 2^m], \end{aligned} \tag{11}$$

and

$$\begin{bmatrix} 1 & g_i \\ g_i^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \text{ for } i \in [1, m]. \tag{12}$$

Denote $H = GQ^{-1}$, then one can obtain

$$\begin{aligned} &(A + B(E_iF + E_i^-H))^{\mathrm{T}}P + P(A + B(E_iF + E_i^-H)) \\ &\quad \leq 0, \forall i \in I[1, 2^m], \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\begin{bmatrix} 1 & h_k(\frac{P}{\rho})^{-1} \\ (\frac{P}{\rho})^{-1}h_k^{\mathrm{T}} & (\frac{P}{\rho})^{-1} \end{bmatrix} \geq 0, k \in I[1, m] \\ &\quad \Leftrightarrow \Psi(P) \subset \Phi(H). \end{aligned} \tag{14}$$

From the above fact, one can easily derive the conclusion that $\Psi(P)$ is invariant, which means that a trajectory starting from x_0 will stay inside of $\Psi(P_j)$ and it is a subset of $co\{\Psi(P_j), j \in I[1, N]\}$. Since that x_0 is a random point inside $co\{\Psi(P_j), j \in I[1, N]\}$, then one can obtain that the convex hull is an invariant set. If "$<$" holds for all the inequalities, then the state trajectory will converge to the origin for all the initial states. □

Theorem 1 shows that if each $\Psi(P_j)$ is invariant, then, their convex hull, $co\{\Psi(P_j), j \in I[1, N]\}$ is also invariant. This theorem provides the sufficient condition for an ellipsoid to be inside the domain of attraction. To maximize the cross-section of the ellipsoid in the state-space $\mathbf{R}^n$, the following type of convex set is considered in this paper:

$$X_R = co\{x_1, x_2, \ldots, x_l\},$$

where X_R represents polyhedrons. Then the problem that how to choose the ellipsoid $\Psi(P^*, \rho)$ with the largest volume from all the ellipsoids $\Psi(P, \rho)$ which satisfies the Theorem 1 is considered. It can be concerned with the maximum quantity of $\lambda_R(\Psi(P, \rho))$ and then it is formulated as

$$\sup_{P_j>0,\rho,H_j} \lambda, \tag{15}$$

$$\begin{aligned} \text{s.t. } &\lambda X_R \subset \Psi(P_j, \rho), &(16) \\ &(A + BE_iF + E_i^-H_j)^{\mathrm{T}}P_j \\ &\quad + P_j(A + BE_iF + E_i^-H_j) < 0, \text{ for } i \in [1, 2^m], &(17) \end{aligned}$$

where $P_j \in \mathbf{R}^{n\times n}$, $H_j \in \mathbf{R}^{m\times n}, j \in I[1, N], \rho = 1$. Inequalities (15)-(17) can be formulated as follows:

$$\inf_{Q_j>0,G} \Lambda, \tag{18}$$

$$\begin{bmatrix} \Lambda & x_{ji}^{\mathrm{T}} \\ x_{ji} & Q_j \end{bmatrix} \geq 0, \text{ for } i \in [1, l], j \in I[1, N], \tag{19}$$

$$\begin{aligned} &Q_jA^{\mathrm{T}} + AQ_j + BE_i^{-1}G_j + G_j^{\mathrm{T}}(BE_i^{-1})^{\mathrm{T}} \\ &\quad + Q_j(BE_iF)^{\mathrm{T}} + (BE_iF)Q_j, \text{ for } i \in [1, 2^m], \end{aligned} \tag{20}$$

where $\Lambda = \frac{1}{\lambda^2}$, $Q_j = (\frac{P_j}{\rho})^{-1}$, $G_j = H_j(\frac{P_j}{\rho})^{-1}$.

IV. Controller Synthesis

In this section, the possibility that a level set can get invariant with controls subject to actuator saturation is investigated. Then, a continuous feedback law that guarantees the invariance of the convex hull of ellipsoids $co\{\Psi(P_j), j \in I[1, N]\} = L_{V_c}(1)$ is constructed.

