The Exp-function Method for Some Time-fractional Differential Equations

Ahmet Bekir, Ozkan Guner, and Adem Cevikel

Abstract—In this article, the fractional derivatives in the sense of modified Riemann-Liouville derivative and the Exp-function method are employed for constructing the exact solutions of nonlinear time fractional partial differential equations in mathematical physics. As a result, some new exact solutions for them are successfully established. It is indicated that the solutions obtained by the Exp-function method are reliable, straightforward and effective method for strongly nonlinear fractional partial equations with modified Riemann-Liouville derivative by Jumarie's. This approach can also be applied to other nonlinear time and space fractional differential equations.

Index Terms—Exact solution, exp-function method, fractional differential equation.

I. INTRODUCTION

F RACTIONAL partial differential equations (FPDEs) have gained much attention as they are widely used to describe various complex phenomena in various applications such as the fluid flow, signal processing, control theory, systems identification, finance and fractional dynamics, physics and other areas. Oldman and Spanier first considered the partial fractional differential equations arising in diffusion problems [1]. The fractional partial differential equations have been investigated by many researchers [2]–[4].

In recent decades, a large amount of literature has been provided to construct the exact solutions of fractional ordinary differential equations and fractional partial differential equations of physical interest. Many powerful and efficient methods have been proposed to obtain exact solutions of fractional partial differential equations, such as the fractional sub-equation method, the first integral method, the (G'/G)-expansion method exp-function method and so on [5]-[19].

The exp-function method [20]–[27] can be used to construct the exact solutions for some time and space fractional differential equations. The present paper investigates for the first time the applicability and effectiveness of the exp-function method on fractional nonlinear partial differential equations. The objective of this paper is to extend the application of the

Manuscript received April 19, 2005; accepted September 17, 2014. Recommended by Associate Editor Dingyü Xue.

Citation: A. Bekir, O. Guner, and A. Cevikel, "The exp-function method for some time-fractional differential equations," *IEEE/CAA Journal of Automatica Sinica*, vol. 4, no. 2, pp. 315–321, Apr. 2017.

A. Bekir is with the Department of Mathematics-computer, Eskisehir Osmangazi University, Eskisehir 26480, Turkey (e-mail: abekir@ogu.edu.tr).
O. Guner is with the Department of International Trade, Cankiri Karatekin

University, Cankiri 18200, Turkey (e-mail: ozkanguner@karatekin.edu.tr).

A. Cevikel is with the Department of Mathematics Education, Yildiz Technical University, Istanbul 34220, Turkey (e-mail: acevikel@yildiz.edu.tr).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/JAS.2016.7510172

exp-function method to obtain exact solutions to some fractional partial differential equations in mathematical physics. These equations include Fitzhugh-Nagumo equation and KdV equation.

This Letter is organized as follows: In Section II, some basic properties of Jumarie's modified Riemann-Liouville derivative are given. The main steps of the exp-function method is given in Section III. In Sections IV and V, we construct the exact solutions of the time fractional Fitzhugh-Nagumo and KdV equations via this method. Some conclusions and discussions are shown in Section VI.

II. MODIFIED RIEMANN-LIOUVILLE DERIVATIVE

In decades years, in order to improve the local behavior of fractional types, a few local versions of fractional derivatives have been proposed, i.e., the Caputo's fractional derivative [28], the Grünwald-Letnikov's fractional derivative [29], the Riemann-Liouville's derivative [29], the Jumarie's modified Riemann-Liouville derivative [30], [31]. The Jumarie's derivative is defined as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} \big[f(\xi) - f(0) \big] d\xi, 0 < \alpha < 1$$
⁽¹⁾

where $f: \mathbb{R} \to \mathbb{R}$, $t \to f(t)$ denotes a continuous (but not necessarily first-order-differentiable) function. We list some properties for the modified Riemann–Liouville derivative as follows:

Property 1:

$$D_t^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0.$$
⁽²⁾

Property 2:

$$D_t^{\alpha}(cf(t)) = cD_t^{\alpha}f(t), \quad c = \text{constant.}$$
(3)

Property 3:

$$D_t^{\alpha}\{af(t) + bg(t)\} = aD_t^{\alpha}f(t) + bD_t^{\alpha}g(t)$$
(4)

where a and b constants.

Property 4:

$$D_t^{\alpha}c = 0, \quad c = \text{constant.}$$
 (5)

III. THE EXP-FUNCTION METHOD

We consider the following general nonlinear FPDE of the type

$$F(u, D_t^{\alpha} u, D_x^{\beta} u, D_y^{\psi} u, D_t^{\alpha} D_t^{\alpha} u, D_t^{\alpha} D_x^{\beta} u, D_x^{\beta} D_x^{\beta} u, D_x^{\beta} D_y^{\psi} u, D_y^{\psi} D_y^{\psi} u, \dots) = 0, \quad 0 < \alpha, \beta, \psi < 1$$
(6)

where u is an unknown function, and F is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of the exp-function method.

