Robust Output Feedback Control for Fractional Order Nonlinear Systems with Time-varying Delays

Changchun Hua, Tong Zhang, Yafeng Li, and Xinping Guan

***Abstract*—Robust controller design problem is investigated for a class of fractional order nonlinear systems with time varying delays. Firstly, a reduced-order observer is designed. Then, an output feedback controller is designed. Both the designed observer and controller are independent of time delays. By choosing appropriate Lyapunov functions, we prove the designed controller can render the fractional order system asymptotically stable. A simulation example is given to verify the effectiveness of the proposed approach.**

***Index Terms*—Fractional order systems, time-varying delays, Laypunov function, backstepping.**

I. Introduction

Fractional calculus is an ancient concept, which can be dated back to the end of 17th century, the time when the classical integer order calculus was established. It is a generalization of the ordinary differentiation and integration to arbitrary order[1]. Although it has a long history, it has not attracted much attention until recently in the control field. It is found that many systems with memory feature or complex material can be more concisely and actually described by fractional order derivatives, such as the diffusion process, the heat transfer process and the effect of the frequency in induction machines. It also has been proved that fractional order controllers, like fractional order PID controllers and fractional order model reference adaptive controllers, can capture much better effect and robustness[2]. For some basic theory of fractional order calculus and fractional order systems, one can refer to [1−6] and the references therein.

Stability analysis is one of the most fundamental and essential issues for the control system. In [7], Matignon firstly studied the stability of fractional-order linear differential systems with the Caputo definition. Since then, many further achievements have been obtained[8−11]. In [8−9], the authors presented the sufficient and necessary conditions for the asymptotical stability of fractional order interval systems with fractional order α satisfying $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively. Reference [10] designed both state and output feedback controllers for fractional order linear systems in triangular form by introducing appropriate transformations of coordinates. Based on Gronwall-Bellman lemma and sector bounded condition, the stability and stabilization of fractional order linear systems subject to input saturation were studied in [11]. For the nonlinear fractional order systems, the stability analysis is much more difficult than that of the linear systems. We can find some sufficient conditions in [12−14]. In [15], the authors introduced Mittag-Leffler stability by using Lyapunov direct method.

Time delay is an inherent phenomenon in the interconnected systems or processes, which makes the stability analysis and controller design challenging. Robust output control problem for a class of nonlinear time-delay systems was studied in [16]. The necessary and sufficient stability conditions for linear fractional-order differential equations and linear time-delayed fractional differential equations have already been obtained in [17−19]. Reference [20] investigated the stability of α-dimensional linear fractional-order differential systems with order $1 < \alpha < 2$. References [21−22] contain the stability analysis of fractional order nonlinear time delay system based on Lyapunov direct method and by using properties of Mittag-Leffler function and Laplace transform.

In recent years, the backstepping technique has attracted much attention as a powerful method for controlling the strict feedback nonlinear systems. There are a few works using backstepping technique to handle fractional order systems. Using Lyapunov indirect method, the authors of [23] presented a new method to design an adaptive backstepping controller for triangular fractional order nonlinear systems. In [24], a new adaptive fractional-order backstepping method is proposed for a class of commensurate fractional order nonlinear systems with uncertain constant parameters. However, for fractional order nonlinear systems with time-varying delays, there is none related work. Motivated by the mentioned situation, we devote to solve the stabilization problem of fractional order nonlinear systems with time-varying delays.

The contributions of this paper are as follows: 1) A reduced-order observer is designed to estimate the state of the system; 2) Based on the backstepping method, we design a robust output feedback controller for a class of fractional order nonlinear time-varying delay systems; 3) With a novel class of fractional Lyapunov functions, we prove the stability of fractional order nonlinear systems.

The remainder of this paper is organized as follows: Section II presents some basic concepts about fractional order calculus and the stability of fractional order nonlinear systems. In Sec-

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tion III, as the main part of this note, an adaptive controller is designed by using the backstepping method for fractional order nonlinear time-varying delayed systems. An example is presented to show the effectiveness of the proposed controller in Section IV. Finally, Section V gives the conclusion of this paper.

Notations . Throughout this paper, $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}^n$ for n-dimensional Euclidean vector space and $\mathbf{R}^{n\times n}$ for the space of $n\times n$ real matrices. X^{T} and X^{-1} represent the transpose and the inverse of matrix X, respectively. I denotes the unit matrix with proper dimensions. For any matrix $A\in\mathbf{R}^{n\times n}$, $\lambda_i(A)$ stands for the i-th eigenvalue of A. For simplicity, ${}_0^CD_t^\alpha$ is mentioned as D^α.