Considering the fractional order system $D^{\alpha}(x) = Ax + BSat(u)$, where $0 < \alpha < 1$. Only the initial condition needs to be specified. For x_0, the state trajectory of system (2) is defined as $\psi(t, x_0)$. Then the *domain of attraction* of the origin is

$$\varPsi := \{x_0 \in \mathbf{R}^n : \lim_{t\to\infty} \psi(t, x_0) = 0\}.$$

Denote $P \in \mathbf{R}^{n\times n}$ be a positive-definite matrix and $V_c(x) = x^{\mathrm{T}}Px$ as Lyapunov function of fractional order system (2). Then, the ellipsoid $\Psi(P, \rho) = \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px \leq \rho\}$ is said to be *(contractively) invariant* if $D^{\alpha}V(x) < 0$ for all $x \in \Psi(P, \rho)\backslash\{0\}$, and then ellipsoid $\Psi(P^*, \rho)$ is called the *invariant set* of fractional order systems (2). Let $\rho = 1$, hence, if the ellipsoid $\Psi(P)$ is contractive invariant set, it is inside the domain of attraction.

Fact 1[25]**.** For a raw vector $f_0 \in \mathbf{R}^{1\times n}$ and matrix $P > 0$, $\Psi(P) \subset \Phi(f_0)$ if and only if,

$$f_0 P^{-1} f_0^{\rm T} \leq 1 \Leftrightarrow \begin{bmatrix} 1 & f_0 P^{-1} \\ P^{-1} f_0^{\rm T} & P^{-1} \end{bmatrix} \geq 0.$$

The equality $f_0 P^{-1} f_0^{\rm T} = 1$ holds if and only if the ellipsoid $\Psi(P)$ touches the hyperplane $f_0 x = 1$ at $x_0 = P^{-1} f_0^{\rm T}$ (the only intersection), namely,

$$1 = f_0 x_0 > f_0 x \quad \forall x \in \Psi(P)\backslash\{x_0\},$$

the ellipsoid $\Psi(P)$ lies strictly between the hyperplane $f_0 x = 1$ and $f_0 x = -1$ without touching them.

Theorem 2. Assuming that each of ellipsoids $\Psi(P_j), j \in I[1, N]$ is contractively invariant. Then, as a result, the level set $L_{V_c}(1)$ is also contractively invariant.

Proof. The proof of invariance is as follows. Let $V_j(x) = x^{\rm T} P_j x$. Assuming there exists a $u_j \in \mathbf{R}^m, |u_j|_\infty \leq 1$ such that

$$D^\alpha(x_j^{\rm T} P_j x_j) \leq 2x_j^{\rm T} P_j (A x_j + BSat(u_j)) \leq 0. \tag{21}$$

As the fact that if $D^\alpha V_j(x) < 0$ for all $x \in \Psi(P_j)\backslash\{0\}$, and then ellipsoid $\Psi(P_j)$ is said to be invariant. From Lemma 2, we know,

$$D^\alpha(X_j^{\rm T} P_j x_j) \leq 2x_j^{\rm T} P_j D^\alpha(x_j),$$

and if the inequality

$$2x_j^{\rm T} P_j D^\alpha(x_j) \leq 0,$$

is specified, the ellipsoid $\Psi(P_j), j \in I[1, N]$ is contractively invariant. Let $r_0 = (P_0 x_0)^{\rm T}$, $x_0 \in \mathcal{L}_{V_c}(1)$, where $\mathcal{L}_{V_c}(1)$ is the edge of $L_{V_c}(1)$. Then we get $r_0 x_0 = x_0^{\rm T} P(\gamma^*(x_0)) x_0 = 1$, which means the hyperplane $r_0 x = 1$ is tangential to the convex set $L_{V_c}(1)$ at x_0. Then the level set $L_{V_c}(1)$ lies between $r_0 x = 1$ and $r_0 x = -1$, therefore

$$\Psi(P_j) \subset \Phi(r_0) \forall j \in I[1, N_0],$$

and

$$1 = r_0 x_0 \geq r_0 x_j.$$

In fact, the $r_0 x_j = 1$ is established for all $r_j \in I[1, N]$. It can be proved as follows: suppose that $r_0 x_j < 1$ for some j, such as $r_0 x_1 < 1$, then

$$\begin{aligned} 1 &= r_0 x_0 = \xi_1 r_0 x_1 + \sum_{j=2}^{N_0} \xi_j r_0 x_j \\ &\leq \xi_1 r_0 x_1 + \sum_{j=2}^{N_0} \xi_j \leq \sum_{j=1}^{N_0} \xi_j = 1, \end{aligned}$$