Step 1: Li and He [32] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$\xi = \frac{u(x, y, t) = U(\xi)}{\Gamma(1+\beta)} + \frac{\delta y^{\psi}}{\Gamma(1+\psi)} + \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}$$
(7)

where τ, δ and λ are non zero arbitrary constants.

By using the chain rule

$$D_x^{\alpha} u = \sigma'_x \frac{dU}{d\xi} D_x^{\alpha} \xi$$
$$D_y^{\alpha} u = \sigma'_y \frac{dU}{d\xi} D_y^{\alpha} \xi$$
$$D_t^{\alpha} u = \sigma'_t \frac{dU}{d\xi} D_t^{\alpha} \xi \tag{8}$$

where σ'_x , σ'_y and σ'_t are called the sigma indexes see [33], [34], without loss of generality we can take $\sigma'_x = \sigma'_y = \sigma'_t = l$, where *l* is a constant.

Substituting (7) with (2) and (8) into (6), we can rewrite (6) in the following nonlinear ODE

$$Q(U, U', U'', U''', \dots) = 0$$
(9)

where the prime denotes the derivation with respect to ξ . If possible, we should integrate (9) term by term one or more times.

Step 2: According to exp-function method, which was developed by He and Wu [35], we assume that the wave solution can be expressed in the following form

$$U(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp\left[n\xi\right]}{\sum_{m=-p}^{q} b_m \exp\left[m\xi\right]}$$
(10)

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite (10) in the following equivalent form.

$$U(\xi) = \frac{a_{-c} \exp\left[-c\xi\right] + \dots + a_d \exp\left[d\xi\right]}{b_{-p} \exp\left[-p\xi\right] + \dots + b_q \exp\left[q\xi\right]}.$$
 (11)

Step 3: This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and p, we balance the linear term of lowest order of equation (9) with the lowest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of highest order of (9) with highest order nonlinear term [36]–[39].

In the remaining sections, we will show the exact solutions to nonlinear time fractional differential equations using expfunction method.

IV. THE TIME FRACTIONAL FITZHUGH-NAGUMO EQUATION

We take into account the time fractional Fitzhugh-Nagumo equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\mu), \quad t > 0; \ 0 < \alpha \le 1; x \in \mathbb{R}$$
(12)

subject to the initial condition

$$u(x,0) = \frac{1}{(1+e^{-\frac{x}{\sqrt{2}}})}$$
(13)

which is an important nonlinear reaction-diffusion equation, applied to model the transmission of nerve impulses [40], [41], and also used in biology and the area of population genetics in circuit theory [42]. When $\mu = -1$, the Fitzhugh-Nagumo equation reduces to the real Newell-Whitehead equation [43].

For our goal, we present the following transformation

$$u(x,t) = U(\xi), \quad \xi = cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}$$
 (14)

where c and $\lambda \neq 0$ are constants.

Then by use of (14) with (2) and (8) into (12), (12) can be turned into an ODE

$$\lambda U' + c^2 U'' + U(1 - U)(U - \mu) = 0$$
 (15)

where $U' = \frac{dU}{d\xi}$.

Balancing the order of U'' and U^3 in (15), we have

$$U^{3} = \frac{c_{1} \exp\left[(3c+p)\xi\right] + \cdots}{c_{2} \exp\left[4p\xi\right] + \cdots}$$
(16)

and

$$U'' = \frac{c_3 \exp\left[(3p+c)\xi\right] + \dots}{c_4 \exp\left[4p\xi\right] + \dots}$$
(17)

where c_i are determined coefficients only for simplicity. Balancing lowest order of exp-function in (16) and (17) we have

$$3p + c = 3c + p \tag{18}$$

which leads to the result

$$p = c. \tag{19}$$

Similarly to determine values of d and q, we balance the linear term of highest order in (15)

$$U'' = \frac{\dots + d_1 \exp\left[-(3q+d)\xi\right]}{\dots + d_2 \exp\left[-4q\xi\right]}$$
(20)

and

$$U^{3} = \frac{\dots + d_{3} \exp\left[-(3d+q)\xi\right]}{\dots + d_{4} \exp\left[-4q\xi\right]}$$
(21)

where d_i are determined coefficients only for simplicity. From (20) and (21), we obtain

$$-(3q+d) = -(3d+q)$$
(22)

and this gives

$$q = d. \tag{23}$$

To simplify, we set p = c = 1 and q = d = 1, so (11) degrades to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}.$$
 (24)

