Numerical Solutions of Fractional Differential Equations by Using Fractional Taylor Basis

Vidhya Saraswathy Krishnasamy, Somayeh Mashayekhi, and Mohsen Razzaghi

Abstract—**In this paper, a new numerical method for solving fractional differential equations (FDEs) is presented. The method is based upon the fractional Taylor basis approximations. The operational matrix of the fractional integration for the fractional Taylor basis is introduced. This matrix is then utilized to reduce the solution of the fractional differential equations to a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of this technique.**

Index Terms—**Caputo derivative, fractional differential equations (FEDs), fractional Taylor basis, operational matrix, Riemann-Liouville fractional integral operator.**

I. INTRODUCTION

THE fractional differential equations (FDEs) have drawn increasing attention and interest due to their important applications in science and engineering. A history of the development of fractional differential operators can be found in [1]−[3].

Many mathematical modelings contain FDEs. To mention a few, fractional derivatives are used in visco-elastic systems [4], economics [5], continuum and statistical mechanics [6], solid mechanics [7], electrochemistry [8], biology [9] and acoustics [10]. Generally speaking, most of the FDEs do not have exact analytic solutions. Therefore, seeking numerical solutions of these equations becomes more and more important. Recently, several numerical methods to solve FDEs have been given, such as Fourier transforms [11], Laplace transforms [12], Adomian decomposition method [13], variational iteration method [14], the power series method [15], truncated fractional power series method [16], fractional differential transform method (FDTM) [17], homotopy analysis method [18], fractional-order Legendre functions method [19], modified homotopy perturbation method (MHPM) [20] and enhanced homotopy perturbation method (EHPM) [21].

Moreover, for solving FDEs in [22], the Bernstein polynomials are used to solve the fractional Riccati type differential equations. In [22], the Bernstein polynomials were first expanded into fractional Taylor polynomials. The operational matrix of fractional differentiation (OMFD) of fractional Taylor polynomials were then used for calculating OMFD for Bernstein polynomials. In addition, the Chebyshev, Legendre and Bernoulli wavelets operational matrices of fractional integration (OMFI) were calculated in [23]−[25], respectively. For obtaining OMFI in [23], [24], these wavelets were first expanded into block-pulse functions. Then, OMFI of block-pulse were used for calculating OMFI for Chebyshev and Legendre wavelets in [23], [24], respectively. In [25], for obtaining the OMFI for Bernoulli wavelets, these wavelets were expanded into Bernoulli polynomials.

In this paper, a new numerical method for solving the initial and boundary value problems for fractional differential equations is presented. The method is based upon the fractional Taylor basis approximations. The OMFI for the fractional Taylor basis is calculated. This matrix is then utilized to reduce the solution of the FDEs to the solution of algebraic equations. This method is applicable for linear equations or nonlinear equations with square nonlinearities.

The outline of this paper is as follows: In Section II, we introduce some necessary definitions and properties of fractional calculus. Section III is devoted to the basic formulation of the fractional Taylor basis. In Section IV, we derive the Fractional Taylor OMFI. In Section V, the problem statement is given. Section VI is devoted to the numerical method for solving the initial and boundary value problems for FDEs and, in Section VII we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering five numerical examples.

II. PRELIMINARIES

A. The Fractional Integral and Derivative

In this section, we present some notations, definitions, and preliminary facts of the fractional calculus theory which will be used further in this work.

Definition 1: The Riemann-Liouville fractional integral operator of order α is defined as [12]

$$I^{\alpha}y(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds, & \alpha>0\\ y(t), & \alpha=0.\end{cases}$$

The Riemann-Liouville fractional integral operator has the following properties:

$$I^{\alpha}\ t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}\ t^{\gamma+\alpha},\quad \alpha\geq 0;\gamma>-1 \tag{1}$$

$$I^{\alpha}I^{\beta}y(t)=I^{\beta}I^{\alpha}y(t)=I^{\alpha+\beta}y(t),\quad \alpha,\ \ \beta>0. \tag{2}$$

Manuscript received August 8, 2015; accepted April 18, 2016. Recommended by Associate Editor YangQuan Chen.

Citation: V. S. Krishnasamy, S. Mashayekhi, and M. Razzaghi, "Numerical solutions of fractional differential equations by using fractional Taylor basis," *IEEE/CAA Journal of Automatica Sinica*, vol. 4, no. 1, pp. 98−106, Jan. 2017.

V. S. Krishnasamy and M. Razzaghi are with the Department of Mathematics and Statistics, Mississippi State University, MS 39762, USA (e-mail: vk81@msstate.edu; razzaghi@math.msstate.edu).

S. Mashayekhi is with the Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA (e-mail: sm2395@msstate.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/JAS.2017.7510337

Also the fractional integral is a linear operator, that is for constants λ_1 and λ_2, we have

$$I^\alpha(\lambda_1 y_1(t) + \lambda_2 y_2(t)) = \lambda_1 I^\alpha y_1(t) + \lambda_2 I^\alpha y_2(t).$$

Definition 2: The Caputo fractional derivative of order α is defined as [12]

$$D^\alpha y(t) = I^{n-\alpha}\left(\frac{d^n}{dt^n} y(t)\right), \quad n-1 < \alpha \le n;\ n \in \mathbb{N}. \tag{3}$$

The fractional integral operator and fractional derivative operator do not commute in general, but we have the following property

$$I^\alpha(D^\alpha y(t)) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(0)\ \frac{t^k}{k!}. \tag{4}$$

III. The Properties of Fractional Taylor Basis

A. Fractional Taylor Basis Vector

In this paper, we define the fractional Taylor basis vector as

$$T_{m\gamma}(t) = [1, t^\gamma, t^{2\gamma}, \dots, t^{m\gamma}]^T \tag{5}$$

where m is a positive integer and $\gamma > 0$, is a real number.

B. Function Approximation

Let $H = L^2[0,1]$, and assume that $T_{m\gamma}(t) \subset H$, S = span $\{1, t^\gamma, t^{2\gamma}, \dots, t^{m\gamma}\}$ and y be an arbitrary element in H. Since S is a finite dimensional vector subspace of H, y has a unique best approximation out of S such as $y_0 \in S$, that is

$$\forall \hat{y} \in S, \quad ||y - y_0|| \le ||y - \hat{y}||.$$

Since $y_0 \in S$, there exist unique coefficients $c_0, c_1, c_2, \dots, c_m$, such that

$$y \simeq y_0 = \sum_{i=0}^{m} c_i t^{i\gamma} = C^T T_{m\gamma}(t) \tag{6}$$

where

$$C^T = [c_0, c_1, c_2, \dots, c_m]. \tag{7}$$

C. Error Bound for the Best Approximation

To obtain the error bound for the best approximation, we use the following formula.