II. Preliminaries

In this section, we provide some basic knowledge of fractional calculus and fractional order systems (details can be found in [1−2]). There are several definitions of fractional order derivatives, among which the Riemann-Liouville and Caputo definitions are well known and most commonly used. In this paper, we choose the Caputo definition for the fractional order derivatives. The Caputo derivative and fractional integral are defined as follows.

Definition 1 (Caputo fractional derivative). The Caputo fractional derivative of order $\alpha\geq 0$ for a function $f:[0,\infty]\rightarrow\mathbf{R}$ is defined as

$$ {}_0D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\mathrm{d}s,\quad t>0,\qquad(1)$$

where n is the first integer that is larger than α and $\Gamma(\cdot)$ is the well known Gamma function which is defined as follows:

$$\Gamma(t)=\int_0^\infty x^{t-1}\mathrm{e}^{-x}\mathrm{d}x.$$

Definition 2 (Fractional integral). The fractional integral of order $\alpha\geq 0$ for a function $f:[0,\infty]\rightarrow\mathbf{R}$ is defined as

$$ {}_0I_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}\mathrm{d}s,\quad t>0.\qquad(2)$$

Definition 3 (Class-K function). A continuous function $\gamma:[0,t)\rightarrow[0,\infty)$ is said to belong to class-K function if it is strictly increasing and $\gamma(0)=0$.

Here are some lemmas we could use in this paper.

Lemma 1 (General Leibnitz's rule). If f and g are differentiable and continuous functions, the product $f\cdot g$ is also differentiable and its α-th ($\alpha\geq 0$) derivative is given by

$$D^\alpha(f\cdot g)=(D^\alpha f)\cdot g+\sum_{k=1}^{\infty}\frac{\Gamma(\alpha+1)D^{\alpha-k}f\cdot D^kg}{\Gamma(k+1)\Gamma(\alpha-k+1)}.\qquad(3)$$

According to Lemma 1, the α-th order time derivative of $V(x)=2x^{\mathrm{T}}x$ can be extended as $D^\alpha V(x)=(D^\alpha x)^{\mathrm{T}}x+x^{\mathrm{T}}D^\alpha x+2\gamma$, where x is a column vector and $\gamma=\sum_{k=1}^\infty\frac{\Gamma(\alpha+1)(D^{\alpha-k}x)^{\mathrm{T}}D^kx}{\Gamma(k+1)\Gamma(\alpha-k+1)}$.

Lemma 2 (Fractional comparison principle). Let $D^\alpha x(t)\geq D^\alpha y(t)$ and $x(0)=y(0)$, where $\alpha\in(0,1)$. Then $x(t)\geq y(t)$. In particular, if $x(t)=c$, where c is a constant, $D^\alpha c=0$ and $y(0)=c$, we will have $y(t)\leq c$.

Theorem 1[15]**.** Let $x=0$ be an equilibrium point for $D^\alpha x(t)=f(t,x)$ and $D\subset\mathbf{R}^n$ be a domain containing $x=0$. Let $V(t,x):[0,\infty)\times D\rightarrow\mathbf{R}$ be a continuously differentiable function such that for $\forall t\geq 0$, $\forall x\in D$, $0<\alpha<1$,

$$\begin{aligned}&W_1(x)\leq V(t,x)\leq W_2(x),\\&D^\alpha V(t,x)\leq -W_3(x),\end{aligned}\qquad(4)$$

where $W_1(x)$, $W_2(x)$, and $W_3(x)$ are class-K functions on D. Then $x=0$ is asymptotically stable.

