Discrete Fractional Order Chaotic Systems Synchronization Based on the Variable Structure Control with a New Discrete Reaching-law

Lilian Huang, Longlong Wang, and Donghai Shi

Abstract—**In this paper, we directly derive a new discrete state space expression of the fractional order chaotic system based on the fractional order Grnwald-Letnikow(G-L) definition and design a variable structure controller with a new faster reaching-law. The new reaching-law has the advantages of weakening the high frequency shake. Firstly, the condition of the discrete sliding mode surface is demonstrated. Then a multi-parametric function for sliding mode surface is constructed for weakening the high frequency shake through improving the Gao discrete reaching-law. Finally, the newly designed variable structure controller is applied to realize the synchronization of two different order discrete fractional chaotic systems. The simulation results show that the designed controller in this paper is effective, as it can achieve the synchronization of the discrete fractional order chaotic systems with external disturbances. Theoretical analysis and simulation results prove the effectiveness and robustness of this control method.**

***Index Terms*—Discrete fractional order chaotic system, Different order system, Sliding mode control, Discrete reaching-law, Chaos synchronization.**

I. Introduction

RECENTLY the fractional calculus is applied widely in image processing neural network, signal processing, robust control and so on[1], because it can more accurately describe the actual dynamic characteristics of the physical system. Through researching on fractional calculus, many researchers generally accepted that fractional order calculus is a generalization of the integer calculus[2], and they also believed that the fractional calculus had many new characteristics of the systematic memory, the dynamic system and so on. The fractional calculus' relationship with the chaos and the fractal theory deeply attracted researchers because new chaotic phenomenon was found in fractional order nonlinear systems[3].

The chaotic synchronization has a great potential of application in the subject field[4−5] of communication, information science, medicine, biological engineering and so on. There are many methods about chaotic synchronization, for example linear feedback control[6], coupled synchronization[7], adaptive synchronization, sliding mode control[8]. The research on chaotic synchronization is usually aiming for the same structure systems[9] with different initial values or known parameters[10] or fractional order hyper chaos system[11]. As far as the synchronization of discrete fractional order chaotic systems[12], many researchers had developed some methods. Liao et al[13] realized the synchronization of Henon map by using sliding-mode control. Hu[14] proposed tracking control and predicted synchronization control on discrete chaotic system. Majidabad et al[15] designed the algorithm of fast synchronization and zero steady-steady error fast synchronization and so on. On the other side, A.Dzielinski[16] proposed the expression of the discrete fractional order state space system. D. Sierociuk[17] obtained some results including the controllability of discrete fractional order state space system and adaptive feedback control. J. A. Tenreiro[18] designed a discrete fractional order controller which could be applied to the linear and nonlinear systems in the time domain. Yao[19] put forward another form of discrete fractional order chaotic system and realized synchronous control, and Gong[20] came up with a different form of expression on discrete fractional order chaotic systems. In all the above, the form of discrete fractional order chaotic systems is obtained indirectly by different discrete methods. In this paper, we derive directly a new discrete state space expression of the fractional order chaotic system based on the fractional order Grünwald-Letnikow[21] definition. We can obtain the scope of the order using bifurcation diagram when the system is chaotic. Then based on the state space analysis method, the synchronization control problem is researched for different structural discrete fractional order chaotic systems by sliding mode control theory.

Based on the fractional order definition of Grünwald-Letnikow, the paper directly derives a new discrete state space expression of the fractional order chaotic system Then the paper designed a new kind of discrete sliding mode controller through improving the Gao's discrete reaching-law. The structure of controller designed is simple and easy to select. For two different structures of discrete fractional order chaotic systems with different dimensions, we can achieve synchronization using the new controller. Simulation results show that two fractional order chaotic systems with different dimensions can still realize synchronization when the driven system has

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disturbance, which proved the controller's effectiveness and feasibility.

II. Discrete Fractional Grünwald-Letnikow (G-L) Definition

The discrete fractional order G-L expression is as following:

$$\nabla^{\alpha}x(k)=\sum_{j=0}^{k}(-1)^{j}\left(\begin{array}{c}\alpha\\ j\end{array}\right)x(k-j)$$
$$\left(\begin{array}{c}\alpha\\ j\end{array}\right)=\begin{cases}1, & j=0\\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & j>0\end{cases} \tag{1}$$

where $\left(\begin{array}{c}\alpha\\ j\end{array}\right)$ is binomial coefficient, α is the order of discrete equations, $\alpha\in R$.