Generalized Taylor formula [15]: Suppose that $D^{k\gamma} y(t) \in \mathbb{C}[0,1]$ for $k = 0, 1, \dots, m$, where $0 < \gamma \le 1$, then

$$y(t) = \sum_{i=0}^{m} \frac{(t)^{i\gamma}}{\Gamma(i\gamma+1)} [D^{i\gamma} y(t)]_{t=0} + R_m^\gamma(t,0) \tag{8}$$

where $D^{i\gamma} = \underbrace{D^\gamma D^\gamma \cdots D^\gamma}_{i \text{ times}}$, with D^γ defined similar to D^α in (3), and

$$R_m^\gamma(t,0) = \frac{(t)^{(m+1)\gamma}}{\Gamma((m+1)\gamma+1)} [D^{(m+1)\gamma} y(t)]_{t=\xi}$$
$$0 \le \xi \le t; \quad \forall t \in [0,1].$$

Theorem 1: Let y_0 be the best approximation of y out of S and suppose $D^{k\gamma} y(t) \in C[0,1]$, $k = 0, 1, \dots,$ then

$$||y(t) - y_0(t)||_{L^2[0,1]} \le \frac{M_\gamma}{\Gamma((m+1)\gamma+1)} \sqrt{\frac{1}{2(m+1)\gamma+1}}$$

where

$$M_\gamma = \sup_{t\in[0,1]} |D^{(m+1)\gamma} y(t)|.$$

Proof: Similar to [19], since y_0 is the best approximation of y out of S, by using (8) we have

$$||y - y_0||^2_{L^2[0,1]} \le \frac{M_\gamma^2}{(\Gamma((m+1)\gamma+1))^2} \int_0^1 (t)^{2(m+1)\gamma} dt$$
$$= \frac{M_\gamma^2}{(\Gamma((m+1)\gamma+1))^2} \frac{1}{2(m+1)\gamma+1}. \tag{9}$$

By using (9), the result can be obtained. ■

D. Error Bound for Fractional Integration

In this section we obtain the error bound for $I^\alpha y(t)$.

Theorem 2: Suppose all the conditions in Theorem 1 are true and $\alpha > 1$, then

$$||I^\alpha y(t) - I^\alpha y_0(t)||_{L^2[0,1]} \le \frac{M_\gamma}{\Gamma((m+1)\gamma+1)\Gamma(\alpha)} \sqrt{\frac{1}{2(m+1)\gamma+1}}.$$

Proof: By using Definition 1, we have

$$\begin{aligned}
&||I^\alpha y(t) - I^\alpha y_0(t)||_{L^2[0,1]} \\
&\quad = ||I^\alpha (y(t) - y_0(t))||_{L^2[0,1]} \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t ||(t-s)^{\alpha-1}(y(s) - y_0(s))||_{L^2[0,1]}\ ds \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^1 ||(1-s)^{\alpha-1}(y(s) - y_0(s))||_{L^2[0,1]}\ ds \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^1 ||(y(s) - y_0(s))||_{L^2[0,1]}\ ds.
\end{aligned} \tag{10}$$

By using (9) and (10), the result can be obtained. ■

IV. The Operational Matrices

A. Operational Matrix of the Fractional Integration

In this section we derive the fractional Taylor operational matrix of the fractional integration.

By using (1) and (5), we have

$$\begin{aligned}
I^\alpha(T_{m\gamma}(t)) &= \left[\frac{1}{\Gamma(\alpha+1)} t^\alpha, \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \right. \\
&\left. \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+\alpha+1)} t^{2\gamma+\alpha}, \dots, \frac{\Gamma(m\gamma+1)}{\Gamma(m\gamma+\alpha+1)} t^{m\gamma+\alpha}\right]^T \\
&= t^\alpha F_\alpha T_{m\gamma}(t)
\end{aligned} \tag{11}$$

where

$$F_\alpha = \text{diag}\left[\frac{1}{\Gamma(\alpha+1)},\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)},\ \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+\alpha+1)},\ \dots,\ \frac{\Gamma(m\gamma+1)}{\Gamma(m\gamma+\alpha+1)}\right].$$

Equation (11) can be rewritten as

$$I^\alpha(T_{m\gamma}(t)) = t^\alpha G_\alpha * T_{m\gamma}(t) \tag{12}$$

where

$$G_\alpha = \left[\frac{1}{\Gamma(\alpha+1)}, \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}, \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+\alpha+1)}, \ldots, \frac{\Gamma(m\gamma+1)}{\Gamma(m\gamma+\alpha+1)}\right]^T$$

and * denotes term by term multiplication of two matrices of the same dimensions.

B. Operational Matrix of Product

The following property of the product of two fractional Taylor vectors will also be used.

$$I^\alpha(T_{m\gamma}(t)T_{m\gamma}^T(t)) = t^\alpha S_\alpha * (T_{m\gamma}(t)T_{m\gamma}^T(t)) \quad (13)$$

where S_α is given by

$$S_\alpha = \begin{bmatrix} \frac{1}{\Gamma(\alpha+1)} & \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} & \cdots & \frac{\Gamma(m\gamma+1)}{\Gamma(m\gamma+\alpha+1)} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} & \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+\alpha+1)} & \cdots & \frac{\Gamma((m+1)\gamma+1)}{\Gamma((m+1)\gamma+\alpha+1)} \\ \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+\alpha+1)} & \frac{\Gamma(3\gamma+1)}{\Gamma(3\gamma+\alpha+1)} & \cdots & \frac{\Gamma((m+2)\gamma+1)}{\Gamma((m+2)\gamma+\alpha+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(m\gamma+1)}{\Gamma(m\gamma+\alpha+1)} & \frac{\Gamma((m+1)\gamma+1)}{\Gamma((m+1)\gamma+\alpha+1)} & \cdots & \frac{\Gamma(2m\gamma+1)}{\Gamma(2m\gamma+\alpha+1)} \end{bmatrix}. \quad (14)$$

To illustrate the calculation procedure, by using (5), we have

$$T_{m\gamma}(t)T_{m\gamma}^T(t) = \begin{bmatrix} 1 & t^\gamma & t^{2\gamma} & \cdots & t^{m\gamma} \\ t^\gamma & t^{2\gamma} & t^{3\gamma} & \cdots & t^{(m+1)\gamma} \\ t^{2\gamma} & t^{3\gamma} & t^{4\gamma} & \cdots & t^{(m+2)\gamma} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{m\gamma} & t^{(m+1)\gamma} & t^{(m+2)\gamma} & \cdots & t^{2m\gamma} \end{bmatrix}. \quad (15)$$

From (1) and (15), we get (16), shown at the bottom of the page.