Lemma 4[12]**.** Let $x(t)\in\mathbf{R}^n$ be a differentiable and continuous function. Then, for $\forall t\geq t_0$ and $\forall\alpha\in(0,1)$

$$\frac{1}{2}D^\alpha(x^{\mathrm{T}}(t)x(t))\leq x^{\mathrm{T}}(t)D^\alpha x(t).\qquad(5)$$

Lemma 5 (Schur complement lemma). The linear matrix inequality (LMI)

$$M=\begin{bmatrix}A&B\\B^{\mathrm{T}}&C\end{bmatrix}<0,$$

where $A=A^{\mathrm{T}}$, $C=C^{\mathrm{T}}$ and C is invertible, is equivalent to

$$C<0,\quad A-BC^{-1}B^{\mathrm{T}}<0.$$

III. Main Results

Consider the following fractional order nonlinear system with $0<\alpha<1$ and time-varying delays:

$$\begin{cases}D^\alpha x_1=x_2+F_1(x_1)+H_1(y(t),y(t-d_1(t))),\\D^\alpha x_i=x_{i+1}+F_i(\bar{x}_i)+H_i(y(t),y(t-d_i(t))),\\D^\alpha x_n=u+F_n(\bar{x}_n)+H_n(y(t),y(t-d_n(t))),\\y=x_1,\end{cases}\qquad(6)$$

where $x(t)=[x_1(t),x_2(t),\ldots,x_n(t)]^{\mathrm{T}}\in\mathbf{R}^n$ is the state and $x_i(\theta)=\phi_i(\theta)$, $\theta\in[-d_i(0),0)$, $i=1,\ldots,n$, $x(0)=0$, $u(t)\in\mathbf{R}$ and $y(t)\in\mathbf{R}$ are the control input and the output of the system, respectively; $\bar{x}_i(t)=[x_1(t),x_2(t),\ldots,x_i(t)]^{\mathrm{T}}$; $F_i(\cdot)$ and $H_i(\cdot)$ are smooth nonlinear functions and $F_i(0,\ldots,0)=H_i(0,0)=0$; $d_i(t)$ is the time-varying delay and there exists positive scalar η_i such that $\dot{d}_i(t)\leq\eta_i<1$.

We impose the following assumptions on system (6).

Assumption 1. Nonlinear functions $H_i(\xi_1,\xi_2)$ $(i=1,2,\ldots,n)$ satisfy the following inequality:

$$|H_i(\xi_1,\xi_2)|\leq\bar{H}_{i1}(\xi_1)\xi_1+\bar{H}_{i2}(\xi_2)\xi_2,\qquad(7)$$

where $\bar{H}_{i1}(\cdot)$ and $\bar{H}_{i2}(\cdot)$ are known functions.

Let $H_{i1}(\xi_1)=2\bar{H}_{i1}^2(\xi_1)\xi_1$, $H_{i2}(\xi_2)=2\bar{H}_{i2}^2(\xi_2)\xi_2$, then we can have the following inequality:

$$|H_i(\xi_1,\xi_2)|^2\leq H_{i1}(\xi_1)\xi_1+H_{i2}(\xi_2)\xi_2.\qquad(8)$$

Assumption 1 is very common in nonlinear time delay systems, by which the term $y(t-d_i(t))$ can be separated from the delay-function, so that we can handle the delay problems.

Assumption 2. For nonlinear functions $F_i(\cdot)$, there exist some positive scalars l_i such that the following inequalities hold for $i=1,2,3,\ldots,n$:

$$|F_i(\bar{\zeta}_i)-F_i(\hat{\bar{\zeta}}_i)|\leq l_i\left\|\bar{\zeta}_i-\hat{\bar{\zeta}}_i\right\|,\qquad(9)$$

where $\bar{\zeta}_i=[\zeta_1,\zeta_2,\ldots,\zeta_i]^{\mathrm{T}}$, $\hat{\bar{\zeta}}_i=[\zeta_1,\hat{\zeta}_2,\ldots,\hat{\zeta}_i]^{\mathrm{T}}$ and l_i is a known positive parameter.

We can use the following expression for $F_i(\bar{x}_i)$

$$F_i(\bar{x}_i) = n_{i1}x_1 + \bar{F}_i(\bar{x}_i), \quad i = 1, \ldots, n, \tag{10}$$

where n_{i1} is a constant which could be zero.

In this paper, we focus on solving the following problem. For system (6) satisfying Assumptions 1 and 2, design a reduced-order observer based memoryless output feedback controller to render the closed-loop system stable.

