Robust Finite-time Synchronization of Non-Identical Fractional-order Hyperchaotic Systems and its Application in Secure Communication

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***Abstract*—This paper proposes a novel adaptive sliding mode control (SMC) method for synchronization of non-identical fractional-order (FO) chaotic and hyper-chaotic systems. Under the existence of system uncertainties and external disturbances, finite-time synchronization between two FO chaotic and hyper-chaotic systems is achieved by introducing a novel adaptive sliding mode controller (ASMC). Here in this paper, a fractional sliding surface is proposed. A stability criterion for FO nonlinear dynamic systems is introduced. Sufficient conditions to guarantee stable synchronization are given in the sense of the Lyapunov stability theorem. To tackle the uncertainties and external disturbances, appropriate adaptation laws are introduced. Particle Swarm Optimization (PSO) is used for estimating the controller parameters. Finally, finite-time synchronization of the FO chaotic and hyper-chaotic systems is applied to secure communication.**

***Index Terms*—Adaptive sliding mode control, chaos synchronization, fractional order, hyper-chaotic system, Lyapunov theorem, secure communication**

I. INTRODUCTION

CHAOTIC behavior is a prevalent phenomenon appearing in nonlinear systems. Chaotic systems have received more attention in the literature during the last three decades. A chaotic system is a nonlinear deterministic system that has complex and unpredictable behavior.

Fractional calculus is a mathematical topic more than three centuries old, but its application to physics and engineering fields have attracted more attention only in recent years[1−3]. This happens because it has been recently found that several physical phenomena can be more adequately described by fractional differential equations rather than integer-order models[4], and it has been found that many FO systems can show complex dynamical behavior such as chaos. The advantages of the FO systems are that there are more degrees of freedom in the model. Also memory is included in FO systems. Many systems in interdisciplinary fields, such as viscoelastic materials[5] and micro-electromechanical systems[6] can be described using fractional calculus methods.

Recently many researchers have recognized that many complex systems, such as FO Lorenz system[7], FO Chen system[8] and FO Arnodo-Coullet system[9], can be described using fractional integrals and derivatives.

Since Pecora and Carroll[10] established a chaos synchronization scheme for two identical chaotic systems with different initial conditions, chaos synchronization has attracted a great attention. The chaotic synchronization occurs whenever the state trajectories of the slave system track the state trajectories of the master system in a given finite-time[11,12]. Chaos synchronization is a contemporary topic in nonlinear science because of its broad and considerable applications in secure communication, automatic control, neural networks and etc.[13−15].

Due to the existence of chaos in real practical systems and many applications in physics and engineering fields, control and synchronization of FO chaotic systems have attracted many researchers attention in the past few years[16−23]. In [24], an active sliding mode approach for synchronization of FO chaotic system is proposed. The FO Novel and Chen hyper-chaotic systems are proposed for synchronization in [25], where the states of the FO hyper-chaotic Novel system are used to control the states of the FO hyper-chaotic Chen system. Several methods have been proposed to achieve chaos synchronization such as adaptive feedback control, adaptive impulsive control, sliding mode control, active control, backstepping design and optimal control[26−36].

Most of the published papers focus on asymptotic stability which leads to infinite-time synchronization, but in practical applications, finite-time synchronization is more valuable than infinite-time synchronization. Also, most of the researches are related to synchronization between two chaotic systems without uncertainty or two identical chaotic systems, but in a real control system, due to the limitations of physical devices and the effect of interference (such as noise, temperature, etc.), uncertainties are unavoidable.

Motivated by the above discussion, a novel adaptive sliding mode control approach for synchronization of a class of new FO chaotic system and a FO hyper-chaotic system is proposed. In our contribution we pursue five main research aims. First, the proposed approach is very simple and easily realized experimentally for secure communication. Second, the proposed controller can be applied for a width range of systems and is more suitable for engineering applications. Third, finite-time convergence to zero and stability of the proposed method are analytically proved, which contains new ideas. Fourth, a fractional sliding surface is presented and stability of the proposed surface is proved. Fifth, the upper

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bound of the system uncertainties and external disturbances are estimated using Lyapunov stability theorem.

The rest of this paper is organized as follows. First, the fractional calculus and the fractional systems stability theory are briefly introduced. Then, the system description and problem statement are given. After that, the design strategy of the proposed ASMC is presented. Then, the simulations for synchronization of non-identical FO chaotic and hyperchaotic systems are done and the application of the proposed synchronization scheme is studied in secure communication. Finally, concluding remarks are addressed.

