Variational Calculus With Conformable Fractional Derivatives

Matheus J. Lazo and Delfim F. M. Torres

Abstract—Invariant conditions for conformable fractional problems of the calculus of variations under the presence of external forces in the dynamics are studied. Depending on the type of transformations considered, different necessary conditions of invariance are obtained. As particular cases, we prove fractional versions of Noether's symmetry theorem. Invariant conditions for fractional optimal control problems, using the Hamiltonian formalism, are also investigated. As an example of potential application in Physics, we show that with conformable derivatives it is possible to formulate an Action Principle for particles under frictional forces that is far simpler than the one obtained with classical fractional derivatives.

Index Terms—Conformable fractional derivative, fractional calculus of variations, fractional optimal control, invariant variational conditions, Noether's theorem.

I. INTRODUCTION

FRACTIONAL calculus is a generalization of (integer) differential calculus, allowing to define integrals and derivatives of real or complex order [1]–[3]. It had its origin in the 1600 s and for three centuries the theory of fractional derivatives developed as a pure theoretical field of mathematics, useful only for mathematicians. The theory took more or less finished form by the end of the 19th century. In the last few decades, fractional differentiation has been "rediscovered" by applied scientists, proving to be very useful in various fields: physics (classic and quantum mechanics, thermodynamics, etc.), chemistry, biology, economics, engineering, signal and image processing, and control theory [4]. One can find in the existent literature several definitions of fractional derivatives, including the Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, Weyl, Grunwald-Letnikov, Hadamard, and Chen derivatives. Recently, a simple solution to the discrepancies between known definitions was presented with the introduction of a new fractional notion, called the conformable derivative [5]. The new definition is a natural extension of the usual derivative, and satisfies the main properties one expects in a derivative: the conformable

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derivative of a constant is zero; satisfies the standard formulas of the derivative of the product and of the derivative of the quotient of two functions; and satisfies the chain rule. Besides simple and similar to the standard derivative, one can say that the conformable derivative combines the best characteristics of known fractional derivatives [6]. For this reason, the subject is now under strong development: see [7]–[10] and references therein.

The fractional calculus of variations was introduced in the context of classical mechanics when Riewe [11] showed that a Lagrangian involving fractional time derivatives leads to an equation of motion with non-conservative forces such as friction. It is a remarkable result since frictional and non-conservative forces are beyond the usual macroscopic variational treatment [12]. Riewe generalized the usual calculus of variations for a Lagrangian depending on Riemann-Liouville fractional derivatives [11] in order to deal with linear non-conservative forces. Actually, several approaches have been developed to generalize the calculus of variations to include problems depending on Caputo fractional derivatives, Riemann-Liouville fractional derivatives, Riesz fractional derivatives and others [13]–[19] (see [20]–[22] for the state of the art). Among theses approaches, recently it was shown that the action principle for dissipative systems can be generalized, fixing the mathematical inconsistencies present in the original Riewe's formulation, by using Lagrangians depending on classical and Caputo derivatives [23].

In this paper we work with conformable fractional derivatives in the context of the calculus of variations and optimal control [20]. In order to illustrate the potential application of conformable fractional derivatives in physical problems we show that it is possible to formulate an action principle with conformable fractional calculus for the frictional force free from the mathematical inconsistencies found in the Riewe original approach and far simpler than the formulations proposed in [23]. Furthermore, we obtain a generalization of Noether's symmetry theorem for fractional variational problems and we also consider conformable fractional optimal control problems. Emmy Noether was the first who proved, in 1918, that the notions of invariance and constant of motion are connected: when a system is invariant under a family of transformations, then a conserved quantity along the Euler-Lagrange extremals can be obtained [24], [25]. All conservation laws of Mechanics, e.g., conservation of energy or conservation of momentum, are easily explained from Noether's theorem. In this paper we study necessary conditions for invariance under a family of continuous transformations,

where the Lagrangian contains a conformable fractional derivative of order $\alpha \in (0,1)$. When $\alpha \to 1$, we obtain some well-known results, in particular the Noether theorem [25]. The advantages of our fractional results are clear. Indeed, the classical constants of motion appear naturally in closed systems while in practical terms closed systems do not exist: forces that do not store energy, so-called nonconservative or dissipative forces, are always present in real systems. Fractional dynamics provide a good way to model nonconservative systems [11]. Nonconservative forces remove energy from the systems and, as a consequence, the standard Noether constants of motion are broken [26]. Our results assert that it is still possible to obtain Noether-type theorems, which cover both conservative and nonconservative cases, and that this is done in a particularly simple and elegant way via the conformable fractional approach. This is in contrast with the approaches followed in [27]–[30].

The paper is organized as follows. In Section II we collect some necessary definitions and results on the conformable fractional calculus needed in the sequel. In Section III we obtain the conformable fractional Euler-Lagrange equation and in Section IV we formulate an action principle for dissipative systems, as an example of application and motivation to study the conformable calculus of variations. In Section V, we present an immediate consequence of the Euler-Lagrange equation that we use later in Sections VI and VII, where we prove, respectively, some necessary conditions for invariant fractional problems and a conformable fractional Noether theorem. We then review the obtained results using the Hamiltonian language in Section VIII. In Section IX, we consider the conformable fractional optimal control problem, where the dynamic constraint is given by a conformable fractional derivative. Using the Hamiltonian language, we provide an invariant condition. In Section X we consider the multi-dimensional case, for several independent and dependent variables.

II. PRELIMINARIES

In this section we review the conformable fractional calculus [5]–[7]. The conformable fractional derivative is a new well-behaved definition of fractional derivative, based on a simple limit definition. We review in this section the generalization of [5] proposed in [6].

Definition 1: The left conformable fractional derivative of order $0 < \alpha \leq 1$ starting from $a \in \mathbb{R}$ of a function $f : [a,b] \to \mathbb{R}$ is defined by

$$\frac{d_a^\alpha}{dx_a^\alpha} f(x) = f_a^{(\alpha)}(x)$$
$$= \lim_{\epsilon \to 0} \frac{f(x + \epsilon(x-a)^{1-\alpha}) - f(x)}{\epsilon}. \quad (1)$$

If the limit (1) exist, then we say that f is left α-differentiable. Furthermore, if $f_a^{(\alpha)}(x)$ exist for $x \in (a,b)$, then

$$f_a^{(\alpha)}(a) = \lim_{x \to a^+} f_a^{(\alpha)}(x)$$

and

$$f_a^{(\alpha)}(b) = \lim_{x \to b^-} f_a^{(\alpha)}(x).$$

The right conformable fractional derivative of order $\alpha \in (0,1]$ terminating at $b \in \mathbb{R}$ of a function $f : [a,b] \to \mathbb{R}$ is defined by

$$\frac{_bd^\alpha}{_bdx^\alpha} f(x) = {_bf^{(\alpha)}}(x)$$
$$= -\lim_{\epsilon \to 0} \frac{f(x + \epsilon(b-x)^{1-\alpha}) - f(x)}{\epsilon}. \quad (2)$$

If the limit (2) exist, then we say that f is right α-differentiable. Furthermore, if ${_bf^{(\alpha)}}(x)$ exist for $x \in (a,b)$, then

$$_bf^{(\alpha)}(a) = \lim_{x \to a^+} {_bf^{(\alpha)}}(x)$$

and

$$_bf^{(\alpha)}(b) = \lim_{x \to b^-} {_bf^{(\alpha)}}(x).$$

It is important to note that for $\alpha = 1$ the conformable fractional derivatives (1) and (2) reduce to first order ordinary derivatives. Furthermore, despite the definition of the conformable fractional derivatives (1) and (2) can be generalized for $\alpha > 1$ [6], we consider only $0 < \alpha \leq 1$ in the present work. Is is also important to note that, differently from the majority of definitions of fractional derivative, including the popular Riemann-Liouville and Caputo fractional derivatives, the fractional derivatives (1) and (2) are local operators and are related to ordinary derivatives if the function is differentiable (see Remark 1). For more on local fractional derivatives, we refer the reader to [31], [32] and references therein.

Remark 1: If $f \in C^1[a,b]$, then we have from (1) that

$$f_a^{(\alpha)}(x) = (x-a)^{1-\alpha} f'(x) \quad (3)$$

and from (2) that

$$_bf^{(\alpha)}(x) = -(b-x)^{1-\alpha} f'(x) \quad (4)$$

where $f'(x)$ stands for the ordinary first order derivative of $f(x)$.

From (3) and (4) it is easy to see that the conformable fractional derivative of a constant is zero, differently from the Riemann-Liouville derivative of a constant, and for the power functions $(x-a)^p$ and $(b-x)^p$ one has

$$\frac{d_a^\alpha}{dx_a^\alpha}(x-a)^p = p(x-a)^{p-\alpha}$$

and

$$\frac{_bd^\alpha}{_bdx^\alpha}(b-x)^p = p(b-x)^{p-\alpha}$$

for all $p \in \mathbb{R}$.

The most remarkable consequence of definitions (1) and (2) is that the conformable fractional derivatives satisfy very simple fractional versions of chain and product rules.

Proposition 1 [5], [6]: Let $0 < \alpha < 1$ and f and g be α-differentiable functions. Then,

1)

$$(c_1 f + c_2 g)_a^{(\alpha)}(x) = c_1 f_a^{(\alpha)}(x) + c_2 g_a^{(\alpha)}(x)$$

and

$$_b(c_1 f + c_2 g)^{(\alpha)}(x) = c_{1b} f^{(\alpha)}(x) + c_{2b} g^{(\alpha)}(x)$$

for all $c_1, c_2 \in \mathbb{R}$.