Considering system (6) with unmeasured state variables, we propose the following reduced-order observer:

$$\begin{cases} D^\alpha \lambda_i(t) = \lambda_{i+1}(t) + k_{i+1}x_1(t) + F_i(\hat{\bar{x}}_i(t)) \\ \qquad\qquad -k_i(\lambda_2(t) + k_2 y(t) + F_1(y(t))), \\ D^\alpha \lambda_n(t) = u(t) + F_n(\hat{\bar{x}}_n(t)) \\ \qquad\qquad -k_n(\lambda_2(t) + k_2 y(t) + F_1(y(t))), \\ \hat{x}_i(t) = \lambda_i + k_i y(t), \quad i = 2, \ldots, n, \end{cases} \tag{11}$$

where $\hat{\bar{x}}_i(t) = [x_1(t), \hat{x}_2(t), \ldots, \hat{x}_i(t)]^{\mathrm{T}}$ and parameters k_i ($i = 2, \ldots, n$) are to be specified later.

Similar to (10), we can change $F_i(\hat{\bar{x}}_i(t))$ into

$$F_i(\hat{\bar{x}}_i(t)) = n_{i1}x_1 + \bar{F}_i(\hat{\bar{x}}_i(t)).$$

Remark 1. In this paper, we introduce the reduced-order observer instead of the full-order one. In this way, some of the states can be derived from the real output, making the results more precise and simplifying the structure and computation complexity. To our best knowledge, it is the first time that the reduced-order observer is introduced to fractional order nonlinear systems.

The estimation errors are defined as

$$e_i(t) = x_i(t) - \hat{x}_i(t). \tag{12}$$

From (11) and (12), we can get

$$D^\alpha e(t) = Ae(t) + \widetilde{F}(t) + \widetilde{H}(t), \tag{13}$$

where

$$e(t) = [e_2(t), \ldots, e_n(t)]^{\mathrm{T}},$$

$$A = \begin{bmatrix} -k_2 & 1 & 0 & \cdots & 0 \\ -k_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_n & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\tilde{F}(t) = [F_2(\bar{x}_2(t)) - F_2(\hat{\bar{x}}_2(t)), \ldots, \\ F_n(\bar{x}_n(t)) - F_n(\hat{\bar{x}}_n(t))]^{\mathrm{T}}, \tag{14}$$

$$\tilde{H}(t) = [H_2(y(t), y(t-d_2))) - k_2 H_1(y(t), y(t-d_1))), \\ \ldots, H_n(y(t), y(t-d_n))) - k_n H_1(y(t), y(t-d_1)))]^{\mathrm{T}}. \tag{15}$$

Next, we extend the backstepping technique to the fractional order nonlinear system with time-varying delays described by (6). The virtual controllers α_i ($i = 1, \ldots, n-1$) are developed at each step. Finally, at step n, the actual controller u is designed. First, we introduce the following transformation of states.

$$z_1(t) = y(t), \\ z_i(t) = \lambda_i(t) - \alpha_{i-1}, \quad i = 2, \ldots, n. \tag{16}$$

Then we choose the Lyapunov function as

$$V = V_e + V_z + V_d, \tag{17}$$

where

$$V_e = I^{1-\alpha} e^{\mathrm{T}} P e, \tag{18}$$

and P is a real symmetric positive matrix.

$$V_z = I^{1-\alpha} \sum_{i=1}^{n} M_i, \tag{19}$$

$$M_i = \frac{1}{2} z_i^2,$$

$$V_d = \left(2\sum_{i=2}^{n} k_i^2 + 1\right) \int_{t-d_1(t)}^{t} \frac{1}{1-\eta_1} H_{12}(y(t))y(t)\mathrm{d}t \\ + 2\sum_{i=2}^{n} \int_{t-d_i(t)}^{t} \frac{1}{1-\eta_i} H_{i2}(y(t))y(t)\mathrm{d}t. \tag{20}$$

Next, we will give the derivative of V_e, V_z, V_d, and V in turn. From Lemma 4, the derivative of V_e is:

$$\dot{V}_e = D^\alpha e(t)^{\mathrm{T}} P e(t) \\ \leq (D^\alpha e(t))^{\mathrm{T}} P e(t) + e^{\mathrm{T}}(t) P D^\alpha e(t) \tag{21}$$

$$= (Ae(t) + \tilde{F}(t) + \tilde{H}(t))^{\mathrm{T}} P e(t) \\ + e^{\mathrm{T}}(t) P (Ae(t) + \tilde{F}(t) + \tilde{H}(t)) \\ = e^{\mathrm{T}}(t)(A^{\mathrm{T}}P + PA)e(t) + \tilde{F}^{\mathrm{T}}(t) P e(t) \\ + e^{\mathrm{T}}(t) P \tilde{F}(t) + \tilde{H}^{\mathrm{T}}(t) P e(t) + e^{\mathrm{T}}(t) P \tilde{H}(t) \tag{22}$$

