Study on Four Disturbance Observers for FO-LTI Systems

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***Abstract*—This paper addresses the problem of designing disturbance observer for fractional order linear time invariant (FO-LTI) systems, where the disturbance includes time series expansion disturbance and sinusoidal disturbance. On one hand, the reduced order extended state observer (ROESO) and reduced order cascade extended state observer (ROCESO) are proposed for the case that the system state can be measured directly. On the other hand, the extended state observer (ESO) and the cascade extended state observer (CESO) are presented for another case when the system state cannot be measured directly. It is shown that combination of ROCESO and CESO can achieve a highly effective observation result. In addition, the way how to tune observer parameters to ensure the stability of the observers and reduce the observation error is presented in this paper. Finally, numerical simulations are given to illustrate the effectiveness of the proposed methods.**

***Index Terms*—Fractional order linear time invariant (FO-LTI) systems, disturbance observer, reduced order, cascade method.**

I. INTRODUCTION

In recent years, fractional order systems (FOSs) have attracted considerable attention from control community, since many engineering plants and processes cannot be described concisely and precisely without the introduction of fractional order calculus[1−6]. Due to the tremendous efforts devoted by researchers, a number of valuable results on stability analysis[7−10] and controller synthesis[11−14] of FOSs have been reported in the literature.

Tracking reference signal and disturbance rejection are two of the challenging and significant tasks in engineering plants and processes. It is important to reject disturbance so as to maintain the controlled system running in a fine manner. Aimed at disturbance rejection to enhance control performance, numerous methods have been presented[15]. Sliding mode control (SMC) is an effective method which involves designing of a sliding surface and reaching motion controller. Reference [16] gives the detailed contents to introduce the SMC technique. Guo et al. developed SMC approach to reject disturbance for the Euler-Bernoulli beam in [17]. However, the chattering in sliding surface is the main drawback of SMC. Adaptive control is another method to reject disturbance by adjusting the control parameters automatically[18−19]. While both SMC and adaptive control suppress disturbance passively by improving the robustness of the controller to reduce the sensitivity to external disturbance in the output channel, rather than by actively obtaining the characteristics of the disturbance in time domain or frequency domain. Therefore, the obvious drawback of the two methods is that there is an undesirable trade-off between reference tracking and disturbance rejection[20].

Another idea for disturbance rejection is to utilize the information of the external disturbance to build the feedback compensation, namely, to reject disturbance actively. Active disturbance rejection control (ADRC)[21] technique is proposed by Han in 1998, in which the uncertainties of system model and external disturbance are regarded totally as an extended state which can be observed by an extended state observer. Internal model control is another method to reject disturbance actively. Fedele et al. employed an orthogonal signals generator based on a second-order generalized integrator (OSG-SOGI) to estimate the frequency of the unknown external sinusoidal disturbance, which can be utilized to build internal model control (IMC) algorithm for the disturbed system[22]. However, how to extract the unknown external disturbance for the OSG-SOGI is a difficult issue. Disturbance observer (DOB) is a popular approach to compensate disturbance actively, which was proposed by Nakao et al.[23] in 1987. Chen et al. investigated the disturbance observer based control and related methods in [24]. Park et al. developed DOB algorithm for industrial robots to compensate external disturbance in [25]. While the main drawback of the DOB is that the inverse dynamics of the system, which may cause cancellation of unstable poles and unstable zeros, is required. Ginoya et al. proposed an extended disturbance observer for unmatched uncertain systems based on the assumption that the system state can be measured accurately[26]. However, the assumption cannot be satisfied in many cases. Therefore, it is an important and meaningful issue to develop a method to observe the external disturbance by utilizing the control input and the measured output of the disturbed system.

Motivated by the discussions above, we develop ROESO and ROCESO to observe the external disturbance under the

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assumption that the state can be measured and then propose a way to improve accuracy of the observations. Furthermore, by considering the case that the state cannot be measured directly, ESO is developed to observe the disturbance, in which only the control input and system output signals are utilized. In addition, the CESO is proposed based on ESO to extend the scope of the observations and to get a more effective performance of the observations than the ESO.

The rest of this paper is structured as follows. Section II provides some background materials and the main problem. ROESO and ROCESO for measurable system state and ESO and CESO for unmeasurable system state are presented in Section III. In Section IV, some numerical simulation examples are provided to illustrate the effectiveness of the proposed methods. Conclusions are given in Section V.

Notations. I_n and 0_n are used to denote a $n\times n$ identity matrix and $n\times n$ zero matrix, respectively. $0_{n\cdot m}$ is used to denote a $n\times m$ zero matrix. $\|e(t)\|$ represents the Euclidean norm of $e(t)$. $\mathrm{sym}(M)=M+M^{\mathrm{T}}$. $\Omega^{n\cdot n}=\{X\tan(\pi\alpha/2)+Y: X,Y\in\mathbf{R}^{n\cdot n}, \begin{bmatrix} X & Y\\ -Y & X\end{bmatrix}>0\}$.

II. Problem Formulation and Preliminaries

Consider the following disturbed FO-LTI system with an assumption that the order α is known in prior:

$$\begin{cases} D^\alpha x(t)=Ax(t)+Bu(t)+Fd(t),\\ y(t)=Cx(t)color1,\end{cases}\tag{1}$$

where the order $0<\alpha<1$; $x(t)\in\mathbf{R}^n$, $u(t)\in\mathbf{R}^m$, $d(t)\in\mathbf{R}^q$ and $y(t)\in\mathbf{R}^p$ are the system state, the control input, the disturbance and the measurable output, respectively; the system matrices A, B, F and C are the constant real matrices with appropriate dimensions, and $F=BJ$ where J is a constant matrix, and $\mathrm{rank}(F)=q$. The definition of the fractional order derivative can be referred to [1].

The following Caputo's definition is adopted for fractional derivatives of order α for function $f(t)$

$$D^\alpha f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}f^{(m)}(\tau)\mathrm{d}\tau,\tag{2}$$

where $m-1<\alpha<m$, $m\in\mathbf{N}$ and $\Gamma(x)=\int_0^\infty \mathrm{e}^{-t}t^{x-1}\mathrm{d}t$. And the Riemann-Liouville's fractional order integral is defined as

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau,\tag{3}$$

where $\alpha>0$.

In this study, the objective is to develop an approach to ensure the designed observer stable and the external disturbance can be observed by the observer under the following assumptions:

1) Pair $\{A,B\}$ is controllable;
2) Pair $\{A,C\}$ is observable.