Considering the general nonlinear discrete systems, the expression for general nonlinear discrete systems is

$$x(k+1)=f(x(k)) \tag{2}$$

Consider the first order integer discrete difference equation as below:

$$\nabla^{1}=x(k+1)-x(k)=f(x(k))-x(k) \tag{3}$$

We generalize the above align toαorder differential align as follows:

$$\nabla^{\alpha}(x(k+1))=f(x(k))-x(k) \tag{4}$$

From the expression (1), we can get

$$\nabla^{\alpha}x(k+1)=$$
$$x(k+1)-\alpha x(k)+\sum_{j=2}^{k+1}(-1)^{j}\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)x(k-j+1) \tag{5}$$

For formula (5), introducing a new parameter m and let $m=j-1$, so $j=m+1$, and obtaining another formula:

$$\begin{aligned}\nabla^{\alpha}x(k+1)&=x(k+1)-\alpha x(k)\\ &\quad+\sum_{m=1}^{k}(-1)^{m+1}\left(\begin{array}{c}\alpha\\ m+1\end{array}\right)x(k-m)\\ &=x(k+1)-\alpha x(k)+\sum_{m=1}^{k}M_{m}x(k-m)\end{aligned} \tag{6}$$

where $M_{m}=(-1)^{m+1}\left(\begin{array}{c}\alpha\\ m+1\end{array}\right)m\in N$.

The general form of discrete fractional order aligns could be obtained from (4) and (6):

$$x(k+1)=f(x(k))+(\alpha-1)x(k)+\sum_{m=1}^{k}M_{m}x(k-m) \tag{7}$$

where $\sum\limits_{m=1}^{k}M_{m}x(k-m)$ is the memory term for align, and it indicates that the value of a certain point is not only related to the function of the point, but also with the previous function value And the farther away from the point, the less influence on that point value The memory term can be replaced by truncation function that is, the above form (7) can be written as follows.

$$x(k+1)=f(x(k))+(\alpha-1)x(k)+\sum_{m=1}^{L}M_{m}x(k-m) \tag{8}$$

where L is the length of the memory, usually $L=20$

For general discrete proportional fractional order system, the state space expression can be written:

$$\left[\begin{array}{c}\nabla^{\alpha}x_{1}(k+1)\\ \nabla^{\alpha}x_{2}(k+1)\\ \vdots\\ \nabla^{\alpha}x_{n}(k+1)\end{array}\right]=\left[\begin{array}{c}f_{1}(x(k))-x_{1}(k)\\ f_{2}(x(k))-x_{2}(k)\\ \vdots\\ f_{n}(x(k))-x_{n}(k)\end{array}\right] \tag{9}$$

where ∇^{α} represents fractional differential factor of system, α is the order of the fraction. The ∇^{α} can be rewritten based on the discrete fractional G-L definition:

$$\left[\begin{array}{c}x_{1}(k+1)\\ x_{2}(k+1)\\ \vdots\\ x_{n}(k+1)\end{array}\right]=$$
$$\left[\begin{array}{c}f_{1}(x(k))+(\alpha-1)x_{1}(k)+\sum\limits_{m=1}^{L}M_{m}x_{1}(k-m)\\ f_{2}(x(k))+(\alpha-1)x_{2}(k)+\sum\limits_{m=1}^{L}M_{m}x_{2}(k-m)\\ \vdots\\ f_{n}(x(k))+(\alpha-1)x_{n}(k)+\sum\limits_{m=1}^{L}M_{m}x_{n}(k-m)\end{array}\right] \tag{10}$$

III. The Synchronization of Discrete Fractional Order Chaotic Systems Based on Sliding Mode Variable Structure Control

A. The design of the new discrete sliding mode reaching law

Selecting the following and regarding as the sliding-mode surface.

$$s(k)=Be(k) \tag{11}$$

where B is an invertible matrix. When the system is in sliding mode, it needs to satisfy the following conditions:

$$s_{i}(k)\rightarrow 0 \tag{12}$$

The basic principles of discrete and continuous sliding modes are nearly same, and they have two stages from the initial state to the stable state, also called two modes. The first stage is a reaching process and the second stage is sliding state, but they also have some differences. Based on the discrete sliding mode variable structure, in order to get the better of discrete sliding mode, firstly select a discrete sliding surface, so that the system has good dynamic characteristics. Secondly, for satisfying reaching condition, design a controller based on the reaching law. So the controller can make the system converge to the discrete sliding mode surface from any point in finite time.