which is a contradiction. Similarly, the $r_0 x_j = 1$ implies that $\Psi(P_j)$ touches the hyperplane $r_0 x = 1$ at $x = x_j$. Then, the hyperplane $r_0 x = 1$ is tangential to $\Psi(P_j)$ at x_j for every $j \in I[1, N_0]$. One can obtain from Fact 1 that

$$r_0^{\rm T} = P_j x_j \quad \forall j \in I[1, N_0].$$

By inequality (21) and the assumption we know, there exists a $u_j \in \mathbf{R}^m, |u_j|_\infty < 1$ such that inequality (21) is satisfied. Let $u_0 = \sum_{j=1}^{N_0} \xi_j u_j$. Then, by $|u_0|_\infty \leq 1$ and by the convexity,

$$D^\alpha(x_0^{\rm T} P_0 x_0) \leq 2r_0(A x_0 + BSat(u_0)) \leq 0.$$

Due to the fact that x_0 is a random point in $L_{V_c}(1)$, hence, the level set $L_{V_c}(1)$ is contractively invariant. □

Theorem 3. Consider ellipsoid $\Psi(P_j)$ and feedback matrices $F_j \in \mathbf{R}^{m\times n}$. For the closed-loop fractional order system (2), if the ellipsoid type convex set X_R is considered, then the state feedback matrix $F_j \in \mathbf{R}^{m\times n}$ can be obtained as $F_j = Y_j Q_j^{-1}$, where Y_j, Q_j are solutions of the following optimization problem:

$$\inf_{Q_j > 0, Y_j, G_j} \Lambda, \tag{22}$$

$$\text{s.t. } (10), (19), \text{ and}$$

$$\begin{aligned} &Q_j A^{\rm T} + A Q_j + B E_i^{-1} G_j + G_j^{\rm T} (B E_i^{-1})^{\rm T} \\ &\quad + Y_j^{\rm T} E_i^{\rm T} B^{\rm T} + B E_i Y_j \leq 0, \text{ for } i \in [1, 2^m], j \in [1, N]. \end{aligned} \tag{23}$$

Let $\gamma^*(x)$ be such that $x^{\rm T} P(\gamma^*(x)) x = V_c(x)$, define,

$$Y(\gamma) = \sum_{j=1}^{N} \gamma_j Y_j, \quad Q(\gamma) = \sum_{j=1}^{N} \gamma_j Q_j, \tag{24}$$

then the fractional order linear closed-loop system is invariant under the feedback $u = Sat(F((\gamma^*(x)x))$, which is continuous while the vector function $\gamma^*(\cdot)$ is continuous.

Proof. Let $G_j = Q_j H_j$, $G(\gamma) = \sum_{j=1}^{N} \gamma_j G_j$ and $H(\gamma) = G(\gamma) Q^{-1}(\gamma)$, give

$$\begin{aligned} &Q(\gamma) A^{\rm T} + A Q(\gamma) + B E_i^{-1} G(\gamma) + G(\gamma)^{\rm T} (B E_i^{-1})^{\rm T} \\ &\quad + Y(\gamma)^{\rm T} E_i^{\rm T} B^{\rm T} + B E_i Y(\gamma) \leq 0, \end{aligned} \tag{25}$$

for all $i \in [1, 2^m]$ and $\gamma \in \Gamma$. The previous inequality can be formulated as follows:

$$\begin{aligned} &(A + B(E_i F(\gamma) + D_i^- H(\gamma)))^{\rm T} P(\gamma) \\ &\quad + P(\gamma)(A + B(E_i F(\gamma) + D_i^- H(\gamma))) \\ &\quad \leq 0, \forall i \in I[1, 2^m]. \end{aligned} \tag{26}$$

By Lemma 6, $\Psi(P(\gamma)) \subset \Phi(F(\gamma))$ and inequality (26) can make sure that the ellipsoid set $\Psi(P(\gamma))$ is invariant under the control of $u = sat(F(\gamma)x)$. Thus, by Theorem 2, one can get the level set $L_{V_c}(1)$ is invariant under the control of $u = Sat(F(\gamma^*(x)))x$. □