In this paper, in order to show the generality of the proposed method, we consider two kinds of disturbances: time series expansion disturbance and sinusoidal disturbance. The time series expansion disturbance has the following form

$$d(t)=\sum_{i=0}^k d_i t^{n_i},\tag{4}$$

where d_i ($i\in\{0,1,\ldots,k\}$) is constant but unknown, $n_{i-1}\le n_i$ ($i\in\{1,\ldots,k\}$) and $n_k<2\alpha$ holds. Based on the relationship between n_k and α, the disturbance can be divided into the following two categories:

1) Slowly varying disturbance $n_k<\alpha$;
2) Slope forms disturbance $\alpha\le n_k<2\alpha$.

This paper aims at designing a proper method to observe the external unknown disturbance. For this purpose, the following lemmas are first introduced.

Lemma 1[27]**.** Let $x(t)\in\mathbf{R}$ be a continuous and differentiable function. Then the α-th derivative of $x^2(t)$ has the following properties

$$D^\alpha x^2(t)\le 2x(t)D^\alpha x(t).\tag{5}$$

Consider a FO-LTI system as follows

$$D^\alpha x(t)=(A+BKC)x(t).\tag{6}$$

Based on the system (6), we give our research result in [28] as a lemma as follows.

Lemma 2[28]**.** The system in (6) with $0<\alpha<1$ is asymptotically stable, if there exist matrices $Z\in\Omega^{n\cdot n}$, $G\in\mathbf{R}^{m\cdot m}$, and $H\in\mathbf{R}^{m\cdot p}$, such that

$$\begin{bmatrix}\Xi & Z^{\mathrm{T}}B+C^{\mathrm{T}}H^{\mathrm{T}}-{K_0}^{\mathrm{T}}G^{\mathrm{T}}\\ * & -\mathrm{sym}(G)\end{bmatrix}<0\tag{7}$$

is feasible, and the matrix K is given by

$$K=G^{-1}H,\tag{8}$$

where $*$ stands for the symmetrical part matrix, $\Xi=\mathrm{sym}(Z^{\mathrm{T}}A+Z^{\mathrm{T}}BK_0)$, and K_0 is an additional initialization matrix, which is derived from $K_0=QP^{-1}$. The matrices $P\in\Omega^{n\cdot n}$ and $Q\in\mathbf{R}^{m\cdot n}$ satisfy following linear matrix inequality (LMI),

$$\mathrm{sym}(AP+BQ)<0.\tag{9}$$

III. Main Results

A. Reduced Order Extended State Observer (ROESO)

If the state of the FO-LTI system (1) can be measured directly, we can utilize the state to design a disturbance observer as follows:

$$\begin{cases}\hat d(t)=\Lambda\varepsilon(t),\\ \varepsilon(t)=F^+x(t)-z(t),\\ D^\alpha z(t)=F^+[Ax(t)+Bu(t)]+\hat d(t),\end{cases}\tag{10}$$

where $F^+=(F^{\mathrm{T}}F)^{-1}F^{\mathrm{T}}$, Λ is a positive definite $q\times q$ diagonal matrix.

Theorem 1. The disturbance can be observed asymptotically by (10), if the disturbance is slowly varying disturbance, namely, $\lim_{t\to\infty}D^\alpha d(t)=0$.

Proof. Define the observation error $e(t)=d(t)-\hat d(t)$ which yields

$$\begin{aligned}D^\alpha e(t)&=D^\alpha d(t)-D^\alpha\hat d(t)\\ &=D^\alpha d(t)-\Lambda D^\alpha\varepsilon(t)\\ &=D^\alpha d(t)-\Lambda e(t).\end{aligned}\tag{11}$$

The Laplace transform of (11) is

$$s^{\alpha}E(s)-s^{\alpha-1}e(0)=s^{\alpha}D(s)-s^{\alpha-1}d(0)-\Lambda E(s), \quad (12)$$

where $E(s)$ and $D(s)$ are the Laplace transforms of $e(t)$ and $d(t)$, respectively. Then using the final-value theorem, yields

$$\begin{aligned} e(\infty) &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0}(sI_q+\Lambda)^{-1}[s^{1+\alpha}D(s)-s^{\alpha}d(0)+s^{\alpha}e(0)]. \end{aligned} \quad (13)$$

Then if $\lim_{s\to 0} s^{1+\alpha}D(s)=0$, that is $\lim_{t\to\infty} D^{\alpha}d(t)=0$, thereby, $\lim_{t\to\infty} e(t)=0$. □

Remark 1. Since the disturbance can be observed asymptotically if $\lim_{t\to\infty} D^{\alpha}d(t)=0$, the constant disturbance, etc. can be observed asymptotically by the ROESO. In addition, square disturbance also satisfies the condition of Theorem 1 when the hopping points are overlooked.

Remark 2. The gain Λ can change the rate of convergence of the observer. The larger value of Λ is, the higher rate of convergence we can get.

B. Reduced Order Cascade Extended State Observer (ROCESO)

In order to expand the scope of the disturbance that can be observed asymptotically based on the system state can be measured directly, we improve the ROESO to ROCESO as follows

$$\begin{cases} \hat{d}(t)=2\Lambda\varepsilon(t)+\Lambda^2 I^{\alpha}\varepsilon(t), \\ \varepsilon(t)=F^{+}x(t)-z(t), \\ D^{\alpha}z(t)=F^{+}\left[Ax(t)+Bu(t)\right]+\hat{d}(t), \end{cases} \quad (14)$$

where Λ is a positive definite $q\times q$ diagonal matrix.

Theorem 2. The disturbance can be observed asymptotically by (14), if the disturbance is of slope form, namely, $\lim_{t\to\infty} D^{2\alpha}d(t)=0$.