Substituting (24) into (15), and by the help of symbolic computation, we have

$$\frac{1}{A} \Big[R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 \\ + R_{-1} \exp(-\xi) + R_{-2} \exp(-2\xi) + R_{-3} \exp(-3\xi) \Big] = 0$$
(25)

where

$$A = (b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi))^3$$

$$R_3 = a_1^2 b_1 + a_1^2 k b_1 - a_1 b_1^2 k - a_1^3$$

 $R_{2} = a_{1}^{2}b_{0} - 3a_{1}^{2}a_{0} + c^{2}a_{0}b_{1}^{2} + 2a_{1}b_{1}a_{0} - \lambda a_{0}b_{1}^{2} + ka_{1}^{2}b_{0}$ $-a_{0}b_{1}^{2}k + 2a_{1}a_{0}kb_{1} - 2a_{1}b_{1}kb_{0} - c^{2}a_{1}b_{1}b_{0} + \lambda a_{1}b_{1}b_{0}$

$$\begin{split} R_1 &= a_1^2 b_{-1} - 3a_1^2 a_{-1} - 3a_1 a_0^2 + a_0^2 b_1 + c^2 a_1 b_0^2 \\ &\quad - 2\lambda a_{-1} b_1^2 - a_1 b_0^2 k + 4c^2 a_{-1} b_1^2 + 2a_1 b_1 a_{-1} \\ &\quad - a_{-1} b_1^2 k + \lambda a_1 b_0^2 + a_1^2 b_{-1} k + a_0^2 k b_1 + 2a_1 b_0 a_0 \\ &\quad - c^2 a_0 b_1 b_0 + 2a_1 b_0 a_0 k - 4c^2 a_1 b_1 b_{-1} - 2a_1 b_1 b_{-1} k \\ &\quad - \lambda a_0 b_1 b_0 - 2a_0 b_1 b_0 k + 2\lambda a_1 b_1 b_{-1} + 2a_1 b_1 a_{-1} k \end{split}$$

$$\begin{aligned} R_0 &= -a_0^3 - a_0 b_0^2 k + a_0^2 k b_0 - 2a_1 b_0 k b_{-1} \\ &+ 2a_1 a_0 k b_{-1} - 2a_{-1} b_1 k b_0 + 2a_1 b_0 a_{-1} + 2a_1 b_{-1} a_0 \\ &+ a_0^2 b_0 + 3\lambda a_1 b_0 b_{-1} - 3\lambda a_{-1} b_1 b_0 + 3c^2 a_1 b_0 b_{-1} \\ &+ 3c^2 a_{-1} b_0 b_1 - 6c^2 a_0 b_1 b_{-1} - 6a_1 a_0 a_{-1} \\ &+ 2a_0 b_1 a_{-1} + 2a_1 a_{-1} k b_0 - 2a_0 b_1 k b_{-1} + 2a_0 a_{-1} k b_1 \end{aligned}$$

$$\begin{split} R_{-1} &= a_0^2 b_{-1} - 3a_{-1} a_0^2 - 3a_1 a_{-1}^2 + b_1 a_{-1}^2 - 2\lambda a_{-1} b_1 b_{-1} \\ &+ 2a_1 a_{-1} k b_{-1} + \lambda a_0 b_{-1} b_0 + 2a_0 a_{-1} k b_0 - c^2 a_0 b_0 b_{-1} \\ &- 2a_{-1} b_1 k b_{-1} - 2a_0 b_0 k b_{-1} - 4c^2 a_{-1} b_1 b_{-1} + 2\lambda a_1 b_{-1}^2 \\ &- \lambda a_{-1} b_0^2 + a_{-1}^2 k b_1 + 2a_1 b_{-1} a_{-1} - a_1 b_{-1}^2 k + 4c^2 a_1 b_{-1}^2 \\ &+ a_0^2 k b_{-1} + 2a_0 b_0 a_{-1} + c^2 a_{-1} b_0^2 - a_{-1} b_0^2 k \end{split}$$

$$\begin{split} R_{-2} &= -3a_0a_{-1}^2 + b_0a_{-1}^2 - 2a_{-1}b_0kb_{-1} + 2a_0a_{-1}kb_{-1} \\ &\quad -c^2a_{-1}b_0b_{-1} - \lambda a_{-1}b_0b_{-1} + 2a_0b_{-1}a_{-1} \\ &\quad -a_0b_{-1}^2k + \lambda a_0b_{-1}^2 + a_{-1}^2kb_0 + c^2a_0b_{-1}^2 \end{split}$$

$$R_{-3} = a_{-1}^2 b_{-1} + a_{-1}^2 k b_{-1} - a_{-1} b_{-1}^2 k - a_{-1}^3.$$

Solving this system of algebraic equations by using Maple, we obtain the following results

Case 1:

$$a_0 = 0, \quad b_{-1} = \frac{1}{5}a_{-1}, \quad b_0 = 0, \quad b_1 = a_1$$

 $\mu = 5, \quad \lambda = 6, \quad c = \pm\sqrt{2}$ (26)

where a_{-1} and a_1 are free parameters. Substituting these results into (24), we obtain the exact solution (27), shown at the bottom of the page.

The evolution of exact solution for (27) with $\alpha = 0.5$ and $\alpha = 1.0$ is shown in Fig. 1.



Fig. 1. The exact solution for (27) with (a) $\alpha = 0.5$ and (b) $\alpha = 1$, respectively, when $a_1 = 1, a_{-1} = -1$.

Case 2 :

$$a_0 = 0, \quad b_{-1} = a_{-1}, \quad b_0 = 0, \quad b_1 = \frac{1}{5}a_1$$

 $\mu = 5, \quad \lambda = -6, \quad c = \pm\sqrt{2}$
(28)

where a_{-1} and a_1 are free parameters. Substituting these results into (24), we obtain the exact solution (29), shown at the bottom of the page.