According to Assumption 2, we can get

$$\tilde{F}^{\mathrm{T}}(t)\tilde{F}(t) = \sum_{i=2}^{n} (F_i(\bar{x}_i(t)) - F_i(\hat{\bar{x}}_i(t)))^2 \\ \leq \sum_{i=2}^{n} l_i^2 \left\|\bar{x}_i - \hat{\bar{x}}_i\right\|^2 \\ \leq \rho \left\|\bar{x}_n - \hat{\bar{x}}_n\right\|^2,$$

where $\rho = \sum_{i=2}^{n} l_i^2$.

$$\dot{V}_e \leq e^{\mathrm{T}}(t)(A^{\mathrm{T}}P + PA)e(t) + \rho e^{\mathrm{T}}(t)e(t) \\ + \tilde{H}^{\mathrm{T}}(t)\tilde{H}(t) + 2e^{\mathrm{T}}(t)PPe(t) \\ \leq e^{\mathrm{T}}(t)(A^{\mathrm{T}}P + PA + \rho I + 2PP)e(t) + \tilde{H}^{\mathrm{T}}(t)\tilde{H}(t) \\ \leq e^{\mathrm{T}}(t)(A^{\mathrm{T}}P + PA + \rho I + 2PP)e(t) \\ + 2\sum_{i=2}^{n} H_i^2 + 2\sum_{i=2}^{n} k_i^2 H_1^2 \tag{23}$$

and P and k_i ($i = 2, \ldots, n$) satisfy

$$A^{\mathrm{T}}P + PA + \rho I + 2PP < -\frac{n}{2\epsilon_1} I. \tag{24}$$

To solve inequality (24), we decompose $A = \bar{A} + kB$ with

$$\bar{A} = \begin{bmatrix} 0 & I_{(n-2)\times(n-2)} \\ 0 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} k_2 \\ \vdots \\ k_n \end{bmatrix}_{(n-1)\times 1},$$

$$B = [-1, 0, \ldots, 0]_{1\times(n-1)}.$$

According to Lemma 5, inequality (24) is equivalent to the following LMI

$$\begin{bmatrix} P\bar{A}+WB+B^{\mathrm{T}}W^{\mathrm{T}}+\bar{A}^{\mathrm{T}}P+(\rho+\frac{n}{2\epsilon_1})I & P \\ P & -0.5I \end{bmatrix} < 0, \tag{25}$$

where $W = Pk$. Further, we can use LMI toolbox in Matlab to obtain P and k.

The derivative of V_z is:

$$\dot{V}_z = \sum_{i=1}^{n} D^\alpha M_i. \tag{26}$$

Next, by the backstepping method, the virtual controllers α_i ($i = 1, 2, \ldots, n-1$) and controller u are designed respectively.

Step 1.

$$\begin{aligned} D^\alpha M_1 =& \frac{1}{2}D^\alpha z_1^2 \leq z_1 D^\alpha z_1 \\ =& z_1(z_2+\alpha_1+k_2x_1+e_2+F_1(y(t)) \\ &+ H_1(y(t), y(t-d_1(t)))) \\ \leq& \left(\frac{1}{4}+\frac{\epsilon_1}{2}\right)z_1^2+\frac{1}{2\epsilon_1}e_2^2+H_1^2(y(t),y(t-d_1(t))) \\ &+ z_1(\alpha_1+k_2x_1+F_1(x_1(t)))+z_1z_2, \end{aligned} \tag{27}$$

where ϵ_1 is a positive constant.

Choose

$$\begin{aligned} \alpha_1 =& -\left(c_1+\frac{1}{4}+\frac{\epsilon_1}{2}+k_2+n_{11}\right)z_1-\bar{F}_1(y(t)) \\ &- \left(2\sum_{i=2}^{n}k_i^2+1\right)\left(H_{11}(y(t))+\frac{1}{1-\eta_1}H_{12}(y(t))\right) \\ &- 2\sum_{i=2}^{n}\left(H_{i1}(y(t))+\frac{1}{1-\eta_i}H_{i2}(y(t))\right) \\ =& -K_1z_1-\bar{\alpha}_1, \end{aligned} \tag{28}$$

where c_i ($i = 1, 2, \ldots, n$) are positive constants,

$$\begin{aligned} K_1 =& c_1+\frac{1}{4}+\frac{\epsilon_1}{2}+k_2+n_{11} \\ \bar{\alpha}_1 =& \bar{F}_1(y(t))+\left(2\sum_{i=2}^{n}k_i^2+1\right)(H_{11}(y(t)) \\ &+ \frac{1}{1-\eta_1}H_{12}(y(t)))+2\sum_{i=2}^{n}(H_{i1}(y(t)) \\ &+ \frac{1}{1-\eta_i}H_{i2}(y(t))). \end{aligned}$$