2)

$$(fg)_a^{(\alpha)}(x) = f_a^{(\alpha)}(x)g(x) + f(x)g_a^{(\alpha)}(x)$$

and

$$_b(fg)^{(\alpha)}(x) = {}_bf^{(\alpha)}(x)g(x) + f(x){}_bg^{(\alpha)}(x).$$

3)

$$\left(\frac{f}{g}\right)_a^{(\alpha)}(x) = \frac{f_a^{(\alpha)}(x)g(x) - f(x)g_a^{(\alpha)}(x)}{g^2(x)}$$

and

$$_b\left(\frac{f}{g}\right)^{(\alpha)}(x) = \frac{_bf^{(\alpha)}(x)g(x) - f(x)_bg^{(\alpha)}(x)}{g^2(x)}.$$

4) If $g(x) \geq a$, then

$$(f \circ g)_a^{(\alpha)}(x) = f_a^{(\alpha)}(g(x))g_a^{(\alpha)}(x)(g(x) - a)^{\alpha - 1}.$$

5) If $g(x) \leq b$, then

$$_b(f \circ g)^{(\alpha)}(x) = {}_bf^{(\alpha)}(g(x)){}_bg^{(\alpha)}(x)(b - g(x))^{\alpha - 1}.$$

6) If $g(x) < a$, then

$$(f \circ g)_a^{(\alpha)}(x) = -{}_af^{(\alpha)}(g(x))g_a^{(\alpha)}(x)(a - g(x))^{\alpha - 1}.$$

7) If $g(x) > b$, then

$$_b(f \circ g)^{(\alpha)}(x) = -f_b^{(\alpha)}(g(x)){}_bg^{(\alpha)}(x)(g(x) - b)^{\alpha - 1}.$$

The simple chain and product rules given in Proposition 1 justify the increasing interest in the study of the conformable fractional calculus, since it enable us to investigate its potential applications as a tool to practical modeling of complex problems in science and engineering.

The conformable fractional integrals are defined as follows [5], [6].

Definition 2: The left conformable fractional integral of order $0 < \alpha \leq 1$ starting from $a \in \mathbb{R}$ of a function $f \in L^1[a, b]$ is defined by

$$I_a^\alpha f(x) = \int_a^x f(u)d_a^\alpha u = \int_a^x \frac{f(u)}{(u - a)^{1-\alpha}} du \qquad (5)$$

and the right conformable fractional integral of order $0 < \alpha \leq 1$ terminating at $b \in \mathbb{R}$ of a function $f \in L^1[a, b]$ is defined by

$$_bI^\alpha f(x) = \int_x^b f(u)_bd^\alpha u = \int_x^b \frac{f(u)}{(b - u)^{1-\alpha}} du. \qquad (6)$$

It is important to mention that the conformable fractional integrals (5) and (6) differ from the traditional fractional Riemann-Liouville integrals [1]–[3] only by a multiplicative constant. Moreover, for $\alpha = 1$, the conformable fractional integrals reduce to ordinary first order integrals.

In addition to these definitions, in the present work we make use of the following properties of conformable fractional derivatives and integrals.

Theorem 1: Let $f \in C[a, b]$ and $0 < \alpha \leq 1$. Then,

$$\frac{d_a^\alpha}{dx_a^\alpha}I_a^\alpha f(x) = f(x)$$

and

$$\frac{_bd^\alpha}{_bdx^\alpha}{}_bI^\alpha f(x) = f(x)$$

for all $x \in [a, b]$.

Theorem 2 (Fundamental theorem of conformable fractional calculus): Let $f \in C^1[a, b]$ and $0 < \alpha \leq 1$. Then,

$$I_a^\alpha f_a^{(\alpha)}(x) = f(x) - f(a)$$

and

$$_bI^\alpha {}_bf^{(\alpha)}(x) = f(x) - f(b)$$

for all $x \in [a, b]$.

Theorem 3 (Integration by parts): Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then,

$$\int_a^b f(x)g_a^{(\alpha)}(x)d_a^\alpha x = f(x)g(x)|_a^b - \int_a^b g(x)f_a^{(\alpha)}(x)d_a^\alpha x \tag{7}$$

and

$$\int_a^b f(x)_bg^{(\alpha)}(x)_bd^\alpha x$$
$$= -f(x)g(x)|_a^b - \int_a^b g(x)_bf^{(\alpha)}(x)_bd^\alpha x. \tag{8}$$

If $f, g : [a, b] \to \mathbb{R}$ are differentiable functions, then

$$\int_a^b f(x)g_a^{(\alpha)}(x)d_a^\alpha x = f(x)g(x)|_a^b + \int_a^b g(x)_bf^{(\alpha)}(x)_bd^\alpha x.$$

The proof of Theorem 1 follows directly from (3), (4), (5) and (6) since $I_a^\alpha f(x)$ and $_bI^\alpha f(x)$ are differentiable. On the other hand, the fundamental theorem of the conformable fractional calculus (Theorem 2) is a direct consequence of (3), (4) and definitions (5) and (6) since $f, g : [a, b] \to \mathbb{R}$ are differentiable functions. Finally, the integration by parts (7) and (8) follow from Proposition 1 and Theorem 1. We also need the following result.

Theorem 4 (Chain rule for functions of several variables): Let $f : \mathbb{R}^N \to \mathbb{R}$ ($N \in \mathbb{N}$) be a differentiable function in all its arguments and $y_1, \ldots, y_N : \mathbb{R} \to \mathbb{R}$ be α-differentiable functions. Then,

$$\frac{d_a^\alpha}{dx_a^\alpha}f(y_1(x), \ldots, y_N(x))$$
$$= \frac{\partial f}{\partial y_1}y_{1a}^{(\alpha)} + \frac{\partial f}{\partial y_2}y_{2a}^{(\alpha)} + \cdots + \frac{\partial f}{\partial y_N}y_{Na}^{(\alpha)} \tag{9}$$

and

$$\frac{_bd^\alpha}{_bdx^\alpha}f(y_1(x), \ldots, y_N(x))$$
$$= \frac{\partial f}{\partial y_1}{}_by_1^{(\alpha)} + \frac{\partial f}{\partial y_2}{}_by_2^{(\alpha)} + \cdots + \frac{\partial f}{\partial y_N}{}_by_N^{(\alpha)}. \tag{10}$$

Proof : For simplicity, we prove (9) only for $N = 2$. The proofs for a general N and of (10) are similar. From (1) we have for $N = 2$ that (by writing $\bar{x} = x + \epsilon(x - a)^{1-\alpha}$ for

simplicity)

$$\frac{d_a^\alpha}{dx_a^\alpha} f(y_1(x), y_2(x))$$

$$= \lim_{\epsilon \to 0} \frac{f(y_1(\overline{x}), y_2(\overline{x})) - f(y_1(x), y_2(x))}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{f(y_1(\overline{x}), y_2(\overline{x})) - f(y_1(x), y_2(\overline{x}))}{y_1(\overline{x}) - y_1(x)} \frac{y_1(\overline{x}) - y_1(x)}{\epsilon}$$

$$+ \lim_{\epsilon \to 0} \frac{f(y_1(x), y_2(\overline{x})) - f(y_1(x), y_2(x))}{y_2(\overline{x}) - y_2(x)} \frac{y_2(\overline{x}) - y_2(x)}{\epsilon}$$

$$= \frac{\partial f}{\partial y_1} y_1{}_a^{(\alpha)} + \frac{\partial f}{\partial y_2} y_2{}_a^{(\alpha)}$$

since f is differentiable. ∎

III. THE CONFORMABLE EULER–LAGRANGE EQUATION

Let us first consider the fractional variational integral

$$\mathcal{J}(y) = \int_a^b L\left(x, y(x), y_a^{(\alpha)}(x)\right) d_a^\alpha x \qquad (11)$$

defined on the set of continuous functions $y : [a, b] \to \mathbb{R}$ such that $y_a^{(\alpha)}$ exists on $[a, b]$, where the Lagrangian L, $L(x, y, y_a^{(\alpha)}) : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$, is of class C^1 in each of its arguments. The fundamental problem of the calculus of variations consists in finding which functions extremize functional (11). In order to obtain a necessary condition for the extremum of (11) we need the following Lemma.

Lemma 1 (Fundamental lemma for the conformable calculus of variations): Let M and η be continuous functions on $[a, b]$. If

$$\int_a^b \eta(x) M(x) d_a^\alpha x = 0 \qquad (12)$$

for any $\eta \in C[a, b]$ with $\eta(a) = \eta(b) = 0$, then

$$M(x) = 0 \qquad (13)$$

for all $x \in [a, b]$.

Proof : We do the proof by contradiction. From (12) we have that

$$\int_a^b \eta(x) M(x) d_a^\alpha x = \int_a^b \eta(x) \frac{M(x)}{(x-a)^{1-\alpha}} dx = 0. \qquad (14)$$

Suppose that there exist an $x_0 \in (a, b)$ such that $M(x_0) \neq 0$. Without loss of generality, let us assume that $M(x_0) > 0$. Since M is continuous on $[a, b]$, there exists a neighborhood $N^\delta(x_0) \subset (a, b)$ such that

$$M(x) > 0 \quad \text{for all} \quad x \in N^\delta(x_0).$$

Let us choose

$$\eta(x) = \begin{cases} (x - x_0 - \delta)^2 (x - x_0 + \delta)^2, & \text{if } x \in N^\delta(x_0) \\ 0, & \text{if } x \notin N^\delta(x_0). \end{cases} \qquad (15)$$