The classic Gao's discrete reaching law is as following:

$$s(k+1) - s(k) = -\varepsilon s(k) - \beta \mathrm{sgn}(s(k)) \tag{13}$$

where ε is the reaching speed, β indicates the reaching speed index. The reaching condition of Gao′s reaching law is that the dynamics of system once moves across the switching surface, the subsequent movements are from the other side of the switching surface and then the dynamics keeps on it. This can ensure strong robustness of sliding mode control, but it also leads to the phenomenon of high frequency chattering. In order to weaken the high frequency chattering, we propose a new reaching-law by considering the two aspects.

The improved discrete reaching-law is shown below.

$$s(k+1) - s(k) = -\beta(s(k))\mathrm{sgn}(s(k)) \tag{14}$$

where β is a function that can be designed as following:

$\beta(s(k)) = \beta + \gamma |s(k)| + \gamma |s(k)| \mathrm{sgn}(|s(k)| - \delta)$ where $0 < \beta < \delta$, $0 < \gamma < 1/2$

According to the existing conditions for the discrete sliding surface, we illustrate the rationality of the new reaching law from two aspects as follows.

1) Since $s(k+1) - s(k) = -\beta(s(k))\mathrm{sgn}(s(k))$

Then $(s(k+1) - s(k))\mathrm{sgn}(s(k)) = -\beta(s(k))\ \mathrm{sgn}^2(s(k)) = -\beta(s(k))$

And $\beta(s(k)) = \beta + \gamma |s(k)| + \gamma |s(k)| \mathrm{sgn}(|s(k)| - \delta)$, $0 < \beta < \delta$, $0 < \gamma < 1/2$

So $\beta(s(k)) > 0$, therefore $(s(k+1) - s(k))\mathrm{sgn}(s(k)) < 0$

2) Since $s(k+1) - s(k) + 2s(k) = -\beta(s(k))\mathrm{sgn}(s(k)) + 2s(k)$

$$\begin{aligned}
&(s(k+1) + s(k))\mathrm{sgn}(s(k)) \\
&= -\beta(s(k))\mathrm{sgn}^2(s(k)) + 2s(k)\mathrm{sgn}(s(k)) \\
&= -\beta(s(k))\mathrm{sgn}^2(s(k)) + 2|s(k)| \\
&= -\beta(s(k)) + 2|s(k)|
\end{aligned}$$

Now compare $\beta(s(k))_{max}$ and $2|s(k)|$. When $|s(k)| > \delta$, $\beta(s(k))$is the max value $\beta(s(k))_{max} = \beta + 2\gamma |s(k)|$, so $\beta < \delta < |s(k)|$ that is $\beta + 2\gamma |s(k)| < |s(k)| + 2\gamma |s(k)|$. Because $\gamma < 1/2$ then $\beta(s(k))_{max} < 2|s(k)|$ so $(s(k+1) + s(k))\mathrm{sgn}(s(k)) > 0$.

From (1) and (2), we can get:

$$\begin{aligned}
(s(k+1) - s(k))\mathrm{sgn}(s(k)) < 0 \\
(s(k+1) + s(k))\mathrm{sgn}(s(k)) > 0
\end{aligned}$$

That is $|s(k+1)| < |s(k)|$. So the discrete sliding mode surface exists under the control of new reaching-law.

In the new reaching-law, β is the function of $s(k)$. By setting the expression of β, it contains the parameters of the reaching-law speed and reaching speed index. In the process of approaching the discrete sliding surface, the parameter δ determines the reaching rate. After reaching the sliding surface, the system will be stable in the neighborhood of sliding surface, at this time β determines reaching speed index.

