State Feedback Control for a Class of Fractional Order Nonlinear Systems

Yige Zhao, Yuzhen Wang, and Haitao Li

***Abstract*—Using the Lyapunov function method, this paper investigates the design of state feedback stabilization controllers for fractional order nonlinear systems in triangular form, and presents a number of new results. First, some new properties of Caputo fractional derivative are presented, and a sufficient condition of asymptotical stability for fractional order nonlinear systems is obtained based on the new properties. Then, by introducing appropriate transformations of coordinates, the problem of controller design is converted into the problem of finding some parameters, which can be certainly obtained by solving the Lyapunov equation and relevant matrix inequalities. Finally, based on the Lyapunov function method, state feedback stabilization controllers making the closed-loop system asymptotically stable are explicitly constructed. A simulation example is given to demonstrate the effectiveness of the proposed design procedure.**

***Index Terms*—Fractional order system, triangular system, asymptotical stabilization, state feedback, Lyapunov function method.**

I. INTRODUCTION

FRACTIONAL order systems have been of great interest in the last two decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications. Apart from diverse areas of mathematics, fractional order systems play an important role in physics, chemistry, engineering and so on[1−2].

As we all know, stability is an essential issue to control systems, certainly including fractional order systems. The earliest study on the stability of fractional differential equations can be traced back to 1960s[3], where it was shown that the stability problem of fractional differential equations comes down to the eigenvalue problem of system matrices. For fractional order systems, there are many papers related to the stability theory[4−10] such as root-locus, asymptotical stability, bounded input bounded output stability, internal stability, external stability, robust stability, finite-time stability, etc.

Recently, the Lyapunov function method has also been used to study the stability of fractional order systems[11−17]. On one hand, some Lyapunov functions were constructed in works related to fractional sliding mode control[13−14], and the classic Lyapunov function method was presented to stabilize fractional order systems. On the other hand, Li et al. investigated the Mittag-Leffler stability and the asymptotical stability of fractional order nonlinear systems by using the fractional Lyapunov's direct method[15−16]. It is usually difficult to construct a positive definite function and calculate its fractional derivative for a given fractional order system. Recently, a new property for Caputo fractional derivative of a quadratic function has been presented in [17]. The result allows the use of classic quadratic Lyapunov functions in the stability analysis of fractional order systems. In some cases, those simple quadratic functions[17] cannot work, and more general quadratic Lyapunov functions should be used instead. These results are very important in the sense that they have provided a basic tool for the stability analysis and controller design of fractional order systems.

However, it should be pointed out that it is usually difficult to construct a positive definite function and calculate its fractional derivative for a given fractional order system. The Leibniz rule for Caputo fractional derivative does not work very well like that for classical derivative. To this end, this work will present some new and useful properties for Caputo fractional derivative which allow finding a simple Lyapunov candidate function for many fractional order systems. Furthermore, to the authors' best knowledge, fewer works have been done to study the stabilization problems for fractional order nonlinear systems in the triangular form. For fractional order nonlinear systems in the triangular form, such as the ones considered in this work, it is difficult or even impossible to solve the feedback stabilizer design problem by the existing approaches.

In this paper, using the Lyapunov function method, we investigate the design of state feedback stabilization controllers for fractional order nonlinear systems in the upper triangular form. The main contributions of this paper are as follows: 1) Some new properties for Caputo fractional derivative are presented, which allow finding a simple Lyapunov candidate function for many fractional order systems. As an application, a sufficient condition of asymptotical stability for fractional order nonlinear systems is obtained based on the new properties. 2) By introducing appropriate transformations of coordinates, the problems of controller design are converted into the problems of finding some parameters, which can be certainly obtained by solving the Lyapunov equation and relevant matrix inequalities. By designing state feedback stabilization controllers for fractional order nonlinear systems in the upper

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triangular form, asymptotical stability for closed-loop systems is considered based on the Lyapunov function method.

