Stability analysis of a class of nonlinear fractional differential systems with Riemann-Liouville derivative

Ruoxun Zhang, Shiping Yang, Shiwen Feng

***Abstract*—This paper investigates the stability of n-dimensional nonlinear fractional differential systems with Riemann-Liouville derivative. By using the Mittag-Leffler function, Laplace transform and the Gronwall-Bellman lemma, one sufficient condition is attained for the asymptotical stability of a class of nonlinear fractional differential systems whose order lies in (0, 2). According to this theory, if the nonlinear term satisfies some conditions, then the stability condition for nonlinear fractional differential systems is the same as the ones for corresponding linear systems. Several examples are provided to illustrate the applications of our result.**

***Index Terms*—Stability, Nonlinear fractional differential system, Riemann-Liouville derivative**

I. Introduction

In this paper, we consider the stability of n-dimensional nonlinear fractional differential systems with Riemann-Liouville derivative:

$$ {}_0^{RL}D_t^{\alpha}x(t) = Ax(t) + f(x(t)) \tag{1} $$

where $0 < \alpha < 2$, $x(t) \in \mathbf{R}^{n\times 1}$is the state vector, ${}_0^{RL}D_t^{\alpha}x(t)$denotes Riemann-Liouville's fractional derivative with the lower limit 0 for the function $x(t)$, $A \in \mathbf{R}^{n\times n}$is the constant parameter matrix and$f(x(t)) \in \mathbf{R}^{n\times 1}$is a nonlinear function vector.

In the last 30 years, fractional calculus has attracted attention of many mathematicians, physicists and engineers. Significant contributions have been made to both the theory and applications of fractional differential equations (see [1] and references there in). Also, fractional differential equations have recently been proved to be valuable tools in modeling of many physical phenomena in various ?elds of science and engineering.

Recently, the stability of fractional differential systems has attracted increasing interest due to its importance in control theory. In 1996, Matignon [2] firstly studied the stability of linear fractional differential systems. Since then, many researchers have studied further on the stability of linear fractional differential systems $[3-5]$. The stability analysis of nonlinear fractional differential systems is much more difficult and only a few available. For example, Li et al. investigated the Mittag-Leffler stability of fractional order nonlinear dynamic systems [6] and proposed Lyapunov direct method to check stability of fractional order nonlinear dynamic systems [7]. Wen et al. [8] and Zhou et al [9] considered the stability of nonlinear fractional differential systems. In [10], Zhang et al proposed a single state adaptive-feedback controller for stabilization of three-dimensional fractional-order chaotic systems. Based on the theory of Linear Matrix Inequality (LMI), Faieghi et al [11] proposed a simple controller for stabilization of a class of fractional-order chaotic systems. Wang et al. present the Ulam -Hyers stability for fractional Langevin equations [12], and Ulam- Hyers-Mittag-Lef?er stability for fractional delay differential equations [13]. The methods which they proposed for stability of a class of fractional differential equations provide us with a very useful method for studying Hyers–Ulam stable system. That is, one does not have to reach the exact solution. What is required is to get a function which satis?es a suitable approximation inequality.

Note that these papers on the stability of the fractional differential systems mainly concentrated on fractional orderαlying in (0, 1). Recently, in Ref [14], Zhang et al considered the stability of nonlinear fractional differential systems with Caputo derivative whose order lies in (0, 2). In this paper, we study the stability of the nonlinear fractional differential systems with Riemann-Liouville derivative whose order lies in (0, 2). By using the Mittag-Leffler function, Laplace transform and the Gronwall-Bellman lemma, a stability theorem is proven theoretically. The stability conditions have no restriction on the norm of the linear parameter matrix A. The paper is outlined as follows. In section II, some definitions and lemmas are introduced. In section III, the stability of a class of nonlinear fractional differential systems with commensurate order $0 < \alpha < 2$ is investigated. The simulation and conclusions are included in section IV and V, respectively.