Case 3:

$$a_{0} = 0, \quad b_{-1} = b_{-1}, \quad b_{0} = 0, \quad b_{1} = a_{1}$$
$$\mu = \frac{a_{-1}}{b_{-1}}, \quad \lambda = \frac{a_{-1}^{2} - b_{-1}^{2}}{4b_{-1}^{2}}, \quad c = \pm \frac{\sqrt{2}}{4b_{-1}} (a_{-1} - b_{-1})$$
(30)

where a_{-1} and b_{-1} are free parameters which exist provided that $b_{-1} \neq 0$ and $a_{-1}^2 \neq b_{-1}^2$. Substituting these results into (24), we obtain the exact solution (31), shown at the bottom of the page.

$$u(x,t) = \frac{a_1 \exp(\pm\sqrt{2}x + \frac{6t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(\pm\sqrt{2}x + \frac{6t^{\alpha}}{\Gamma(1+\alpha)}))}{\frac{a_1}{5} \exp(\pm\sqrt{2}x + \frac{6t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(\pm\sqrt{2}x + \frac{6t^{\alpha}}{\Gamma(1+\alpha)}))}$$
(27)

$$u(x,t) = \frac{a_1 \exp(\pm\sqrt{2}x + \frac{6t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(\pm\sqrt{2}x + \frac{6t^{\alpha}}{\Gamma(1+\alpha)}))}{\frac{a_1}{5} \exp(\pm\sqrt{2}x + \frac{6t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(\pm\sqrt{2}x + \frac{6t^{\alpha}}{\Gamma(1+\alpha)}))}$$
(29)

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}{a_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + b_{-1} \exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}$$
(31)

Case 4:

$$a_0 = 0, \qquad b_{-1} = a_{-1}, \qquad b_0 = 0, \qquad b_1 = b_1$$

 $\mu = \frac{a_1}{b_1}, \qquad \lambda = -\frac{a_1^2 - b_1^2}{4b_1^2}, \qquad c = \pm \frac{\sqrt{2}}{4b_1} (a_1 - b_1)$
(32)

where a_1 and b_1 are free parameters which exist provided that $b_1 \neq 0$ and $a_1^2 \neq b_1^2$. Substituting these results into (24), we obtain the exact solution (33), shown at the bottom of the page. *Case 5*:

$$a_{0} = 0, \qquad b_{-1} = 2a_{-1}, \qquad b_{0} = \sqrt{-a_{1}a_{-1}}, \qquad b_{1} = a_{1}$$
$$\mu = \frac{1}{2}, \qquad \lambda = -\frac{3}{8}, \qquad c = \pm \frac{\sqrt{2}}{4}$$
(27)

where a_1 and a_{-1} are free parameters. Substituting these results into (24), we obtain the exact solution (35), shown at the bottom of the page.

Obtained exact solution is described in Fig. 2.



Fig. 2. The exact solution for (35) with (a) $\alpha = 0.5$ and (b) $\alpha = 1$, respectively, when $a_1 = 1, a_{-1} = -1$.

Case 6 :

$$a_{0} = 0, \qquad \lambda = \frac{a_{-1}^{2} - b_{-1}^{2}}{2b_{-1}^{2}}$$

$$b_{-1} = b_{-1}, \qquad b_{0} = \pm \sqrt{-\frac{a_{1}}{a_{-1}}} (a_{-1} - b_{-1}), \quad b_{1} = a_{1}$$

$$\mu = \frac{a_{-1}}{b_{-1}}, \qquad c = \pm \frac{\sqrt{2}}{2b_{-1}} (a_{-1} - b_{-1})$$
(36)

where a_1 and a_{-1} are free parameters which exist provided that $b_{-1} \neq 0$ and $a_{-1}^2 \neq b_{-1}^2$. Substituting these results into (24), we obtain the exact solution (37), shown at the bottom of the page.

Case 7 :

$$a_0 = 0, \quad b_{-1} = a_{-1}, \quad b_0 = \frac{\sqrt{-a_1 a_{-1}}}{2}, \quad b_1 = \frac{a_1}{2}$$

 $\mu = 2, \quad \lambda = -\frac{3}{2}, \qquad c = \pm \frac{\sqrt{2}}{2}$
(38)

where a_1 and a_{-1} are free parameters. Substituting these results into (24), we obtain the exact solution (39), shown at the bottom of the page.

Case 8 :

$$a_{0} = 0, \qquad \lambda = \frac{a_{1}^{2} - b_{1}^{2}}{2b_{1}^{2}}$$

$$b_{-1} = a_{-1}, \qquad b_{0} = \pm \sqrt{-\frac{a_{-1}}{a_{1}}} (a_{1} - b_{1}), \qquad b_{1} = b_{1}$$

$$\mu = \frac{a_{1}}{b_{1}}, \qquad c = \pm \frac{\sqrt{2}}{2b_{1}} (a_{1} - b_{1}) \qquad (40)$$

where a_1 and a_{-1} are free parameters which exist provided that $b_1 \neq 0$ and $a_1^2 \neq b_1^2$. Substituting these results into (24), we obtain the exact solution (41), shown at the bottom of the page.