Therefore from (13) and (15), we get S_α in (14).

V. Problem Statement

In this paper we focus on the following FDE problems [24].

A. Problem a

Caputo fractional differential equation

$$D^\alpha y(t) = f(t, y(t), D^\beta y(t)) \quad 0 \le t \le 1;\quad 0 < \alpha \le 2;\quad 0 \le \beta \le \alpha \quad (17)$$

with the initial conditions

$$y(0) = Y_0, \quad y'(0) = Y_1. \quad (18)$$

The existence and uniqueness results for solution of this problem are given in [26].

B. Problem b

Caputo fractional differential equation in (17) with the boundary conditions

$$y(0) = Y_0, \quad y(1) = \bar{Y}_1. \quad (19)$$

For this problem, we have the following Lemma 1.

Lemma 1: Assume that $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous. Then $y(t)\in\mathbb{C}[0,1]$ is a solution of the boundary value problem in (17) and (19) if and only if $y(t)$ is the solution of [24].

$$y(t) = I^\alpha f(t, y(t), D^\beta y(t)) - tI^\alpha f(1, y(1), D^\beta y(1)) + (\bar{Y}_1 - Y_0)t + Y_0. \quad (20)$$

The existence and uniqueness results for solution of this problem are given in [24].

VI. The Numerical Method

In this section, we use the fractional Taylor vector in (5) for solving Problem a given in (17) and (18) and Problem b given in (17) and (19).

A. Problem a

In this case, by using (4) and (17), we have

$$y(t) - \sum_{k=0}^{n-1} y^k(0)\ \frac{t^k}{k!} = I^\alpha f(t, y(t), D^\beta y(t)). \quad (21)$$

$$I^\alpha(T_{m\gamma}(t)T_{m\gamma}^T(t)) = \begin{bmatrix} \frac{1}{\Gamma(\alpha+1)}t^\alpha & \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\gamma+\alpha} & \cdots & \frac{\Gamma(m\gamma+1)}{\Gamma(m\gamma+\alpha+1)}t^{m\gamma+\alpha} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\gamma+\alpha} & \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+\alpha+1)}t^{2\gamma+\alpha} & \cdots & \frac{\Gamma((m+1)\gamma+1)}{\Gamma((m+1)\gamma+\alpha+1)}t^{(m+1)\gamma+\alpha} \\ \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+\alpha+1)}t^{2\gamma+\alpha} & \frac{\Gamma(3\gamma+1)}{\Gamma(3\gamma+\alpha+1)}t^{3\gamma+\alpha} & \cdots & \frac{\Gamma((m+2)\gamma+1)}{\Gamma((m+2)\gamma+\alpha+1)}t^{(m+2)\gamma+\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(m\gamma+1)}{\Gamma(m\gamma+\alpha+1)}t^{m\gamma+\alpha} & \frac{\Gamma((m+1)\gamma+1)}{\Gamma((m+1)\gamma+\alpha+1)}t^{(m+1)\gamma+\alpha} & \cdots & \frac{\Gamma(2m\gamma+1)}{\Gamma(2m\gamma+\alpha+1)}t^{2m\gamma+\alpha} \end{bmatrix} \quad (16)$$

Substituting (6) and (18) in (21), we obtain

$$C^T T_{m\gamma}(t) - Y_0 - Y_1 t = I^\alpha f\left(t, C^T T_{m\gamma}(t), D^\beta(C^T T_{m\gamma}(t)\right). \tag{22}$$

Next, we use the operational matrices obtained in Section 4 as needed and collocate (22) at the following equidistant nodes t_i given by

$$t_i = \frac{i}{m}, \quad i = 0, 1, 2, \ldots m. \tag{23}$$

These equations give $m+1$ algebraic equations, which can be solved for the unknown vector C^T using Newton's iterative method. It is known that the initial guess for Newton's iterative method is very important. According to the conditions in (18) the solution $y(t)$ will pass through the point $(0, Y_0)$ and have a slope Y_1 at this point. We choose our initial guess $y_0(t)$ such that $y_0(t) = Y_1 t + Y_0$.

B. *Problem b*

For Problem b, by substituting (6) in (20) we get (24), shown at the bottom of the page.

By using the operational matrices obtained in Section IV wherever needed and collocating (24) at the equidistant nodes t_i, given in (23), we get a system of algebraic equations, which can be solved for the unknown vector C^T using Newton's iterative method. In this case, the initial values required to start Newton's iterative method have been chosen by taking $y(t)$ as a linear function between the initial value $y(0) = Y_0$ and the final value $y(1) = \bar{Y}_1$.

VII. Illustrative Examples

In this section, five examples are given to demonstrate the applicability and accuracy of our method. Examples 1−4 are initial value problems and Example 5 is a boundary value problem. Example 1 is an initial value FDE, which was first considered in [19]. The exact solution of Example 1 is a polynomial, and the exact solution can be obtained using the proposed method. Examples 2 and 3 are FDEs describing the fractional Riccati equation, which were first considered in [20] by using modified homotopy perturbation method, it was also studied in [21] by applying the enhanced homotopy perturbation method, in [22] by using Bernstein polynomials and in [25] by applying Bernoulli wavelets. For Examples 2 and 3, we compare our findings with the numerical results in [20]−[22], [25]. Example 4 was first considered in [27] by using a predictor corrector approach; it was also solved in [28] by converting the FDE to a Volterra type integral equation and in [24] by using Legendre wavelet method. For Example 4 we compare our method with [24] which has been shown to be comparable or superior to [27], [28]. Example 5 was solved in [24] by using Legendre wavelet. For Example 5, we compare our results with [24]. In Examples 2−5 the package of Mathematica ver. 9.0 has been used to solve the test problems. Here, we first give a method for selecting γ in (5) for our examples. We select $\gamma = 1$ if $\alpha = 1$ or $\alpha = 2$. Otherwise, we select $\gamma = \alpha$. For Example 5, similar to [29] we have also used $\gamma = \alpha - \lfloor\alpha\rfloor$, and we get better results than α. Here $\lfloor\alpha\rfloor$ is the floor function which is the greatest integer less than or equal to the α.