B. The synchronization of discrete fractional order chaotic systems

Consider the following two discrete fractional order chaotic systems, as the drive system and response system, respectively:

$$\nabla^{\alpha} X(k+1) = f(X(k)) - X(k) \tag{15}$$
$$\nabla^{\alpha} Y(k+1) = g(Y(k)) - Y(k) + U(k) \tag{16}$$

where ∇^{α} is fractional order differential factor, $\alpha \in R$. Based on the discrete fractional G-L definition, the differential factor is expanded as follows.

$$\begin{aligned}
X(k+1) &= f(X(k)) + (\alpha - 1)X(k) + \sum_{m=1}^{L} M_m X(k-m) \\
&= R_f(X(k)) + (\alpha - 1)X(k) \qquad (17) \\
Y(k+1) &= g(Y(k)) + (\alpha - 1)Y(k) \\
&\quad + \sum_{m=1}^{L} M_m Y(k-m) + U(k) \\
&= R_g(Y(k)) + (\alpha - 1)Y(k) + U(k) \qquad (18)
\end{aligned}$$

where $R_f(X(k))$ is the function of $f(X(k))$ and $\sum\limits_{m=1}^{L} M_m X(k-m)$, also the same as $R_g(Y(k))$ $\sum\limits_{m=1}^{L} M_m X(k-m)$ and $\sum\limits_{m=1}^{L} M_m Y(k-m)$ are the memory terms for drive system and response system. $X(k) \in R^m$ and $Y(k) \in R^n$ are the state variables of drive system and response system, respectively. α is the order value of systems. $U(k)$ is the controller for the response system to be designed.

The purpose of designing controller $U(k)$ is to guarantee the synchronization of the drive-response system and to have strong robustness, i.e. $\lim\limits_{k\to\infty} \|e(k)\| = 0$ where$e(k)$ is the generalized synchronization state error, and $e(k) = Y(k) - CX(k)$, $e(k) = (e_1(k), e_2(k), ...e_n(k))^{\mathrm{T}}$, $C \in R^{n\times m}$, so the state error dynamic system can be written:

$$\begin{aligned}
e(k+1) &= Y(k+1) - CX(k+1) \\
&= R_g(Y(k)) - CR_f(X(k)) + U(k) \qquad (19)
\end{aligned}$$

C. Controller design

Theorem: To achieve the synchronization of system (17) and (18), the following controller is designed.

$$U(k) = U_0 + CR_f(X(k)) - R_g(Y(k)) + Y(k) - CX(k)$$

where $U_0 = [u_1, u_2, \cdots, u_n]$, $u_i = -\sum\limits_{j=0}^{n} a_{ij}(\beta(s_i(k))$ $\mathrm{sgn}(s_i(k)))$, $A = B^{-1}\beta(s_i(k)) = \beta + \gamma |s_i(k)| + \gamma |s_i(k)| \mathrm{sgn}(|s_i(k)| - \delta)$, $s_i(k) = \sum\limits_{j=1}^{n} b_{ij} e_j(k)$, $|B| \neq 0$.

Proof: Choosing Lyapunov function as follows:

$$\begin{aligned} v(k) &= (s(k))^{\mathrm{T}} s(k) \\ \Delta v(k) &= v(k+1) - v(k) \\ &= (s(k+1))^{\mathrm{T}} s(k+1) - s(k)^{\mathrm{T}} s(k) \\ &= (-\beta(s(k))\mathrm{sgn}(s(k)) + s(k))^{\mathrm{T}}(-\beta(s(k))\mathrm{sgn}(s(k)) \\ &\quad + s(k)) - (s(k))^{\mathrm{T}} s(k) \\ &= (-\beta(s(k))\mathrm{sgn}(s(k)))^{\mathrm{T}}(-\beta(s(k))\mathrm{sgn}(s(k))) \\ &\quad + (-\beta(s(k))\mathrm{sgn}(s(k)))^{\mathrm{T}} s(k) \\ &\quad + s(k)^{\mathrm{T}}(-\beta(s(k))\mathrm{sgn}(s(k))) \\ &\quad + (s(k))^{\mathrm{T}} s(k) - (s(k))^{\mathrm{T}} s(k) \end{aligned}$$

Since $-\beta(s(k))\mathrm{sgn}(s(k))$ and $s(k)$ are all column vectors.