V. THE TIME FRACTIONAL KDV EQUATION

We consider the time fractional KdV equation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u + 6u\frac{\partial u}{\partial x} + \frac{\partial^{3}u}{\partial x^{3}} = 0, \quad t > 0; \ 0 < \alpha \le 1; x \in \mathbb{R}$$
(42)

subject to the initial condition:

$$u(x,0) = \frac{1}{2}\operatorname{sech}^{2}\left(\frac{1}{2}x\right)$$
(43)

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}{b_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}$$
(33)

$$u(x,t) = \frac{a_1 \exp(\pm \frac{\sqrt{2}}{4}x + \frac{3t^{\alpha}}{8\Gamma(1+\alpha)}) + a_{-1} \exp(-(\pm \frac{\sqrt{2}}{4}x + \frac{3t^{\alpha}}{8\Gamma(1+\alpha)}))}{a_1 \exp(\pm \frac{\sqrt{2}}{4}x + \frac{3t^{\alpha}}{8\Gamma(1+\alpha)}) + \sqrt{-a_1a_{-1}} + 2a_{-1}\exp(-(\pm \frac{\sqrt{2}}{4}x + \frac{3t^{\alpha}}{8\Gamma(1+\alpha)}))}$$
(35)

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}{a_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + \sqrt{-a_1 a_{-1}} + 2a_{-1} \exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}$$
(37)

$$u(x,t) = \frac{a_1 \exp(\pm \frac{\sqrt{2}}{2}x + \frac{3t^{\alpha}}{2\Gamma(1+\alpha)}) + a_{-1} \exp(-(\pm \frac{\sqrt{2}}{2}x + \frac{3t^{\alpha}}{2\Gamma(1+\alpha)}))}{\frac{a_1}{2} \exp(\pm \frac{\sqrt{2}}{2}x + \frac{3t^{\alpha}}{2\Gamma(1+\alpha)}) + \frac{\sqrt{-a_1a_{-1}}}{2} + a_{-1} \exp(-(\pm \frac{\sqrt{2}}{2}x + \frac{3t^{\alpha}}{2\Gamma(1+\alpha)}))}$$
(39)

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + a_{-1} \exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}{b_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) \pm \sqrt{-\frac{a_{-1}}{a_1}} (a_1 - b_1) + a_{-1} \exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}$$
(41)

where α is a parameter describing the order of the fractional time-derivative. The function u(x,t) is assumed to be a causal function of time.

For our purpose, we introduce the following transformations;

$$u(x,t) = U(\xi), \quad \xi = cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}$$
(44)

where c and λ are non-zero constants.

Substituting (44) with (2) and (8) into (42), we can show that (42) reduced into following ODE

$$-\lambda U' + 6cUU' + c^3 U''' = 0$$
 (45)

where $U' = \frac{dU}{d\xi}$.

Integrating (45) with respect to ξ yields

$$-\lambda U + 3cU^2 + c^3 U'' + \xi_0 = 0 \tag{46}$$

where ξ_0 is a constant of integration.

By the same procedure as illustrated in Section III, we can determine values of c and p by balancing terms U^2 and U'' in (46). By symbolic computation, we have

$$U'' = \frac{c_1 \exp\left[(3p+c)\xi\right] + \dots}{c_2 \exp\left[4p\xi\right] + \dots}$$
(47)

and

$$U^{2} = \frac{\dots + c_{3} \exp\left[2c\xi\right]}{\dots + c_{4} \exp\left[2p\xi\right]}$$
(48)

where c_i are determined coefficients only for simplicity. According to exp-function method, balancing lowest order of (47) and (48), we have

$$3p + c = 2c + 2p \tag{49}$$

that gives

$$p = c. (50)$$

In the same way, we balance the linear term of highest order in (46)

$$U'' = \frac{\dots + d_1 \exp\left[-(3q+d)\xi\right]}{\dots + d_2 \exp\left[-4q\xi\right]}$$
(51)

and

$$U^{2} = \frac{d_{3} \exp\left[-2d\xi\right] + \cdots}{d_{4} \exp\left[-2q\xi\right] + \cdots}$$
(52)

where d_i are determined coefficients only for simplicity. From

(51) and (52), we get

$$-(3q+d) = -(2d+2q)$$
(53)

and this gives

$$q = d. \tag{54}$$

For simplicity, we set p = c = 1 and q = d = 1, so (11) reduces to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}.$$
 (55)