Clearly, $\eta(x)$ given by (15) is continuous and satisfy $\eta(a) = 0$ and $\eta(b) = 0$. Inserting (15) into (14), we obtain that

$$\int_a^b \eta(x) M(x) d_a^\alpha x$$

$$= \int_{x_0-\delta}^{x_0+\delta} (x - x_0 - \delta)^2 (x - x_0 + \delta)^2 \frac{M(x)}{(x-a)^{1-\alpha}} dx > 0$$

which contradicts our hypothesis. Thus,

$$\frac{M(x)}{(x-a)^{1-\alpha}} > 0 \quad \text{for all} \quad x \in (a, b).$$

Since $(x-a)^{1-\alpha} > 0$ for $x \in (a, b)$, and since $M \in C[a, b]$, we get

$$M(x) = 0 \quad \text{for all} \quad x \in [a, b].$$

∎

Theorem 5 (The conformable fractional Euler–Lagrange equation): Let $\mathcal{J}$ be a functional of form (11) with $L \in C^1\left([a, b] \times \mathbb{R}^2\right)$, and $0 < \alpha \leq 1$. Let $y : [a, b] \to \mathbb{R}$ be a α-differentiable function with $y(a) = y_a$ and $y(b) = y_b$, y_a, $y_b \in \mathbb{R}$. Furthermore, let $y \frac{\partial L}{\partial y_a^{(\alpha)}}$ be a differentiable function and $\frac{\partial L}{\partial y_a^{(\alpha)}}$ be α-differentiable. If y is an extremizer of $\mathcal{J}$, then y satisfies the following fractional Euler–Lagrange equation:

$$\frac{\partial L}{\partial y} - \frac{d_a^\alpha}{dx_a^\alpha}\left(\frac{\partial L}{\partial y_a^{(\alpha)}}\right) = 0. \qquad (16)$$

Proof : Let y^* give an extremum to (11). We define a family of functions

$$y(x) = y^*(x) + \epsilon \eta(x) \qquad (17)$$

where ϵ is a constant and η is an arbitrary α-differentiable function satisfying $\eta \frac{\partial L}{\partial y^*{}_a^{(\alpha)}} \in C^1$ and the boundary conditions $\eta(a) = \eta(b) = 0$ (weak variations). From (17), the boundary conditions $\eta(a) = \eta(b) = 0$, and the fact that $y^*(a) = y_a$ and $y^*(b) = y_b$, it follows that function y is admissible: y is α-differentiable with $y(a) = y_a$, $y(b) = y_b$, and $y \frac{\partial L}{\partial y^*{}_a^{(\alpha)}}$ is differentiable. Let the Lagrangian L be $C^1([a, b] \times \mathbb{R}^2)$. Because y^* is an extremizer of functional $\mathcal{J}$, the Gateaux derivative $\delta \mathcal{J}(y^*)$ needs to be identically null. For the functional (11),

$$\delta \mathcal{J}(y^*) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_a^b L\left(x, y, y_a^{(\alpha)}\right) d_a^\alpha x \right.$$

$$\left. - \int_a^b L\left(x, y^*, y^*{}_a^{(\alpha)}\right) d_a^\alpha x \right)$$

$$= \int_a^b \left(\eta(x) \frac{\partial L\left(x, y^*, y^*{}_a^{(\alpha)}\right)}{\partial y^*} \right.$$

$$\left. + \eta_a^{(\alpha)}(x) \frac{\partial L\left(x, y^*, y^*{}_a^{(\alpha)}\right)}{\partial y^*{}_a^{(\alpha)}} \right) d_a^\alpha x = 0.$$

Using the integration by parts formula (7) ($\eta \frac{\partial L}{\partial y^*{}_a^{(\alpha)}}$ is differentiable), we get

$$\delta \mathcal{J}(y^*) = \int_a^b \eta(x) \left(\frac{\partial L\left(x, y^*, y^*{}_a^{(\alpha)}\right)}{\partial y^*} \right.$$

$$\left. - \frac{d_a^\alpha}{dx_a^\alpha} \frac{\partial L\left(x, y^*, y^*{}_a^{(\alpha)}\right)}{\partial y^*{}_a^{(\alpha)}} \right) d_a^\alpha x = 0 \qquad (18)$$

since $\eta(a) = \eta(b) = 0$. The fractional Euler–Lagrange equation (16) follows from (18) by using the fundamental Lemma 1. ∎

Definition 3: A continuous function y solution of (16) is said to be an extremal of (11).

Remark 2: For $\alpha = 1$, the functional $\mathcal{J}$ given by (11) reduces to the classical variational functional

$$\mathcal{J}(y) = \int_0^1 L\left(x, y(x), y'(x)\right) dx$$

and the associated Euler–Lagrange equation (16) is

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0. \qquad (19)$$

Let us now consider the more general case where the Lagrangian depends on both integer order and fractional order derivatives. In this case the following theorem holds.

Theorem 6 (The generalized conformable fractional Euler–Lagrange equation): Let $\mathcal{J}$ be a functional of form

$$\mathcal{J}(y) = \int_a^b L\left(x, y(x), y'(x), y_a^{(\alpha)}(x)\right) dx \qquad (20)$$

with $L \in C^1\left([a,b] \times \mathbb{R}^3\right)$, and $0 < \alpha \le 1$. Let $y : [a,b] \to \mathbb{R}$ be a differentiable function with $y(a) = y_a$ and $y(b) = y_b$, $y_a, y_b \in \mathbb{R}$. If y is an extremizer of $\mathcal{J}$, then y satisfies the following fractional Euler–Lagrange equation:

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{1}{(x-a)^{1-\alpha}}\frac{d_a^\alpha}{dx_a^\alpha}\left(\frac{\partial \tilde{L}}{\partial y_a^{(\alpha)}}\right) = 0 \quad (21)$$

where $\tilde{L}\left(x, y, y', y_a^{(\alpha)}\right) = (x-a)^{1-\alpha} L\left(x, y, y', y_a^{(\alpha)}\right)$.

Proof : Let y^* give an extremum to (20). We define a family of functions as in (17) but with $y \in C^1[a,b]$. From (17) and the boundary conditions $\eta(a) = \eta(b) = 0$, and the fact that $y^*(a) = y_a$ and $y^*(b) = y_b$, it follows that function y is admissible. Because y^* is an extremizer of $\mathcal{J}$, the Gateaux derivative $\delta\mathcal{J}(y^*)$ needs to be identically null. For the functional (20) we have

$$\delta\mathcal{J}(y^*) = \lim_{\epsilon \to 0}\frac{1}{\epsilon}\left(\int_a^b L\left(x, y, y', y_a^{(\alpha)}\right) dx\right.$$
$$\left. - \int_a^b L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right) dx\right)$$
$$= \int_a^b \left(\eta(x)\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y^*}\right.$$
$$\left. + \eta'(x)\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y'^*}\right) dx$$
$$+ \int_a^b \eta_a^{(\alpha)}(x)\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y_a^{*(\alpha)}} dx$$

$$= \int_a^b \eta(x)\left(\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y^*}\right.$$
$$\left. - \frac{d}{dx}\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y'^*}\right) dx$$
$$+ \int_a^b \eta_a^{(\alpha)}(x)\frac{\partial \tilde{L}\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y_a^{*(\alpha)}} d_a^\alpha x = 0$$

where we performed an integration by parts in the second term in the first integral (since $\eta(a) = \eta(b) = 0$), and we rewrote the second integral as a conformable integral by using definition (5). Using the integration by parts formula (7) ($\eta\frac{\partial L}{\partial y_a^{*(\alpha)}}$ is differentiable), we get

$$\delta\mathcal{J}(y^*) = \int_a^b \eta(x)\left(\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y^*}\right.$$
$$\left. - \frac{d}{dx}\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y'^*}\right) dx$$
$$- \int_a^b \eta(x)\frac{d_a^\alpha}{dx_a^\alpha}\frac{\partial \tilde{L}\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y_a^{*(\alpha)}} d_a^\alpha x$$
$$= \int_a^b \eta(x)\left((x-a)^{1-\alpha}\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y^*}\right.$$
$$- (x-a)^{1-\alpha}\frac{d}{dx}\frac{\partial L\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y'^*}$$
$$\left. - \frac{d_a^\alpha}{dx_a^\alpha}\frac{\partial \tilde{L}\left(x, y^*, y'^*, y_a^{*(\alpha)}\right)}{\partial y_a^{*(\alpha)}}\right) d_a^\alpha x = 0 \qquad (22)$$

since $\eta(a) = \eta(b) = 0$. The fractional Euler–Lagrange equation (21) follows from (22) by using the fundamental Lemma 1. ∎

IV. LAGRANGIAN FORMULATION FOR FRICTIONAL FORCES

As an example of potential application of the variational calculus with conformable fractional derivatives, we formulate an action principle for dissipative systems free from the mathematical inconsistencies found in the Riewe approach [23] and far simpler than the formulation proposed in [23]. The action principle we propose states that the equation of motion for dissipative systems is obtained by taking the limit $a \to b$ in the extremal of the action

$$S = \int_a^b L\left(x, x', x_a^{(\alpha)}\right) dt \qquad (23)$$

that satisfy the fractional Euler–Lagrange equation (see (21))

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'} - \frac{1}{(t-a)^{1-\alpha}}\frac{d_a^\alpha}{dt_a^\alpha}\frac{\partial \tilde{L}}{\partial x_a^{(\alpha)}} = 0 \qquad (24)$$

where $\tilde{L}\left(x, x', x_a^{(\alpha)}\right) = (t-a)^{1-\alpha} L\left(x, x', x_a^{(\alpha)}\right)$, $x(t)$ is the path of the particle and t is the time. It is important to emphasize that the condition $a \to b$ (also considered in the original Riewe's approach) applied to the action principle does not imply any restrictions for conservative systems, since in this case $x(t)$ is the action's extremal for any time interval $[a, b]$, even when $a \to b$. Furthermore, our action principle is simpler than the formulation in [23] and free from the mathematical inconsistencies present in Riewe's approach (see [23] for a detailed discussion). In order to show that our method provides us with physical Lagrangians, let us consider the simple problem of a particle under a frictional force proportional to velocity. A quadratic Lagrangian for a particle under a frictional force proportional to the velocity is given by

$$L\left(x, x', x_a^{\left(\frac{1}{2}\right)}\right) = \frac{1}{2}m\left(x'\right)^2 - U(x) + \frac{\gamma}{2}\left(x_a^{\left(\frac{1}{2}\right)}\right)^2 \quad (25)$$

where the three terms in (25) represent the kinetic energy, potential energy, and the fractional linear friction energy, respectively. Note that, differently from Riewe's Lagrangian [11], our Lagrangian (25) is a real function with a linear friction energy, which is physically meaningful. Since the equation of motion is obtained in the limit $a \to b$, if we consider the last term in (25) up to first order in $\Delta t = t - a$, we get

$$\frac{\gamma}{2}\left(x_a^{\left(\frac{1}{2}\right)}\right)^2 = \frac{\gamma}{2}\left(x'\Delta t^{\frac{1}{2}}\right)^2 \approx \frac{\gamma}{2}x'\Delta x$$

that coincides, apart from the multiplicative constant $1/2$, with the work from the frictional force $\gamma x'$ in the displacement $\Delta x \approx x'\Delta t$. The appearance of an additional multiplicative constant is a consequence of the use of fractional derivatives in the Lagrangian and does not appear in the equation of motion after we apply the action principle [23].