II. PRELIMINARIES

Definition 2.1 [15]**.** The Riemann-Liouville derivative with $\alpha$of function $x(t)$ is defined as follows

$$ {}_{t_0}^{RL}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{x(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \quad (n-1 \le \alpha < n) \tag{2} $$

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The Laplace transform of the Riemann-Liouville fractional derivative ${}_{t_0}^{RL}D_t^{\alpha}x(t)$ is

$$\int_0^{\infty} e^{-st}{}_{t_0}^{RL}D_t^{\alpha}x(t)\mathrm{d}t = s^{\alpha}X(t) - \sum_{k=0}^{n-1} s^k[D^{\alpha-k-1}x(t)]_{t=t_0} \tag{3}$$

Definition 2.2 [15]**.** The two-parameter Mittag- Leffler function is defined as

$$E_{\alpha,\,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\,,\ (Re\alpha > 0,\ \beta \in C, z \in C). \tag{4}$$

The Laplace transform of Mittag-Leffler function can be found to be

$$\int_0^{\infty} e^{-st}t^{\alpha k-\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha})\mathrm{d}t = \frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}},\ (R(s) > |a|^{\frac{1}{n}}). \tag{5}$$

Definition 2.3 [16]**.** By analogy with *Definition* 2.2, for $A \in C^{n\times n}$, a matrix Mittag-Leffler function is defined as: $E_{\alpha,\beta}(A) = \sum\limits_{k=0}^{\infty}\frac{A^k}{\Gamma(\alpha k+\beta)}, \quad (\beta \in C, R(\alpha) > 0)$

Lemma 2.1 [8]**.** If $A \in C^{n\times n},\ 0 < \alpha < 2,\ \beta$ is an arbitrary real number, μ is such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$ and $C_1$is a real constant, then

$$||E_{\alpha,\beta}(A)|| \le \frac{C_1}{1+||A||}, \tag{6}$$

where $\mu \le \arg(\lambda(A)) \le \pi,\ \lambda(A)$ denotes the eigenvalues of matrix A and $||\cdot||$denotes the l_2-norm.

Lemma 2.2 [17]**.** (Gronwall-Bellman lemma) If

$$\varphi(t) \le h(t) + \int_{t_0}^{t} g(\tau)\varphi(\tau)\mathrm{d}\tau, \quad t_0 \le t \le t_1. \tag{7}$$

where $g(t),\ h(t)$ and $\varphi(t)$ are continuous on $[t_0,\,t_1]$, $t_1 \to \infty$, $t_0 \le t \le t_1$ and $g(t) \ge 0$. Then $\varphi(t)$ satisfies

$$\phi(t) \le h(t) + \int_{t_0}^{t} h(\tau)g(\tau)\exp[\int_{\tau}^{t} g(s)\mathrm{d}s]\mathrm{d}\tau. \tag{8}$$

In addition, if $h(t)$ is nondecreasing, then

$$\phi(t) \le h(t)\exp[\int_{t_0}^{t} g(s)\mathrm{d}s]\mathrm{d}\tau. \tag{9}$$

III. Stability Theory of N-Dimensional Nonlinear Fractional Differential Systems

In this section, based on the definition and lemma in section 2, we present the stability theorem for a class of nonlinear fractional differential systems such as system (1).

Theorem 1. Consider the system (1). Let $\lambda_i(A)$ $(i = 1, 2, \cdots, n)$be the eigenvalues of matrix A. If

1) $|\arg(\lambda_i(A))| > \alpha\pi/2$;
2) The nonlinear function $f(x(t))$satisfies

$$\lim_{||x(t)||\to 0}\frac{||f(x(t))||}{||x(t)||} = 0. \tag{10}$$

Then the zero solution of (1) is locally asymptotically stable.

Proof. a) The case $0 < \alpha < 1$

In this case, the initial condition is

$${}_{0}^{RL}D_t^{\alpha-1}x(t)|_{t=0} = x_0 \tag{11}$$

Taking Laplace transform of (1), we have

$$X(s) = (Is^{\alpha} - A)^{-1}(x_0 + L[f(x(t))]) \tag{12}$$

where I is an $n\times n$ identity matrix.

Then taking inverse Laplace transform for (12), it yields

$$x(t) = x_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f(x(\tau))\mathrm{d}\tau \tag{13}$$

By the condition (10), there exist $C_0 > 0$ and $\delta > 0$, such that

$$||f(x(t))|| < \frac{\alpha||A||}{2C_0}||x(t)||\mathrm{as}||x(t)|| < \delta \tag{14}$$