Substituting (55) into (46), and by the help of Maple, we have

$$\frac{1}{A} \Big[R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 \\ + R_{-1} \exp(-\xi) + R_{-2} \exp(-2\xi) + R_{-3} \exp(-3\xi) \Big] = 0$$
(56)

where

$$A = (b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi))^3$$

$$R_3 = -\lambda a_1 b_1^2 + k b_1^3 + 3c a_1^2 b_1$$

$$R_2 = c^3 a_0 b_1^2 + 3k b_1^2 b_0 - \lambda a_0 b_1^2 + 3c a_1^2 b_0 - 2\lambda a_1 b_1 b_0 + 6c a_1 a_0 b_1 - c^3 a_1 b_1 b_0$$

$$R_{1} = -2\lambda a_{0}b_{1}b_{0} + 6ca_{1}b_{0}a_{0} - c^{3}a_{0}b_{1}b_{0} + 3kb_{1}b_{0}^{2} - \lambda a_{1}b_{0}^{2} + 3ca_{0}^{2}b_{1} + c^{3}a_{1}b_{0}^{2} + 4c^{3}a_{-1}b_{1}^{2} + 3kb_{1}^{2}b_{-1} + 3ca_{1}^{2}b_{-1} - \lambda a_{-1}b_{1}^{2} - 2\lambda a_{1}b_{1}b_{-1} + 6ca_{1}a_{-1}b_{1} - 4c^{3}a_{1}b_{1}b_{-1}$$

$$\begin{aligned} R_0 &= 3ca_0^2b_0 + 6ca_0a_{-1}b_1 + kb_0^3 - \lambda a_0b_0^2 + 6kb_{-1}b_1b_0 \\ &+ 3c^3a_1b_0b_{-1} + 3c^3a_{-1}b_1b_0 - 6c^3a_0b_1b_{-1} - 2\lambda a_1b_0b_{-1} \\ &- 2\lambda a_0b_1b_{-1} - 2\lambda a_{-1}b_1b_0 + 6ca_1a_0b_{-1} + 6ca_1a_{-1}b_0 \end{aligned}$$

$$\begin{aligned} R_{-1} &= -2\lambda a_0 b_{-1} b_0 + 6ca_0 a_{-1} b_0 - c^3 a_0 b_0 b_{-1} + 3k b_0^2 b_{-1} \\ &- \lambda a_{-1} b_0^2 + 3c a_0^2 b_{-1} + c^3 a_{-1} b_0^2 + 3c a_{-1}^2 b_1 + 4c^3 a_1 b_{-1}^2 \\ &+ 3k b_1 b_{-1}^2 - \lambda a_1 b_{-1}^2 - 2\lambda a_{-1} b_1 b_{-1} + 6c a_1 a_{-1} b_{-1} \\ &- 4c^3 a_{-1} b_1 b_{-1} \end{aligned}$$

$$\begin{split} R_{-2} &= c^3 a_0 b_{-1}^2 + 3k b_0 b_{-1}^2 - \lambda a_0 b_{-1}^2 + 3c b_0 a_{-1}^2 - 2\lambda a_2 b_0 b_{-1} \\ &+ 6c a_0 a_{-1} b_{-1} - c^3 a_{-1} b_0 b_{-1} \end{split}$$

$$R_{-3} = -\lambda a_{-1}b_{-1}^2 + 3ca_{-1}^2b_{-1} + kb_{-1}^3.$$
(57)

Solving this system of algebraic equations by using Maple, we obtain the following results

$$a_{1} = a_{1}, \qquad a_{0} = \frac{b_{0}}{b_{1}}(c^{2}b_{1} + a_{1}), \qquad a_{-1} = \frac{a_{1}b_{0}^{2}}{4b_{1}^{2}}$$

$$b_{1} = b_{1}, \qquad b_{0} = b_{0}, \qquad b_{-1} = \frac{b_{0}^{2}}{4b_{1}}$$

$$\xi_{0} = \frac{a_{1}c(c^{2}b_{1} + 3a_{1})}{b_{1}^{2}}, \qquad \lambda = \frac{c(c^{2}b_{1} + 6a_{1})}{b_{1}}$$
(58)

where a_1 , b_0 and b_1 are free parameters which exist provided that $b_1 \neq 0$ and $c^2b_1 + 6a_1 \neq 0$. Substituting these results into (56), we obtain the exact solution (59), shown at the bottom of the page.

Also, u(x,t) in (59) is represented in Fig. 3.

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + \frac{b_0}{b_1}(c^2b_1 + a_1) + \frac{a_1b_0^2}{4b_1^2}\exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}{b_1 \exp(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}) + b_0 + \frac{b_0^2}{4b_1}\exp(-(cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}$$
(59)

Fig. 3. The exact solution for (59) with (a) $\alpha = 0.5$ and (b) $\alpha = 1$ respectively, when $c = 1, a_1 = 0, b_1 = 1, b_0 = 2$.

Comparing our results with the results [45], it can be seen that our results are new.

VI. CONCLUSION

exp-function method known as very powerful and an effective method for solving nonlinear problems and ordinary, partial, difference, fractional equations and so many other equations. The basic idea described in this paper is expected to be further employed to solve other similar nonlinear equations in fractional calculus. To our knowledge, these new solutions have not been reported in former literature, they may be of significant importance for the explanation of some special physical phenomena. As a result, many new and more rational solitary wave solutions are obtained, from which hyperbolic function and trigonometric function solutions.