Remark 3: It is important to stress that the order of the fractional derivative should be fixed to $\alpha = 1/2$ in order to obtain, by a fractional Lagrangian, a correct equation of motion of a dissipative system. For α different from $1/2$, the Lagrangian does not describe a frictional system under a frictional force proportional to the velocity. Consequently, the fractional linear friction energy makes sense only for $\alpha = 1/2$.

The Lagrangian (25) is physical in the sense it provides physically meaningful relations for the momentum and the Hamiltonian. If we define the canonical variables

$$q_1 = x', \quad q_{\frac{1}{2}} = x_a^{\left(\frac{1}{2}\right)}$$

and

$$p_1 = \frac{\partial L}{\partial q_1} = mx', \quad p_{\frac{1}{2}} = \frac{\partial L}{\partial q_{\frac{1}{2}}} = \gamma x_a^{\left(\frac{1}{2}\right)}$$

we obtain the Hamiltonian

$$H = q_1 p_1 + q_{\frac{1}{2}} p_{\frac{1}{2}} - L = \frac{1}{2}m\left(x'\right)^2 + U(x) + \frac{\gamma}{2}\left(x_a^{\left(\frac{1}{2}\right)}\right)^2. \quad (26)$$

From (26) we can see that the Lagrangian (25) is physical in the sense it provides us a correct relation for the momentum $p_1 = m\dot{x}$ and a physically meaningful Hamiltonian (it is the sum of all energies). Furthermore, the additional fractional momentum $p_{\frac{1}{2}} = \gamma x_a^{\left(\frac{1}{2}\right)}$ goes to zero when we take the limit

$a \to b$, since $x \in C^2[a, b]$.

Finally, the equation of motion for the particle is obtained by inserting our Lagrangian (25) into the Euler–Lagrange equation (24),

$$mx'' + \gamma(t-a)^{-\frac{1}{2}}\frac{d_a^{\frac{1}{2}}}{dt_a^{\frac{1}{2}}}\left[(t-a)^{\frac{1}{2}}x_a^{\left(\frac{1}{2}\right)}\right]$$
$$= mx'' + \gamma x' + \gamma(t-a)x'' = F(x) \quad (27)$$

where we have used (3) since $x \in C^2[a, b]$ and

$$F(x) = -\frac{d}{dx}U(x)$$

is the external force. By taking the limit $a \to b$ with $t \in [a, b]$, we finally obtain the correct equation of motion for a particle under a frictional force:

$$mx'' + \gamma x' = F(x).$$

V. THE CONFORMABLE DUBOIS–REYMOND CONDITION

In the remainder of the present work, we are going to consider only the simplest case where we have no mixed integer and fractional derivatives. We now present the DuBois–Reymond condition in the conformable fractional context. It is an immediate consequence of the chain rule (9) and the Euler–Lagrange equation (16).

Theorem 7 (The conformable fractional DuBois-Reymond condition): If y is an extremal of $\mathcal{J}$ as in (11), then

$$\frac{d_a^\alpha}{dx_a^\alpha}\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right) = \frac{\partial L}{\partial x}\cdot(x-a)^{1-\alpha}. \quad (28)$$

Proof : By the chain rule (9) and the Leibniz rule in Proposition 1:

$$\frac{d_a^\alpha}{dx_a^\alpha}\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right)$$
$$= \frac{\partial L}{\partial x}x_a^{(\alpha)} + \frac{\partial L}{\partial y}y_a^{(\alpha)} + \frac{\partial L}{\partial y_a^{(\alpha)}}\frac{d_a^\alpha}{dx_a^\alpha}y_a^{(\alpha)}$$
$$- \frac{d_a^\alpha}{dx_a^\alpha}\left(\frac{\partial L}{\partial y_a^{(\alpha)}}\right)y_a^{(\alpha)} - \frac{\partial L}{\partial y_a^{(\alpha)}}\frac{d_a^\alpha}{dx_a^\alpha}y_a^{(\alpha)}$$
$$= \frac{\partial L}{\partial x}x_a^{(\alpha)} + y_a^{(\alpha)}\left[\frac{\partial L}{\partial y} - \frac{d_a^\alpha}{dx_a^\alpha}\left(\frac{\partial L}{\partial y_a^{(\alpha)}}\right)\right]$$
$$= \frac{\partial L}{\partial x}\cdot(x-a)^{1-\alpha}.$$

∎

Corollary 1: If (11) is autonomous, that is, if $L = L(y, y_a^{(\alpha)})$ does not depend on x, then

$$\frac{d_a^\alpha}{dx_a^\alpha}\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right) = 0$$

along any extremal y.

Remark 4: When $\alpha = 1$ and $y \in C^1$, Theorem 7 is the classical DuBois–Reymond condition: if $y \in C^1$ is an extremal of $\mathcal{J}(y) = \int_0^1 L(x, y, y')dx$ (i.e., y satisfies (19)), then

$$\frac{d}{dx}\left(L - \frac{\partial L}{\partial y'}y'\right) = \frac{\partial L}{\partial x}.$$

VI. FRACTIONAL INVARIANT CONDITIONS

We consider invariance transformations in the (x, y)-space, depending on a real parameter ϵ. To be more precise, we consider transformations of type

$$\begin{cases} \overline{x} = x + \epsilon\tau(x, y(x)) \\ \overline{y} = y + \epsilon\xi(x, y(x)) \end{cases} \tag{29}$$

where the generators τ and ξ are such that $\overline{x} \geq a$ and there exist $\tau_a^{(\alpha)}$ and $\xi_a^{(\alpha)}$.

Definition 4: We say that the fractional variational integral (11) is invariant under the family of transformations (29) up to the Gauge term Λ, if a function $\Lambda = \Lambda(x, y)$ exists such that for any function y and for any real $x \in [a, b]$, we have

$$L\left(\overline{x}, \overline{y}, \frac{d_a^\alpha \overline{y}}{d\overline{x}_a^\alpha}\right)\frac{d_a^\alpha \overline{x}}{d_a^\alpha x} = L(x, y, y_a^{(\alpha)}) + \epsilon\frac{d_a^\alpha \Lambda}{dx_a^\alpha}(x, y) + o(\epsilon) \tag{30}$$

for all ϵ in some neighborhood of zero, where $\frac{d_a^\alpha \overline{x}}{d_a^\alpha x}$ stands for

$$\frac{\frac{d_a^\alpha \overline{x}}{dx_a^\alpha}}{\frac{d_a^\alpha x}{dx_a^\alpha}} = 1 + \epsilon\frac{\tau_a^{(\alpha)}}{(x - a)^{1-\alpha}}. \tag{31}$$

We note that for $\alpha = 1$ our Definition 4 coincides with the standard approach (see, e.g., [33]). When $\Lambda \equiv 0$, one obtains the concept of absolute invariance. The presence of a new function Λ is due to the presence of external forces in the dynamical system, like friction. The function Λ is called a Gauge term. In fact, many phenomena are nonconservative and this has to be taken into account in the conservation laws [26], [34]. We give an example.

Example 1: Consider the transformation

$$\begin{cases} \overline{x} = x \\ \overline{y} = y + \epsilon\frac{1}{2\alpha}(x - a)^\alpha \end{cases} \tag{32}$$

and the functional

$$\mathcal{J}(y) = \int_a^b \left(y_a^{(\alpha)}(x)\right)^2 d_a^\alpha x. \tag{33}$$

Since

$$\frac{d_a^\alpha}{dx_a^\alpha}\frac{1}{2\alpha}(x - a)^\alpha = \frac{1}{2}$$

it is easy to verify that (33) is invariant under (32) up to the Gauge function $\Lambda = y$.

Definition 5: Given a function $C = C(x, y, y_a^{(\alpha)})$, we say that C is a conserved quantity for (11) if

$$\frac{d_a^\alpha C}{dx_a^\alpha}(x, y(x), y_a^{(\alpha)}(x)) = 0 \tag{34}$$

along any solution y of (16) (i.e., along any extremal of (11)).

Remark 5: Applying the conformable integral (5) to both sides of equation (34), Definition 5 is equivalent to $C(x, y(x), y_a^{(\alpha)}(x)) \equiv$ const.

We now provide a necessary condition of invariance.