The rest of this paper is organized as follows: Section II presents some necessary preliminaries. Section III gives new properties on Caputo fractional derivative and presents a sufficient condition of asymptotical stability for fractional order nonlinear systems. Section IV investigates the design of state feedback controller for the upper triangular fractional order nonlinear systems. Section V gives an illustrative example to illustrate our new results, which is followed by the conclusion in Section VI.

Notation. $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbf{R}^{n\times n}$ denotes the set of $n\times n$ real matrices. For real symmetric matrices X and Y, the notation $X>Y$ ($X\geq Y$) means that matrix $X-Y$ is positive definite (positive semi-definite), and similarly, $X<Y$ ($X\leq Y$) means that matrix $X-Y$ is negative definite (negative semi-definite). I is the identity matrix with appropriate dimension. $X^{\rm T}$ and X^{-1} represent the transpose and the inverse of matrix X, respectively. $\|\cdot\|$ denotes the Euclidean norm for a vector, or the induced Euclidean norm for a matrix.

II. PRELIMINARIES

In this section, we first give some definitions and properties for Caputo fractional derivative, and then present a Lyapunov-based stability theorem for fractional order systems. Throughout this paper, we use Caputo fractional derivative as our main tools, which is given in [2].

Definition1[2] **.** Caputo fractional derivative of order $\alpha>0$ of a continuous function $f(t)$ is given by

$$ {}_0^CD_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}{\rm d}s, $$

where n is the smallest integer greater than or equal to α and $\Gamma(\cdot)$ denotes the Gamma function, provided that the right side is pointwise defined on $(0,+\infty)$.

The Leibniz rule for Caputo fractional derivative is the following.

Lemma 1[2]**.** If φ and f along with all its derivatives are continuous in $(0,+\infty)$, then the Leibniz rule for Caputo fractional derivative takes the form

$$ \begin{aligned} &{}_0^CD_t^\alpha\left(\varphi(t)f(t)\right)\\ &=\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)}\varphi^{(k)}(t)\ {}_0^CD_t^{\alpha-k}f(t), \end{aligned} $$

where $\alpha\in(0,1)$, $\Gamma(\cdot)$ denotes the Gamma function.

Remark 1. By Lemma 1, we can easily see that

$$ {}_0^CD_t^\alpha\left(\varphi(t)f(t)\right)\neq {}_0^CD_t^\alpha\varphi(t)f(t)+\varphi(t){}_0^CD_t^\alpha f(t), $$

where $\alpha\in(0,1)$. Obviously, the Leibniz rule for Caputo fractional derivative does not have the form like that for classical derivative.

The following lemma is the property for Caputo fractional derivative of a matrix.

Lemma 2. Let $A(t)=(a_{i,j}(t))_{n\times n}$ be a time-varying matrix and $a_{i,j}(t)$ be continuous and derivable functions, and $Q\in\mathbf{R}^{n\times n}$. Then the following equalities:

$$ \begin{aligned} &{}_0^CD_t^\alpha A(t)=\left({}_0^CD_t^\alpha a_{i,j}(t)\right)_{n\times n},\\ &{}_0^CD_t^\alpha(QA(t))=Q{}_0^CD_t^\alpha A(t) \end{aligned} $$

hold, where ${}_0^CD_t^\alpha$ is Caputo fractional derivative, $\alpha\in(0,1]$.

Proof. The proof is straightforward. □

The property for Caputo fractional derivative of a quadratic function is the following.