A. *Example 1*

Consider the following linear fractional differential equation given in [19].

$$\begin{aligned} &D^2 y(t) + D^{\frac{3}{2}} y(t) + y(t) = 1 + t \\ &0 < t \le 1; \quad y(0) = 1; \quad y'(0) = 1. \end{aligned} \tag{25}$$

The exact solution of this problem is

$$y(t) = 1 + t.$$

Here, we solve this problem by using the proposed method with $\gamma = 1$ and $m = 1$.

Let

$$y(t) \cong C^T T_{m\gamma}(t) = [c_0,\ c_1]\begin{bmatrix} 1 \\ t \end{bmatrix}. \tag{26}$$

By using (1)−(4) and (25), we have

$$y(t) - 1 - t + I^{\frac{1}{2}}\left(I^{\frac{3}{2}} D^{\frac{3}{2}} y(t)\right) + I^2 y(t) = \frac{t^2}{2} + \frac{t^3}{6}. \tag{27}$$

By substituting (26) in (27), we get

$$\begin{aligned} &C^T T_{m\gamma}(t) - 1 - t + I^{\frac{1}{2}}\left(C^T T_{m\gamma}(t) - y(0) - y'(0)t\right) \\ &+ I^2 C^T T_{m\gamma}(t) = \frac{t^2}{2} + \frac{t^3}{6}. \end{aligned}$$

From (12), we have (28), shown at the bottom of the page. where

$$G_{\frac{1}{2}} = [\frac{1}{\Gamma(\frac{3}{2})}, \frac{1}{\Gamma(\frac{5}{2})}]^T, \qquad G_2 = [\frac{1}{2}, \frac{1}{6}]^T. \tag{29}$$

Substituting (29) in (28) and collocating the resulting equation at $t_0 = 0$ and $t_1 = 1$, we get

$$c_0 = 1, \quad c_1 = 1.$$

Then, by using (26), we get $y(t) = 1 + t$, which is the exact solution.

B. *Example 2*

Consider the fractional Riccati differential equation [22].

$$D^\alpha y(t) + y^2(t) = 1, \quad y(0) = 0; \quad 0 < \ \alpha \ \le 1. \tag{30}$$

$$C^T T_{m\gamma}(t) - I^\alpha f(t, C^T T_{m\gamma}(t), D^\beta C^T T_{m\gamma}(t)) + tI^\alpha\left(\ f(1, C^T T_{m\gamma}(t), D^\beta C^T T_{m\gamma}(t))|_{t=1}\right) - (\bar{Y}_1 - Y_0)t - Y_0 = 0 \tag{24}$$

$$C^T T_{m\gamma}(t) - 1 - t + t^{\frac{1}{2}} C^T\left(G_{\frac{1}{2}} * T_{m\gamma}(t) - \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \ - \frac{t^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\right) + t^2 C^T(G_2 * T_{m\gamma}(t)) = \frac{t^2}{2} + \frac{t^3}{6} \tag{28}$$

TABLE I
COMPARISON OF NUMBERICAL RESULTS FOR $\alpha = 0.75$

t_i	BPM [22], $N = 8$	Proposed method, $m = 8$	BPM [22], $N = 11$	Proposed method, $m = 11$	IABMM [21]	EHPM [21]	MHPM [20]
0	0	0	0	0	0	0	0
0.2	0.30996891	0.30997496	0.30997552	0.30997528	0.3117	0.3214	0.3138
0.4	0.48162749	0.48163161	0.48163184	0.48163169	0.4855	0.5077	0.4929
0.6	0.59777979	0.59778262	0.59778277	0.59778267	0.6045	0.6259	0.5974
0.8	0.67884745	0.67884945	0.67884957	0.67884949	0.6880	0.7028	0.6604
1	0.73684181	0.73683663	0.73683686	0.73683667	0.7478	0.7542	0.7183

TABLE II
COMPARISON OF NUMBERICAL RESULTS FOR $\alpha = 0.9$

t_i	BPM [22], $N = 8$	Proposed method, $m = 8$	BPM [22], $N = 11$	Proposed method, $m = 11$	IABMM [21]	EHPM [21]	MHPM [20]
0	0	0	0	0	0	0	0
0.2	0.23878798	0.23878894	0.23878915	0.23878913	0.2393	0.2647	0.2391
0.4	0.42258214	0.42258305	0.42258309	0.42258308	0.4234	0.4591	0.4229
0.6	0.56617082	0.56617156	0.56617157	0.56617156	0.5679	0.6031	0.5653
0.8	0.67462642	0.67462706	0.67462700	0.67462699	0.6774	0.7068	0.6740
1	0.75460256	0.75458885	0.75458901	0.75458880	0.7584	0.7806	0.7569

TABLE III
COMPARISON OF ABSOLUTE ERROR FOR $\alpha = 1$

t_i	BPM [22], $N = 5$	Proposed method, $m = 5$	BPM [22], $N = 11$	Proposed method, $m = 11$	Proposed method, $m = 20$	MHPM [20]
0	0	0	0	0	0	0
0.2	5.1734E−05	1.1440E−06	2.6847E−10	4.3055E−11	5.5511E−17	3.2022E−7
0.4	2.5969E−05	8.4839E−08	2.5057E−10	1.2536E−11	1.1102E−16	4.9622E−6
0.6	4.0657E−05	1.0711E−06	2.1577E−10	1.4442E−11	0	0.0001925
0.8	1.2390E−05	1.0920E−06	2.9392E−10	5.7991E−11	1.1102E−16	0.0023307
1	7.5141E−04	5.8350E−06	6.8444E−08	1.8625E−10	1.1102E−16	0.0155622

The exact solution of this problem for $\alpha = 1$ is

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

To compare the proposed method with [20]−[22], we solve (30) for $\alpha = 0.75$, $\alpha = 0.9$, and $\alpha = 1$. Now, we solve (30), with $m = 3$ and $\gamma = \alpha = 0.75$.