Step 2.

$$\begin{aligned} D^\alpha M_2 =& \frac{1}{2}D^\alpha z_2^2 \leq z_2D^\alpha z_2 \\ =& z_2(z_3+\alpha_2+k_3x_1+F_2(\hat{\bar{x}}_2(t)) \\ &- k_2(\lambda_2+k_2x_1+F_1(x_1))-D^\alpha\alpha_1) \\ \leq& z_2z_3+\frac{1}{2\epsilon_1}e_2^2+z_2(\alpha_2+k_3x_1+F_2(\hat{\bar{x}}_2(t)) \\ &- k_2(\lambda_2+k_2x_1+F_1(x_1))+D^\alpha\bar{\alpha}_1 \\ &+ K_1(z_2+\alpha_1+k_2x_1+F_1(y(t)))+\frac{K_1\epsilon_1}{2}z_2^2. \end{aligned} \tag{29}$$

Choose

$$\begin{aligned} \alpha_2 =& -z_1-\left(c_2+\frac{K_1\epsilon_1}{2}\right)z_2-k_3x_1-n_{21}z_1 \\ &- \bar{F}_2(\hat{\bar{x}}_2(t))-K_1(z_2+\alpha_1+k_2x_1+F_1(y(t))) \\ &+ k_2(\lambda_2+k_2x_1(t)+F_1(x_1(t)))-D^\alpha\bar{\alpha}_1 \\ =& -K_2z_1-\bar{\alpha}_2, \end{aligned} \tag{30}$$

where

$$\begin{aligned} K_2 =& 1+(K_1-k_2)(n_{11}+k_2)+n_{21}+k_3 \\ \bar{\alpha}_2 =& \left(c_2+\frac{K_1\epsilon_1}{2}\right)z_2+\bar{F}_2(\hat{\bar{x}}_2(t)) \\ &+ (K_1-k_2)(z_2+\alpha_1+\bar{F}_1(y(t)))+D^\alpha\bar{\alpha}_1. \end{aligned}$$

Step *i*.

$$\begin{aligned} D^\alpha M_i =& \frac{1}{2}D^\alpha z_i^2 \leq z_iD^\alpha z_i \\ =& z_i(z_{i+1}+\alpha_i+k_{i+1}x_1+F_i(\hat{\bar{x}}_i(t)) \\ &- k_i(\lambda_2+k_2x_1(t)+F_1(x_1(t)))-D^\alpha\alpha_{i-1}) \\ =& z_iz_{i+1}+\frac{1}{2\epsilon_1}e_2^2+z_i(\alpha_i+k_{i+1}x_1+F_i(\hat{\bar{x}}_i(t))) \\ &- k_i(\lambda_2+k_2x_1(t)+F_1(x_1(t)))+D^\alpha\bar{\alpha}_{i-1} \\ &+ K_{i-1}(z_2+\alpha_1+k_2x_1+F_1(y(t))))+\frac{K_{i-1}\epsilon_1}{2}z_i^2. \end{aligned} \tag{31}$$

Choose

$$\begin{aligned} \alpha_i =& -z_{i-1}-\left(c_i+\frac{K_{i-1}\epsilon_1}{2}\right)z_i-k_{i+1}x_1-n_{i1}z_1 \\ &- \bar{F}_i(\hat{\bar{x}}_i(t))+k_i(\lambda_2+k_2x_1(t)+F_1(x_1(t))) \\ &- D^\alpha\bar{\alpha}_{i-1}-K_{i-1}(z_2+\alpha_1+k_2x_1+F_1(y(t))) \\ =& -K_iz_1-\bar{\alpha}_i, \end{aligned} \tag{32}$$

where

$$\begin{aligned} K_i =& (K_{i-1}-k_i)(n_{11}+k_2)+n_{i1}+k_{i+1} \\ \bar{\alpha}_i =& z_{i-1}+\left(c_i+\frac{K_{i-1}\epsilon_1}{2}\right)z_i+\bar{F}_i(\hat{\bar{x}}_i(t)) \\ &+ (K_{i-1}-k_i)(z_2+\alpha_1+\bar{F}_1(y(t)))+D^\alpha\bar{\alpha}_{i-1}. \end{aligned}$$