So $(-\beta(s(k))\mathrm{sgn}(s(k)))^{\mathrm{T}} s(k) = s(k)^{\mathrm{T}}(-\beta(s(k))\cdot \mathrm{sgn}(s(k)))$ is constant. Then

$$\begin{aligned} \Delta v(k) =& (-\beta(s(k))\mathrm{sgn}(s(k)))^{\mathrm{T}}(-\beta(s(k))\mathrm{sgn}(s(k))) \\ &+ 2(-\beta(s(k))\mathrm{sgn}(s(k)))^{\mathrm{T}} s(k) \\ =& (-\beta(s(k))\mathrm{sgn}(s(k)))^{\mathrm{T}}(-\beta(s(k))\mathrm{sgn}(s(k)) + 2s(k)) \end{aligned}$$

Now assuming that $\beta(s(k))\mathrm{sgn}(s(k)) = (m_1, m_2, \cdots, m_n)^T$, $2s(k) = (n_1, n_2, \cdots, n_n)^T$

So

$$\begin{aligned} \Delta v(k) &= (m_1, m_2, \cdots, m_n)(m_1 - n_1, m_2 - n_2, \cdots, m_n - n_n)^T \\ &= (m_1^2 - m_1 n_1) + (m_2^2 - m_2 n_2) + \cdots + (m_n^2 - m_n n_n) \end{aligned}$$

Otherwise $\beta(s(k))_{\max} < 2\,|s(k)|$,

So $\beta(s(k))\mathrm{sgn}(s(k)) < 2\,|s(k)|\,\mathrm{sgn}(s(k)) = 2s(k)$ that is $m_i < n_i, i = 1, 2, \cdots, n$

it is easy to obtain:

$$\Delta v(k) < 0$$

According to Lyapunov stability theory, the original Lyapunov function is positive definite $v(k) > 0$ and its first derivative is negative definite $\Delta v(k) < 0$ then the error system $e(k)$ converges to zero. So the expression (17) and (18) achieve synchronization finally.

IV. Simulation

A. Case 1: The dimension of drive system is bigger than that of the response system

The generalized discrete Henon chaotic system as drive system is,

$$\begin{cases} x_1(k+1) = a x_3(k) \\ x_2(k+1) = b x_1(k) + a x_3(k) \\ x_3(k+1) = 1 + x_2(k) - c x_3^2(k) \end{cases} \tag{20}$$

According to the above, the fractional order of this system is:

$$\begin{cases} \nabla^{\alpha} x_1(k+1) = a x_3(k) - x_1(k) \\ \nabla^{\alpha} x_2(k+1) = b x_1(k) + a x_3(k) - x_2(k) \\ \nabla^{\alpha} x_3(k+1) = 1 + x_2(k) - c x_3^2(k) - x_3(k) \end{cases} \tag{21}$$

From the above the equation, every differential align contains fractional differential factor ∇^{α} and α is the order, which would be set to make the system be chaotic. By using the definition of G-L, the ∇^{α} can be expanded to get the driving system as follows:

$$\begin{cases} x_1(k+1) = a x_3(k) + (\alpha - 1) x_1(k) + \sum\limits_{m=1}^{L} M_m x_1(k-m) \\ \qquad = f_1(x(k)) + \Delta f(x(k)) \\ x_2(k+1) = b x_1(k) + a x_3(k) + (\alpha - 1) x_2(k) \\ \qquad + \sum\limits_{m=1}^{L} M_m x_2(k-m) \\ \qquad = f_2(x(k)) \\ x_3(k+1) = 1 + x_2(k) - c x_3^2(k) + (\alpha - 1) x_3(k) \\ \qquad + \sum\limits_{m=1}^{L} M_m x_3(k-m) \\ \qquad = f_3(x(k)) \end{cases} \tag{22}$$

where a, b, c are the parameters of drive system, $\Delta f(x(k))$ indicates external disturbance. The bifurcation diagram of the system can be obtained when selecting the parameter $a = 0.358$, $b = 1.3$, $c = 1.07$ and the initial value is $(0.45, 0.3, 0.4)$.

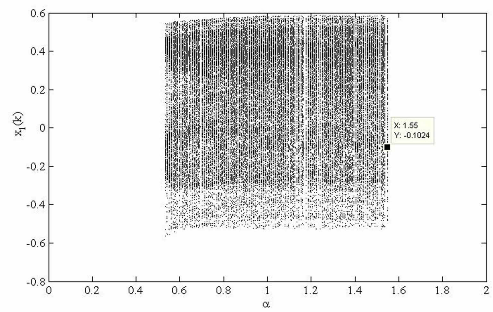

Fig. 1. Bifurcation diagram of generalized discrete fractional Henon chaotic system

From the Fig.(1), the drive system is chaotic when $0.534 < \alpha < 1.55$. In this simulation we choose $\alpha = 0.8$.