Let

$$y(t) \cong C^T T_{m\gamma}(t) \tag{31}$$

where

$$C^T = [c_0,\ c_1,\ c_2,\ c_3]$$

and

$$T_{m\gamma}(t) = [1,\ t^{0.75},\ t^{1.5},\ t^{2.25}]^T.$$

By using (22), (30), and (31), we get

$$C^T T_{m\gamma}(t) + t^{\alpha}\ C^T (S_{\alpha} * (T_{m\gamma}(t) T_{m\gamma}^T(t)) C - \frac{t^{\alpha}}{\Gamma(\alpha+1)} = 0 \tag{32}$$

where

$$T_{m\gamma}(t).T_{m\gamma}^T(t) = \begin{bmatrix} 1 & t^{0.75} & t^{1.5} & t^{2.25} \\ t^{0.75} & t^{1.5} & t^{2.25} & t^3 \\ t^{1.5} & t^{2.25} & t^3 & t^{3.75} \\ t^{2.25} & t^3 & t^{3.75} & t^{4.5} \end{bmatrix}$$

and

$$S_{\alpha}(t) = \begin{bmatrix} 1.08807 & 0.691367 & 0.521462 & 0.424876 \\ 0.691367 & 0.521462 & 0.424876 & 0.361746 \\ 0.521462 & 0.424876 & 0.361746 & 0.316877 \\ 0.424876 & 0.361746 & 0.316877 & 0.283147 \end{bmatrix}.$$

By collocating (32) at the nodes given in (23), and solving the resulting equations we get,

$$c_0 = 0,\ c_1 = 1.03094,\ c_2 = -0.165321,\ \text{and}\ c_3 = -0.1044.$$

Then, by using (31), we have

$$y(t) = 1.03094\ t^{0.75} - 0.165321\ t^{1.5} - 0.1044\ t^{2.25}.$$

In Tables I and II, we compare our results with the solutions of the modified homotopy perturbation method (MHPM) in [20], the improved Adams-Bashforth-Moulton method (IABMM) in [21], the enhanced homotopy perturbation method (EHPM) in [21] and with the Bernstein polynomials method (BPM) in [22] for $\gamma = \alpha = 0.75$ and $\gamma = \alpha = 0.9$ for different values of m. In Table III, we compare the absolute error of our method for $\gamma = \alpha = 1$ with MHPM [20] and BPM [22], for different values of m. In Tables I−III, N represents the degree of the Bernstein polynomial used in [22]. Also, Fig. 1 shows the approximate solutions obtained for different values of α using the proposed method with $m = 5$. From these results, it is seen that the approximate solutions converge to the exact solution for α=1. In addition, the absolute difference between the exact and approximate solutions for $\alpha = 1$ with $m = 5$ is plotted in Fig. 2. The absolute difference between the exact and approximate solutions for $k = 1$ and $M = 5$ or $\hat{m} = 2^{k-1}M = 5$ and $\alpha = 1$ by Bernoulli wavelets method is plotted in [25]. Here, k and M are the order of wavelets and Bernoulli polynomials respectively. From our figures and those in [25], we can conclude that the result obtained by the proposed method has less error compared to Bernoulli wavelets method.

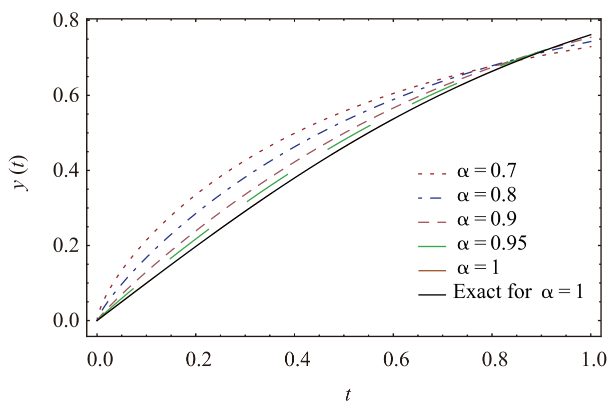


Fig. 1. Comparison of the computed solutions for different values of α with exact solution for $\gamma = \alpha = 1$ for Example 2 with $m = 5$.

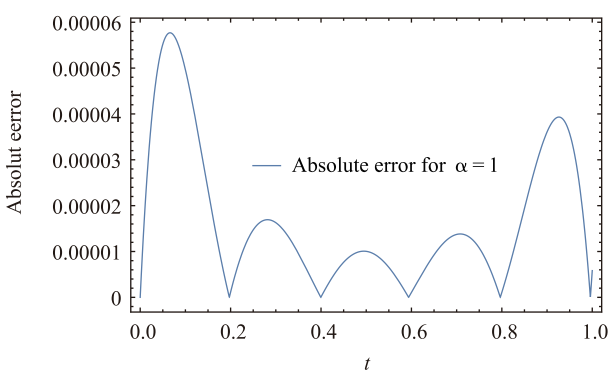


Fig. 2. The absolute error for $\gamma = \alpha = 1$ for Example 2 with $m = 5$.

C. Example 3

Consider the following Riccati fractional differential equation given in [22].

$$D^{\alpha}y(t) = 2\,y(t) - y^2(t) + 1, \quad y(0) = 0; \quad 0 < \alpha \leq 1. \tag{33}$$

To solve this problem by using the proposed method, we let

$$y(t) \cong C^T T_{m\gamma}(t) \tag{34}$$

where C^T and $T_{m\gamma}(t)$ are given in (5) and (7) respectively. Using (22), (33), and (34), we have

$$C^T T_{m\gamma}(t) - 2t^{\alpha}C^T(G_{\alpha} * T_{m\gamma}(t)) + t^{\alpha}C^T(S_{\alpha} * (T_{m\gamma}(t)T^T_{m\gamma}(t)))C - \frac{t^{\alpha}}{\Gamma(\alpha+1)} = 0. \tag{35}$$

Now, by collocating (35) at the nodes given in (23), we get $m + 1$ nonlinear algebraic equations which can be solved for the unknown vector C^T using Newton's iterative method. It is well known that the initial guesses for Newton's iterative method are very important. For this problem, by using $y(0) = 0$, and (34), we choose the initial guesses such that $C^T T_{m\gamma}(0) = 0$. The exact solution of this problem for α = 1 is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\ln(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right).$$

Table IV shows the comparison of our numerical results with [20]−[22] for $\gamma = \alpha = 0.9$. In Table V, we compare the absolute error of our numerical method with [20] and [22] for α=1. Also, Fig. 3 shows the approximate solutions obtained for different values of α using the proposed method with $m = 5$. From these results, it is seen that the approximate solutions converge to the exact solution for $\alpha = 1$. From Table V it is seen that our results with $m = 18$ has less error than the results in the table given in [25], with $k = 2$ and $M = 10$ or $\hat{m} = 2^{k-1}M = 20$ using Bernoulli wavelets method.