Step *n*.

$$\begin{aligned} D^\alpha M_n =& \frac{1}{2}D^\alpha z_n^2 \leq z_nD^\alpha z_n \\ =& z_n(u+F_n(\hat{\bar{x}}_n(t))-k_n(\lambda_2+k_2x_1(t) \\ &+ F_1(x_1(t)))-D^\alpha\alpha_{n-1}) \\ =& \frac{1}{2\epsilon_1}e_2^2+\frac{K_{n-1}\epsilon_1}{2}z_n^2+z_n(u+F_n(\hat{\bar{x}}_n(t)) \\ &+ D^\alpha\bar{\alpha}_{n-1}-k_n(\lambda_2+k_2x_1(t)+F_1(x_1(t))) \\ &+ K_{n-1}(z_2+\alpha_1+k_2x_1+F_1(y(t)))). \end{aligned} \tag{33}$$

Then

$$\begin{aligned} u =& -z_{n-1}-\left(c_n+\frac{K_{n-1}\epsilon_1}{2}\right)z_n-F_n(\hat{\bar{x}}_n(t)) \\ &+ k_n(\lambda_2+k_2x_1(t)+F_1(x_1(t))) \\ &- K_{n-1}(z_2+\alpha_1+k_2x_1+F_1(x_1(t)))-D^\alpha\bar{\alpha}_{n-1}. \end{aligned} \tag{34}$$

The derivative of V_d is:

$$\begin{aligned}\dot{V}_d =& \left(2\sum_{i=2}^{n} k_i^2+1\right)\\ &\times\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{t-d_1(t)}^{t}\frac{1}{1-\eta_1}H_{12}(y(t))y(t)\mathrm{d}t\right)\right)\\ &+2\sum_{i=2}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{t-d_i(t)}^{t}\frac{1}{1-\eta_i}H_{i2}(y(t))y(t)\mathrm{d}t\right)\right)\\ \le& \left(2\sum_{i=2}^{n} k_i^2+1\right)\left(\frac{1}{1-\eta_1}H_{12}(y(t)\right)y(t)\\ &-H_{12}(y(t-d_1(t)))y(t-d_1(t)))\\ &+2\sum_{i=2}^{n}(\frac{1}{1-\eta_i}H_{i2}(y(t))y(t)\\ &-H_{i2}(y(t-d_i(t)))y(t-d_i(t))).\end{aligned} \tag{35}$$

Then, we have

$$\begin{aligned}\dot{V} =& \dot{V}_e+\dot{V}_z+\dot{V}_d\\ \le& e^{\mathrm{T}}(t)\left(A^{\mathrm{T}}P+PA+\left(\rho+\frac{n}{2\epsilon_1}\right)I+2PP\right)e(t)\\ &-\sum_{i=1}^{n}c_i z_i^2.\end{aligned} \tag{36}$$

So

$$\dot{V} \le -W(e(t),\bar{z}_n), \tag{37}$$

where $\bar{z}_i=[z_1,z_2,\ldots,z_n]$ and $W(\cdot)$ and $W_1(\cdot)$ are class-K functions.

According to the definition of the Caputo fractional derivative, Lemma 2 and (37)

$$D^{\alpha}V = I^{1-\alpha}\dot{V} \le -I^{1-\alpha}W(e(t),\bar{z}_n) \le -W_1(e(t),\bar{z}_n)$$

Finally, we present the main result of this paper as follows:

Theorem 2. For a system described by (6) satisfying Assumptions 1 and 2, controller (34) can render the closed-loop system asymptotically stable.

IV. Simulation

In this section, an example is given to show the effectiveness of the proposed controller.