REFERENCES

- [1] K. B. Oldham and J. Spanier, The Fractional Calculus. New York: Academic Press, 1974.
- [2] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: Wiley, 1993.
- [3] I. Podlubny, Fractional Differential Equations. San Diego: Academic Press, 1999.
- [4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations. Amsterdam: Elsevier, 2006.
- [5] S. Zhang and H. O. Zhang, "Fractional sub-equation method and its applications to nonlinear fractional PDEs," Phys. Lett. A, vol. 375, no. 7, pp. 1069-1073, Feb. 2011.
- [6] B. Tong, Y. N. He, L. L. Wei, and X. D. Zhang, "A generalized fractional sub-equation method for fractional differential equations with variable coefficients," Phys. Lett. A, vol. 376, no. 38-39, pp. 2588-2590, Aug. 2012.
- [7] S. M. Guo, L. Q. Mei, Y. Li, and Y. F. Sun, "The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics," Phys. Lett. A, vol. 376, no. 4, pp. 407-411, Jan. 2012.
- [8] H. Jafari, M. Ghorbani, and C. M. Khalique, "Exact travelling wave solutions for isothermal magnetostatic atmospheres by Fan subequation method," Abstr & Appl. Anal., vol. 2012, pp. 1395-1416, Nov. 2012.
- [9] Y. F. Zhang and Q. H. Feng, "Fractional Riccati equation rational expansion method for fractional differential equations," Appl. Math. Inform. Sci., vol. 7, no. 4, pp. 1575-1584, Jul. 2013.

- [10] B. Lu, "The first integral method for some time fractional differential equations," J. Math. Anal. Appl., vol. 395, no. 2, pp. 684-693, Nov. 2012.
- [11] O. Güner, A. Bekir, and A. C. Cevikel, "A variety of exact solutions for the time fractional Cahn-Allen equation," Eur. Phys. J. Plus, vol. 130, pp. 146, Jul. 2015.
- [12] A. Bekir, Ö. Güner, and Ö. Ünsal, "The first integral method for exact solutions of nonlinear fractional differential equations," J. Comput. & Nonlinear Dynam., vol. 10, no. 2, pp. 021020, Mar. 2015.
- [13] B. Zheng, "(G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics," Commun. Theor. Phys., vol. 58, no. 5, pp. 623-630, Aug. 2012.
- [14] A. Bekir, Ö. Güner, A. H. Bhrawy, and A. Biswas, "Solving nonlinear fractional differential equations using Exp-function and (G'/G)expansion methods," Rom. J. Phys., vol. 60, no. 3-4, pp. 360-378, Jan. 2015.
- [15] K. A. Gepreel and S. Omran, "Exact solutions for nonlinear partial fractional differential equations," Chin. Phys. B, vol. 21, no. 11, pp. 110204, May 2012.
- [16] A. Bekir and O. Guner, "Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method," Chin. Phys. B, vol. 22, no. 11, pp. 110202, Apr. 2013.
- [17] S. Zhang, Q. A. Zong, D. Liu, and Q. Gao, "A generalized Exp-function method for fractional Riccati differential equations," Commun. Fract. Calc., vol. 1, no. 1, pp. 48-51, 2010.
- [18] A. Bekir, Ö. Güner, and A. C. Cevikel, "Fractional complex transform and Exp-function methods for fractional differential equations," Abstr. Appl. Anal., vol. 2013, pp. 426462, Mar. 2013.
- [19] Ö. Güner and A. C. Cevikel, "A procedure to construct exact solutions of nonlinear fractional differential equations," Sci. World J., vol. 2014, pp. 489495, Mar. 2014.
- [20] S. Zhang, "Application of Exp-function method to high-dimensional nonlinear evolution equation," Chaos Solit. Fract., vol. 38, no. 1, pp. 270-276, Oct. 2008.
- [21] A. Bekir and A. C. Cevikel, "New solitons and periodic solutions for nonlinear physical models in mathematical physics," Nonlinear Anal. Real World Appl., vol. 11, no. 4, pp. 3275-3285, Aug. 2010.
- [22] S. Zhang, "Application of Exp-function method to a KdV equation with variable coefficients," Phys. Lett. A, vol. 365, no. 5-6, pp. 448-453, Jun. 2007.
- [23] S. A. El-Wakil, M. A. Madkour, and M. A. Abdou, "Application of Exp-function method for nonlinear evolution equations with variable coefficients," Phys. Lett. A, vol. 369, no. 1-2, pp. 62-69, Sep. 2007.
- [24] S. D. Zhu, "Exp-function method for the hybrid-lattice system," Int. J. Nonlinear Sci. Numer. Simul., vol. 8, no. 3, pp. 461-464, May 2007.
- [25] A. Bekir, "Application of the Exp-function method for nonlinear differential-difference equations," Appl. Math. Comput., vol. 215, no. 11, pp. 4049-4053, Feb. 2010.
- [26] C. Q. Dai and J. L. Chen, "New analytic solutions of stochastic coupled KdV equations," Chaos Solit. Fract., vol. 42, no. 4, pp. 2200-2207, Nov. 2009.
- [27] A. C. Cevikel and A. Bekir, "New solitons and periodic solutions for (2+1)-dimensional Davey-Stewartson equations," Chin. J. Phys., vol. 51, no. 1, pp. 1–13, Feb. 2013.