Proof. Define the observation error $e(t)=d(t)-\hat{d}(t)$, it yields

$$\begin{aligned} D^{\alpha}e(t) &= D^{\alpha}d(t)-D^{\alpha}\hat{d}(t) \\ &= D^{\alpha}d(t)-D^{\alpha}\left[2\Lambda\varepsilon(t)+\Lambda^2 I^{\alpha}\varepsilon(t)\right] \\ &= D^{\alpha}d(t)-2\Lambda e(t)-\Lambda^2\varepsilon(t). \end{aligned} \quad (15)$$

Based on (15), the 2α-th derivative of $e(t)$ can be expressed as

$$D^{2\alpha}e(t)=D^{2\alpha}d(t)-2\Lambda D^{\alpha}e(t)-\Lambda^2 e(t). \quad (16)$$

If $0<\alpha\le 0.5$, the Laplace transform of (16) is

$$\begin{aligned} &s^{2\alpha}E(s)-s^{2\alpha-1}e(0) \\ &\quad= s^{2\alpha}D(s)-s^{2\alpha-1}d(0)-2\Lambda s^{\alpha}E(s) \\ &\qquad+2\Lambda s^{\alpha-1}e(0)-\Lambda^2 E(s), \end{aligned} \quad (17)$$

where $E(s)$ and $D(s)$ are the Laplace transforms of $e(t)$ and $d(t)$, respectively. Then using the final-value theorem, yields

$$\begin{aligned} e(\infty) &= \lim_{s\to 0}(s^{2\alpha}I_q+2\Lambda s^{\alpha}+\Lambda^2)^{-1} \\ &\quad\times[s^{1+2\alpha}D(s)-s^{2\alpha}d(0)+s^{2\alpha}e(0)+2\Lambda s^{\alpha}e(0)]. \end{aligned} \quad (18)$$

Then if $\lim_{s\to 0} s^{1+2\alpha}D(s)=0$, that is $\lim_{t\to\infty} D^{2\alpha}d(t)=0$, thereby, $\lim_{t\to\infty} e(t)=0$. If $0.5<\alpha\le 1$, the Laplace transform of (16) is

$$\begin{aligned} E(s)\,-\,&s^{2\alpha-1}e'(0)-s^{2\alpha-2}e(0) \\ &= s^{2\alpha}D(s)-s^{2\alpha-1}d'(0)-s^{2\alpha-2}d(0) \\ &\quad-2\Lambda s^{\alpha}E(s)+2\Lambda s^{\alpha-1}e(0)-\Lambda^2 E(s), \end{aligned} \quad (19)$$

and it is easy to obtain the same conclusion. All of these discussions establish the Theorem 2. □

Remark 3. Since the disturbance can be observed by ROCESO if $\lim_{t\to\infty} D^{2\alpha}d(t)=0$, the scope of the disturbance observation is expanded from slowly varying disturbance to time series expansion disturbance, which is defined as (2). Therefore, not only the square disturbance, but also the sawtooth disturbance can be observed asymptotically, if $0.5<\alpha\le 1$.

C. Extended State Observer (ESO)

In the former contents, we have developed ROESO and ROCESO to observe time series expansion disturbance asymptotically based on the ability to directly measure the system state. While, in most cases, the system state cannot be measured directly. And now we are in the position to utilize the control input and output signals only to develop ESO to observe slowly varying disturbance asymptotically and to observe other disturbances with bounded error based on $\lim_{t\to\infty}|D^{\alpha}d(t)|\le\mu$, with $\mu>0$. In addition, we will propose a way to lower the boundary of the observation error.

Theorem 3. $\{\bar{A},\bar{C}\}$ is observable, if $\{A,C\}$ is observable and $\mathrm{rank}(F^{\mathrm{T}}A^{(n-1)\mathrm{T}}[\begin{array}{cccc} C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} & \ldots & A^{(p-1)\mathrm{T}}C^{\mathrm{T}}\end{array}])\ge q$, where $\bar{A}=\left[\begin{array}{cc} A & F \\ 0_{q\cdot n} & 0_q\end{array}\right]$, $\bar{C}=[\begin{array}{cc} C & 0_{p\cdot q}\end{array}]$.

Proof. According to [1], $\{A,C\}$ is observable, if and only if the observability matrix M_O is full column rank, where $M_O=[\begin{array}{cccc} C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} & \ldots & A^{(n-1)\mathrm{T}}C^{\mathrm{T}}\end{array}]^{\mathrm{T}}$.

As a result, the observability matrix $\bar{M}_o$ related to $\{\bar{A},\bar{C}\}$ can be described as

$$\begin{aligned} \bar{M}_o &= [\begin{array}{cccc} \bar{C}^{\mathrm{T}} & \bar{A}^{\mathrm{T}}\bar{C}^{\mathrm{T}} & \ldots & \bar{A}^{(n+p-1)\mathrm{T}}\bar{C}^{\mathrm{T}}\end{array}]^{\mathrm{T}} \\ &= \left[\begin{array}{cccc} C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} & \ldots & A^{(n+p-1)\mathrm{T}}C^{\mathrm{T}} \\ 0_{q\cdot p} & F^{\mathrm{T}}C^{\mathrm{T}} & \ldots & F^{\mathrm{T}}A^{(n+p-2)\mathrm{T}}C^{\mathrm{T}}\end{array}\right]^{\mathrm{T}}, \end{aligned} \quad (20)$$

which implies that

$$\begin{aligned} &\mathrm{rank}(\bar{M}_o) \\ &\quad= \mathrm{rank}\left(\left[\begin{array}{cccc} C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} & \ldots & A^{(n+p-1)\mathrm{T}}C^{\mathrm{T}} \\ 0 & F^{\mathrm{T}}C^{\mathrm{T}} & \ldots & F^{\mathrm{T}}A^{(n+p-2)\mathrm{T}}C^{\mathrm{T}}\end{array}\right]\right) \\ &\quad\ge \mathrm{rank}([\begin{array}{cccc} C^{\mathrm{T}} & A^{\mathrm{T}}C^{\mathrm{T}} & \cdots & A^{(n-1)\mathrm{T}}C^{\mathrm{T}}\end{array}]) \\ &\qquad+\ \mathrm{rank}\left([\begin{array}{ccc} F^{\mathrm{T}}A^{(n-1)\mathrm{T}}C^{\mathrm{T}} & \ldots & F^{\mathrm{T}}A^{(n+p-2)\mathrm{T}}C^{\mathrm{T}}\end{array}]\right) \\ &\quad= n+\mathrm{rank}(F^{\mathrm{T}}A^{(n-1)\mathrm{T}}\left[\begin{array}{ccc} C^{\mathrm{T}} & \ldots & A^{(p-1)\mathrm{T}}C^{\mathrm{T}}\end{array}\right]) \\ &\quad\ge n+q. \end{aligned} \quad (21)$$