Theorem 8: If $\mathcal{J}$ given by (11) is invariant under a family of transformations (29), then

$$\begin{aligned} &\frac{\partial L}{\partial x}\tau + \frac{\partial L}{\partial y}\xi \\ &+ \frac{\partial L}{\partial y_a^{(\alpha)}}\left[\xi_a^{(\alpha)} - y_a^{(\alpha)}\left((\alpha - 1)\frac{\tau}{(x - a)} + \frac{\tau_a^{(\alpha)}}{(x - a)^{1-\alpha}}\right)\right] \\ &+ L\frac{\tau_a^{(\alpha)}}{(x - a)^{1-\alpha}} = \frac{d_a^\alpha \Lambda}{dx_a^\alpha}. \end{aligned} \tag{35}$$

Proof : By the fractional chain rule (see Proposition 1),

$$\begin{aligned} \frac{d_a^\alpha \overline{y}}{d\overline{x}_a^\alpha} &= \frac{\frac{d_a^\alpha \overline{y}}{dx_a^\alpha}}{(\overline{x} - a)^{\alpha-1}\frac{d_a^\alpha \overline{x}}{dx_a^\alpha}} \\ &= \frac{y_a^{(\alpha)} + \epsilon\xi_a^{(\alpha)}}{(x + \epsilon\tau - a)^{\alpha-1}[(x - a)^{1-\alpha} + \epsilon\tau_a^{(\alpha)}]}. \end{aligned}$$

Substituting this formula into (30), differentiating with respect to ϵ and then putting $\epsilon = 0$, we obtain relation (35). ∎

Remark 6: Allowing α to be equal to 1, for $\Lambda \equiv 0$ our equation (35) becomes the standard necessary condition of invariance (cf., e.g., [24]):

$$\frac{\partial L}{\partial x}\tau + \frac{\partial L}{\partial y}\xi + \frac{\partial L}{\partial y'}(\xi' - y'\tau') + L\tau' = 0.$$

For $\alpha = 1$ and an arbitrary Λ, see [33].

In particular, if we consider "time invariance" (i.e., $\tau \equiv 0$), we obtain the following result.

Corollary 2: Let $\overline{y} = y + \epsilon\xi(x, y(x))$ be a transformation that leaves invariant $\mathcal{J}$ in the sense that

$$L(x, \overline{y}, \overline{y}_a^{(\alpha)}) = L(x, y, y_a^{(\alpha)}) + \epsilon\frac{d_a^\alpha \Lambda}{dx_a^\alpha}(x, y) + o(\epsilon).$$

Then,

$$\frac{\partial L}{\partial y}\xi + \frac{\partial L}{\partial y_a^{(\alpha)}}\xi_a^{(\alpha)} = \frac{d_a^\alpha \Lambda}{dx_a^\alpha}.$$

VII. THE CONFORMABLE NOETHER THEOREM

Noether's theorem is a beautiful result with important implications and applications in optimal control [35]−[37]. We provide here a conformable fractional Noether theorem in the context of the calculus of variations. Later, in Section IX, we provide a conformable fractional optimal control version (see Theorem 11).

Theorem 9 (The conformable fractional Noether theorem): If $\mathcal{J}$ given by (11) is invariant under (29) and if y is an extremal of $\mathcal{J}$, then

$$\begin{aligned} &\frac{d_a^\alpha}{dx_a^\alpha}\left[\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right)\tau + \frac{\partial L}{\partial y_a^{(\alpha)}}\xi(x - a)^{1-\alpha}\right] \\ &= (1 - \alpha)\frac{\partial L}{\partial y_a^{(\alpha)}}\left[\xi(x - a)^{1-2\alpha} - \frac{y_a^{(\alpha)}\tau}{(x - a)^\alpha}\right] \\ &+ \frac{d_a^\alpha \Lambda}{dx_a^\alpha}(x - a)^{1-\alpha}. \end{aligned} \tag{36}$$

Proof : From Theorem 8, and using the conformable fractional Euler-Lagrange equation (16) and the DuBois-Reymond condition (28), we deduce successively that

$$\frac{d_a^\alpha \Lambda}{dx_a^\alpha}(x-a)^{1-\alpha}$$

$$= \left[\frac{d_a^\alpha}{dx_a^\alpha}\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right)\frac{\tau}{(x-a)^{1-\alpha}}\right.$$

$$\left. + \frac{d_a^\alpha}{dx_a^\alpha}\left(\frac{\partial L}{\partial y_a^{(\alpha)}}\right)\xi + \frac{\partial L}{\partial y_a^{(\alpha)}}\xi_a^{(\alpha)}\right](x-a)^{1-\alpha}$$

$$- \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\left[\frac{(\alpha-1)\tau}{(x-a)^\alpha} + \tau_a^{(\alpha)})\right] + L\tau_a^{(\alpha)}$$

$$= \left[\frac{d_a^\alpha}{dx_a^\alpha}\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right)\tau + \frac{d_a^\alpha}{dx_a^\alpha}\left(\frac{\partial L}{\partial y_a^{(\alpha)}}\xi\right)(x-a)^{1-\alpha}\right]$$

$$- \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\left[\frac{(\alpha-1)\tau}{(x-a)^\alpha} + \tau_a^{(\alpha)})\right] + L\tau_a^{(\alpha)}$$

$$= \frac{d_a^\alpha}{dx_a^\alpha}\left[\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right)\tau + \frac{\partial L}{\partial y_a^{(\alpha)}}\xi(x-a)^{1-\alpha}\right]$$

$$- \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\left[\frac{(\alpha-1)\tau}{(x-a)^\alpha} + \tau_a^{(\alpha)}\right]$$

$$+ L\tau_a^{(\alpha)} - \left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right)\tau_a^{(\alpha)}$$

$$- \frac{\partial L}{\partial y_a^{(\alpha)}}\xi(1-\alpha)(x-a)^{1-2\alpha}$$

$$= \frac{d_a^\alpha}{dx_a^\alpha}\left[\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right)\tau + \frac{\partial L}{\partial y_a^{(\alpha)}}\xi(x-a)^{1-\alpha}\right]$$

$$+ \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\frac{(1-\alpha)\tau}{(x-a)^\alpha} - \frac{\partial L}{\partial y_a^{(\alpha)}}\xi(1-\alpha)(x-a)^{1-2\alpha}.$$

Thus, we obtain equation (36). ∎

Remark 7: When $\alpha = 1$, equation (36) is simply Noether's conservation law in the presence of external forces: for any extremal of $\mathcal{J}$ and for any family of transformations $(\overline{x}, \overline{y})$ for which $\mathcal{J}$ is invariant, the conservation law

$$\left(L - \frac{\partial L}{\partial y'}y'\right)\tau + \frac{\partial L}{\partial y'}\xi = \Lambda + \text{constant}$$

holds [33, Theorem 2.1]. In addition, if system is conservative ($\Lambda \equiv 0$), then one has the classical Noether theorem

$$\left(L - \frac{\partial L}{\partial y'}y'\right)\tau + \frac{\partial L}{\partial y'}\xi = \text{constant}.$$

Corollary 3 (The conformable fractional Noether theorem under the presence of an external force f): If $\mathcal{J}$ given by (11) is invariant under (29), y is an extremal of $\mathcal{J}$, and the function $f = f(x, y, y_a^{(\alpha)})$ satisfies the equation

$$\frac{d_a^\alpha f}{dx_a^\alpha} = (1-\alpha)\frac{\partial L}{\partial y_a^{(\alpha)}}\left[\xi(x-a)^{1-2\alpha} - \frac{y_a^{(\alpha)}\tau}{(x-a)^\alpha}\right]$$

$$+ \frac{d_a^\alpha \Lambda}{dx_a^\alpha}(x-a)^{1-\alpha}$$

then

$$\left(L - \frac{\partial L}{\partial y_a^{(\alpha)}}y_a^{(\alpha)}\right)\tau + \frac{\partial L}{\partial y_a^{(\alpha)}}\xi(x-a)^{1-\alpha} - f$$

is a conserved quantity.

Corollary 4: If $\mathcal{J}$ given by (11) is invariant under the transformation $\overline{x} = x$, $\overline{y} = y + \epsilon\xi(x, y(x))$, and if y is an extremal of $\mathcal{J}$, then

$$\frac{\partial L}{\partial y_a^{(\alpha)}}\xi - \Lambda$$

is a conserved quantity.

Proof : Given that

$$\frac{d_a^\alpha(x-a)^{1-\alpha}}{dx_a^\alpha} = (1-\alpha)(x-a)^{1-2\alpha}$$

the result follows immediately from Theorem 9. ∎

VIII. THE HAMILTONIAN FORMALISM

The Hamiltonian formalism is related to the Lagrangian one by the so called Legendre transformation, from coordinates and velocities to coordinates and momenta. Let the momenta be given by

$$p(x) = \frac{\partial L}{\partial y_a^{(\alpha)}}(x, y(x), y_a^{(\alpha)}(x)) \qquad (37)$$

and the Hamiltonian function by

$$H(x, y, v, \psi) = -L(x, y, v) + \psi v. \qquad (38)$$

To simplify notation, $[y](x)$ and $\{y\}(x)$ will denote $(x, y(x), y_a^{(\alpha)}(x))$ and $(x, y(x), y_a^{(\alpha)}(x), p(x))$, respectively. Differentiating (38), and using definition (37), it follows that

$$\frac{d_a^\alpha H}{dx_a^\alpha}\{y\}(x)$$

$$= -\frac{\partial L}{\partial x}[y](x)x_a^{(\alpha)} - \frac{\partial L}{\partial y}[y](x) \cdot y_a^{(\alpha)}(x)$$

$$- \frac{\partial L}{\partial v}[y](x) \cdot \frac{d_a^\alpha}{dx_a^\alpha}y_a^{(\alpha)}(x) + p_a^{(\alpha)}(x) \cdot y_a^{(\alpha)}(x)$$

$$+ \frac{\partial L}{\partial v}[y](x) \cdot \frac{d_a^\alpha}{dx_a^\alpha}y_a^{(\alpha)}(x)$$

$$= -\frac{\partial L}{\partial x}[y](x) \cdot (x-a)^{1-\alpha} - \frac{\partial L}{\partial y}[y](x) \cdot y_a^{(\alpha)}(x)$$

$$+ p_a^{(\alpha)}(x) \cdot y_a^{(\alpha)}(x). \qquad (39)$$

On the other hand, by the definition of Hamiltonian (38), one has immediately that

$$\begin{cases} \dfrac{\partial H}{\partial x}(x, y, v, \psi) = -\dfrac{\partial L}{\partial x}(x, y, v) \\[2mm] \dfrac{\partial H}{\partial y}(x, y, v, \psi) = -\dfrac{\partial L}{\partial y}(x, y, v) \\[2mm] \dfrac{\partial H}{\partial \psi}(x, y, v, \psi) = v \end{cases}$$

and so we can write (39) in the form

$$\frac{d_a^\alpha H}{dx_a^\alpha}\{y\}(x)$$
$$= \frac{\partial H}{\partial x}\{y\}(x)(x-a)^{1-\alpha}$$
$$+ \frac{\partial H}{\partial y}\{y\}(x) \cdot y_a^{(\alpha)}(x) + \frac{\partial H}{\partial \psi}\{y\}(x) \cdot p_a^{(\alpha)}(x). \quad (40)$$