- [28] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent-II," *Geophys. J. Int.*, vol. 13, no. 5, pp. 529–539, May 1967.
- [29] S. G. Samko, A. A. Kilbas, and O. I. Marichev O I, Fractional Integrals and Derivatives: Theory and Applications. Switzerland: Gordon and Breach Science Publishers, 1993.
- [30] G. Jumarie, "Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results," *Comput. Math. Appl.*, vol. 51, no. 9–10, pp. 1367–1376, May 2006.
- [31] G. Jumarie, "Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions," *Appl. Math. Lett.*, vol. 22, no. 3, pp. 378–385, Mar. 2009.
- [32] J. H. He, S. K. Elegan, and Z. B. Li, "Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus," *Phys. Lett. A*, vol. 376, no. 4, pp. 257–259, Jan. 2012.
- [33] M. Saad, S. K. Elagan, Y. S. Hamed, and M. Sayed, "Using a complex transformation to get an exact solution for fractional generalized coupled MKDV and KDV equations," *Int. J. Basic Appl. Sci.*, vol. 13, no. 1, pp. 23–25, Jan. 2013.
- [34] T. Elghareb, S. K. Elagan, Y. S. Hamed, and M. Sayed, "An exact solutions for the generalized fractional Kolmogrove-Petrovskii Piskunov equation by using the generalized (G'/G)-expansion method," Int. J. Basic Appl. Sci., vol. 13, no. 1, pp. 19–22, Feb. 2013.
- [35] J. H. He and X. H. Wu, "Exp-function method for nonlinear wave equations," *Chaos Solit. Fract.*, vol. 30, no. 3, pp. 700–708, Nov. 2006.
- [36] J. H. He and M. A. Abdou, "New periodic solutions for nonlinear evolution equations using Exp-function method," *Chaos Solit. Fract.*, vol. 34, no. 5, pp. 1421–1429, Dec. 2007.
- [37] A. Ebaid, "Exact solitary wave solutions for some nonlinear evolution equations via Exp-function method," *Phys. Lett. A*, vol. 365, no. 3, pp. 213–219, May 2007.
- [38] S. Kutluay and A. Esen, "Exp-function method for solving the general improved KdV equation," *Int. J. Nonlinear Sci. Numer. Simul.*, vol. 10, no. 6, pp. 717–725, Jun. 2009.
- [39] A. Bekir, "The Exp-function method for Ostrovsky equation," Int. J. Nonlinear Sci. Numer. Simul., vol. 10, no. 6, pp. 735–739, Jun. 2009.
- [40] R. FitzHugh, "Impulses and physiological states in theoretical models of nerve membrane," *Biophys. J.*, vol. 1, no. 6, pp. 445–466, Jul. 1961.
- [41] J. Nagumo, S. Arimoto, and S. Yoshizawa, "An active pulse transmission line simulating nerve axon," *Proc. IRE*, vol. 50, no. 10, pp. 2061–2070, Oct. 1962.

- [42] M. Shih, E. Momoniat, and F. M. Mahomed, "Approximate conditional symmetries and approximate solutions of the perturbed Fitzhugh-Nagumo equation," J. Math. Phys., vol. 46, no. 2, pp. 023503, Jan. 2005.
- [43] M. Merdan, "Solutions of time-fractional reaction-diffusion equation with modified Riemann-Liouville derivative," *Int. J. Phys. Sci.*, vol. 7, no. 15, pp. 2317–2326, Apr. 2012.
- [44] Z. Odibat and S. Momani, "The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics," *Comput. Math. Appl.*, vol. 58, no. 11–12, pp. 2199–2208, Dec. 2009.
- [45] S. Momani, "An explicit and numerical solutions of the fractional KdV equation," *Math. Comput. Simul.*, vol. 70, no. 2, pp. 110–118, Sep. 2005.



Ahmet Bekir is currently full professor of Mathematics-Computer Department, Eskisehir Osmangazi University. He obtained his Ph.D. degree from Eskisehir Osmangazi University. His research interests include theory and exact solutions of partial differential equations in mathematical physics. His favorites in mathematics are ODEs, PDEs, fractional differential equations, integral equations, and analytic methods. He has published more than 150 articles.



Ozkan Guner is currently an assistant professor at Cankiri Karatekin University. He obtained his Ph.D. degree from Eskisehir Osmangazi University. His research interests include exact solution of nonlinear ODEs, PDEs, FDEs equations. He has over 50 research articles that published in international journals.



Adem Cevikel is an associate professor of mathematics at Yildiz Technical University. He obtained his Ph.D. degree from Yildiz Technical University. His research interests include exact solutions of nonlinear ODEs, PDEs, FDEs equations and has significant research studies on applied mathematics. He has over 30 research articles that published in international journals. Corresponding author of this paper.