Since $\bar{M}_O\in\mathbf{R}^{p(n+q)\cdot(n+q)}$, we have

$$\begin{aligned} \mathrm{rank}\left(\bar{M}_O\right) &\le \min\left(p(n+q),(n+q)\right) \\ &= n+q. \end{aligned}$$

Proceeding forward, we have

$$\operatorname{rank}\left(\bar{M}_O\right) = n + q. \tag{22}$$

□

Theorem 4. If the α-th order derivative of the disturbance satisfy $\lim_{t\to\infty} D^\alpha d(t) = 0$, the disturbance can be observed asymptotically by following observer

$$\begin{cases} D^\alpha \hat{x}(t) = A\hat{x}(t) + Bu(t) + F\hat{d}(t) + L_1[\hat{y}(t) - y(t)], \\ \hat{y}(t) = C\hat{x}(t), \\ D^\alpha \hat{d}(t) = L_2[\hat{y}(t) - y(t)]. \end{cases} \tag{23}$$

Proof. Defining state observation error $e_x(t) = x(t) - \hat{x}(t)$ and disturbance observation error $e_d(t) = d(t) - \hat{d}(t)$, we can easily get the following equation

$$\begin{aligned} \begin{bmatrix} D^\alpha e_x(t) \\ D^\alpha e_d(t) \end{bmatrix} &= \left(\begin{bmatrix} A & F \\ 0_{q\cdot n} & 0_q \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} C & 0_{p\cdot q} \end{bmatrix} \right) \\ &\quad \times \begin{bmatrix} e_x(t) \\ e_d(t) \end{bmatrix} + \begin{bmatrix} 0_{n\cdot q} \\ I_q \end{bmatrix} w_d(t), \end{aligned} \tag{24}$$

where $w_d(t) = D^\alpha d(t)$. If we define the augment state $e(t) = \begin{bmatrix} e_x^{\mathrm{T}}(t) & e_d^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, then (24) can be written as

$$D^\alpha e(t) = (\bar{A} + L\bar{C})e(t) + Gw_d(t), \tag{25}$$

where

$$\bar{A} = \begin{bmatrix} A & F \\ 0_{q\cdot n} & 0_q \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad G = \begin{bmatrix} 0_{n\cdot q} \\ I_q \end{bmatrix},$$

$$\bar{C}^{\mathrm{T}} = \begin{bmatrix} C^{\mathrm{T}} \\ 0_{q\cdot p} \end{bmatrix}.$$

By using Theorem 3, we can easily search suitable L to make the system (25) asymptotically stable by LMI if $\lim_{t\to\infty} D^\alpha d(t) = 0$, that is, the disturbance can be observed accurately. This establishes Theorem 4. □

By using LMI, we can select L such that the eigenvalues of $\bar{A} + L\bar{C}$ are in the left half plane (LHP), and thereby we can find a positive definite matrix P such that

$$(\bar{A} + L\bar{C})^{\mathrm{T}} P + P(\bar{A} + L\bar{C}) = -Q, \tag{26}$$

for any positive definite matrix Q.

Theorem 5. If the α-th derivative of $d(t)$ is bounded, the ESO can observe the disturbance and state with bounded error and the norm of the estimation error is bounded by

$$\|e(t)\| \le \frac{2\mu \left\|G^{\mathrm{T}} P\right\|}{\lambda_s}, \tag{27}$$

where λ_s is the smallest eigenvalue of Q.

Proof. Consider a Lyapunov function

$$V(t) = e^{\mathrm{T}}(t) P e(t). \tag{28}$$

Seeking for the α-th derivative of $V(t)$, yields

$$\begin{aligned} D^\alpha V(t) &\le e^{\mathrm{T}}(t)[(\bar{A} + L\bar{C})^{\mathrm{T}} P + P(\bar{A} + L\bar{C})]e(t) \\ &\quad + 2w_d(t) G^{\mathrm{T}} P e(t) \\ &\le -e^{\mathrm{T}}(t) Q e(t) + 2w_d(t) G^{\mathrm{T}} P e(t) \\ &\le -\lambda_s \|e(t)\|^2 + 2\mu \left\|G^{\mathrm{T}} P\right\| \|e(t)\| \\ &\le -\|e(t)\| \left(\lambda_s \|e(t)\| - 2\mu \left\|G^{\mathrm{T}} P\right\|\right). \end{aligned} \tag{29}$$

We can easily get that the norm of the observation error is bounded by $\frac{2\mu\|G^T P\|}{\lambda_s}$. □

Remark 4. Theorem 5 shows that the norm of the observation error is bounded by $\frac{2\mu\|G^T P\|}{\lambda_s}$, therefore, we can decrease the observation error by increasing the λ_s, which can be realized by placing all of the eigenvalues far from imaginary axis. Then ESO can observe more kinds of disturbance, such as time series expansion disturbance, sinusoidal disturbance.

D. Cascade Extended State Observer (CESO)

From the former contents, the ESO can observe the disturbance $d(t)$ asymptotically if $\lim_{t\to\infty} D^\alpha d(t) = 0$. However, it is invalid for $\lim_{t\to\infty} D^\alpha d(t) \ne 0$. In order to extend the scope of the disturbance observation and get more accurate observation results, the CESO is developed as follows:

$$\begin{cases} D^\alpha \hat{x}(t) = A\hat{x}(t) + Bu(t) + F\hat{d}(t) \\ \qquad + L_1[\hat{y}(t) - y(t)] + L_2 I^\alpha[\hat{y}(t) - y(t)], \\ \hat{y}(t) = C\hat{x}(t), \\ D^\alpha \hat{d}(t) = L_3[\hat{y}(t) - y(t)] + L_4 I^\alpha[\hat{y}(t) - y(t)]. \end{cases} \tag{30}$$

Theorem 6. $\{\bar{A}, \bar{B}\}$ is controllable if $\{A, B\}$ is controllable; $\{\bar{A}, \bar{C}\}$ is observable if $\{A, C\}$ is observable, where

$$\bar{A} = \begin{bmatrix} A & F & 0_n \\ 0_{q\cdot n} & 0_{q\cdot q} & 0_{q\cdot n} \\ I_n & 0_{n\cdot q} & 0_n \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} I_n & 0_{n\cdot q} \\ 0_{q\cdot n} & I_q \\ 0_n & 0_{n\cdot q} \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} I_n & 0_{n\cdot q} & 0_n \\ 0_n & 0_{n\cdot q} & I_n \end{bmatrix}.$$