From (14) and Lemma 2.1 , (13) gives

$$\begin{aligned}||x(t)|| \le{}& \frac{C_0||x_0||t^{\alpha-1}}{1+||At^{\alpha}||} + \int_0^t \frac{||(t-\tau)^{\alpha-1}||C_0}{1+||A(t-\tau)^{\alpha}||}\frac{\alpha||A||}{2C_0}||x(\tau)||\mathrm{d}\tau = \\ & \frac{C_0||x_0||t^{\alpha-1}}{1+||A||t^{\alpha}} + \int_0^t \frac{\alpha||A||(t-\tau)^{\alpha-1}}{2(1+||A||(t-\tau)^{\alpha})}||x(\tau)||\mathrm{d}\tau\end{aligned}$$

According to Lemma 2.2, we obtain

$$\begin{aligned}||x(t)|| \le{}& \frac{C_0||x_0||t^{\alpha-1}}{1+||A||t^{\alpha}} + \int_0^t \frac{C_0||x_0||\tau^{\alpha-1}}{1+||A||\tau^{\alpha}} \\ & \times \frac{\alpha||A||(t-\tau)^{\alpha-1}}{2(1+||A||(t-\tau)^{\alpha})} \\ & \exp\Big(\int_{\tau}^{t}\frac{\alpha||A||(t-s)^{\alpha-1}}{2(1+||A||(t-s)^{\alpha})}\mathrm{d}s\Big)\mathrm{d}\tau \\ & = \frac{C_0||x_0||}{t^{1-\alpha}+||A||t} \\ & + \int_0^t \frac{\alpha C_0||x_0||\tau^{\alpha-1}||A||(t-\tau)^{\alpha-1}}{2(1+||A||\tau^{\alpha})(1+||A||(t-\tau)^{\alpha})^{0.5}}\mathrm{d}\tau \\ & \le \frac{C_0||x_0||}{t^{1-\alpha}+||A||t} \\ & + \int_0^t \frac{\alpha C_0||x_0||\tau^{\alpha-1}||A||(t-\tau)^{\alpha-1}}{2(1+||A||\tau^{\alpha})^{0.5}(1+||A||(t-\tau)^{\alpha})^{0.5}}\mathrm{d}\tau \\ & \le \frac{C_0||x_0||}{t^{1-\alpha}+||A||t} \\ & + 0.5\alpha C_0||x_0||\int_0^t \tau^{0.5\alpha-1}(t-\tau)^{0.5\alpha-1}\mathrm{d}\tau \\ & \le \frac{C_1||x_0||}{t^{1-\alpha}+||A||t} \\ & + 0.5\alpha C_0||x_0||\frac{\Gamma(0.5\alpha)\Gamma(0.5\alpha)}{\Gamma(\alpha)}t^{\alpha-1} \to 0 \\ & \text{as } t \to \infty.\end{aligned}$$

So, the zero solution of (1) is locally asymptotically stable.

2) The case $1<\alpha<2$

In this case, the initial condition is

$$ {}_0^{RL}D_t^{\alpha-k}x(t)|_{t=0}=x_{k-1},\ (k=1,\ 2) \tag{15}$$

We can get the solution of (1) with the initial condition (15) by using the Laplace transform and Laplace inverse transform:

$$\begin{aligned} x(t)&=x_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})+t^{\alpha-2}x_1E_{\alpha,\alpha-1}(At^{\alpha})\\ &+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f(x(\tau))\mathrm{d}\tau \end{aligned}\tag{16}$$

By the condition (10), there exist $C_0>0$ and $\delta>0$, such that

$$||f(x(t))||<\frac{(\alpha-1)||A||}{2C_0}||x(t)||\text{as}||x(t)||<\delta. \tag{17}$$

From (17) and Lemma 2.1 , (16) gives

$$\begin{aligned} ||x(t)||&\le\frac{C_0||x_0||t^{\alpha-1}}{1+||At^{\alpha}||}+\frac{C_1||x_1||t^{\alpha-2}}{1+||At^{\alpha}||}\\ &+\int_0^t \frac{||(t-\tau)^{\alpha-1}||C_0}{1+||A(t-\tau)^{\alpha}||}\frac{(\alpha-1)||A||}{2C_0}||x(\tau)||\mathrm{d}\tau\\ &=\frac{C_0||x_0||t^{\alpha-1}}{1+||A||t^{\alpha}}+\frac{C_1||x_1||t^{\alpha-2}}{1+||A||t^{\alpha}}\\ &+\int_0^t \frac{||(t-\tau)^{\alpha-1}||}{1+||A||(t-\tau)^{\alpha}}\frac{(\alpha-1)||A||}{2}||x(\tau)||\mathrm{d}\tau \end{aligned}\tag{18}$$