V. Numerical Examples

A. Example 1. Comparative Example

We use an example of [22] to illustrate the effectiveness of our method. For fractional order linear systems (2), let $\alpha = 0.7$, and

$$A = \begin{bmatrix} 0.7 & -1.4 \\ -0.2 & 1.5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The following reference polyhedrons is chosen as the prescribed convex set:

$$X_R = co\{x, -x\},$$

where $x_1 = [1 \quad 0]^{\mathrm{T}}, x_2 = [0 \quad 1]^{\mathrm{T}}$. Then, according the approach we proposed in [21], the following two ellipsoids can be obtained

$$P_1 = \begin{bmatrix} 4.7515 & -0.3347 \\ -0.3347 & 2.2345 \end{bmatrix}, P_2 = \begin{bmatrix} 0.2985 & -0.5538 \\ -0.5538 & 2.9894 \end{bmatrix}.$$

The boundaries of the two ellipsoids are plotted in red solid curves, while the dotted curves are the boundaries of $\Psi(P(\gamma))$ as γ varies in the set Γ. The shape of $L_{V_c}(1) = \bigcup_{\gamma\in\Gamma} \Psi(P(\gamma))$ can be obtained by those blue dotted curves in Fig. 2. To illustrate the effectiveness of the proposed method, the following closed-ball $B_\epsilon := \{x \in \mathbf{R}^n : x^{\mathrm{T}}x \leq 0.4326\}$, who is proposed in [22] is also demonstrated in Fig. 2.

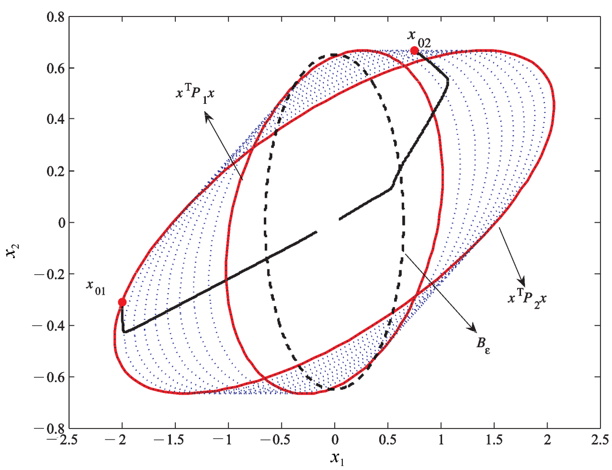

Fig. 2. Comparison between proposed method and existing method[22].

The convex combination of P_1 and P_2 in Fig. 2 shows that the proposed method provides a satisfactory estimation on the domain of attraction and also can obtain better estimation that the approach in [22].

Remark 5. From Fig. 2 we know, the closed-ball B_ϵ which proposed in [22] is almost contained in the convex hull of the proposed method. Two state trajectories started from initial points $x_{01} = [-2 \ -0.3127], x_{02} = [0.750.6665]$ on the boundary of $L_{V_c(1)}$ shows that, the trajectories are convergent to origin , and the proposed method provides better estimation on the domain of attraction than method in [22]. This method extend the domain of attraction. The control signals $u_1(t)$, $u_2(t)$ and the Lyapunov function $V_c(x(t))$ of example 1 are illustrated in Fig. 3. Comparative example indicates that the proposed method is less conservative than [22].

B. Example 2. Continuous State Feedback Control Law Synthesis

In this paper, we consider the fractional order system with

$$\alpha = 0.9, A = \begin{bmatrix} 1.6 & -0.5 \\ -0.6 & 0.6 \end{bmatrix}, B = \begin{bmatrix} 5 \\ -5 \end{bmatrix}.$$

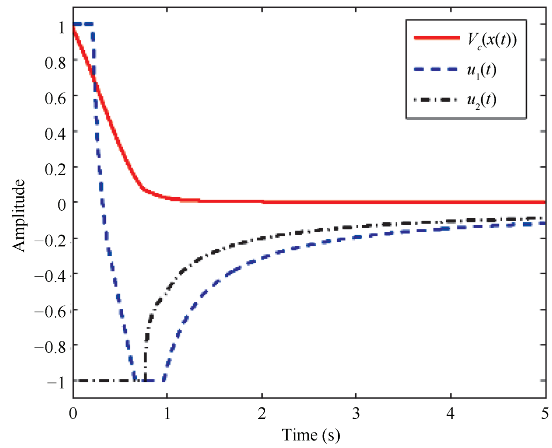

Fig. 3. Signals $u_1(t)$, $u_2(t)$ and the Lyapunov function $V_c(x(t))$.