Proof. According to [1], $\{A, B\}$ is controllable, if and only if the controllability matrix M_C is full row rank, where $M_C = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$. $\{A, C\}$ is observable, if and only if the observability matrix M_O is full column rank, where $M_O = \begin{bmatrix} C^{\mathrm{T}} & A^{\mathrm{T}} C^{\mathrm{T}} & \dots & A^{(n-1)\mathrm{T}} C^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. □

As a result, the controllability matrix $\bar{M}_c$ related to $\{\bar{A}, \bar{B}\}$ can be described as

$$\begin{aligned} \bar{M}_c &= \begin{bmatrix} \bar{B} \,|\, \bar{A}\bar{B} \,|\, \cdots \,|\, \bar{A}^{2n+q-1}\bar{B} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc|c|cc} I_n & 0_{n\cdot q} & A & F & & A^{2n+q-1} & A^{2n+q-2}F \\ 0_{q\cdot n} & I_q & 0_{q\cdot n} & 0_q & \cdots & 0_{q\cdot n} & 0_q \\ 0_n & 0_{n\cdot q} & I_n & 0_{n\cdot q} & & A^{2n+q-2} & 0_{n\cdot q} \end{array}\right], \end{aligned} \tag{31}$$

which implies that

$$\begin{aligned} \operatorname{rank}\left(\bar{M}_c\right) &\ge \operatorname{rank}\left(\begin{bmatrix} I_n & 0_{n\cdot q} & A \\ 0_{q\cdot n} & I_q & 0_{q\cdot n} \\ 0_n & 0_{n\cdot q} & I_n \end{bmatrix} \right) \\ &= 2n + q. \end{aligned} \tag{32}$$

By virtue of $\bar{M}_c \in \mathbf{R}^{(2n+q)\cdot((n+q)(2n+q))}$,

$$\begin{aligned} \operatorname{rank}(\bar{M}_c) &\le \min\left((2n+q), (2n+q)(n+q)\right) \\ &= 2n + q. \end{aligned} \tag{33}$$

All of the above stated facts lead to the following

$$\operatorname{rank}(\bar{M}_c) = 2n + q. \tag{34}$$

In the other words, $\{\bar{A}, \bar{B}\}$ is controllable.

The corresponding observability matrix $\bar{M}_O$ satisfies

$$\begin{aligned}\bar{M}_O &= [\ \bar{C}^{\mathrm{T}} \mid \bar{A}^{\mathrm{T}}\bar{C}^{\mathrm{T}} \mid \cdots \mid \bar{A}^{(2n+q)\mathrm{T}}\bar{C}^{\mathrm{T}}\ ]^{\mathrm{T}} \\ &= \left[\begin{array}{ccc} I_n & 0_{n\cdot q} & 0_n \\ 0_n & 0_{n\cdot q} & I_n \\ \hline A & F & 0_n \\ I_n & 0_{n\cdot q} & 0_n \\ \hline \vdots & \vdots & \vdots \\ \hline A^{2n+q-1} & A^{2n+q-2}F & 0_n \\ A^{2n+q-2} & A^{2n+q-3}F & 0_n \end{array}\right],\end{aligned} \tag{35}$$

which implies that

$$\begin{aligned}\mathrm{rank}(\bar{M}_O) &\ge \mathrm{rank}\left(\left[\begin{array}{ccc} I_n & 0_{n\cdot q} & 0_n \\ 0_n & 0_{n\cdot q} & I_n \\ A & F & 0_n \end{array}\right]\right) \\ &= 2n+q.\end{aligned} \tag{36}$$

Since $\bar{M}_O \in \mathbf{R}^{2n(2n+q)\cdot(2n+q)}$, we have

$$\begin{aligned}\mathrm{rank}\left(\bar{M}_O\right) &\le \min\left(2n(2n+q), (2n+q)\right) \\ &= 2n+q.\end{aligned} \tag{37}$$

Proceeding forward, it follows

$$\mathrm{rank}\left(\bar{M}_O\right) = 2n+q. \tag{38}$$

Consequently, $\{\bar{A}, \bar{C}\}$ is observable. □

Theorem 7. The CESO can observe the state of the disturbed system and α-th order derivative of the disturbance asymptotically, if $\lim_{t\to\infty} D^{2\alpha}d(t) = 0$.

Proof. Defining the observation error

$$e(t) = \left[\begin{array}{c} e_x(t) \\ e_d(t) \end{array}\right] = \left[\begin{array}{c} x(t)-\hat{x}(t) \\ d(t)-\hat{d}(t) \end{array}\right]. \tag{39}$$

Considering the α-th order derivative of $e(t)$, yields

$$\begin{aligned}&\left[\begin{array}{c} D^{\alpha}e_x(t) \\ D^{\alpha}e_d(t) \end{array}\right] \\ &\quad= \left[\begin{array}{c} (A+L_1C)e_x(t) + Fe_d(t) + L_2CI^{\alpha}e_x(t) \\ D^{\alpha}d(t) + L_3Ce_x(t) + L_4CI^{\alpha}e_x(t) \end{array}\right].\end{aligned} \tag{40}$$

Based on (38), considering the 2α-th order derivative of $e(t)$, yields

$$\begin{aligned}&\left[\begin{array}{c} D^{2\alpha}e_x(t) \\ D^{2\alpha}e_d(t) \end{array}\right] \\ &\quad= \left[\begin{array}{cc} A+L_1C & F \\ L_3C & 0_q \end{array}\right]\left[\begin{array}{c} D^{\alpha}e_x(t) \\ D^{\alpha}e_d(t) \end{array}\right] \\ &\qquad+ \left[\begin{array}{cc} L_2C & 0_{n\cdot q} \\ L_4C & 0_q \end{array}\right]\left[\begin{array}{c} e_x(t) \\ e_d(t) \end{array}\right] + \left[\begin{array}{c} 0_{n\cdot q} \\ I_q \end{array}\right] w_d(t),\end{aligned} \tag{41}$$

where $w_d(t) = D^{2\alpha}d(t)$. Defining the augmented state $\hat{e}(t) = [\ D^{\alpha}e_x^{\mathrm{T}}(t) \quad D^{\alpha}e_d^{\mathrm{T}}(t) \quad e_x^{\mathrm{T}}(t) \quad e_d^{\mathrm{T}}(t)\ ]^{\mathrm{T}}$, then (41) can be rewritten as

$$D^{\alpha}\hat{e}(t) = \hat{A}\hat{e}(t) + Gw_d(t), \tag{42}$$

where

$$\hat{A} = \left[\begin{array}{cccc} A+L_1C & F & L_2C & 0_{n\cdot q} \\ L_3C & 0_q & L_4C & 0_q \\ I_n & 0_{n\cdot q} & 0_n & 0_{n\cdot q} \\ 0_{q\cdot n} & I_q & 0_{q\cdot n} & 0_q \end{array}\right], \quad G = \left[\begin{array}{c} 0_{n\cdot q} \\ I_q \\ 0_{n\cdot q} \\ 0_q \end{array}\right].$$