II. Derivative and Stability Theorem on FO System

The Caputo fractional derivative of order α of m order continuous function $f(t)$ with respect to t is defined by

$$ {}_{t_0}^{C}D_t^{\alpha}f(t)=I^{m-\alpha}f^{(m)}(t),\quad \alpha>0 \tag{1}$$

where m is the smallest integer number, larger than , and I^{β} is the Riemann-Liouville integral operator of order β which is described as follows

$$ {}_{t_0}I_t^{\beta}f(t)=\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{1-\beta}}\mathrm{d}\tau,\quad \beta>0 \tag{2}$$

In (2), $\Gamma(\cdot)$ is the Gamma function which is given by

$$\Gamma(\beta)=\int_0^{\infty}t^{\beta-1}e^{-t}\mathrm{d}t \tag{3}$$

The numerical simulation of a fractional differential equation is not as simple as that of an ordinary differential equation. Recently, many approaches have been investigated for solving nonlinear FO differential equations. Throughout this paper, we choose the fractional Adams-Bashforth-Moulton method as a representative numerical scheme[37,38]. In order to explain this method, the following differential equation is considered

$$\begin{cases} D_t^{\alpha}y(t)=r(t,y(t)),\ 0\le t\le T,\\ y^{(k)}(0)=y_0^{(k)},\ k=0,1,\cdots,m-1.\end{cases} \tag{4}$$

The differential equation (4) is equivalent to Volterra integral equation which is as follows

$$y(t)=\sum_{k=0}^{\lceil\alpha\rceil-1}y_0^{(k)}\frac{t^k}{k!}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}r(s,y(s))\mathrm{d}s. \tag{5}$$

Now, set $h=T/N$, $t_n=nh$, $n=0,1,\cdots,N$. The integral equation can be discretized as

$$\begin{aligned} y_h(t_{n+1})=&\sum_{k=0}^{\lceil\alpha\rceil-1}y_0^{(k)}\frac{t^k}{k!}+\frac{h^{\alpha}}{\Gamma(\alpha+2)}r\left(t_{n+1},y_h^p(t_{n+1})\right)\\ &+\frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{n}a_{j,n+1}r(t_j,y_h(t_j)) \end{aligned}\tag{6}$$

where

$$y_h^p(t_{n+1})=\sum_{k=0}^{\lceil\alpha\rceil-1}y_0^{(k)}\frac{t^k}{k!}+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}b_{j,n+1}r(t_j,y_h(t_j)) \tag{7}$$

and

$$a_{j,n+1}=\begin{cases} n^{\alpha+1}-(n-\alpha)(n+1)^{\alpha}, & j=0\\ (n-j+2)^{\alpha+1}+(n-j)^{\alpha+1}-2(n-j+1)^{\alpha+1}, & \\ & 1\le j\le n\\ 1, & j=n+1\end{cases} \tag{8}$$

$$b_{j,n+1}=\frac{h^{\alpha}}{\alpha}((n+1-j)^{\alpha}-(n-j)^{\alpha}) \tag{9}$$

The error of this approximation is described as follows

$$\max_{j=0,1,\cdots,N}|y(t_j)-y_h(t_j)|=O(h^p)$$

where $p=\min(2,1+\alpha)$.

In this paper, the operator D^{α} is generally called the "Caputo differential operator of order α".

Remark 1. In this paper, let us define $\|f(t)\|=\sqrt{f_1(t)^2+f_2(t)^2+\cdots+f_n(t)^2}$, and $\|f(t)\|_1=|f_1(t)|+|f_2(t)|+\cdots+|f_n(t)|$, where $f(t)=(f_1(t),f_2(t),\cdots,f_n(t))^{\rm T}$ is a vector of continuous functions.

Property 1. For the Caputo derivative, we have[1,39]

$$ {}_{t_0}^{C}D_t^{1-\alpha}({}_{t_0}^{C}D_t^{\alpha}f(t))={}_{t_0}^{C}D_t^{1}=\dot f(t) \tag{10}$$

Property 2. For the Caputo derivative, the following equality holds[1,39]

$$ {}_{t_0}^{C}D_t^{\alpha_1}({}_{t_0}^{C}D_t^{-\alpha_2}f(t))={}_{t_0}^{C}D_t^{\alpha_1-\alpha_2}f(t) \tag{11}$$

where $\alpha_1\ge\alpha_2\ge0$.

Property 3. For the Caputo derivative, if $f(t)\in C^1[0,T]$ for some $T>0$, then we have[39]

$$\begin{aligned} &{}_{t_0}^{C}D_t^{\alpha_1}\,{}_{t_0}^{C}D_t^{\alpha_2}f(t)={}_{t_0}^{C}D_t^{\alpha_2}\,{}_{t_0}^{C}D_t^{\alpha_1}f(t)={}_{t_0}^{C}D_t^{\alpha_1+\alpha_2}f(t),\\ &t\in[0,T]\end{aligned} \tag{12}$$

where $\alpha_1,\ \alpha_2\in R^+$ and $\alpha_1+\alpha_2\le1$.

III. FO Chaotic System Description

Consider a general form of nonlinear master and slave systems as follows. The master system is

$$D^{\alpha}X=f(X)+\Delta f(X)+d(X) \tag{13}$$

where $\alpha\in(0,1]$ is the FO operator, $X\in R^n$ is the state vector of the master system, $f(X)\in R^n$ is the continuous nonlinear vector functions of the master system, $\Delta f(X)\in R^n$ and $d(X)\in R^n$ are the system uncertainties and external disturbances of the master system, respectively. And the slave system is

$$D^{\alpha}Y=g(Y)+\Delta g(Y)+d(Y)+u(t) \tag{14}$$

where $Y\in R^n$ is the state vector of the slave system, $g(Y)\in R^n$ is the continuous nonlinear vector functions of the slave system, $\Delta g(Y)\in R^n$ and $d(Y)\in R^n$ are the system uncertainties and external disturbances of the slave system, respectively. Also, $u(t)\in R^n$ is the vector of control inputs.