Selecting the discrete map Ikeda [22] as the response system is,

$$\begin{cases} y_1(k+1) = a' + b'(y_1(k)\cos(\theta) - y_2(k)\sin(\theta)) \\ y_2(k+1) = b'(y_1(k)\sin(\theta) - y_2(k)\cos(\theta)) \end{cases} \tag{23}$$

where $\theta = y_1^2(k) + y_2^2(k), a', b'$ are the parameters of the system. We can obtain its fractional order expression from above.

$$\begin{cases} \nabla^{\alpha} y_1(k+1) = a' + b'(y_1(k)\cos(\theta) - y_2(k)\sin(\theta)) - y_1(k) \\ \nabla^{\alpha} y_2(k+1) = b'(y_1(k)\sin(\theta) - y_2(k)\cos(\theta)) - y_2(k) \end{cases} \tag{24}$$

By using the definition of G-L, the ∇^{α} can be simplified. Then the above expression can be rewritten as following:

$$\begin{cases} y_1(k+1) = a' + b'(y_1(k)\cos(\theta) - y_2(k)\sin(\theta)) \\ \qquad + (\alpha - 1) y_1(k) + \sum\limits_{m=1}^{L} M_m y_1(k-m) + u_1(k) \\ \qquad = g_1(y(k)) + u_1(k) \\ y_2(k+1) = b'(y_1(k)\sin(\theta) - y_2(k)\cos(\theta)) \\ \qquad + (\alpha - 1) y_2(k) + \sum\limits_{m=1}^{L} M_m y_2(k-m) + u_2(k) \\ \qquad = g_2(y(k)) + u_2(k) \end{cases} \tag{25}$$

$u_1(k)$, $u_2(k)$ are the controllers for response system. Selecting transfer matrix $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ the state error dynamic system $e(k) = y(k) - Cx(k)$ can be rewritten:

$$\left\{ \begin{array}{l} e_1(k) = y_1(k) - x_1(k) \\ e_2(k) = y_2(k) - (x_2(k) + x_3(k)) \end{array} \right. \tag{26}$$

From the details in subsection **(3.3)** $U(k)$ can be designed as following:

$$U(k) = \left[\begin{array}{l} u_{01}(k) + f_1(x(k)) - g_1(y(k)) + y_1(k) - x_1(k) \\ u_{02}(k) + f_2(x(k)) + f_3(x(k)) - g_2(y(k)) \\ +y_2(k) - (x_2(k) + x_3(k)) \end{array} \right] \tag{27}$$

where $u_{0i}(k), i = 1, 2$ is

$$\left\{ \begin{array}{l} u_{01}(k) = -a_{11}\beta(s_1(k))\mathrm{sgn}(s_1(k)) - a_{12}\beta(s_2(k))\mathrm{sgn}(s_2(k)) \\ u_{02}(k) = -a_{21}\beta(s_1(k))\mathrm{sgn}(s_1(k)) - a_{22}\beta(s_2(k))\mathrm{sgn}(s_2(k)) \end{array} \right. \tag{28}$$

where $\beta(s_i(k)) = \beta + \gamma\,|s_i(k)| + \gamma\,|s_i(k)|\,\mathrm{sgn}(|s_i(k)| - \delta)$, the sliding surface is chosen as below:

$$\left\{ \begin{array}{l} s_1(k) = b_{11}e_1(k) + b_{12}(e_1(k) + e_2(k)) \\ s_2(k) = b_{21}e_1(k) + b_{22}(e_1(k) + e_2(k)) \end{array} \right. \tag{29}$$

The matrix A and B meet the conditions of $A = B^{-1}$ based on the **Theorem.** The parameters of the drive system are $a = 0.358$, $b = 1.3$, $c = 1.07$, $\alpha = 0.8$ and its initial value is $(0.45, 0.3, 0.4)$. The parameters of the response system are $a' = 1.5$, $b' = 0.2$, $\alpha = 0.8$ and its initial value is $(0.9, 0.2)$. The parameters of the sliding surface are $(0.7, 2.4)(0.2, 3.6)$ Selecting the controller's parameters $\beta = 0.04$, $\gamma = 0.4$, $\delta = 0.8$. Assuming that the external disturbance $\Delta f(x(k)) = 0.01\sin(0.04k\pi)$. The result of simulation is shown in Fig.(2).