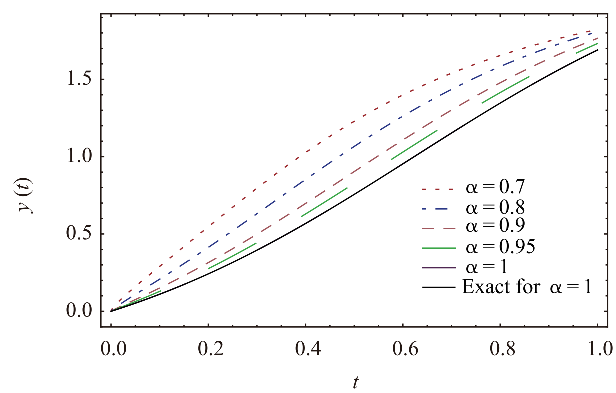


Fig. 3. Comparison of the computed solutions for different values of α with exact solution for $\alpha = 1$ for Example 3 with $m = 5$.

D. Example 4

Consider the FDE [24]

$$D^{\alpha}y(t) + y(t) = 0, \quad 0 < \alpha \leq 2 \tag{36}$$

with $y(0) = 1$ and $y'(0) = 0$. The condition $y'(0) = 0$ is for $1 < \alpha \leq 2$ only.

The exact solution of this problem is $y(t) = E_a(-t^{\alpha})$ [24], where

$$E_a(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + 1)}$$

is the Mittag-Leffler function with order α. To solve this problem by using the proposed method, similar to (34) in Example 3 we let

$$y(t) \cong C^T T_{m\gamma}(t). \tag{37}$$

Using (22), (36), and (37), we have

$$C^T T_{m\gamma}(t) - 1 + t^{\alpha}C^T(G_{\alpha} * T_{m\gamma}(t)) = 0.$$

TABLE IV
COMPARISON OF NUMBERICAL RESULTS FOR $\alpha = 0.9$

t_i	BPM [22], $N = 8$	Proposed method, $m = 8$	BPM [22], $N = 11$	Proposed method, $m = 11$	IABMM [21]	EHPM [21]	MHPM [20]
0	0	0	0	0	0	0	0
0.2	0.31488815	0.31485423	0.31486902	0.31486367	-	-	-
0.4	0.69756771	0.69751826	0.69754441	0.69753816	-	-	-
0.5	0.90369502	0.90364539	0.90367312	0.90366680	0.8621	1.4614	0.9010
0.6	1.10789047	1.10783899	1.10786695	1.10786083	-	-	-
0.8	1.47772823	1.47768008	1.47770748	1.47770236	-	-	-
1	1.76452008	1.76525852	1.76529044	1.76527469	1.7356	2.0697	1.8720

TABLE V
COMPARISON OF ABSOLUTE ERROR FOR $\alpha = 1$

t_i	BPM [22], $N = 5$	Proposed method, $m = 5$	BPM [22], $N = 11$	Proposed method, $m = 11$	Proposed method, $m = 18$	MHPM [20]
0	0	0	0	0	0	0
0.2	6.9332E−04	1.7402E−04	2.3521E−07	5.7111E−08	5.2301E−12	0.00001
0.4	6.2509E−04	1.9080E−04	3.0542E−07	6.3894E−08	6.7869E−12	0.00030
0.6	9.1370E−04	2.1689E−04	3.3836E−07	7.0059E−08	7.4832E−12	0.00469
0.8	3.9346E−04	2.0848E−04	3.4201E−07	7.1896E−08	7.0779E−12	0.01887
1	7.1367E−03	1.3840E−04	1.1799E−05	3.1903E−08	5.9701E−12	0.03431

TABLE VI
COMPARISON OF ABSOLUTE ERROR WITH [24] FOR $\alpha = 1.5$

t_i	LWM [24], $\hat{M} = 384$	Proposed method, $m = 3$	Proposed method, $m = 4$	Proposed method, $m = 5$	Proposed method, $m = 10$
0.1	4.207E−7	3.06933E−6	3.60303E−8	2.8731E−10	2.5259E−17
0.2	1.944E−7	4.52543E−6	2.57570E−8	1.17022E−11	2.3353E−16
0.3	5.705E−8	1.45914E−6	3.43268E−8	3.89751E−10	7.0728E−17
0.4	4.605E−8	4.53425E−6	6.35744E−8	3.84075E−12	3.6902E−18
0.5	1.282E−7	8.96162E−6	1.91067E−10	6.05008E−10	7.2132E−17
0.6	1.944E−7	6.66709E−6	1.10701E−7	2.24834E−11	2.6174E−16
0.7	2.471E−7	5.03679E−6	1.02038E−7	1.58912E−9	6.2367E−17
0.8	2.878E−7	2.24058E−5	1.90389E−7	2.93859E−11	3.0854E−16
0.9	3.176E−7	3.05511E−5	5.93256E−7	7.70141E−9	2.8515E−16

TABLE VII
ABSOLUTE ERROR OF OUR METHOD FOR DIFFERENT VALUES OF α WITH $m = 10$

t_i	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$
0.1	1.95877E−16	4.86487E−17	2.52589E−17	6.61786E−17	8.17143E−18	5.5437E−17
0.2	1.48822E−16	2.00823E−16	2.33527E−16	2.28488E−16	1.70503E−16	1.02528E−18
0.3	2.9433E−16	1.60699E−17	7.07283E−17	1.87274E−17	1.90848E−17	2.66293E−17
0.4	4.38831E−17	3.31082E−17	3.69019E−18	1.86397E−17	9.881E−17	7.59506E−17
0.5	2.29268E−16	1.72008E−16	7.21322E−17	6.88831E−17	1.32945E−16	4.27994E−17
0.6	4.46312E−16	1.77538E−18	2.61743E−16	3.10642E−17	1.78043E−17	3.43499E−17
0.7	4.04245E−16	1.85026E−16	6.23670E−17	9.06418E−17	1.30585E−16	1.05295E−16
0.8	4.16117E−16	1.31649E−16	3.08539E−16	2.75084E−16	7.1246E−17	1.84848E−17
0.9	5.21935E−16	1.84856E−17	2.85145E−16	1.22344E−16	2.10206E−17	6.86984E−17
1	9.45424E−17	1.87242E−16	1.70220E−17	8.38901E−18	1.07569E−16	5.90071E−17

By collocating at the points given in (23) we get $m + 1$ algebraic equations, which can be solved for the unknown vector C^T. Table VI shows the absolute error obtained for different values of t and for $\alpha = 1.5$ by using the proposed method with different values of m and the Legendre wavelets method (LWM) in [24], with $k = 8$ and $M_1 = 3$ or $\hat{M} = 2^{k-1}M_1 = 384$. Here, M_1 shows the order of Legendre polynomials. In Table VII, the absolute error obtained using the proposed method for different values of α with $m = 10$ is given.