The tracking error can be defined as

$$e(t)=Y(t)-X(t) \tag{15}$$

By subtracting (13) from (14), the error dynamics are obtained as

$$D^{\alpha}e(t) =(g(Y)+\Delta g(Y)+d(Y))-(f(X)+\Delta f(X) + d(X))+u(t)$$

Then one can conclude that

$$D^{\alpha}e_i(t) =(g_i(Y)+\Delta g_i(Y)+d_i(Y))-(f_i(X)+\Delta f_i(X) + d_i(X))+u_i(t),\quad i=1,2,\cdots,n \tag{16}$$

Chaos synchronization problem can be defined as follows; Design an appropriate robust sliding mode controller for the slave system (14) whose its state trajectories track the state trajectories of the master system (13) in finite-time.

In this paper it will be proved that for any defined master system (13) and slave system (14) with system uncertainties and external disturbances, a suitable control input $u(t)$ is derived such that the finite-time stability of the resulting error dynamics by (16) can be obtained in the sense of

$$\lim_{t\to T}\|e(t)\|=0,\ \|e(t)\|=0 \text{ for } t>T \tag{17}$$

Assumption 1. It is assumed that the system uncertainties $\Delta f(X)$, $\Delta g(Y)$ and external disturbances $d(X)$, $d(Y)$ are bounded by

$$\begin{array}{ll}\|\Delta f(X)\|_1\le\tau_1, & \|\Delta g(Y)\|_1\le\tau_2,\\ \|d(X)\|_1\le\varphi_1, & \|d(Y)\|_1\le\varphi_2,\end{array} \tag{18}$$

Then one can conclude that

$$\|\Delta g(Y)-\Delta f(X)\|_1<\gamma,\ \|d(Y)-d(X)\|_1<\delta$$

Therefore we have

$$\begin{array}{l}|(\Delta g_i(Y)-\Delta f_i(X))|<\gamma_i,\ i=1,2,\cdots,n\\ |(d_i(Y)-d_i(X))|<\delta_i,\ i=1,2,\cdots,n\end{array} \tag{19}$$

where τ_1, τ_2, φ_1, φ_2, γ and δ are positive constants; then, γ_i, $i=1,2,\cdots,n$ and δ_i, $i=1,2,\cdots,n$ are positive constants. Also $|\cdot|$ is absolute value.

IV. Robust Adaptive Sliding Mode Control

A. Design of FO Sliding Surface

Design of a sliding mode control law may be divided into two phases: First, choosing an adequate FO sliding surface to achieve the control objective. Second, designing a discontinuous control law which forces the system trajectories to reach the sliding surface in a finite-time. We used the following FO sliding surface

$$\sigma_i(t)=a_iD^{\alpha-1}(e_i(t)),\ i=1,2,\cdots,n \tag{20}$$

where a_i is a positive constant. Then we have

$$D^{\alpha}\sigma_i(t)=a_iD^{2\alpha-1}(e_i(t)) \tag{21}$$

When the FO system (16) operates in the sliding mode, the derivative of the sliding surface must satisfy $\dot\sigma_i(t)=0$[40].

This step concerns the design of control scheme for steering the system (16) in finite-time onto the sliding surface (20). The task is not trivial due to, both, the presence of the unknown disturbance and the FO nature of the system dynamics[41]. Taking the integer-order derivative of (20) yields

$$\dot\sigma_i(t)=a_iD^{\alpha}(e_i(t)),\ i=1,2,\cdots,n \tag{22}$$

By substituting (16) into (22), we have

$$\dot\sigma_i(t)=a_i\Big((g_i(Y)+\Delta g_i(Y)+d_i(Y))-(f_i(X)+\Delta f_i(X) + d_i(X))+u_i(t)\Big),\quad i=1,2,\cdots,n \tag{23}$$

The finite-time stability of system (23) with the control law (25) is proven by Lyapunov analysis in Theorem 1.

B. Design of Robust Control Scheme

After establishing a suitable fractional sliding surface (20), the sliding mode controller is designed in a way so that the system trajectories drive onto the sliding mode $\sigma_i(t)=0$, in finite-time.

Using (23) and $\dot\sigma_i(t)=0$, the equivalent control law can be derived as follows

$$u_{eq_i}(t)=(f_i(X)-g_i(Y))$$

In order to improve the robustness against uncertainties, we design the reaching control law, which drives the system trajectories onto the sliding surface $\sigma_i(t)=0$.

$$u_{r_i}(t)=-\Big(k_i\sigma_i(t)+(\omega_i+\gamma_i+\delta_i)\mathrm{sgn}(\sigma_i(t))\Big) \tag{24}$$

where

$$\mathrm{sgn}(\sigma_i(t))=\begin{cases}+1, & \sigma_i(t)>0\\ 0, & \sigma_i(t)=0\\ -1, & \sigma_i(t)<0\end{cases}$$

k_i, ω_i are positive switching gains.