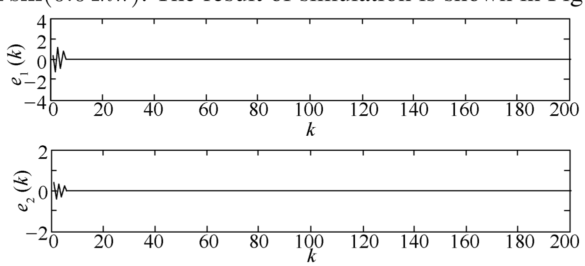

Fig. 2. Synchronization error curves

B. Case 2: The dimension of the drive system is smaller than that of the response system

Selecting the discrete Henon map as drive system is,

$$\left\{ \begin{array}{l} x_1(k+1) = 1 - ax_1^2(k) + x_2(k) + \Delta f(x(k)) \\ x_2(k+1) = bx_1(k) \end{array} \right. \tag{30}$$

where $\Delta f(x(k))$ is an external disturbing perturbations. The fractional order of drive system is below:

$$\left\{ \begin{array}{l} x_1(k+1) = 1 - ax_1^2(k) + x_2(k) + (\alpha - 1)x_1(k) \\ \qquad + \sum\limits_{m=1}^{L} M_m x_1(k-m) \\ \quad = f_1(x(k)) + \Delta f(x(k)) \\ x_2(k+1) = bx_1(k) + (\alpha - 1)x_2(k) + \sum\limits_{m=1}^{L} M_m x_2(k-m) \\ \quad = f_2(x(k)) \end{array} \right. \tag{31}$$

Selecting the parameters of system $a = 1.4$, $b = 0.3$ and its initial value is $(0.5, 0.2)$. The Bifurcation diagram of drive system can be obtained without the external disturbance.

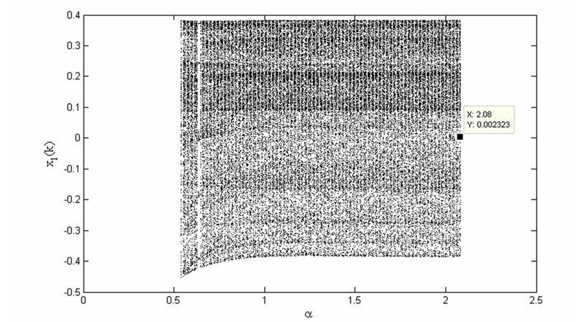

Fig. 3. Bifurcation diagram of discrete fractional Henon chaotic system

The drive system is chaotic when $0.54 < \alpha < 2.08$ from the Fig.(3). The generalized discrete Henon chaotic system as response system is:

$$\left\{ \begin{array}{l} y_1(k+1) = a'y_3(k) + (\alpha - 1)y_1(k) \\ \qquad + \sum\limits_{m=1}^{L} M_m y_1(k-m) + u_1(k) \\ \qquad = g_1(y(k)) + u_1(k) \\ y_2(k+1) = b'y_1(k) + a'y_3(k) + (\alpha - 1)y_2(k) \\ \qquad + \sum\limits_{m=1}^{L} M_m y_2(k-m) + u_2(k) \\ \qquad = g_2(y(k)) + u_2(k) \\ y_3(k+1) = 1 + y_2(k) - c'y_3^2(k) + (\alpha - 1)y_3(k) \\ \qquad + \sum\limits_{m=1}^{L} M_m y_3(k-m) + u_3(k) \\ \qquad = g_3(y(k)) + u_3(k) \end{array} \right. \tag{32}$$

From the content of (**4.1**), the response system is chaotic when $0.534 < \alpha < 1.55$. Selecting transfer matrix $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, so the state error dynamic system $e(k) = y(k) - Cx(k)$ can be rewritten:

$$\left\{ \begin{array}{l} e_1(k) = y_1(k) - x_1(k) \\ e_2(k) = y_2(k) - x_2(k) \\ e_3(k) = y_3(k) - (x_1(k) + x_2(k)) \end{array} \right. \tag{33}$$