If y is an extremal of $\mathcal{J}$, then by the conformable fractional Euler-Lagrange equation (16) one has

$$\frac{\partial L}{\partial y}[y](x) - \frac{d_a^\alpha}{dx_a^\alpha}\left(\frac{\partial L}{\partial v}[y]\right)(x) = -\frac{\partial H}{\partial y}\{y\}(x) - p_a^{(\alpha)}(x) = 0$$

and we can write

$$\begin{cases} y_a^{(\alpha)}(x) = \dfrac{\partial H}{\partial \psi}\{y\}(x) \\[2mm] p_a^{(\alpha)}(x) = -\dfrac{\partial H}{\partial y}\{y\}(x). \end{cases} \quad (41)$$

The system (41) is nothing else than the conformable fractional Euler-Lagrange equation in Hamiltonian form. Substituting the expressions of (41) into (40), we get the analog to the DuBois–Reymond condition (28) in Hamiltonian form:

$$\frac{d_a^\alpha H}{dx_a^\alpha}\{y\}(x) = \frac{\partial H}{\partial x}\{y\}(x)(x-a)^{1-\alpha}. \quad (42)$$

If the Lagrangian L is autonomous, i.e., L does not depend on x, then

$$\frac{\partial L}{\partial x} = 0$$

and, consequently, by (42) H is a conserved quantity. If the Lagrangian L does not depend on y, then

$$\frac{\partial L}{\partial y} = -\frac{\partial H}{\partial y} = 0$$

and so $p_a^{(\alpha)} = 0$, i.e., p is a conserved quantity.

We now exhibit Corollary 3 within the Hamiltonian framework.

Theorem 10 (Conformable fractional Noether's theorem in Hamiltonian form under the presence of an external force f): If $\mathcal{J}$ given by (11) is invariant under (29), y is an extremal of $\mathcal{J}$, and function $f = f(x, y(x), y_a^{(\alpha)}(x))$ satisfies the equation

$$\frac{d_a^\alpha f}{dx_a^\alpha}(x, y(x), y_a^{(\alpha)}(x))$$
$$= (1-\alpha)p(x)\left[\xi(x-a)^{1-2\alpha} - \frac{y_a^{(\alpha)}(x)\tau}{(x-a)^\alpha}\right]$$
$$+ \frac{d_a^\alpha \Lambda}{dx_a^\alpha}(x, y(x))(x-a)^{1-\alpha}$$

then

$$p(x)\xi(x-a)^{1-\alpha} - H\{y\}(x)\tau - f(x, y(x), y_a^{(\alpha)}(x))$$

is a conserved quantity.

IX. CONFORMABLE FRACTIONAL OPTIMAL CONTROL

The conformable fractional optimal control problem is stated as follows: find a pair of functions $(y(\cdot), v(\cdot))$ that

minimizes

$$\mathcal{J}(y, v) = \int_a^b L(x, y(x), v(x)) \, d_a^\alpha x \quad (43)$$

when subject to the (nonautonomous) fractional control system

$$y_a^{(\alpha)}(x) = \varphi(x, y(x), v(x)). \quad (44)$$

A pair $(y(\cdot), v(\cdot))$ that minimizes functional (43) subject to (44) is called an optimal process. The reader interested on the fractional optimal control theory is referred to [28], [29], [38]. Here we note that if $\alpha = 1$, then (43) and (44) is the standard optimal control problem: to minimize

$$\mathcal{J}(y, v) = \int_a^b L(x, y(x), v(x)) \, dx$$

subject to the control system

$$y'(x) = \varphi(x, y(x), v(x)).$$

We assume that the Lagrangian L and the velocity vector φ are functions at least of class C^1 in their domain $[a, b] \times \mathbb{R}^2$. Also, the admissible state trajectories y are such that $y_a^{(\alpha)}$ exist.

Remark 8: In case $\varphi \equiv v$, the previous problem (43) and (44) reduces to the fundamental problem of the conformable fractional variational calculus (11), as stated in Section III.

Following the standard approach [37], [39], we consider the augmented conformable fractional functional

$$\mathcal{I}(y, v, p) = \int_a^b [\, L(x, y(x), v(x)) + p(x)(y_a^{(\alpha)}(x)$$
$$- \varphi(x, y(x), v(x)))] \, d_a^\alpha x \quad (45)$$

where p is such that $p_a^{(\alpha)}$ exists. Consider a variation vector of type $(y + \epsilon y_1, v + \epsilon v_1, p + \epsilon p_1)$ with $|\epsilon| \ll 1$. For convenience, we restrict ourselves to the case $y_1(a) = y_1(b) = 0$. If $(y(\cdot), v(\cdot))$ is an optimal process, then the first variation is zero when $\epsilon = 0$. Thus, using the conformable fractional integration by parts formula (Theorem 3), we obtain that

$$0 = \int_a^b \left[\frac{\partial L}{\partial y}y_1 + \frac{\partial L}{\partial v}v_1 + p_1(y_a^{(\alpha)} - \varphi)\right.$$
$$\left. + p\left(y_{1a}^{(\alpha)} - \frac{\partial \varphi}{\partial y}y_1 - \frac{\partial \varphi}{\partial v}v_1\right)\right] d_a^\alpha x$$
$$= \int_a^b \left[y_1\left(\frac{\partial L}{\partial y} - p\frac{\partial \varphi}{\partial y} - p_a^{(\alpha)}\right) + v_1\left(\frac{\partial L}{\partial v} - p\frac{\partial \varphi}{\partial v}\right)\right.$$
$$\left. + p_1(y_a^{(\alpha)} - \varphi)\right] d_a^\alpha x.$$

By the arbitrariness of the the variation functions, we obtain the following system, called the Euler–Lagrange equations for the conformable fractional optimal control problem:

$$\begin{cases} y_a^{(\alpha)}(x) = \varphi(x, y(x), v(x)) \\[2mm] p_a^{(\alpha)}(x) = \dfrac{\partial L}{\partial y}(x, y(x), v(x)) - p(x)\dfrac{\partial \varphi}{\partial y}(x, y(x), v(x)) \\[2mm] \dfrac{\partial L}{\partial v}(x, y(x), v(x)) - p(x)\dfrac{\partial \varphi}{\partial v}(x, y(x), v(x)) = 0. \end{cases}$$
$$(46)$$

These equations give necessary conditions for finding the

optimal solutions of problem (43) and (44). We remark that they are similar to the standard ones, in case of integer order derivatives, but in this case they contain conformable fractional derivatives, as expected. The solution can be stated using the Hamiltonian formalism. Consider the Hamiltonian function

$$H(x, y, v, p) = -L(x, y, v) + p(x)\varphi(x, y, v). \quad (47)$$

Then (46) gives:

1) The fractional Hamiltonian system

$$\begin{cases} y_a^{(\alpha)}(x) = \dfrac{\partial H}{\partial p}(x, y, v, p) \\[2mm] p_a^{(\alpha)}(x) = -\dfrac{\partial H}{\partial y}(x, y, v, p). \end{cases} \quad (48)$$

2) The stationary condition

$$\frac{\partial H}{\partial v}(x, y, v, p) = 0. \quad (49)$$

Definition 6: Any triplet (y, v, p) satisfying system (48) and (49) is called a conformable fractional Pontryagin extremal.

Remark 9: In the particular case $\varphi \equiv v$, i.e., when the conformable fractional optimal control problem is reduced to the fundamental conformable fractional problem of the calculus of variations, we obtain

$$H = -L(x, y, v) + pv, \quad y_a^{(\alpha)} = v$$

and the equations

$$p_a^{(\alpha)} = -\frac{\partial H}{\partial y} = \frac{\partial L}{\partial y}, \quad p = \frac{\partial L}{\partial v}.$$

Therefore, we obtain the conformable fractional Euler–Lagrange equation (16):

$$\frac{\partial L}{\partial y} = \frac{d_a^\alpha}{dx_a^\alpha}\left(\frac{\partial L}{\partial y_a^{(\alpha)}}\right).$$

Let us now considerer the augmented fractional variational functional (45) written in the Hamiltonian form:

$$\mathcal{I}(y, v, p) = \int_0^1 \left(-H(x, y(x), v(x), p(x)) + p(x)y_a^{(\alpha)}(x)\right) d_a^\alpha x \quad (50)$$

where H is given by expression (47). For a parameter ϵ, with $|\epsilon| \ll 1$, consider the family of transformations

$$\begin{cases} \overline{x} = x + \epsilon\tau(x, y(x), v(x), p(x)) \\ \overline{y} = y + \epsilon\xi(x, y(x), v(x), p(x)) \\ \overline{v} = v + \epsilon\sigma(x, y(x), v(x), p(x)) \\ \overline{p} = p + \epsilon\pi(x, y(x), v(x), p(x)). \end{cases} \quad (51)$$

We now define the notion of invariance of (43)−(44) in terms of the Hamiltonian H and the augmented conformable fractional variational functional (50).