Finally, the control input law can be obtained as follows

$$u_i(t)=(f_i(X)-g_i(Y)) -\Big(k_i\sigma_i(t)+(\omega_i+\gamma_i+\delta_i)\mathrm{sgn}(\sigma_i(t))\Big) \tag{25}$$

C. Stability Analysis

In this section, Lyapunov theorem is used to analyze the stability of the system. The basic philosophy of Lyapunovs direct method is the mathematical extension of a principal physical observation: If all of the energy of a mechanical (or electrical) system is continuously reduced, then the system, that may be linear or nonlinear, must move to an equilibrium point at last. Thus, the stability of a system by examining the variation of a single Lyapunov function can be analyzed[42].

Theorem 1. If the uncertain FO system (16) is controlled by the control input (25), then the system trajectories will converge to the sliding surface $\sigma_i(t)=0$ in a finite-time t_i.

Proof. Selecting a positive Lyapunov function candidate $v_i(t)=\frac{1}{2}\sigma_i^2(t)$ and taking its time derivative, results

$$\dot v_i(t)=\sigma_i(t)\Big(a_iD^{\alpha}(e_i(t))\Big) \tag{26}$$

Inserting (16) in (26), results

$$\dot v_i(t)=a_i\sigma_i(t)\Big((g_i(Y)+\Delta g_i(Y)+d_i(Y))$$

$$-\left(f_i(X)+\Delta f_i(X)+d_i(X)\right)+u_i(t)\Big) \quad (27)$$

By substituting (25) into (27) and using Assumption 1, then

$$\begin{aligned}\dot{v}_i(t)\leq & a_i\sigma_i(t)(|\Delta g_i(Y)-\Delta f_i(X)|+|d_i(Y)-d_i(X)|)\\ & -a_i\sigma_i(t)\Big(k_i\sigma_i(t)+(\omega_i+\gamma_i+\delta_i)\mathrm{sgn}(\sigma_i(t))\Big)\end{aligned} \quad (28)$$

Hence the above inequality can be written as

$$\dot{v}_i(t)\leq -(2a_ik_iv_i(t)+\sqrt{2}a_i\omega_iv_i(t)^{0.5}) \quad (29)$$

Multiplying both sides of (29) by $v_i(t)^{-0.5}$, results

$$v_i(t)^{-0.5}\dot{v}_i(t)+2a_ik_iv_i(t)^{0.5}\leq -\sqrt{2}a_i\omega_i \quad (30)$$

Multiplying (30) by $(1/2)e^{a_ik_it}$ and then integrating at both sides from zero to t, one obtains

$$\begin{aligned}v_i(t)^{0.5}\leq & \Big((\sqrt{2}/2)(\omega_i/k_i)+v_i(0)^{0.5}\Big)e^{-a_ik_it}\\ & -(\sqrt{2}/2)(w_i/k_i)\end{aligned} \quad (31)$$

then one can get

$$t\leq (1/a_ik_i)\ln\left(1+\sqrt{2}(k_i/\omega_i)v_i(0)^{0.5}\right) \quad (32)$$

Hence, the proof is achieved. i.e., according to the inequality (31), the state trajectories of the error system (16) will converge to $\sigma_i(t)=0$ in a finite-time

$$t_i=(1/a_ik_i)\ln\left(1+(k_i/\omega_i)|\sigma_i(0)|\right)$$

D. Adaptation Law Synthesis

In the previous sections, it has been shown knowing the bounds of system uncertainties and external disturbances is vital to guarantee the system stability. However, in practice it is not convenient to determine these bounds precisely. In what follows, we develop an adaptation laws to overcome this problem. In order to estimate the unknown controller parameters, appropriate update laws are derived as follow:

$$\dot{\widehat{k}}_i=\mu_i\sigma_i(t)^2,\dot{\widehat{\omega}}_i=\rho_i|\sigma_i(t)|,\dot{\widehat{\gamma}}_i=\kappa_i|\sigma_i(t)|,\dot{\widehat{\delta}}_i=\xi_i|\sigma_i(t)| \quad (33)$$

Theorem 2. If the chaotic system of this paper is controlled by the discontinuous control law (25) with the adaptation laws (33), then the system trajectories will converge to the sliding surface $\sigma_i(t)=0$.