For this example, we choose the following reference polyhedrons as the prescribed convex set:

$$X_R = co\{x_i, -x_i\}, \ i = 1, 2,$$

where $x_1 = [1 \quad 0]^{\mathrm{T}}, x_2 = [0 \quad 1]^{\mathrm{T}}$, then respectively, the following two feedback matrices can be designed with the approach proposed in [21],

$$F_1 = [-0.0414 \quad -0.2897], \quad F_2 = [0.2161 \quad -0.3540],$$

along with two ellipsoids $\Psi(P_1)$ and $\Psi(P_2)$, where,

$$P_1 = \begin{bmatrix} 0.7985 & 0.1881 \\ 0.1881 & 0.1270 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.6258 & 0.1881 \\ 0.1881 & 0.0443 \end{bmatrix}.$$

The matrix P_1 and F_1 are designed such that the closed-loop system (2) is locally Mittag-Leffler stable and the estimated ellipsoid invariant set is maximized with respect to the bounded convex set X_R. Similarly, the matrix P_2 and F_2 are designed such that $\lambda X_R \subset \Psi(P)$. The boundaries of the two ellipsoids are plotted in red solid curves, while the dotted curves are the boundaries of $\Psi(P(\gamma))$ as γ varies in the set Γ. The shape of $L_{V_c}(1) = \bigcup_{\gamma\in\Gamma} \Psi(P(\gamma))$ can be obtained by those blue dotted curves in Fig. 4. The $\gamma^*(x)$ can be obtained by computing the optimization problem (4) or the formula in (6).

From Theorem 2. we know,

$$F(\gamma^*) = (\gamma^* Y_1 + (1-\gamma^*)Y_2)(\gamma^* Q_1 + (1-\gamma^*)Q_2)^{-1},$$

$$Y_1 = \begin{bmatrix} -0.9000 & 1.7921 \end{bmatrix}, Y_2 = \begin{bmatrix} -0.9000 & 5.9397 \end{bmatrix},$$

and

$$Q_1 = \begin{bmatrix} 1.9228 & -2.8469 \\ -2.8469 & 12.0873 \end{bmatrix}, Q_2 = \begin{bmatrix} 1.2087 & -5.1320 \\ -5.1320 & 44.3652 \end{bmatrix}.$$

Fig. 5 shows that the state trajectories with different initial conditions. Obviously, the closed-loop system (2) is asymptotically Mittag-Leffler stable. To obtain the control signal and the composite quadratic function, the state trajectory from initial point $x_{01} = [1.2300 \ -3.9533]$ and $x_{02} = [-1.200 \ 4.2699]$

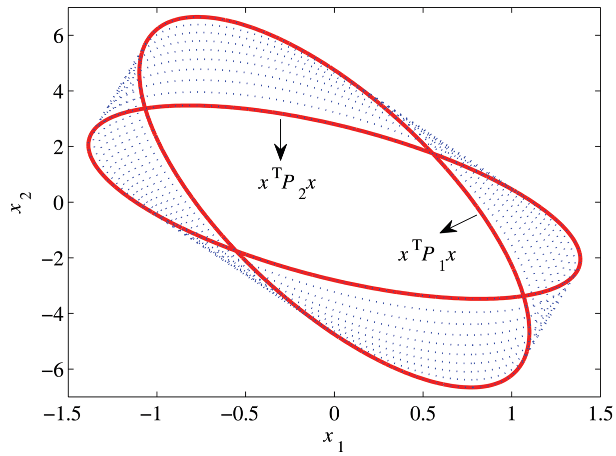

Fig. 4. Convex hull of two provided ellipsoids.

started from the boundary of level set $L_{V_c}(1)$ is illustrated in Fig. 5. The control signal $u(t)$ and the Lyapunov function $V_c(x(t))$ of trajectory started from x_{01} are demonstrated in Fig. 6, and the value of $\gamma^*(x)$ is plotted in Fig. 7 as $x(t)$ varies along the time.