where $C_1\quad>0$. According to Lemma 2.2, we obtain

$$\begin{aligned} ||x(t)|&\le\frac{C_0||x_0||\tau^{\alpha-1}}{1+||A||\tau^{\alpha}}+\\ &\frac{C_1||x_1||\tau^{\alpha-2}}{1+||A||\tau^{\alpha}}+\int_0^t \Big(\frac{C_0||x_0||\tau^{\alpha-1}}{1+||A||\tau^{\alpha}}\\ &+\frac{C_1||x_1||\tau^{\alpha-2}}{1+||A||\tau^{\alpha}}\Big)\frac{(\alpha-1)||A||(t-\tau)^{\alpha-1}}{2(1+||A||(t-\tau)^{\alpha})}\\ &\times\exp\Big(\int_\tau^t\frac{(\alpha-1)||A||(t-s)^{\alpha-1}}{2(1+||A||(t-s)^{\alpha})}\mathrm{d}s\Big)\mathrm{d}\tau\\ &=\frac{C_0||x_0||t^{\alpha-1}}{1+||A||t^{\alpha}}+\frac{C_1||x_1||t^{\alpha-2}}{1+||A||t^{\alpha}}\\ &+\int_0^t \Big(\frac{C_0||x_0||t^{\alpha-1}}{1+||A||t^{\alpha}}+\frac{C_1||x_1||t^{\alpha-2}}{1+||A||t^{\alpha}}\Big)\\ &\times\frac{(\alpha-1)||A||(t-\tau)^{\alpha-1}}{2(1+||A||(t-\tau)^{\alpha})^{1-\frac{\alpha-1}{2\alpha}}}\mathrm{d}\tau\\ &\le\frac{C_0||x_0||t^{\alpha-1}}{1+||A||t^{\alpha}}+\frac{C_1||x_1||t^{\alpha-2}}{1+||A||t^{\alpha}}\\ &+\int_0^t \frac{C_0||x_0||\tau^{\alpha-1}}{(1+||A||\tau^{\alpha})^{0.75}}\\ &\times\frac{(\alpha-1)||A||(t-\tau)^{\alpha-1}}{2(1+||A||(t-\tau)^{\alpha})^{1-\frac{\alpha-1}{2\alpha}}}\mathrm{d}\tau\\ &+\int_0^t \frac{C_1||x_1||\tau^{\alpha-2}}{(1+||A||\tau^{\alpha})^{\frac{\alpha-1}{2\alpha}}}\\ &\times\frac{(\alpha-1)||A||(t-\tau)^{\alpha-1}}{2(1+||A||(t-\tau)^{\alpha})^{1-\frac{\alpha-1}{2\alpha}}}\mathrm{d}\tau\\ &\le\frac{C_0||x_0||t^{\alpha-1}}{1+||A||t^{\alpha}}+\frac{C_1||x_1||t^{\alpha-2}}{1+||A||t^{\alpha}}\\ &+\frac{(\alpha-1)C_0||x_0||}{2||A||^{0.25+1/(2\alpha)}}\int_0^t \tau^{0.25\alpha-1}(t-\tau)^{0.5\alpha-1.5}\mathrm{d}\tau\\ &+\frac{(\alpha-1)C_1||x_1||}{2}\int_0^t \tau^{0.5\alpha-1.5}(t-\tau)^{0.5\alpha-1.5}\mathrm{d}\tau\\ &\le\frac{C_0||x_0||}{||A||t}+\frac{C_1||x_1||}{||A||t^2}\\ &+\frac{(\alpha-1)C_0||x_0||}{2||A||^{0.25+1/(2\alpha)}}\frac{\Gamma(0.25\alpha)\Gamma(0.5\alpha-0.5)}{\Gamma(0.75\alpha-0.5)t^{0.75(2-\alpha)}}\\ &+\frac{(\alpha-1)C_1||x_1||}{2}\frac{\Gamma(0.5\alpha)\Gamma(0.5\alpha-0.5)}{\Gamma(\alpha-0.5)t^{2-\alpha}}\\ &\to 0\ as\ t\to\infty. \end{aligned}\tag{19}$$

So, the zero solution of (1) is locally asymptotically stable.