Proof. Consider the Lyapunov function candidate as

$$\begin{aligned}v_i(t)= & \frac{1}{2}\sigma_i(t)^2+\frac{1}{2}\Big(\mu_i^{-1}\widetilde{k}_i^2+\rho_i^{-1}\widetilde{\omega}_i^2\\ & +\kappa_i^{-1}\widetilde{\gamma}_i^2+\xi_i^{-1}\widetilde{\delta}_i^2\Big),\ i=1,2,\cdots,n\end{aligned} \quad (34)$$

where $\widetilde{k}_i=k_i-\widehat{k}_i$, $\widetilde{\omega}_i=\omega_i-\widehat{\omega}_i$, $\widetilde{\gamma}_i=\gamma_i-\widehat{\gamma}_i$, and $\widetilde{\delta}_i=\delta_i-\widehat{\delta}_i$. In this case, k_i, ω_i, γ_i, and δ_i are the actual values of $\widehat{k}_i$, $\widehat{\omega}_i$, $\widehat{\gamma}_i$, and $\widehat{\delta}_i$, respectively. Also μ_i, ρ_i, κ_i and ξ_i are rates of adaptation. Taking derivative of both sides of (34) with respect to time, yields

$$\begin{aligned}\dot{v}_i(t)= & \sigma_i(t)\dot{\sigma}_i(t)-\mu_i^{-1}\widetilde{k}_i(\dot{\widehat{k}}_i)-\rho_i^{-1}\widetilde{\omega}_i(\dot{\widehat{\omega}}_i)\\ & -\kappa_i^{-1}\widetilde{\gamma}_i(\dot{\widehat{\gamma}}_i)-\xi_i^{-1}\widetilde{\delta}_i(\dot{\widehat{\delta}}_i)\end{aligned} \quad (35)$$

Using Property 1 and then inserting (21) in (35), one obtains

$$\begin{aligned}\dot{v}_i(t)= & \sigma_i(t)D^{1-\alpha}\Big(a_iD^{2\alpha-1}(e_i(t))\Big)-\mu_i^{-1}\widetilde{k}_i(\dot{\widehat{k}}_i)\\ & -\rho_i^{-1}\widetilde{\omega}_i(\dot{\widehat{\omega}}_i)-\kappa_i^{-1}\widetilde{\gamma}_i(\dot{\widehat{\gamma}}_i)-\xi_i^{-1}\widetilde{\delta}_i(\dot{\widehat{\delta}}_i)\end{aligned} \quad (36)$$

Using Properties 2 and 3, one gets

$$\begin{aligned}\dot{v}_i(t)= & a_i\sigma_i(t)D^{\alpha}(e_i(t))-\mu_i^{-1}\widetilde{k}_i(\dot{\widehat{k}}_i)\\ & -\rho_i^{-1}\widetilde{\omega}_i(\dot{\widehat{\omega}}_i)-\kappa_i^{-1}\widetilde{\gamma}_i(\dot{\widehat{\gamma}}_i)-\xi_i^{-1}\widetilde{\delta}_i(\dot{\widehat{\delta}}_i)\end{aligned} \quad (37)$$

Substituting (16) into (37) and using Assumption 1, we have

$$\begin{aligned}&\dot{v}_i(t)\leq (\gamma_i+\delta_i)|\sigma_i(t)|+\sigma_i(t)(g_i(Y)-f_i(X)+u_i(t))\\ &-\mu_i^{-1}\widetilde{k}_i(\dot{\widehat{k}}_i)-\rho_i^{-1}\widetilde{\omega}_i(\dot{\widehat{\omega}}_i)-\kappa_i^{-1}\widetilde{\gamma}_i(\dot{\widehat{\gamma}}_i)-\xi_i^{-1}\widetilde{\delta}_i(\dot{\widehat{\delta}}_i)\end{aligned} \quad (38)$$

By assuming that the parameters of the controller (25) are unknown, then

$$\begin{aligned}&\dot{v}_i(t)\leq -\widehat{k}_i|\sigma_i(t)|^2-\widehat{\omega}_i|\sigma_i(t)|+(\widetilde{\gamma}_i+\widetilde{\delta}_i)|\sigma_i(t)|\\ &-\mu_i^{-1}\widetilde{k}_i(\dot{\widehat{k}}_i)-\rho_i^{-1}\widetilde{\omega}_i(\dot{\widehat{\omega}}_i)-\kappa_i^{-1}\widetilde{\gamma}_i(\dot{\widehat{\gamma}}_i)-\xi_i^{-1}\widetilde{\delta}_i(\dot{\widehat{\delta}}_i)\end{aligned} \quad (39)$$

Introducing the adaptation laws (33) in (39), will lead to

$$\dot{v}_i(t)\leq -k_i|\sigma_i(t)|^2-\omega_i|\sigma_i(t)| \quad (40)$$

Hence, the motion on the sliding surface is asymptotically stable. Therefore, the output can track the desired reference.

E. Particle Swarm Optimization (PSO)

In this section, the parameters of the ASMC are estimated using PSO algorithm. There are a lot of optimal techniques for optimization. One of the simple approaches for optimization is PSO. PSO was introduced by Kennedy and Eberhart[43], and is useful for continuous space. PSO algorithm imitates the behavior of birds and others like fishes for searching the best solution in the space. PSO has been found to be robust in solving problems featuring nonlinearity, multiple optima, and high dimensionality through adaptation, which is derived from the social-psychological theory. In this technique, every particle can be illustrated by two vectors[44]. These vectors are position vector and velocity vector that can be updated with this algorithm to get the best parameters of the controller. The PSO algorithm, at each time step, changes the speed of each particle moving towards its *pBest* and *gBest* locations. Speed is weighted by random terms, with separate random numbers being generated for acceleration toward *pBest* and *gBest* locations, respectively.