Definition 7: The conformable fractional optimal control problem (43) and (44) is invariant under the transformations (51) up to the Gauge term Λ, if a function $\Lambda = \Lambda(x, y)$ exists such that for any functions y, v and p, and for any real

$x \in [0, 1]$, the following equality holds:

$$\left[-H(\overline{x}, \overline{y}, \overline{v}, \overline{p}) + \overline{p}\,\frac{d_a^\alpha \overline{y}}{d\overline{x}_a^\alpha}\right]\frac{d_a^\alpha \overline{x}}{d_a^\alpha x}$$

$$= -H(x, y, v, p) + py_a^{(\alpha)} + \epsilon\frac{d_a^\alpha \Lambda}{dx_a^\alpha}(x, y) + o(\epsilon) \quad (52)$$

for all ϵ in some neighborhood of zero, where as in Definition 4 $\frac{d_a^\alpha \overline{x}}{d_a^\alpha x}$ stands for (31).

Theorem 11 (Fractional Noether's theorem for the fractional optimal control problem (43)−(44)): If (43) and (44) is invariant under (51) in the sense of Definition 7, and if (y, v, p) is a conformable fractional Pontryagin extremal, then

$$\frac{d_a^\alpha}{dx_a^\alpha}(p\xi) - \tau\left(\frac{\partial H}{\partial x} + (\alpha - 1)\frac{py_a^{(\alpha)}}{x - a}\right)$$

$$- H\frac{\tau_a^{(\alpha)}}{(x - a)^{1-\alpha}} = \frac{d_a^\alpha \Lambda}{dx_a^\alpha}. \quad (53)$$

Proof : Differentiating (52) with respect to ϵ, then choosing $\epsilon = 0$, we get

$$-\frac{\partial H}{\partial x}\tau - \frac{\partial H}{\partial y}\xi - \frac{\partial H}{\partial v}\sigma - \frac{\partial H}{\partial p}\pi + \pi y_a^{(\alpha)}$$

$$+ p\left[\xi_a^{(\alpha)} - y_a^{(\alpha)}\left((\alpha - 1)\frac{\tau}{x - a} + \frac{\tau_a^{(\alpha)}}{(x - a)^{1-\alpha}}\right)\right]$$

$$+ \left[-H + py_a^{(\alpha)}\right]\frac{\tau_a^{(\alpha)}}{(x - a)^{1-\alpha}} = \frac{d_a^\alpha \Lambda}{dx_a^\alpha}.$$

Equation (53) follows because (y, v, p) is a conformable fractional Pontryagin extremal. ∎

Remark 10: When $\alpha = 1$ and $\Lambda = 0$, equation (53) becomes

$$\frac{d}{dx}(p\xi) - \tau\frac{\partial H}{\partial x} - H\tau' = 0.$$

Using relations (48) and (49) with $\alpha = 1$, we deduce that

$$-H\tau + p\xi \equiv \text{constant}$$

which is the optimal control version of Noether's theorem [35]−[37]. For $\alpha \in (0, 1)$, Theorem 11 extends the main result of [28].

X. THE MULTI-DIMENSIONAL CASE

In this section, we show a necessary condition of invariance, when the Lagrangian depends on two independent variables x_1 and x_2 and on m functions $y_1, \ldots, y_m$. First, we define conformable fractional partial derivatives and conformable multiple fractional integrals in a natural way, similarly as done in the integer case. In addition, we are going to use the following properties.

Theorem 12 (Conformable Green's theorem for a rectangle): Let f and g be two continuous and α-differentiable functions whose domains contain $R = [a, b] \times [c, d] \subset \mathbb{R}^2$. Then,

$$\int_a^b (f(x_1, c) - f(x_1, d))\, d_a^\alpha x_1 + \int_c^d (g(b, x_2) - g(a, x_2))\, d_c^\alpha x_2$$

$$= \int_R \left(\frac{\partial_a^\alpha}{\partial x_{1a}^\alpha}g(x_1, x_2) - \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha}f(x_1, x_2)\right) d_a^\alpha x_1 d_c^\alpha x_2.$$

$$(54)$$

Proof : By Theorem 2, we have

$$f(x_1, d) - f(x_1, c) = \int_c^d \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} f(x_1, x_2) d_c^\alpha x_2$$

and

$$g(b, x_2) - g(a, x_2) = \int_a^b \frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} g(x_1, x_2) d_a^\alpha x_1.$$

Therefore,

$$\int_a^b (f(x_1, c) - f(x_1, d)) \, d_a^\alpha x_1 + \int_c^d (g(b, x_2) - g(a, x_2)) \, d_c^\alpha x_2$$

$$= -\int_a^b \int_c^d \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} f(x_1, x_2) d_c^\alpha x_2 d_a^\alpha x_1$$

$$+ \int_c^d \int_a^b \frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} g(x_1, x_2) d_a^\alpha x_1 d_c^\alpha x_2$$

$$= \int_R \left(\frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} g(x_1, x_2) - \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} f(x_1, x_2) \right) d_a^\alpha x_1 d_c^\alpha x_2.$$

∎

Remark 11: From Definition 2 and Remark 1, it is easy to verify that for C^1 functions our fractional Green's theorem over a rectangular domain (Theorem 12) reduces to the conventional Green's identity for

$$\tilde{f}(x_1, x_2) = f(x_1, x_2)(x_1 - a)^{\alpha - 1}$$

and

$$\tilde{g}(x_1, x_2) = g(x_1, x_2)(x_2 - a)^{\alpha - 1}.$$

Lemma 2: Let F, G and h be α-differentiable continuous functions whose domains contain $R = [a, b] \times [c, d]$. If $h = 0$ on the boundary ∂R of R, then

$$\int_R \left(G(x_1, x_2) \frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} h(x_1, x_2) \right.$$

$$\left. - F(x_1, x_2) \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} h(x_1, x_2) \right) d_a^\alpha x_1 d_c^\alpha x_2$$

$$= -\int_R \left(\frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} G(x_1, x_2) - \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} F(x_1, x_2) \right)$$

$$\times h(x_1, x_2) d_a^\alpha x_1 d_c^\alpha x_2. \qquad (55)$$

Proof : By choosing $f = Fh$ and $g = Gh$ in Green's formula (54), we obtain that

$$\int_a^b (F(x_1, c) h(x_1, c) - F(x_1, d) h(x_1, d)) \, d_a^\alpha x_1$$

$$+ \int_c^d (G(b, x_2) g(b, x_2) - G(a, x_2) h(a, x_2)) \, d_c^\alpha x_2$$

$$= \int_R \left(\frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} G(x_1, x_2) \right.$$

$$\left. - \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} F(x_1, x_2) \right) h(x_1, x_2) d_a^\alpha x_1 d_c^\alpha x_2$$

$$+ \int_R \left(G(x_1, x_2) \frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} h(x_1, x_2) \right.$$

$$\left. - F(x_1, x_2) \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} h(x_1, x_2) \right) d_a^\alpha x_1 d_c^\alpha x_2.$$

Since $h = 0$ on the boundary ∂R of R, we have

$$\int_R \left(G(x_1, x_2) \frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} h(x_1, x_2) \right.$$

$$\left. - F(x_1, x_2) \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} h(x_1, x_2) \right) d_a^\alpha x_1 d_c^\alpha x_2$$

$$= -\int_R \left(\frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} G(x_1, x_2) - \frac{\partial_c^\alpha}{\partial x_{2c}^\alpha} F(x_1, x_2) \right)$$

$$\times h(x_1, x_2) d_a^\alpha x_1 d_c^\alpha x_2.$$

∎

Remark 12: In the very recent and general paper [40], a vector calculus with deformed derivatives (as the conformable derivative) is formally introduced. We refer the reader to [40] for a detailed discussion of a vector calculus with deformed derivatives and more properties on the multi-dimensional conformable calculus.

Let us now consider the fractional variational integral

$$\mathcal{J}(y) = \int_R L \left(x, y, \frac{\partial_a^\alpha y}{\partial x_a^\alpha} \right) d_a^\alpha x \qquad (56)$$

where for simplicity we choose $R = [a, b] \times [a, b]$, and where $x = (x_1, x_2)$, $y = (y_1, \ldots, y_m)$, $d_a^\alpha x = d_a^\alpha x_1 d_a^\alpha x_2$, and

$$\frac{\partial_a^\alpha y}{\partial x_a^\alpha} = \left(\frac{\partial_a^\alpha y_1}{\partial x_{1a}^\alpha}, \frac{\partial_a^\alpha y_1}{\partial x_{2a}^\alpha}, \ldots, \frac{\partial_a^\alpha y_m}{\partial x_{1a}^\alpha}, \frac{\partial_a^\alpha y_m}{\partial x_{2a}^\alpha} \right).$$

We are assuming that the Lagrangian

$$L = L(x_1, x_2, y_1, \ldots, y_m, v_{1,1}, v_{1,2}, \ldots, v_{m,1}, v_{m,2})$$

is at least of class C^1, that the domains of y_k, $k \in \{1, \ldots, m\}$, contain R, and that all these partial conformable fractional derivatives exist.

Theorem 13 (The multi-dimensional fractional Euler-Lagrange equation): Let y be an extremizer of (56) with

$$y|_{\partial R} = \psi(x_1, x_2)$$

for some given function $\psi = (\psi_1, \ldots, \psi_m)$. Then, the following equation holds:

$$\frac{\partial L}{\partial y_k} - \frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} \left(\frac{\partial L}{\partial v_{k,1}} \right) - \frac{\partial_a^\alpha}{\partial x_{2a}^\alpha} \left(\frac{\partial L}{\partial v_{k,2}} \right) = 0 \qquad (57)$$

for all $k \in \{1, \ldots, m\}$.