Then we extract the state $D^{\alpha}e_x(t)$, $D^{\alpha}e_d(t)$ and $e_x(t)$ from $\hat{e}(t)$ to compose a new state $\bar{e}(t)$. Then the state space equation related to $\bar{e}(t)$ can be written as

$$\begin{aligned}&D^{\alpha}\bar{e}(t) \\ &\quad= \left[\begin{array}{ccc} A+L_1C & F & L_2C \\ L_3C & 0_q & L_4C \\ I_n & 0_{n\cdot q} & 0_n \end{array}\right]\bar{e}(t) + \left[\begin{array}{c} 0_{n\cdot q} \\ I_q \\ 0_{n\cdot q} \end{array}\right] w_d(t) \\ &\quad= \left(\bar{A} + \bar{B}\bar{L}\bar{C}\right)\bar{e}(t) + \bar{G}w_d(t),\end{aligned} \tag{43}$$

where

$$\bar{A} = \left[\begin{array}{ccc} A & F & 0_n \\ 0_{q\cdot n} & 0_q & 0_{q\cdot n} \\ I_n & 0_{n\cdot q} & 0_n \end{array}\right], \quad \bar{B} = \left[\begin{array}{cc} I_n & 0_{n\cdot q} \\ 0_{q\cdot n} & I_q \\ 0_n & 0_{n\cdot q} \end{array}\right],$$

$$\bar{G} = \left[\begin{array}{c} 0_{n\cdot q} \\ I_q \\ 0_{n\cdot q} \end{array}\right], \quad \bar{L} = \left[\begin{array}{cc} L_1C & L_2C \\ L_3C & L_4C \end{array}\right],$$

$$\bar{C} = \left[\begin{array}{ccc} I_{n\cdot n} & 0 & 0 \\ 0 & 0_{n\cdot q} & I_{n\cdot n} \end{array}\right].$$

Based on Theorem 4, we get that $\{\bar{A}, \bar{B}\}$ is controllable and $\{\bar{A}, \bar{C}\}$ is observable. Then based on Lemma 2, we can search $\bar{L}$ by using the MATLAB LMI toolbox to make the system (43) asymptotically stable, if $\lim_{t\to\infty} D^{2\alpha}d(t) = 0$. And the matrix L is given by

$$L = \bar{L}\tilde{C}^{-}, \tag{44}$$

where $L = \left[\begin{array}{cc} L_1 & L_2 \\ L_3 & L_4 \end{array}\right]$, $\tilde{C} = \left[\begin{array}{cc} C & 0_{p\cdot n} \\ 0_{p\cdot n} & C \end{array}\right]$ and $\tilde{C}^{-}$ is the pseudoinverse of $\tilde{C}$, which can be got by command $\mathrm{pinv}(C)$ in MATLAB. That means, the state of the system in (1) and α-th order derivative of disturbance can be observed accurately by CESO if only $\lim_{t\to\infty} D^{2\alpha}d(t) = 0$. This establishes Theorem 7. □

Remark 5. Although the CESO cannot observe disturbance asymptotically if $\lim_{t\to\infty} D^{2\alpha}d(t) = 0$, the state of disturbed system and α-th derivative of disturbance can be observed asymptotically. Therefore, we can combine the CESO and ROCESO to observe the disturbance asymptotically. Hereinafter, we denote this observation method as CESO + ROCESO.

Remark 6. The disturbance can be observed by CESO + ROCESO asymptotically if $\lim_{t\to\infty} D^{2\alpha}d(t) = 0$, thereby, the CESO + ROCESO can asymptotically observe more disturbance than the ESO. And if $\lim_{t\to\infty} D^{2\alpha}d(t) \le \mu$, the CESO + ROCESO can observe it with bounded error, and Theorem 4 and Remark 4 have given the way to reduce the observation error.

Remark 7. Four disturbance observers have been designed in this paper. From the corresponding proofs of Theorem 1, Theorem 2, Theorem 4, and Theorem 7, we can arrive that if and only if these related matrices (Λ in (13) and (18), $\bar{A}+L\bar{C}$)

in (25), and $\bar{A}+\bar{B}\bar{L}\bar{C}$ in (43)) are designed stable and the conditions about the disturbance are satisfied, the convergence of the relevant observation errors are not affected by the initial value of these observation errors.

IV. Illustrative Examples

All the numerical examples illustrated in this paper are implemented via the piecewise numerical approximation algorithm. For more information about the algorithm one can refer to [29].

Example 1. Consider a disturbed fractional order gas-turbine system[30] as follows:

$$\begin{cases} D^{0.84}x(t)=Ax(t)+Bu(t)+Fd(t), \\ y(t)=Cx(t), \end{cases} \tag{45}$$

where

$$A=\begin{bmatrix} 0 & 1 \\ -136.24 & -18.4741 \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$F=\begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad C^{\mathrm{T}}=\begin{bmatrix} 0 \\ 14164.9 \end{bmatrix},$$

and assume that the state can be measured directly.

We get that the disturbed system is completely controllable and observable. Considering the disturbance is square wave (amplitude $=1$ and frequency $=0.5$). Fig. 1 shows the disturbance observed by ROESO. And in Fig. 1, $d(t)$, $\hat{d}_1(t)$, $\hat{d}_2(t)$, and $\hat{d}_3(t)$ are the primary disturbance and the observed results with $\Lambda=20,50,100$, respectively, and $e_1(t)$, $e_2(t)$, $e_3(t)$ are the corresponding observation errors. The observation results illustrate that the ROESO can asymptotically observe the slowly varying disturbance (square wave). And the bigger value of Λ is, the faster observation convergence rate we can get.