Remark 1. The nonlinear term of many fractional order chaotic systems satisfy (12). For example, fractional-order Lorenz system [17], fractional-order Chen system [18], fractional-order Lü system [19], fractional-order Liu system [20], fractional-order Arneodo system [21], fractional-order Chua system [22] and fractional-order hyperchaotic Chen system [23] etc. So, Theorem 1 can be applicable to control chaos in a large class of generalized fractional-order chaotic or hyperchaotic systems via a linear feedback controller. See Example 3 in Section 4.

Remark 2. Theorem 1 provides us with a simple procedure for determining the stability of the fractional order nonlinear systems with Caputo derivative with order $0<\alpha<2$. If the nonlinear term $f(x(t))$satisfies Eq.(10), then one does not have to reach the exact solution. What is required is to calculate the eigenvalues of the matrix A, and test their arguments. If $|\arg(\lambda_i(A))|>\alpha\pi/2$for all i, we conclude that the origin is asymptotically stable.

IV. Three Illustrative Examples

The following illustrative examples are provided to show the effectiveness of the stability theorem. When numerically solving fractional differential equations, we adopt the method introduced in [24].

Example 1. Consider the nonlinear fractional differential systems

$$\begin{aligned} {}_0^{RL}D_t^{\alpha}x_1 &= x_1 + x_2 + x_3 + x_2x_3 \\ {}_0^{RL}D_t^{\alpha}x_2 &= -x_1 + x_2 - x_3 + x_2^2 \\ {}_0^{RL}D_t^{\alpha}x_3 &= x_1x_2 - x_3 \end{aligned} \tag{20}$$

System (20) can be rewritten as (1), in which

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad f(x(t)) = \begin{pmatrix} x_2x_3 \\ x_2^2 \\ x_1x_2 \end{pmatrix} \tag{21}$$

Obviously, it is easy to verify that

$$\begin{aligned} \lim_{||x(t)||\to 0} \frac{||f(x(t)||}{||x(t)||} &= \lim_{||x(t)||\to 0} \frac{\sqrt{(x_2x_3)^2 + x_2^4 + (x_1x_2)^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ &\le \lim_{||x(t)||\to 0} \frac{\sqrt{(x_2x_3)^2 + x_2^4 + (x_1x_2)^2}}{\sqrt{x_2^2}} \\ &\le \lim_{||x(t)||\to 0} \sqrt{x_3^2 + x_2^2 + x_1^2} = 0, \end{aligned}$$

which implies that $f(x(t)$ satisfies Condition (2) in Theorem 1. By using simple calculation, the eigenvalues

of A are $\lambda_{1,2} = 1 \pm i$ and $\lambda_3 = -1$. According to Theorem 1, if $\alpha < 0.5$, the zero solution of (20) is asymptotically stable. Simulation results are displayed in Figs. 1–3. Fig. 1 and Fig. 2 show the zero solution of system (20) is asymptotically stable with $\alpha = 0.4$ and $\alpha = 0.49$, respectively. Fig. 3 shows the zero solution of the system (20) is unstable with $\alpha = 0.5$.

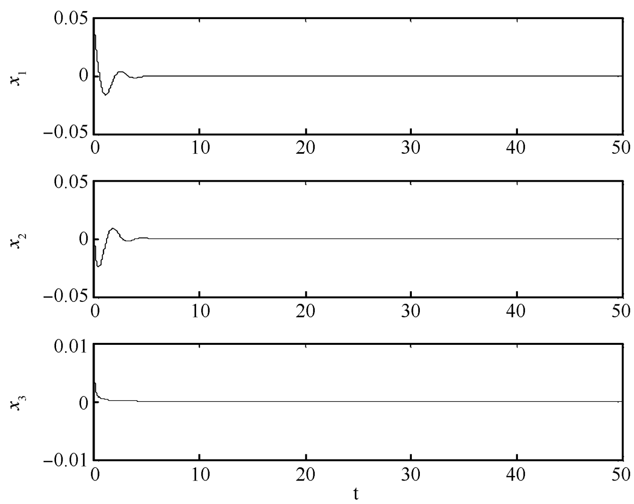

Fig. 1. System (20) is asymptotically stable with $\alpha = 0.4$

Example 2. Consider the nonlinear fractional differential systems

$$\begin{aligned} {}_0^{RL}D_t^{\alpha}x_1 &= -x_1 + x_2x_3 \\ {}_0^{RL}D_t^{\alpha}x_2 &= x_3 \\ {}_0^{RL}D_t^{\alpha}x_3 &= x_1 - x_2 - x_3 - x_1x_2 \end{aligned} \tag{22}$$