Our aim is to have low tracking error; hence the following cost function (Mean Squared Error) is used

$$MSE=\frac{1}{N}\sum_{i=0}^{N}\Big(e_k(i)\Big)^2 \quad (41)$$

where, $e_k(i)$ is the kth error state variable. N is the length of every error state variable.

The procedure for implementing PSO algorithm for estimating the controller parameters is given by the following steps:

i Initialize a (population) of particles with random positions and velocities in the n-dimensional problem space using a uniform probability distribution function;

ii For each particle in swarm, evaluate its fitness value;

iii Compare each particles fitness evaluation with the current particles *pBest*. If current value is better than *pBest*, set its *pBest* value to the current value and the *pBest* location to the current location in n-dimensional space;

iv Compare the fitness evaluation with the populations overall previous best. If current value is better than *gBest*, then reset *gBest* to the current particles array index and value;

v During this process, the position vector and velocity vector of each particle are updated to tend the best position as follows:

$$\begin{aligned} V_i(t+1) =& wV_i(t) + c_1\text{rand}(0,1)(pBest_i(t) - X_i(t)) \\ &+ c_2\text{rand}(0,1)(gBest_i(t) - X_i(t)) \\ X_i(t+1) =& X_i(t) + V_i(t+1) \end{aligned} \tag{42}$$

where $i = 1, 2, \cdots, n$ is the particles index, t is the time (iteration or generation).

In this case, the position and speed vectors are with dimensions d. c_1 and c_2 are acceleration coefficients, w is the inertia weight. In (42), $pBest_i$ is the position with the best fitness found by the ith particle, and $gBest_i$ is the best fitness position in neighborhood.

V. Applications and Numerical Experiments

In this section, an illustrative example is presented to show the feasibility and applicability of the proposed nonsingular sliding mode approach and to confirm the theoretical results. In this example, numerical simulation for two non-identical FO chaotic and FO hyper-chaotic systems is presented. Fourth-order Runge-Kutta method is used with a step time of 0.001 in order to solve the FO differentials.

A. Synchronization of Non-Identical FO Chaotic and FO Hyper-chaotic Systems

In this section, numerical simulations are presented to validate the robustness and effectiveness of the proposed ASMC, when the controller parameters are estimated by PSO algorithm. These values are obtained in order to minimize the synchronization errors. The FO chaotic system[45] as master system drives the FO hyper-chaotic system[46] as slave system.

The master system is

$$\underbrace{\begin{bmatrix} D^\alpha x_1 \\ D^\alpha x_2 \\ D^\alpha x_3 \\ D^\alpha x_4 \end{bmatrix}}_{D^\alpha X} = \underbrace{\begin{bmatrix} 5(x_2 - x_1) + x_4 \\ -x_1 x_3 \\ -90 + x_1 x_2 \\ -10x_1 \end{bmatrix}}_{f(X)} + \underbrace{\begin{bmatrix} 0.2\cos(x_2) \\ 0.3\cos(x_1) \\ 0.25\sin(x_4) \\ 0.35\sin(x_3) \end{bmatrix}}_{\Delta f(X)} + \underbrace{\begin{bmatrix} 0.3\cos(t) \\ 0.25\sin(t) \\ 0.3\cos(t) \\ 0.2\cos(t) \end{bmatrix}}_{d(X)} \tag{43}$$

and the slave system is

$$\underbrace{\begin{bmatrix} D^\alpha y_1 \\ D^\alpha y_2 \\ D^\alpha y_3 \\ D^\alpha y_4 \end{bmatrix}}_{D^\alpha Y} = \underbrace{\begin{bmatrix} 10(y_2 - y_1) \\ 40y_1 + y_1 y_3 + 2y_4 \\ -2y_1^2 - 2y_2^2 - 2.5y_3 \\ -5y_2 \end{bmatrix}}_{g(Y)} + \underbrace{\begin{bmatrix} 0.3\sin(y_2) \\ 0.25\cos(y_3) \\ 0.25\cos(y_1) \\ 0.2\sin(y_2) \end{bmatrix}}_{\Delta g(X)} + \underbrace{\begin{bmatrix} 0.25\cos(t) \\ 0.3\sin(t) \\ 0.3\sin(t) \\ 0.3\cos(t) \end{bmatrix}}_{d(Y)} + \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}}_{u(t)} \tag{44}$$

The FO operator (α) is set to 0.95 to ensure the existence of chaos for the system. Assume, the initial states of the master and slave systems are selected as $(x_1(0), x_2(0), x_3(0), x_4(0))^{\rm T} = (2.5, 0.5, 1, 0.5)^{\rm T}$ and $(y_1(0), y_2(0), y_3(0), y_4(0))^{\rm T} = (4, 2.5, 3.5, 3)^{\rm T}$, respectively. μ_i, ρ_i, κ_i and ξ_i are rates of adaptation which are supposed to be 5, 3, 5 and 2 for $(i = 1, \cdots, 4)$, respectively. The control input suffers high chattering. In order to reduce this drawback of the controller we have used the saturation function instead of the sign function. The time responses of the synchronized states are depicted in Fig. 1. Fig. 2 shows the synchronization errors between two FO chaotic and hyper-chaotic systems. The time response of $\widehat{k}_i$ and $\widehat{\omega}_i$ for $(i = 1, \cdots, 4)$ are depicted in Fig. 3. Besides, the time response of $\widehat{\gamma}_i$ and $\widehat{\delta}_i$ for $(i = 1, \cdots, 4)$ are depicted in Fig. 4. In Table. 1, the controller parameters are depicted before optimization and after that.