Lemma 3[17]**.** Let $x(t)\in\mathbf{R}$ be a continuous and derivable function. Then, for any time instant $t\geq 0$,

$$ \frac{1}{2}{}_0^CD_t^\alpha x^2(t)\leq x(t){}_0^CD_t^\alpha x(t),\qquad\forall\ \alpha\in(0,1). $$

Remark 2[17]**.** In the case when $x(t)\in\mathbf{R}^n$, Lemma 3 is still valid. That is, for $\forall\ \alpha\in(0,1)$ and $\forall\ t\geq 0$,

$$ \frac{1}{2}{}_0^CD_t^\alpha\left(x^{\rm T}(t)x(t)\right)\leq x^{\rm T}(t){}_0^CD_t^\alpha x(t). $$

Finally, we recall a useful result on the Lyapunov-based stability theorem for fractional order systems[15−16].

Lemma 4. Let $\tilde{x}=0$ be an equilibrium point of fractional order systems

$$ {}_0^CD_t^\alpha x(t)=f(t,x),\quad x_0\in\mathbf{R}^n, \tag{1} $$

where ${}_0^CD_t^\alpha$ denotes Caputo fractional derivative, $0<\alpha<1$. Assume that there exists a Lyapunov function $V(t,x(t))$ and class-$\mathcal{K}$ functions β_i $(i=1,\ 2,\ 3)$ satisfying

$$ \begin{aligned} &\beta_1(\|x\|)\leq V(t,x(t))\leq\beta_2(\|x\|),\\ &{}_0^CD_t^\alpha V(t,x(t))\leq-\beta_3(\|x\|). \end{aligned} $$

Then the equilibrium point of the system (1) is asymptotically stable.

III. NEW PROPERTIES FOR CAPUTO FRACTIONAL DERIVATIVE

In this section, we give some new properties for Caputo fractional derivative. To this end, we need the following lemma, which is about the decomposition of a positive definite matrix.

Lemma 5. Let $A\in\mathbf{R}^{n\times n}$ be a positive definite matrix. Then there exists a positive definite matrix $B\in\mathbf{R}^{n\times n}$, such that $A=B^2$.

Proof. The proof is straightforward. □

According to this lemma, some new properties for Caputo fractional derivative of a general quadratic function are given in the following.

Theorem 1. Let $x(t)=(x_1(t),\ x_2(t),\ \ldots,\ x_n(t))^{\rm T}\in\mathbf{R}^n$, $x_i(t)$ $(i=1,\ 2,\ \ldots,\ n)$ be continuous and derivable functions, and $\alpha\in(0,1]$. Then, for any time instant $t\geq 0$, there exists a positive definite matrix $P\in\mathbf{R}^{n\times n}$ such that

$$ \frac{1}{2}{}_0^CD_t^\alpha\left(x^{\rm T}(t)Px(t)\right)\leq x^{\rm T}(t)P{}_0^CD_t^\alpha x(t). \tag{2} $$

Proof. For convenience, we divide the proof into two cases.

Case 1. $\alpha=1$.

This case corresponds to the chain rule for the integer order derivatives, which states that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(x^{\mathrm{T}}(t)Px(t)\right) = x^{\mathrm{T}}(t)P\frac{\mathrm{d}}{\mathrm{d}t}x(t).$$

Case 2. $0 < \alpha < 1$.

By Lemma 5, there exists a positive definite matrix $Q \in \mathbf{R}^{n\times n}$ such that $P = Q^2$. Then we have

$$\frac{1}{2}{}_0^C D_t^\alpha\left(x^{\mathrm{T}}(t)Px(t)\right) = \frac{1}{2}{}_0^C D_t^\alpha\left(x^{\mathrm{T}}(t)Q^{\mathrm{T}}Qx(t)\right).$$

Let $y(t) = Qx(t)$. From Lemma 2 and Remark 2, we obtain

$$\begin{aligned}\frac{1}{2}{}_0^C D_t^\alpha\left(x^{\mathrm{T}}(t)Px(t)\right) &= \frac{1}{2}{}_0^C D_t^\alpha\left(x^{\mathrm{T}}(t)Q^{\mathrm{T}}Qx(t)\right)\\ &= \frac{1}{2}{}_0^C D_t^\alpha\left(y^{\mathrm{T}}(t)y(t)\right)\\ &\le y^{\mathrm{T}}(t){}_0^C D_t^\alpha y(t)\\ &= x^{\mathrm{T}}(t)Q^{\mathrm{T}}{}_0^C D_t^\alpha(Qx(t))\\ &= x^{\mathrm{T}}(t)Q^{\mathrm{T}}Q{}_0^C D_t^\alpha x(t)\\ &= x^{\mathrm{T}}(t)P{}_0^C D_t^\alpha x(t).\end{aligned}$$