The controller of $U(k)$ is designed as following:

$$U(k) = \left[\begin{array}{l} u_{01}(k) + f_1(x(k)) - g_1(y(k)) + y_1(k) - x_1(k) \\ u_{02}(k) + f_2(x(k)) - g_2(y(k)) + y_2(k) - x_2(k) \\ u_{03}(k) + f_2(x(k)) + f_1(x(k)) - g_2(y(k)) \\ \quad +y_2(k) - (x_1(k) + x_2(k)) \end{array} \right] \tag{34}$$

The $u_{0i}(k)$, $i = 1, 2, 3$ is,

$$\left\{ \begin{array}{l} u_{01}(k) = a_{11}\beta(s_1(k)) + a_{12}\beta(s_2(k)) + a_{13}\beta(s_3(k)) \\ u_{02}(k) = a_{21}\beta(s_2(k)) + a_{22}\beta(s_2(k)) + a_{23}\beta(s_3(k)) \\ u_{03}(k) = a_{31}\beta(s_1(k)) + a_{32}\beta(s_2(k)) + a_{33}\beta(s_3(k)) \end{array} \right. \tag{35}$$

where $\beta(s_i(k)) = \beta + \gamma|s_i(k)| + \gamma|s_i(k)|\,\mathrm{sgn}(|s_i(k)| - \delta)$ the sliding surface is chosen as below:

$$\begin{cases} s_1(k) = b_{11}e_1(k) + b_{12}e_2(k) + b_{13}e_3(k) \\ s_2(k) = b_{21}e_1(k) + b_{22}e_2(k) + b_{23}e_3(k) \\ s_3(k) = b_{31}e_1(k) + b_{32}e_2(k) + b_{33}e_3(k) \end{cases} \tag{36}$$

The matrix A and B meet the conditions of $A = B^{-1}$ based on the **Theorem** and selecting the parameters of the drive system $a = 1.5$, $b = 0.2$, $\alpha = 0.8$, and its initial value is $(1.2, 0.8)$. The parameters of response system are $a' = 0.358b' = 1.3$, $c' = 1.07$, $\alpha = 0.8$, and its initial value is $(0.1, 0.2, 0.1)$. The parameters of the sliding surface will be $(0.2, 1.1, 2.4)$, $(0.8, 0.3, 3.6)$, $(0.1, 0.8, 3.9)$. The length of the memory $L = 20$. Setting the parameters of controller$\beta = 0.02$, $\delta = 0.8$,$\gamma = 0.3$ Assuming that the drive system of external disturbance is $\Delta f(x(k)) = 0.05\sin(0.4k\pi)$. The result of simulation is shown in Fig.(4).

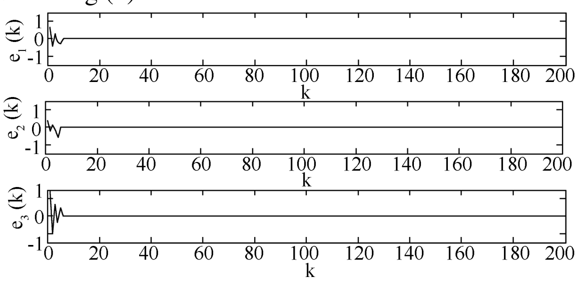

Fig. 4. Synchronization error curves

The simulation results show that the synchronization state error converges to the origin asymptotically in finite time and stabilize the origin eventually with external disturbance. The results demonstrate that the different dimensional structure discrete fractional order chaotic systems achieved synchronization under the action of designed controller.

V. Conclusions

In this paper, a new general state space expression of discrete fractional order chaotic system is obtained based on the fractional order definition of Grünwald-Letnikow. A new discrete sliding mode reaching-law control strategy which has the advantage of weakening the high frequency chatting is proposed by improving the Gao's discrete reaching-law. Based on a novel strategy, a new controller is designed, which would guarantee the different dimensional structure discrete fractional order chaotic systems achieving synchronization. When the systems are with external disturbances, they can still achieve the synchronization of the discrete fractional order chaotic systems. Simulation results verify the effectiveness of the proposed methods and demonstrate the rationality of the designed controller.

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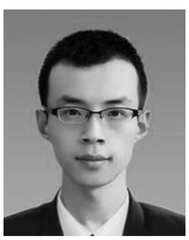

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