TABLE IX
ABSOLUTE ERROR FOR DIFFERENT α WITH $M = 3$

t_i	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$
0.1	1.33357E−17	5.01444E−18	2.71728E−18	7.76899E−18	5.36765E−18	8.84514E−19
0.2	1.04083E−17	6.07153E−18	1.22515E−17	1.85399E−17	1.55041E−17	2.19551E−18
0.3	6.93889E−18	0	3.1225E−17	2.38524E−17	2.32019E−17	2.1684E−19
0.4	2.77556E−17	1.04083E−17	5.89806E−17	1.9082E−17	2.34188E−17	1.04083E−17
0.5	5.55112E−17	1.38778E−17	9.71445E−17	0	1.04083E−17	3.1225E−17
0.6	5.55112E−17	0	1.38778E−16	2.77556E−17	1.38778E−17	6.93889E−17
0.7	5.55112E−17	0	1.38778E−16	5.55112E−17	2.77556E−17	9.71445E−17
0.8	0	5.55112E−17	1.66533E−16	5.55112E−17	5.55112E−17	1.66533E−16
0.9	1.11022E−16	1.11022E−16	1.11022E−16	0	1.11022E−16	1.11022E−16
1	3.33067E−16	3.33067E−16	0	0	3.33067E−16	0

E. Example 5

Consider the following FDE with boundary value conditions [24]

$$D^{\alpha}y(t) + ay^{n}(t) = g(t), \quad y(0) = 0; \quad y(1) = 1 \tag{38}$$

where $1 < \alpha \le 2$, $a = \exp(-2\pi)$ and $n = 2$. For $\alpha = 1.5$ and $g(t) = 105\sqrt{\pi}t^2/32 + \exp(-2\pi)t^7$, the exact solution is given by

$$y(t) = t^{\frac{7}{2}}.$$

Using (20) and (38), we have

$$\begin{aligned} y(t) = I^{\alpha}g(t) - aI^{\alpha}y^2(t) - t(I^{\alpha}g(t))|_{t=1} \\ - a\ t(I^{\alpha}y^2(t))|_{t=1} + t. \end{aligned} \tag{39}$$

Similar to (34) in Example 3, we let

$$y(t) \cong C^T T_{m\gamma}(t). \tag{40}$$

Using (39) and (40), we get

$$\begin{aligned} &C^T T_{m\gamma}(t) - \frac{105\sqrt{\pi}\Gamma(3)}{32\Gamma(3+\alpha)}t^{2+\alpha} - \frac{\exp(-2\pi)\Gamma(8)}{\Gamma(8+\alpha)}t^{7+\alpha} \\ &\quad + \exp(-2\pi)t^{\alpha}C^T(S_{\alpha} * T_{m\gamma}(t)\ T^T_{m\gamma}(t))C \\ &\quad + \ t\frac{105\sqrt{\pi}\Gamma(3)}{32\Gamma(3+\alpha)} + \ t\ \frac{\exp(-2\pi)\Gamma(8)}{\Gamma(8+\alpha)} \\ &\quad + \ \exp(-2\pi)tC^T(S_{\alpha} * T_{m\gamma}(1)\ T^T_{m\gamma}(1))C - \ t = 0. \end{aligned}$$

By collocating at the points given in (23) we get $m+1$ algebraic equations, which can be solved for the unknown vector C^T. Table VIII shows the absolute error obtained for different values of t and for $m = 10$ by using the proposed method for $\alpha = 1.5$ with $\gamma = \alpha$ and $\gamma = \alpha - \lfloor \alpha \rfloor$, together with the absolute error obtained by the Legendre wavelets method in [24], with $k = 3$ and $M_1 = 3$ or $\hat{M} = 2^{k-1}M_1 = 12$. Here, M_1 shows the order of Legendre polynomials.

More generally, the exact solution for (38) with

$$g(t) = \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + \exp(-2\pi)t^{6\alpha}$$

and keeping the other coefficients the same is

$$y(t) = t^{3\alpha}.$$

In Table IX, we show the absolute error of our numerical results for different values of α with $m = 3$.

TABLE VIII
COMPARISON OF ABSOLUT ERROR FOR $\alpha = 1.5$

t_i	LWM[24] $\hat{M} = 12$	Proposed method $m = 10$ $\gamma = \alpha$	Proposed method $m = 10$ $\gamma = \alpha - \lfloor \alpha \rfloor$
0.1	9.6996E−5	1.34431E−9	1.06794E−17
0.2	9.3927E−4	2.68845E−9	1.38778E−17
0.3	1.5087E−3	4.03315E−9	8.67362E−18
0.4	3.3989E−4	5.37645E−9	4.16334E−17
0.5	2.4163E−3	6.72324E−9	2.77556E−17
0.6	3.1023E−4	8.05974E−9	2.77556E−17
0.7	1.4799E−3	9.43278E−9	0
0.8	6.3407E−4	1.06306E−8	1.66533E−16
0.9	4.6701E−3	1.30547E−8	2.22045E−16

VIII. CONCLUSION

In the present work the fractional Taylor basis is used to solve FDEs. The integral operational matrix F_{α} and S_{α} have been derived. The error bounds are also included. The problem has been reduced to a problem of solving a system of algebraic equations. Illustrative examples are solved by using the proposed method to show that this approach can solve the problem effectively.

REFERENCES

[1] K. B. Oldham and J. Spanier, *The Fractional Calculus*. New York: Academic Press, 1974.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional calculus and Fractional Differential Equations*. New York: Wiley, 1993.

[3] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 16, no. 3, pp. 1140−1153, Mar. 2011.

[4] R. L. Bagley and P. J. Torvik, "Fractional calculus in the transient analysis of viscoelastically damped structures," *AIAA J.*, vol. 23, no. 6, pp. 918−925, Jun. 1985.

[5] R. T. Baillie, "Long memory processes and fractional integration in econometrics," *J. Econometr.*, vol. 73, no. 1, pp. 5−59, Jul. 1996.

[6] F. Mainardi, "Fractional calculus: some basic problems in continuum and statistical mechanics," *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds. Vienna: Springer-Verlag, 1997, pp. 291−348.