□

Remark 3. In the case when $P = I$, the conclusion of Theorem 1 turns to be the conclusion of Remark 2.

Remark 4. Inequality (2) is equivalent to any one of the following inequalities:

$$\begin{aligned}&\frac{1}{2}{}_0^C D_t^\alpha\left(x^{\mathrm{T}}(t)Px(t)\right) \le \left({}_0^C D_t^\alpha x(t)\right)^{\mathrm{T}} Px(t),\\ &{}_0^C D_t^\alpha\left(x^{\mathrm{T}}(t)Px(t)\right) \le \left({}_0^C D_t^\alpha x(t)\right)^{\mathrm{T}} Px(t) + x^{\mathrm{T}}(t)P{}_0^C D_t^\alpha x(t).\end{aligned} \tag{3}$$

As an application of Theorem 1 and inequality (3), we present a sufficient condition of stability for fractional order nonlinear system by Lyapunov function method.

Consider the fractional order nonlinear system with Caputo fractional derivative

$${}_0^C D_t^\alpha x(t) = f(t, x(t)), \tag{4}$$

where $\alpha \in (0, 1]$, $x(t) \in \mathbf{R}^n$ is the state, $f: \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$, f_i $(i = 1, 2, \dots, n)$ are continuous functions.

Theorem 2. The system (4) is asymptotically stable if there exists a positive definite matrix $P \in \mathbf{R}^{n\times n}$ and a class-$\mathcal{K}$ function γ such that for $\forall\ x(t) \in \mathbb{R}^n$, $x^T(t)Pf(t, x(t)) < -\gamma(\|x\|)$.

Proof. Let $V(t) = x^{\mathrm{T}}(t)Px(t)$. Because $P \in \mathbf{R}^{n\times n}$ is a positive definite matrix, then V is positive definite. By using (3), we have

$$\begin{aligned}{}_0^C D_t^\alpha V(t)|_4 &= {}_0^C D_t^\alpha x^{\mathrm{T}}(t)Px(t)\\ &\le \left({}_0^C D_t^\alpha x(t)\right)^{\mathrm{T}} Px(t) + x^{\mathrm{T}}(t)P{}_0^C D_t^\alpha x(t)\\ &= f^{\mathrm{T}}(t, x(t))Px(t) + x^{\mathrm{T}}(t)Pf(t, x(t))\\ &= 2x^{\mathrm{T}}(t)Pf(t, x(t)) < -2\gamma(\|x\|).\end{aligned}$$

Thus, according to Lemma 4, the system (4) is asymptotically stable. □

IV. State feedback stabilizers design

In this section, state feedback stabilizers are designed for the upper triangular fractional order nonlinear system.

Consider the following fractional order nonlinear system in the upper triangular form:

$$\begin{cases}{}_0^C D_t^\alpha x_1(t) = x_2(t) + \phi_1(t, x(t)),\\ {}_0^C D_t^\alpha x_2(t) = x_3(t) + \phi_2(t, x(t)),\\ \qquad\vdots\\ {}_0^C D_t^\alpha x_{n-2}(t) = x_{n-1}(t) + \phi_{n-2}(t, x(t)),\\ {}_0^C D_t^\alpha x_{n-1}(t) = x_n(t),\\ {}_0^C D_t^\alpha x_n(t) = u(t),\end{cases} \tag{5}$$

where $\alpha \in (0, 1]$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}$ is the control input. The arguments of the functions will be omitted or simplified whenever no confusion can arise from the context. In this paper, $x_i(t)$ and $z_i(t)$ are always denoted by x_i and z_i. The functions $\phi_i : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$, $i = 1, 2, \dots, n-2$ are continuous, and satisfy the following growth condition:

Assumption 1.

$$\begin{aligned}&|\phi_i(t, x)| \le c(|x_{i+2}| + |x_{i+3}| + \cdots + |x_n|),\\ &i = 1, 2, \dots, n-2,\end{aligned} \tag{6}$$

where $c \ge 0$ is a constant.

Remark 5. It is noted that the condition (6) was widely used in the synthesis of nonlinear triangular systems in the literatures[18−20].

In the following, we consider the state feedback controller design for system (5).

Theorem 3. Under the Assumption 1, constants a_i $(i = 1, 2, \dots, n)$ and r can be chosen, such that the system (5) is globally asymptotically stable by a linear state feedback controller of the form

$$u = -\sum_{i=1}^{n}\left(\frac{a_i}{r^{n-i+1}}x_i\right).$$

Proof. For the convenience of readers, we divide the proof into two parts.

Part 1. State transformation of nonlinear system.

Introduce a state transformation for (5):

$$z_i = \frac{x_i}{r^{n-i+1}}, \quad i = 1, 2, \dots, n, \tag{7}$$

where $r > 1$ is a parameter to be determined later. System (5) can be converted into the following system:

$${}_0^C D_t^\alpha z = \frac{1}{r}\Omega z + \frac{1}{r}Gu + \Phi, \tag{8}$$

where

$$z=\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_{n-1}\\ z_n \end{pmatrix},\quad \Omega=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$G=\begin{pmatrix} 0\\ 0\\ \vdots\\ 0\\ 1 \end{pmatrix},\quad \Phi=\begin{pmatrix} \frac{\phi_1}{r^n}\\ \frac{\phi_2}{r^{n-1}}\\ \vdots\\ \frac{\phi_{n-2}}{r^3}\\ 0\\ 0 \end{pmatrix}.$$

Let $a_j>0$ $(j=1,2,\ldots,n)$ be coefficients of the Hurwitz polynomial

$$q(s)=s^n+a_ns^{n-1}+\cdots+a_2s+a_1.$$

Next, choose $r>1$ such that the closed-loop system (8) with

$$u=-(a_1z_1+a_2z_2+\cdots+a_nz_n) \tag{9}$$

is globally asymptotically stable at the equilibrium $z=0$.

The closed-loop system consisting of (8) and (9) is

$${}_0^CD_t^\alpha z=\frac{1}{r}Az+\Phi, \tag{10}$$

where

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{pmatrix}.$$

Up to now, the problem of designing controller for (5) is converted into that of finding an appropriate r, such that the system (10) is asymptotically stable at $z=0$.

Part 2. Stability analysis.

Since $q(s)$ is a Hurwitz polynomial, it can be concluded that A is a stable matrix. Therefore, there exists a positive definite matrix $P>0$ such that

$$PA+A^{\mathrm{T}}P=-I.$$

Choose Lyapunov function $V=z^{\mathrm{T}}Pz$. Observing Assumption 1, the change of coordinate (7) and $r>1$, gives, for any $i(i=1,2,\ldots,n-2)$,

$$\begin{aligned}\left|\frac{\phi_i}{r^{n-i+1}}\right| &\le \frac{c}{r^{n-i+1}}(|x_{i+2}|+|x_{i+3}|+\cdots+|x_n|)\\ &\le \frac{c}{r^2}\sum_{j=1}^{n}\frac{|x_j|}{r^{n-j+1}}\\ &=\frac{c}{r^2}\sum_{i=1}^{n}|z_j|\le\frac{c\sqrt{n}}{r^2}\|z\|.\end{aligned}$$