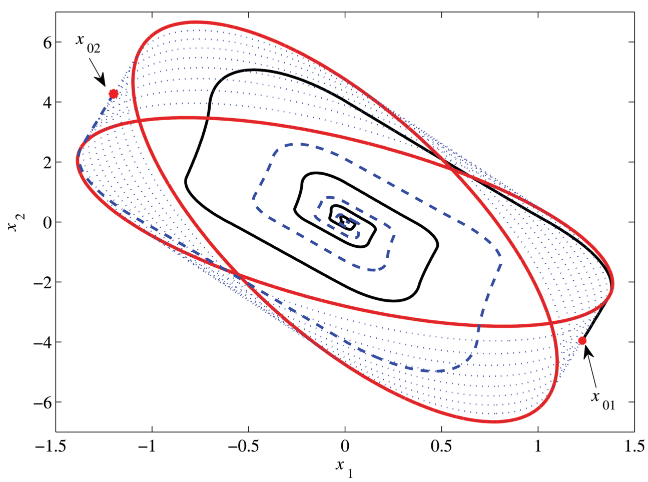

Fig. 5. State trajectory with different initial conditions.

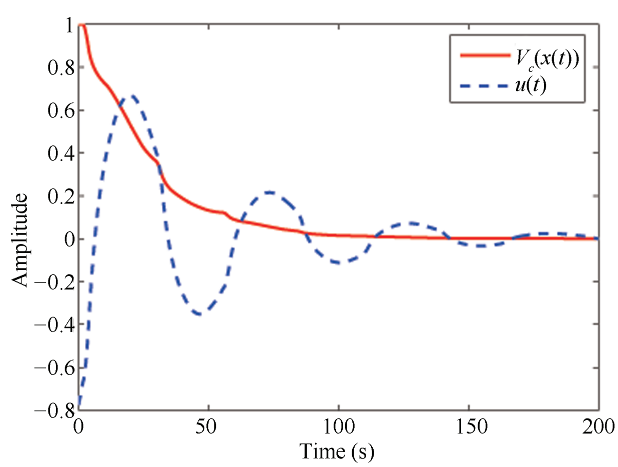

Fig. 6. Control signal $u(t)$ and the Lyapunov function $V_c(x(t))$.

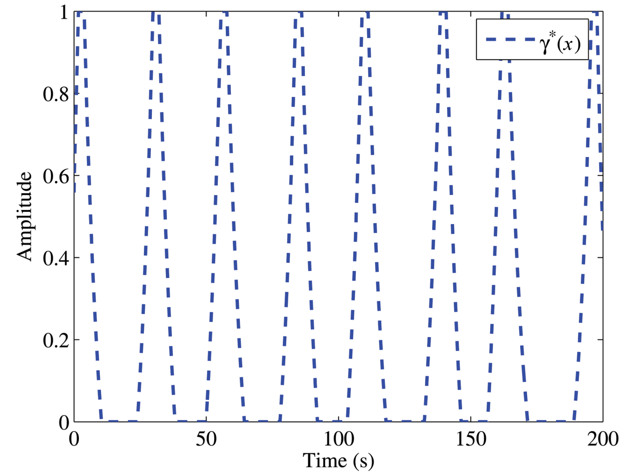

Fig. 7. $\gamma^*(x)$ varies in $[0, 1]$.

VI. Conclusion

This paper provides a method for the estimation of the domain of attraction and state feedback synthesis utiize the convex combination form for fractional order systems subject to actuator saturation with the fractional order $0 < \alpha < 1$. The ellipsoidal invariant set of fractional order systems subject to actuator saturation is investigated by convex hull form for the first time. Then we demonstrate that the convex hull of ellipsoids can be made invariant for fractional order linear systems subject to actuator saturation if each ellipsoid in a set with a bounded control of the saturating actuators is invariant. The results show that the proposed method is effective to handle saturation nonlinearity. The composite quadratic Lyapunov function and Lyapunov direct approach are applied in this paper to estimate the invariant ellipsoids for fractional order systems. In particular, by using a set of feedback laws to make the convex hull of a set of ellipsoid invariant, one proper method is proposed to construct a continuous feedback law.

In order to facilitate analysis, the case $N = 2$ is chosen in this paper. A more effective way needs to be specified to compute the condition $N > 2$. Nevertheless, the stability analysis problem for fractional order system subject to actuator saturation with the fractional order $1 < \alpha < 2$ is still unsolved. The stability conditions for fractional order systems is concerned with the fractional order, thus our future work is related to the problem of order-dependent estimation on the domain of attraction.

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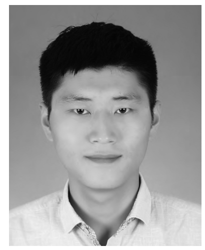

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