System (22) can be rewritten as (1), in which

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad f(x(t)) = \begin{pmatrix} x_2x_3 \\ 0 \\ -x_1x_2 \end{pmatrix} \tag{23}$$

Obviously, it is easy to verify that $\lim_{||x(t)||\to 0} \frac{||f(x(t)||}{||x(t)||} = 0$, which implies that $f(x(t)$ satisfies Condition (2) in Theorem 1. By using simple calculation, the eigenvalues of A are $\lambda_{1,2} = -1/2 \pm \sqrt{3}i/2$ and $\lambda_3 = -1$. According to Theorem 1, if $\alpha < 4/3$, the zero solution of (22) is asymptotically stable. Simulation results are displayed in Figs. 4–7. Figs. 4–6 show the zero solution of the system (22) is asymptotically stable with $\alpha = 1.1\alpha = 1.3$ and $\alpha = 1.33$, respectively. Fig. 7 shows the zero solution of the system (22) is not stable with $\alpha = 1.34$.

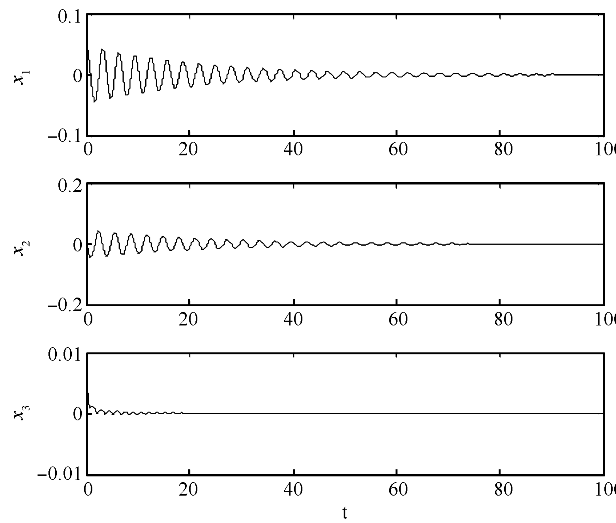

Fig. 2. System (20) is asymptotically stable with $\alpha = 0.49$

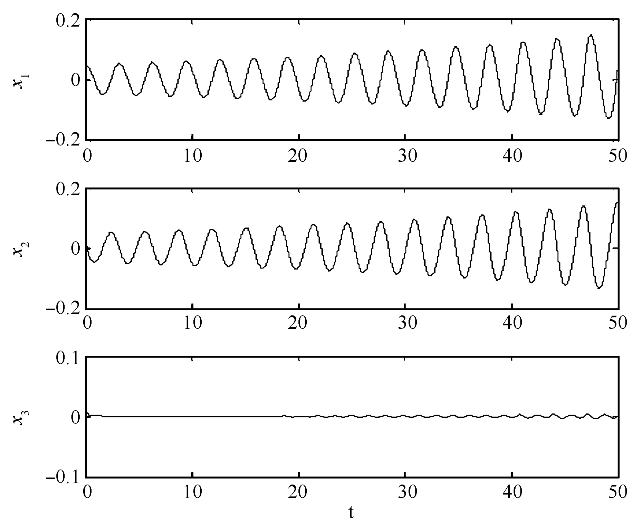