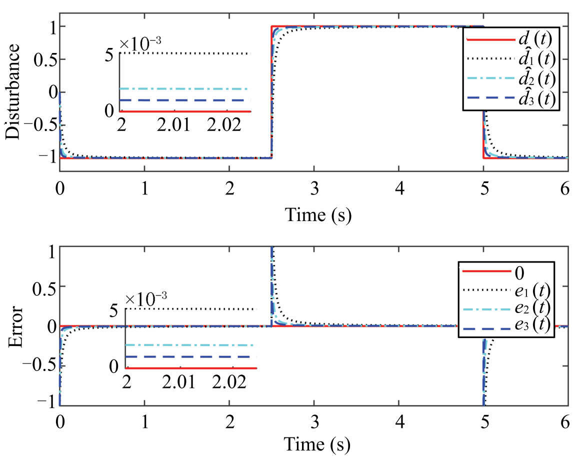

Fig. 1. Observed results for square disturbance of Example 1.

Example 2. Consider system (45) is disturbed by sawtooth wave (amplitude $=1$ and frequency $=0.1$) and assume that the state can be measured directly.

We choose the ROCESO to observe the sawtooth wave and show the observed results as Fig. 2. In Fig. 2, $d(t)$, $\hat{d}_1(t)$, $\hat{d}_2(t)$, and $\hat{d}_3(t)$ are the primary disturbance and the observed results with $\Lambda=20,50,100$, respectively, and $e_1(t)$, $e_2(t)$, $e_3(t)$ are the corresponding observation errors. The observed results illustrate that the ROCESO can asymptotically observe the slope forms disturbance (sawtooth), and then the slowly varying disturbance can be observed by ROCESO certainly. And the bigger value of Λ is, the faster observation convergence rate we can get.

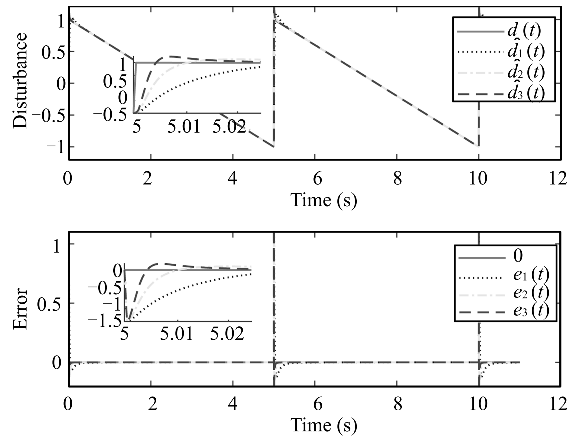

Fig. 2. Observed results for sawtooth disturbance of Example 2.

Example 3. Considering the disturbances in system (45) are square wave (amplitude $=1$ and frequency $=0.1$) and sinusoidal wave as $1+\sin(2t)+2.5\cos(3t)$, and assume that the state cannot be measured directly.

Figs.3-6 show these states and disturbances observed by ESO, respectively. And in these figures, $x_1(t)$, $\hat{x}_{11}(t)$, $\hat{x}_{12}(t)$, $\hat{x}_{13}(t)$ are the system state 1 and the corresponding observed state with $L_1^{\mathrm{T}}=[-0.007 -0.111]$, $[-0.005 -0.158]$, $[-0.022 - 2.977]$ and $L_2=-0.811$, -1.186, -49.370, respectively. And $e_{11}(t)$, $e_{12}(t)$, and $e_{13}(t)$ are the corresponding state observation errors. $\hat{d}_1(t)$, $\hat{d}_2(t)$, and $\hat{d}_3(t)$ are the corresponding observation disturbance and $e_1(t)$, $e_2(t)$, $e_3(t)$ are the corresponding observation errors. The eigenvalues of $\bar{A}+L\bar{C}$ in (23) with respect to the different matrixes $L^{\mathrm{T}}=[L_1^{\mathrm{T}},L_2^{\mathrm{T}}]$ are shown in Table I. Table I and the simulation results illustrate that the larger distance between the eigenvalues and the imaginary axis is, the more accurate observed result we can get, which demonstrate Theorem 5 and Remark 4 in numerical simulation.

TABLE I
The eigenvalues of $\bar{A}+L\bar{C}$ with respect to the different matrixes L

L_1^{T}	$[-0.007-0.111]$	$[-0.005-0.158]$	$[-0.022-2.977]$
L_2	-0.811	-1.186	-49.370
	$-20.34-7.06\mathrm{i}$	$-33.97-30.82\mathrm{i}$	$-130.24-112.63\mathrm{i}$
eig[1]	$-20.34+7.06\mathrm{i}$	$+33.97+30.82\mathrm{i}$	$-130.24+112.63\mathrm{i}$
	-74.34	-23.96	-70.76

Example 4. Consider another disturbed system as follows:

$$\begin{cases} D^{0.8}x(t)=Ax(t)+Bu(t)+Fd(t), \\ y(t)=Cx(t), \end{cases} \tag{46}$$

[1]The eigenvalues of $\bar{A}+L\bar{C}$

where

$$A=\begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix},\quad B=\begin{bmatrix}1\\ 1\end{bmatrix},$$
$$F=\begin{bmatrix}3\\ 3\end{bmatrix},\quad C^{\mathrm{T}}=\begin{bmatrix}1\\ 0\end{bmatrix},$$

and assume that the state cannot be measured directly.

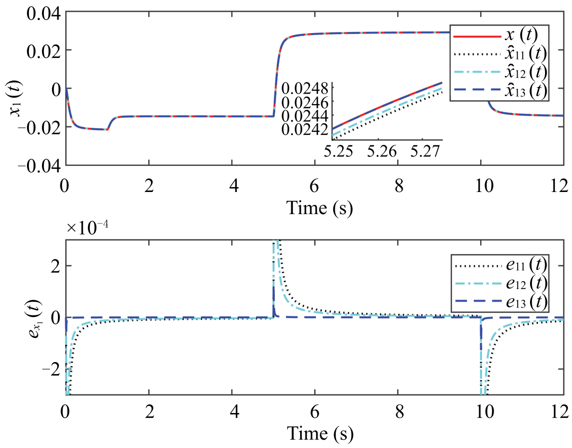

Fig. 3. Observed results for $x_1(t)$ disturbed by square disturbance of Example 3.