Proof : Let $y^* = (y_1^*, \ldots, y_m^*)$ give an extremum to (56). We define m families of functions

$$y_k(x_1, x_2) = y_k^*(x_1, x_2) + \epsilon \eta_k(x_1, x_2) \qquad (58)$$

where $k \in \{1, \ldots, m\}$, ϵ is a constant, and η_k is an arbitrary α-differentiable function satisfying the boundary conditions $\eta_k|_{\partial R} = 0$ (weak variations). From (58), the boundary conditions $\eta_k|_{\partial R} = 0$ and $y_k|_{\partial R} = \psi_k(x_1, x_2)$, it follows that function y_k is admissible. Let the Lagrangian L be C^1. Because y^* is an extremizer of functional $\mathcal{J}$, the Gateaux derivative

$\delta\mathcal{J}(y^*)$ needs to be identically null. For the functional (56),

$$\delta\mathcal{J}(y^*) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\int_R L\left(x, y, \frac{\partial_a^\alpha y}{\partial x_a^\alpha}\right) d_a^\alpha x \right.$$

$$- \left. \int_R L\left(x, y^*, \frac{\partial_a^\alpha y^*}{\partial x_a^\alpha}\right) d_a^\alpha x \right)$$

$$= \sum_{k=1}^m \int_R \left(\eta_k(x_1, x_2) \frac{\partial L\left(x, y^*, \frac{\partial_a^\alpha y^*}{\partial x_a^\alpha}\right)}{\partial y_k^*} \right.$$

$$+ \frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} \eta_k(x_1, x_2) \frac{\partial L\left(x, y^*, \frac{\partial_a^\alpha y^*}{\partial x_a^\alpha}\right)}{\partial v_{k,1}}$$

$$+ \left. \frac{\partial_a^\alpha}{\partial x_{2a}^\alpha} \eta_k(x_1, x_2) \frac{\partial L\left(x, y^*, \frac{\partial_a^\alpha y^*}{\partial x_a^\alpha}\right)}{\partial v_{k,2}} \right) d_a^\alpha x$$

$$= 0.$$

Using (55), we get that

$$\sum_{k=1}^m \int_R \eta_k(x_1, x_2)\left(\frac{\partial L\left(x, y^*, \frac{\partial_a^\alpha y^*}{\partial x_a^\alpha}\right)}{\partial y_k^*} \right.$$

$$- \frac{\partial_a^\alpha}{\partial x_{1a}^\alpha} \frac{\partial L\left(x, y^*, \frac{\partial_a^\alpha y^*}{\partial x_a^\alpha}\right)}{\partial v_{k,1}}$$

$$- \left. \frac{\partial_a^\alpha}{\partial x_{2a}^\alpha} \frac{\partial L\left(x, y^*, \frac{\partial_a^\alpha y^*}{\partial x_a^\alpha}\right)}{\partial v_{k,2}} \right) d_a^\alpha x = 0 \qquad (59)$$

since $\eta_k|_{\partial R} = 0$. The fractional Euler–Lagrange equation (57) follows from (59) by using the fundamental lemma of the conformable fractional calculus of variations (Lemma 1). ∎

Let ϵ be a real, and consider the following family of transformations:

$$\begin{cases} \overline{x}_i = x_i + \epsilon\tau_i(x, y(x)), & i \in \{1, 2\}, \\ \overline{y}_k = y_k + \epsilon\xi_k(x, y(x)), & k \in \{1, \ldots, m\} \end{cases} \qquad (60)$$

where τ_i and ξ_k are such that there exist $\frac{\partial_a^\alpha \tau_i}{\partial x_{j a}^\alpha}$ and $\frac{\partial_a^\alpha \xi_k}{\partial x_{j a}^\alpha}$ for all $i, j \in \{1, 2\}$ and all $k \in \{1, \ldots, m\}$. Denote by $\left[\frac{\partial_a^\alpha \overline{x}}{\partial_a^\alpha x}\right]$ the matrix

$$\begin{bmatrix} \dfrac{\partial_a^\alpha \overline{x}_1}{\partial x_{1a}^\alpha} \dfrac{\partial_a^\alpha x_1}{\partial x_{1a}^\alpha} & \dfrac{\partial_a^\alpha \overline{x}_1}{\partial x_{2a}^\alpha} \dfrac{\partial_a^\alpha x_2}{\partial x_{2a}^\alpha} \\ \dfrac{\partial_a^\alpha \overline{x}_2}{\partial x_{1a}^\alpha} \dfrac{\partial_a^\alpha x_1}{\partial x_{1a}^\alpha} & \dfrac{\partial_a^\alpha \overline{x}_2}{\partial x_{2a}^\alpha} \dfrac{\partial_a^\alpha x_2}{\partial x_{1a}^\alpha} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \dfrac{\epsilon}{(x_1 - a)^{1-\alpha}} \dfrac{\partial_a^\alpha \tau_1}{\partial x_{1a}^\alpha} & \dfrac{\epsilon}{(x_2 - a)^{1-\alpha}} \dfrac{\partial_a^\alpha \tau_1}{\partial x_{2a}^\alpha} \\ \dfrac{\epsilon}{(x_1 - a)^{1-\alpha}} \dfrac{\partial_a^\alpha \tau_2}{\partial x_{1a}^\alpha} & 1 + \dfrac{\epsilon}{(x_2 - a)^{1-\alpha}} \dfrac{\partial_a^\alpha \tau_2}{\partial x_{2a}^\alpha} \end{bmatrix}.$$

Definition 8: Functional $\mathcal{J}$ as in (56) is invariant under the family of transformation (60) if for all y_k and for all $x_i \in [0, 1]$ we have

$$L\left(\overline{x}, \overline{y}, \frac{\partial_a^\alpha \overline{y}}{\partial \overline{x}_a^\alpha}\right) \det\left[\frac{\partial_a^\alpha \overline{x}}{\partial_a^\alpha x}\right]$$

$$= L\left(x, y, \frac{\partial_a^\alpha y}{\partial x_a^\alpha}\right) + \epsilon \frac{d_a^\alpha \Lambda}{dx_a^\alpha}(x, y) + o(\epsilon)$$

for all ϵ in some neighborhood of zero.

Using the same techniques as in the proof of Theorem 8, we obtain a necessary condition of invariance for the fractional variational problem (56).

Theorem 14: If $\mathcal{J}$ given by (56) is invariant under transformations (60), then

$$\sum_{i=1}^2 \frac{\partial L}{\partial x_i} \tau_i + \sum_{k=1}^m \frac{\partial L}{\partial y_k} \xi_k + \sum_{k=1}^m \sum_{i=1}^2 \frac{\partial L}{\partial v_{k,i}} \left[\frac{\partial_a^\alpha \xi_k}{\partial x_{i a}^\alpha} \right.$$

$$- \frac{\partial_a^\alpha y_k}{\partial x_{i a}^\alpha}\left((\alpha - 1)\frac{\tau_i}{x_i - a} + \frac{1}{(x_i - a)^{1-\alpha}} \frac{\partial_a^\alpha \tau_i}{\partial x_{i a}^\alpha}\right) \right]$$

$$+ L\left(\frac{1}{(x_1 - a)^{1-\alpha}} \frac{\partial_a^\alpha \tau_1}{\partial x_{1a}^\alpha} + \frac{1}{(x_2 - a)^{1-\alpha}} \frac{\partial_a^\alpha \tau_2}{\partial x_{2a}^\alpha}\right) = \frac{d^\alpha \Lambda}{dx^\alpha}. \tag{61}$$

Proof : Using relations

$$\frac{\partial_a^\alpha \overline{y}_k}{\partial \overline{x}_{i a}^\alpha} = \frac{\dfrac{\partial_a^\alpha y_k}{\partial x_{i a}^\alpha} + \epsilon \dfrac{\partial_a^\alpha \xi_k}{\partial x_{i a}^\alpha}}{(x_i + \epsilon\tau_i - a)^{\alpha-1}\left[(x_i - a)^{1-\alpha} + \epsilon\dfrac{\partial_a^\alpha \tau_i}{\partial x_{i a}^\alpha}\right]}$$

and

$$\frac{d}{d\epsilon}\det\left[\frac{\partial^\alpha \overline{x}}{\partial^\alpha x}\right]\Bigg|_{\epsilon=0}$$

$$= \frac{1}{(x_1 - a)^{1-\alpha}} \frac{\partial_a^\alpha \tau_1}{\partial x_{1a}^\alpha} + \frac{1}{(x_2 - a)^{1-\alpha}} \frac{\partial_a^\alpha \tau_2}{\partial x_{2a}^\alpha}$$

we conclude that (61) holds. ∎

Remark 13: When $\alpha = 1$ and $\Lambda \equiv 0$, Theorem 14 reduces to the standard one [24]: equality (61) simplifies to

$$\sum_{i=1}^2 \frac{\partial L}{\partial x_i} \tau_i + \sum_{k=1}^m \frac{\partial L}{\partial y_k} \xi_k + \sum_{k=1}^m \sum_{i=1}^2 \frac{\partial L}{\partial v_{k,i}} \left[\frac{\partial \xi_k}{\partial x_i} - \frac{\partial y_k}{\partial x_i} \frac{\partial \tau_i}{\partial x_i} \right]$$

$$+ L\left(\frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2}\right) = 0.$$

Corollary 5: If $\mathcal{J}$ given by (56) is invariant under (60), $\tau_1 \equiv 0 \equiv \tau_2$, and no Gauge term is involved (i.e., $\Lambda \equiv 0$), then

$$\sum_{k=1}^m \frac{\partial L}{\partial y_k} \xi_k + \sum_{k=1}^m \sum_{i=1}^2 \frac{\partial L}{\partial v_{k,i}} \frac{\partial_a^\alpha \xi_k}{\partial x_{i a}^\alpha} = 0.$$

It remains an open question how to obtain a Noether constant of motion for the conformable fractional multi-dimensional case.

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