[7] Y. A. Rossikhin and M. V. Shitikova, "Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids," *Appl. Mech. Rev.*, vol. 50, no. 1, pp. 15−67, Jan. 1997.

[8] K. B. Oldham, "Fractional differential equations in electrochemistry," *Adv. Eng. Softw.*, vol. 41, no. 1, pp. 9−12, Jan. 2010.

[9] V. S. Ertürk, Z. M. Odibat, and S. Momani, "An approximate solution of a fractional order differential equation model of human T-cell lymphotropic virus I (HTLV-I) infection of $CD4^+$ T-cells," *Comput. Math. Appl.*, vol. 62, no. 3, pp. 996−1002, Aug. 2011.

[10] S. A. El-Wakil, E. M. Abulwafa, E. K. El-Shewy, and A. A. Mahmoud, "Ion-acoustic waves in unmagnetized collisionless weakly relativistic plasma of warm-ion and isothermal-electron using time-fractional KdV equation," *Adv. Space Res.*, vol. 49, no. 12, pp. 1721−1727, Jun. 2012.

[11] L. Gaul, P. Klein, and S. Kemple, "Damping description involving fractional operators," *Mech. Syst. Signal Proc.*, vol. 5, no. 2, pp. 81−88, Mar. 1991.

[12] I. Podlubny, *Fractional Differential Equations: an Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*. New York: Academic Press, 1998.

[13] S. Momani and K. Al-Khaled, "Numerical solutions for systems of fractional differential equations by the decomposition method," *Appl. Math. Comput.*, vol. 162, pp. 1351−1365, Mar. 2005.

[14] Z. M. Odibat and S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," *Int. J. Nonlinear Sci. Numer. Simul.*, vol. 7, no. 1, pp. 27−34, Mar. 2006.

[15] Z. M. Odibat and N. T. Shawagfeh, "Generalized Taylor's formula," *Appl. Math. Comput.*, vol. 186, no. 1, pp. 286−293, Mar. 2007.

[16] J. S. Duan, T. Chaolu, and R. Rach, "Solutions of the initial value problem for nonlinear fractional ordinary differential equations by the Rachdomianeyers modified decomposition method," *Appl. Math. Comput.*, vol. 218, no. 17, pp. 8370−8392, May 2012.

[17] A. Arikoglu and I. Ozkol, "Solution of fractional integro-differential equations by using fractional differential transform method," *Chaos Solit. Fract.*, vol. 40, no. 2, pp. 521−529, Apr. 2009.

[18] I. Hashim, O. Abdulaziz, and S. Momani, "Homotopy analysis method for fractional IVPs," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 14, no. 3, pp. 674−684, Mar. 2009.

[19] S. Kazem, S. Abbasbandy, and S. Kumar, "Fractional-order Legendre functions for solving fractional-order differential equations," *Appl. Math. Model.*, vol. 37, no. 7, pp. 5498−5510, Apr. 2013.

[20] Z. Odibat and S. Momani, "Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order," *Chaos Solit. Fract.*, vol. 36, no. 1, pp. 167−174, Apr. 2008.

[21] S. H. Hosseinnia, A. Ranjbar, and S. Momani, "Using an enhanced homotopy perturbation method in fractional differential equations via deforming the linear part," *Comput. Math. Appl.*, vol. 56, no. 12, pp. 3138−3149, Dec. 2008.

[22] S. Yüzbaşi, "Numerical solutions of fractional Riccati type differential equations by means of the Bernstein polynomials," *Appl. Math. Comput.*, vol. 219, no. 11, pp. 6328−6343, Feb. 2013.

[23] Y. L. Li, "Solving a nonlinear fractional differential equation using Chebyshev wavelets," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 15, no. 9, pp. 2284−2292, Sep. 2010.

[24] M. ur Rehman and R. Ali Khan, "The Legendre wavelet method for solving fractional differential equations," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 16, no. 11, pp. 4163−4173, Nov. 2011.

[25] E. Keshavarz, Y. Ordokhani, and M. Razzaghi, "Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations," *Appl. Math. Model.*, vol. 38, no. 24, pp. 6038−6051, Dec. 2014.

[26] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier, 2006.

[27] K. Diethelm, N. J. Ford, and A. D. Freed, "A predictor-corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dyn.*, vol. 29, no. 1−4, pp. 3−22, Jul. 2002.

[28] P. Kumar and O. P. Agrawal, "An approximate method for numerical solution of fractional differential equations," *Signal Proc. -Fract. Calcul. Appl. Signal. Syst.*, vol. 86, no. 10, pp. 2602−2610, Oct. 2006.

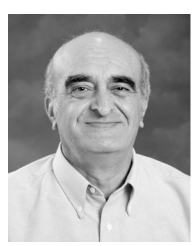
[29] X. H. Ma and C. M. Huang, "Numerical solution of fractional integro-differential equations by a hybrid collocation method," *Appl. Math. Comput.*, vol. 219, no. 12, pp. 6750−6760, Feb. 2013.

Vidhya Saraswathy Krishnasamy received her undergraduate and master's degree in mathematics from Sri G.V.G. Vishalakshi College for Women, Udumalpet, India. She graduated from Mississippi State University (MSU), USA with her Ph.D. degree in August, 2016.

She has diverse research interests, which include fractional calculus, orthogonal functions and its applications to dynamical systems, graph theory, number theory, cryptography and its applications.

Somayeh Mashayekhi graduated from Mississippi State University (MSU), USA, in 2015. She received her first Ph.D. degree from Alzahra University in 2013, and her second Ph.D. degree from MSU in 2015. She is, currently, a post doctoral research associate in computational science and engineering, Department of Mathematics, Florida State University.

Her research interests include optimal control, delay system, fractional calculus, orthogonal functions and its applications to dynamical systems.



Mohsen Razzaghi received his undergraduate degree in mathematics from the University of Sussex in England. He went to Canada and received his master's degree in applied mathematics from the University of Waterloo. He returned to the University of Sussex and obtained his Ph.D.. After that, he has taught and served in several administrative positions in Iran and in the USA. Since 1986, he has been at the Department of Mathematics and Statistics at Mississippi State University, where he is currently a professor and the department head. During the academic years 2011-2012 and 2015-2016, he was a Fulbright Scholar at the Department of Mathematics and Computer Science at the Technical University of Civil Engineering in Bucharest, Romania.

His area of research centers on optimal control, orthogonal functions and wavelets in dynamical system, and fractional calculus. He has over 150 refereed journal publications in mathematics, mathematical physics, and engineering. Corresponding author of this paper.