Hence,

$$\begin{aligned}{}_0^CD_t^\alpha V\big|_{(10)} &= {}_0^CD_t^\alpha\left(z^{\mathrm{T}}Pz\right)\\ &\le \left({}_0^CD_t^\alpha z\right)^{\mathrm{T}}Pz+z^{\mathrm{T}}P{}_0^CD_t^\alpha z\\ &=(\frac{1}{r}Az+\Phi)^{\mathrm{T}}Pz+z^{\mathrm{T}}P(\frac{1}{r}Az+\Phi)\\ &\le -\frac{1}{r}\|z\|^2+2\|z\|\cdot\|P\|\cdot\|\Phi\|\\ &\le -\frac{1}{r}\|z\|^2+2\|z\|\cdot\|P\|(\frac{c\sqrt{n}}{r^2}\|z\|)\\ &\quad\cdot\left\|(1,1,\cdots,1,0,0)^{\mathrm{T}}\right\|\\ &\le -\frac{1}{r}\|z\|^2+\frac{2nc}{r^2}\|P\|\cdot\|z\|^2\\ &=-\frac{1}{r^2}\left(r-2nc\|P\|\right)\cdot\|z\|^2.\end{aligned}$$

Choose

$$r>\max\{1,2nc\|P\|+\eta\},$$

where $\eta>0$. By Lemma 4, we can get ${}_0^CD_t^\alpha V\big|_{(10)}<-\frac{\eta}{r^2}\|z\|^2$ which indicates that (10) is asymptotically stable at $z=0$. Therefore, the closed-loop system consisting of (8) and (9) is asymptotically stable at $z=0$.

Noticing (9) and the change of coordinate (7), we can get the state feedback controller of system (5):

$$u=-\frac{1}{r^n}\left(a_1x_1+a_2rx_2+a_3r^2x_3+\cdots+a_nr^{n-1}x_n\right).$$

□

Remark 5. The conclusion of Theorem 3 also holds for the following fractional order nonlinear system:

$$\begin{cases} {}_0^CD_t^\alpha x_1(t)=d_1x_2(t)+\phi_1(t,x(t)),\\ {}_0^CD_t^\alpha x_2(t)=d_2x_3(t)+\phi_2(t,x(t)),\\ \qquad\vdots\\ {}_0^CD_t^\alpha x_{n-2}(t)=d_{n-2}x_{n-1}(t)+\phi_{n-2}(t,x(t)),\\ {}_0^CD_t^\alpha x_{n-1}(t)=d_{n-1}x_n(t),\\ {}_0^CD_t^\alpha x_n(t)=d_nu(t), \end{cases} \tag{11}$$

where $d_i,\ i=1,2,\ldots,n$ are known nonzero real constants. In fact, by introducing an appropriate state transformation, system (11) can be converted into another system having the same form as system (5).

Remark 6. It should be pointed out that the recent novel work presented in [21] investigated the state feedback H_∞ control problem for commensurate fractional order linear time-invariant systems. When $w=0$, the system (3) in [21] is reduced to the general fractional order linear system. The advantage of [21] was dealing with exogenous disturbance input w for commensurate linear fractional order systems by introducing a new flexible matrix variable. Compared with [21], the main feature of this paper is to deal with the nonlinear terms in fractional order nonlinear systems by the Lyapunov function method (also see Remark 8).

V. EXAMPLE

In this section, we present an example to illustrate the main results.

Example 1. Consider the following fractional order nonlinear system:

$$\begin{cases} {}_0^C D_t^\alpha x_1 = x_2 + \frac{\sin x_1}{7+7\mathrm{e}^{-t}} x_3, \\ {}_0^C D_t^\alpha x_2 = x_3, \\ {}_0^C D_t^\alpha x_3 = u, \end{cases} \tag{12}$$

where $\alpha \in (0,1]$.

It is easy to see that Assumption 1 is satisfied with $c = 1/7$.