Fig. 3. System (20) is unstable with $\alpha = 0.50$

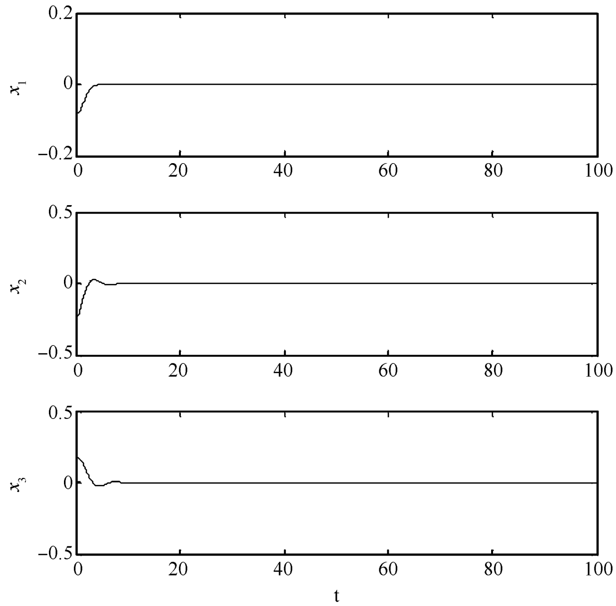

Fig. 4. System (22) is asymptotically stable with $\alpha = 1.1$

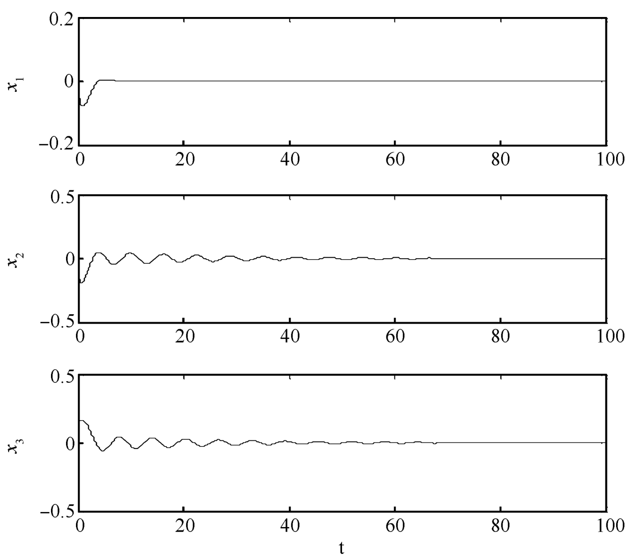

Fig. 5. System (22) is asymptotically stable with $\alpha = 1.30$

Example 3. The fractional-order hyperchaotic Chen system can be written as

$$\begin{aligned}
{}_0^{RL}D_t^{\alpha}x_1 &= a(x_2 - x_1) + x_4\\
{}_0^{RL}D_t^{\alpha}x_2 &= dx_1 + cx_2 - x_1x_3\\
{}_0^{RL}D_t^{\alpha}x_3 &= x_1x_2 - bx_3\\
{}_0^{RL}D_t^{\alpha}x_4 &= x_2x_3 + rx_4
\end{aligned} \tag{24}$$

where a, b, c,d and r are five parameters. When $a = 35,\ b = 3,\ c = 12,\ d = 7, r = 0.3$ and $\alpha = 1.1$ system (24) displays a chaotic attractor, as shown in Fig. 8.

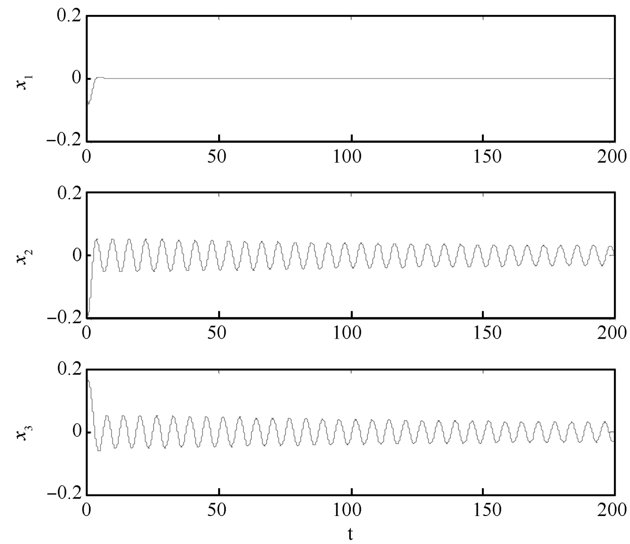

Fig. 6. System (22) is asymptotically stable with $\alpha = 1.33$

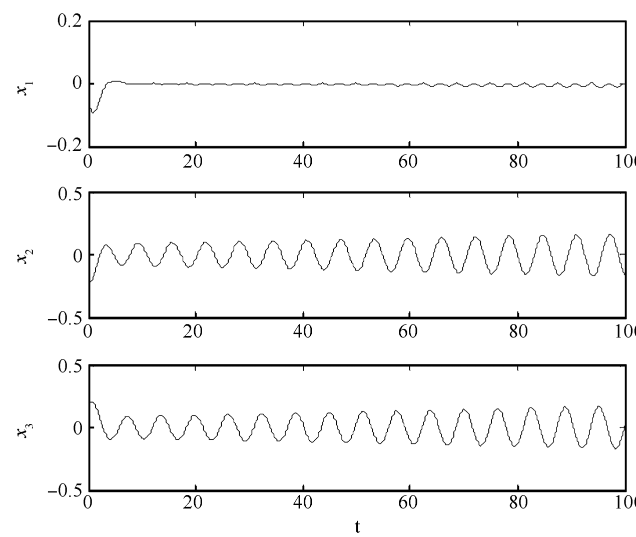

Fig. 7. System (22) is not stable with $\alpha = 1.34$

System (24) can be rewritten as a controlled system:

$$\begin{aligned}
{}_0^{RL}D_t^{\alpha}x_1 &= a(x_2 - x_1) + x_4\\
{}_0^{RL}D_t^{\alpha}x_2 &= dx_1 + cx_2 - x_1x_3 + u_1\\
{}_0^{RL}D_t^{\alpha}x_3 &= x_1x_2 - bx_3\\
{}_0^{RL}D_t^{\alpha}x_4 &= x_2x_3 + rx_4 + u_2
\end{aligned}$$

When $\alpha = 1.1,\ a = 35,\ \ b = 3,\ c = 12,\ d = 7, r = 0.3$, we select the linear state feedback controller $u_1 = -22x_2$, $u_2 = -x_4$. Then, the two conditions of Theorem 1 are satisfied well. It concludes that the zero solution of the controlled system is asymptotically stable. The results of simulation are shown in Fig. 9, while the feedback is activated at time$t$ =10 s.

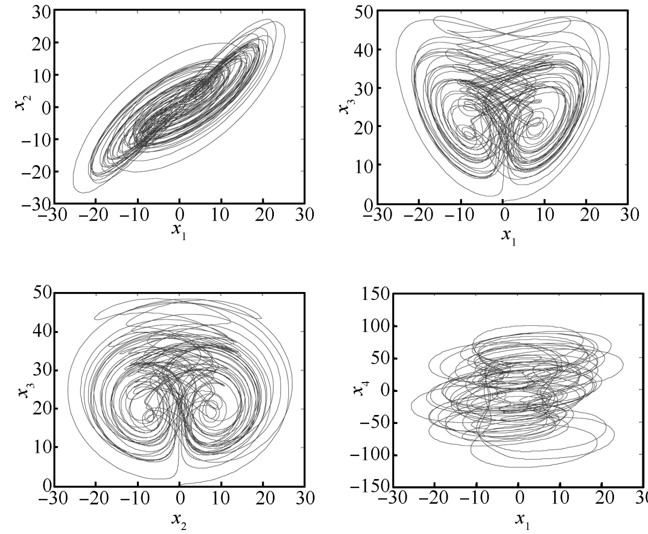

Fig. 8. Attractor of fractional order hyperchaotic Chen system with order $\alpha = 1.1$ ($a = 35,\ b = 3,\ c = 12,\ d = 7, r = 0.3$)

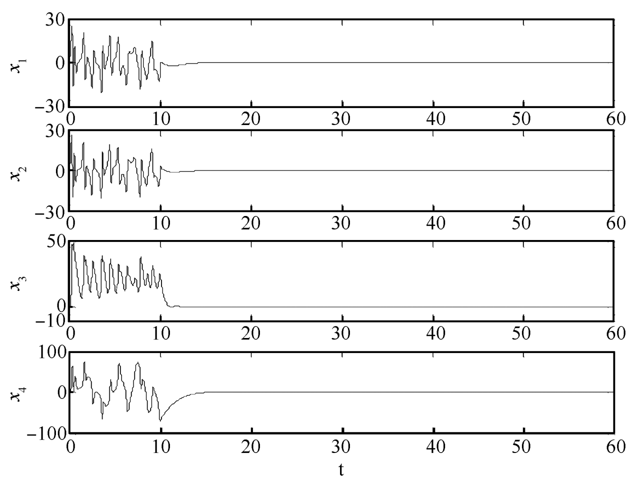

Fig. 9. Asymptotical stabilization of fractional order hyperchaotic Chen system with order $\alpha = 1.1$

V. Conclusions

In this paper, we have studied the local asymptotic stability of the zero solution of n-dimensional nonlinear fractional differential systems with with Riemann-Liouville derivative. The results are obtained in terms of the Mittag-Leffler function, Laplace transform and the Gronwall-Bellman lemma. Compare of the current results with the results in Ref.[14] shows that the stability condition of Riemann-Liouville fractional differential system is same as that for Caputo fractional differential systems. Three numerical examples are given to demonstrate the effectiveness of the proposed approach.

VI. Acknowledgement

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