PSO parameters are set as follow:

Population size$= 20$, *Iterations*$= 40$, $c_1 = 2.0$, $c_2 = 2.0$, *weighting factor*$= 1$, *Inertia weight*$= 0.999$.

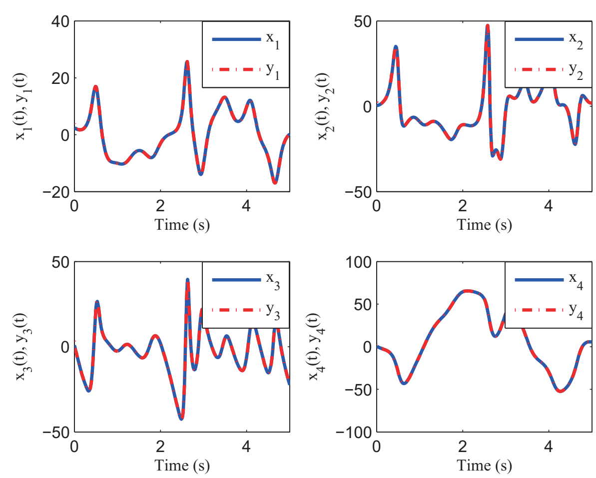

Fig. 1. Time response of signals for master system and slave system.

VI. A Secure Communication Scheme

A secure communication system involves the development of a signal that contains the information which is to remain undetectable by others within a carrier signal. In this section, a popular application of chaotic synchronization in the area

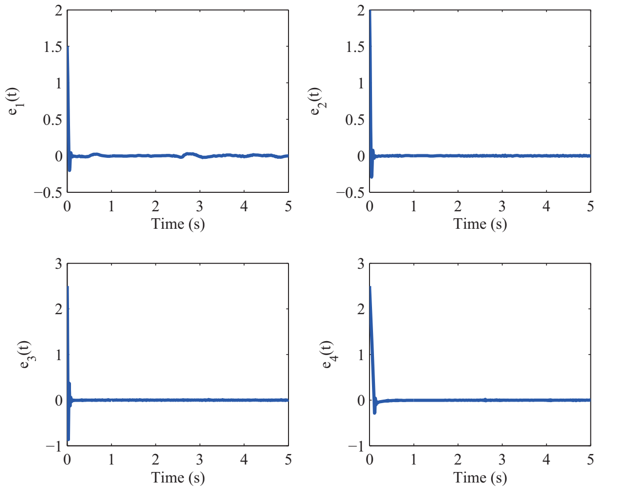

Fig. 2. Time response of the synchronization errors between two non-identical FO chaotic and hyper-chaotic systems.

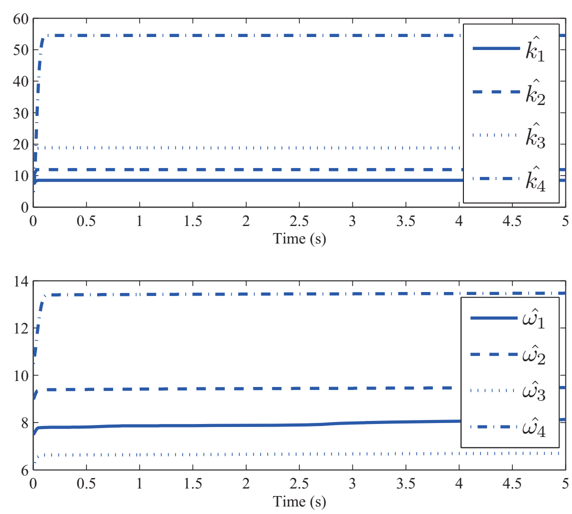

Fig. 3. Time response of the controller parameters.

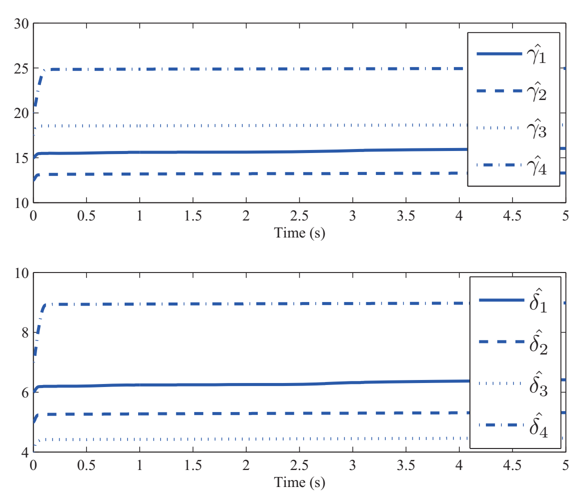

Fig. 4. Time response of the controller parameters.