Let $a_j > 0,\ j = 1,2,3$ be the coefficients of the Hurwitz polynomial

$$q(s) = s^3 + a_3 s^2 + a_2 s + a_1.$$

Choose $a_1 = 6$, $a_2 = 11$, $a_3 = 6$. Then

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}.$$

Solving the Lyapunov equation

$$A^{\mathrm{T}} P + PA = -I$$

leads to

$$P = \begin{pmatrix} 23/15 & -1/2 & -7/10 \\ -1/2 & 7/10 & -1/2 \\ -7/10 & -1/2 & 17/10 \end{pmatrix} > 0.$$

Choose $r = 2 > 6c\|P\| = 1.9911$, we can get the linear state feedback controller for the system (12),

$$u = -\frac{6}{r^3} x_1 - \frac{11}{r^2} x_2 - \frac{6}{r} x_3. \tag{13}$$

Figure 1 shows the state response of the closed-loop system consisting of (12) and (13) with $\alpha = 0.8$ for the initial condition $(x_1(0), x_2(0), x_3(0)) = (1, 4, 3)$, which clearly demonstrates the asymptotic stability of the closed-loop system.

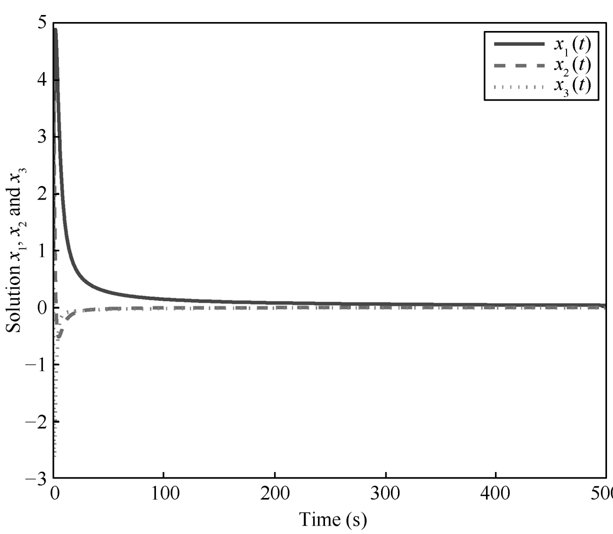

Fig. 1 The state of the closed-loop system consisting of (12) and (13) with $\alpha = 0.8$.

Remark 7. In Example 1, the nonlinear terms $\left|\frac{\sin x_1}{7+7\mathrm{e}^{-t}} x_3\right| < \frac{1}{7}|x_3|$, which implies that the condition (6) is satisfied with $c = 1/7$.

Remark 8. In Example 1, we can easily deal with the nonlinear terms "$\frac{\sin x_1}{7+7\mathrm{e}^{-t}} x_3$" by the method presented in this paper. However, it is clear that one cannot deal with these nonlinear terms by the method presented in [21].

VI. CONCLUSION

In this paper, we have investigated the design of state feedback stabilization controllers for fractional order nonlinear systems in upper triangular form by the Lyapunov function method. We have presented some new properties for Caputo fractional derivative to allow finding a simple Lyapunov candidate function for many fractional order systems. As an application, we have given a sufficient condition of asymptotical stability for fractional order nonlinear systems based on the new properties. By introducing appropriate transformations of coordinates, we have converted the problem of controller design into the problem of finding some parameters, which could be certainly obtained by solving the Lyapunov equation and relevant matrix inequalities. In addition, based on the Lyapunov function method, asymptotical stability for closed-loop systems has been considered by designing state feedback stabilization controllers for fractional order nonlinear systems in upper triangular form. The study of an illustrative example has shown that the new results presented in this paper are very effective.

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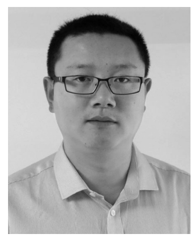

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