Consider the following system:

$$\begin{cases} D^{\alpha}x_1(t) = x_2(t)-0.8x_1(t)+0.5x_1^2(t-d_1(t))\sin t,\\ D^{\alpha}x_2(t) = u-0.8x_2(t)+0.5x_1^3(t-d_2(t))\sin t,\end{cases}$$

where $d_1(t)=0.5(1+\sin t)$, $d_2(t)=0.5(1+\cos t)$. We can see that the aforementioned system satisfies the above assumptions with $P=I$, $\eta_1=\eta_2=0.5$, $l=0.8$, $\rho=0.64$, $n_{11}=-0.8$, $\bar{F}_1(x_1)=0$, $n_{21}=0$, $\bar{F}_2(\bar{x}_2)=-0.8x_2$, $H_{11}(t)=H_{21}(t)=0$, $H_{12}(t-d_1(t))=0.25x_1^3(t-d_1(t))$ and $H_{22}(t-d_1(t))=0.25\,x_1^5(t-d_1(t))$. Choosing $\epsilon_1=1$, $k_2=2$, $c_1=0.05$ and $c_2=0.8$, we can obtain the reduced-order observer and the function $\beta_1(x_1(t))$:

$$\begin{aligned} D^{\alpha}\lambda_2(t) &= u-2.8\lambda_2(t)-4x_1(t),\\ \alpha_1(x_1(t)) &= -2x_1(t)-4.5x_1^3(t)-x_1^5(t).\end{aligned}$$

Then, $K_1=2$, $\bar{\alpha}_1=4.5x_1^3+x_1^5$, the controller can be designed as

$$u(t)=0.6x_1(t)-\lambda_2(t)+1.8\alpha_1(x_1(t))-D^{\alpha}\bar{\alpha}_1.$$

The simulation results are shown in Figs. 1 and 2, from which we can see that the constructed controller renders the closed-loop system stable.

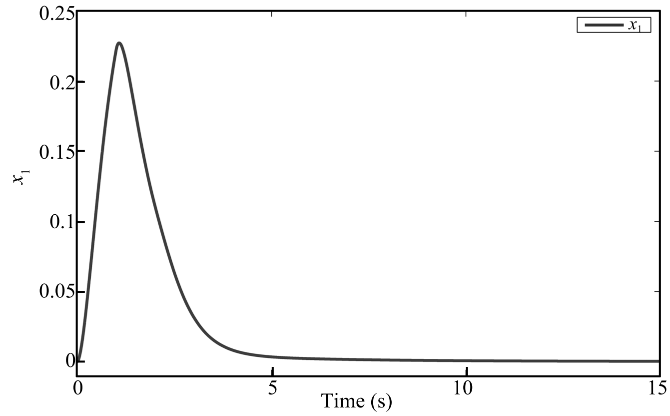

Fig. 1. The output response of the closed-loop system.

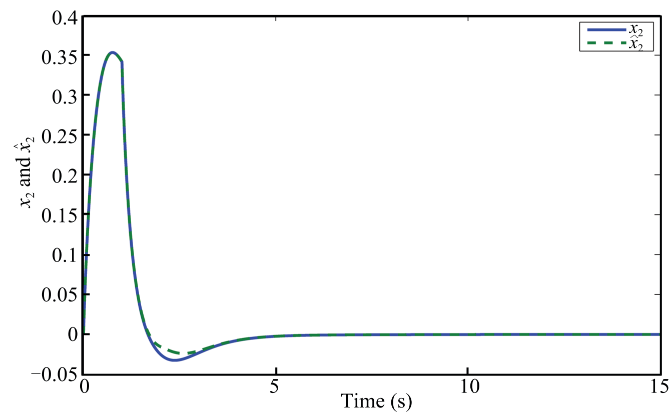

Fig. 2. The trajectories of x_2 and $\hat{x}_2$.

V. Conclusion

In this paper, we study the controller design problem for fractional order nonlinear time-varying delay systems, using the well known backstepping method. Also, we extend the Lyapunov method to fractional order systems. Both the designed observer and controller are independent of time delays. Through the simulation presented in Section IV, the effectiveness of the proposed controller has been verified. As put in [25], fractional order systems have a memory feature, which could make difficulties in the process of controller design. In the future, we will further consider the memory feature and its influence. Based on the result of this paper, we will study the stability and stabilization problems for fractional nonlinear delayed systems with fractional order $1<\alpha<2$ and the stability of fractional order nonlinear systems with fractional order $\alpha>2$.

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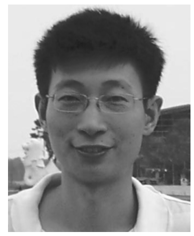

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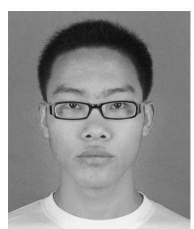

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