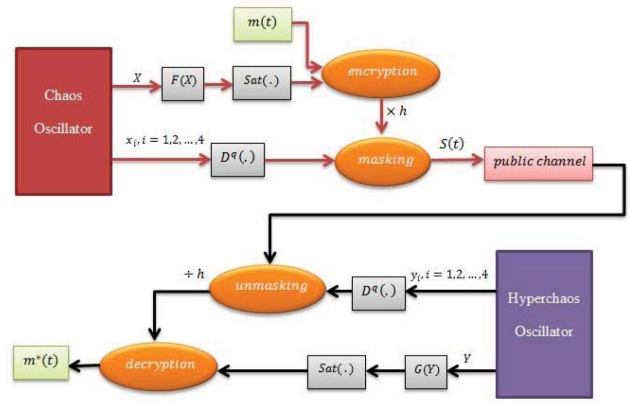

Fig. 5. The secure communication scheme based on the synchronization of FO chaotic and hyper-chaotic systems.

TABLE I
CONTROLLER PARAMETERS BEFORE AND AFTER OPTIMIZATION AND THE COST FUNCTION VALUES

	a_1	a_2	a_3	a_4	Cost
Before optimization	5.0000	6.0000	3.5000	7.5000	8.8959
After optimization	3.5728	4.0087	9.7267	8.1469	7.7656

of secure communications is presented. The useful signal has been modulated two times to improve the security of the system, encrypted by secret key firstly and masked secondly by the FO derivative of chaos variable. Fig. 5 depicts a sketch designed for our communication scheme.

In the transmitter, two chaotic variables of the chaos oscillator are employed to construct a function $F(X)$ which is used to generate secret key $k(t)$. The secret key $k(t)$ is added to the proposed useful signal $m(t)$ in order to encrypt the useful signal. The encrypted useful signal is masked by the FO derivative of chaos variable x_i. Then, the encrypted and masked useful signal is transmitted to the receiver through public channel. In the receiver, first the received signal is unmasked by the FO derivative of hyperchaos variable y_i. Then, the unmasked signal is decrypted by the secret key $k^*(t)$. It is impossible to extract the useful signal $m(t)$ from the transmitted signal $S(t)$ without the dynamics of X. Therefore, when the control signal (25) is designed in the receiver, then the synchronization between chaos oscillator and hyper-chaos oscillator will be obtained and X will converge to Y in finite-time.

The simulation results above are based on discrete useful signal. In the transmitter, the nonlinear function $F(X) = (x_2x_4)^2$ is transmitted through the saturation function to generate the secret key. The FO of the chaos state variable x_3 is used to mask the encrypted message. Demodulation process is inverse operation to modulation. So $G(Y) = (y_2y_4)^2$ and the FO of hyper-chaos variable y_3 is used to unmask the received signal. The notation $D^q(\cdot)$ denotes the FO derivative, where the FO is selected as $q = 0.5$. Also, h is a small constant which is supposed to be 3. By using a small constant h, the security of the transmitted signal in a public channel can be increased.

The useful signal $m(t)$ is shown in Fig. 6-a; chaotic signal

$S(t)$ which is transmitted to the receiver is illustrated in Fig. 6-b; the comparison between demodulated useful signal $m^*(t)$ and sent useful signal $m(t)$ is shown in Fig. 6-c. As a result of the simulation, demodulated signal and useful signal can quickly implement synchronization as a short transient. The error between the demodulated signal and the useful signal is depicted in Fig. 6-d.

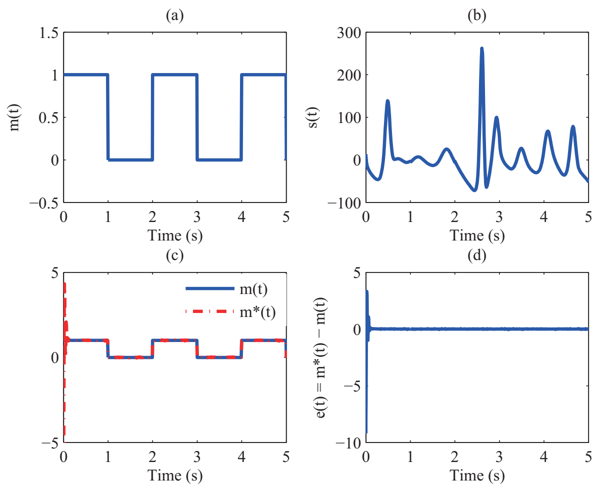

Fig. 6. Simulation results of the proposed secure communication scheme using finite-time synchronization of FO chaotic and hyper-chaotic systems.

VII. Conclusion

In this paper, the proposed novel sliding mode controller is shown to be robust against high uncertainties and variation of the parameters. Suitable adaptive laws are proposed to tackle the unknown parameters and PSO algorithm is used in this paper for optimization of the controller parameters. Finally, the proposed scheme is applied in secure communication. The simulation results show that the synchronization time is very short and the recovered signal is close to the useful signal and it can realize secret communication successfully, having strong security and practicability.

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