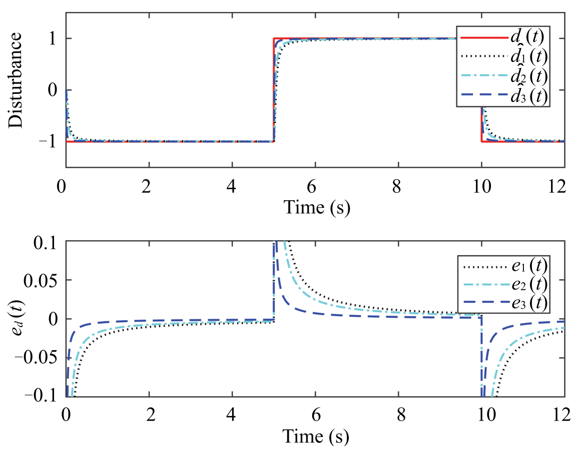

Fig. 4. Observed results for square disturbance of Example 3.

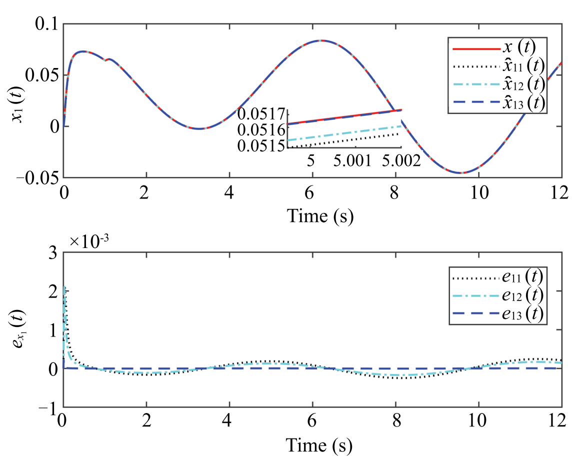

Fig. 5. Observed results for $x_1(t)$ disturbed by sinusoidal disturbance of Example 3.

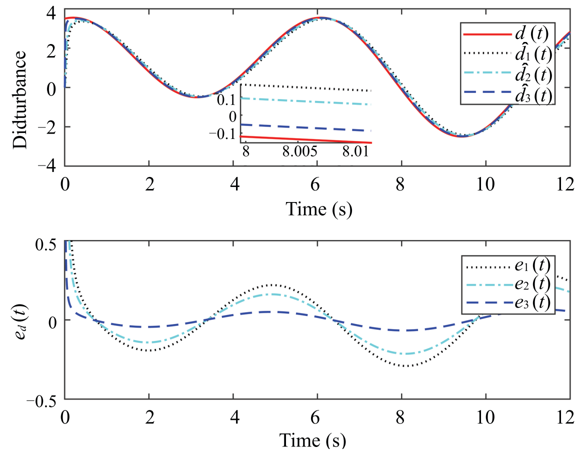

Fig. 6. Observed results for sinusoidal disturbance of Example 3.

It is completely controllable and observable. We can easily get that the system is unstable with poles equal to -2.4142, 0.4142. Consider the disturbances are sawtooth (amplitude $=$ 1 and frequency $=0.2$) and sinusoidal as $\sin(t)+0.5\sin(1.5t)$. The matrix L of ESO is set as $[-56.59\ 4135.56\ \ -1950.61]$; the Λ of ROCESO is set as 15; the matrixes L_1, L_2, L_3, and L_4 in CESO are set as $[-59.81\ -21.91]^{\mathrm{T}}$, $[-764.93\ -320.81]^{\mathrm{T}}$, -141.98 and -2223.81, respectively. The L in ESO and L_i $(i\in 1,2,3,4)$ in CESO can be sought by MATLAB LMI tool box. Fig. 7 and Fig. 8 show observation results of CESO for the system state, which is disturbed by sawtooth disturbance and sinusoidal disturbance, respectively. In Fig. 9 and Fig. 10, $\hat{d}_1(t)$, $\hat{d}_2(t)$, and $\hat{d}_3(t)$ are the disturbance observation results of ESO, CESO, and ROCESO + CESO, respectively, and $e_1(t)$, $e_2(t)$, and $e_3(t)$ are the corresponding observation errors of $\hat{d}_1(t)$, $\hat{d}_2(t)$, and $\hat{d}_3(t)$. The simulation results show that no matter what the kinds of the disturbance is and no matter the disturbed system whether or not stable, the ROCESO + CESO, which utilize the control input and output signals only, can observe the external unknown disturbance more accurately than ESO and CESO.

V. Conclusion

In this article, disturbance observer design for FO-LTI

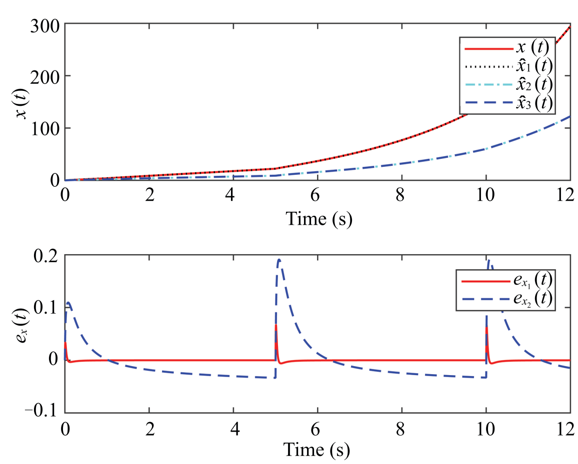

Fig. 7. Observed results of CESO for system state disturbed by sawtooth disturbance of Example 4.

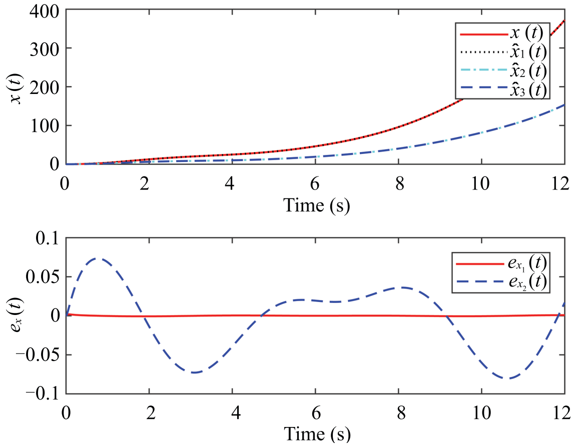

Fig. 8. Observed results of CESO for system state disturbed by sinusoidal disturbance of Example 4.

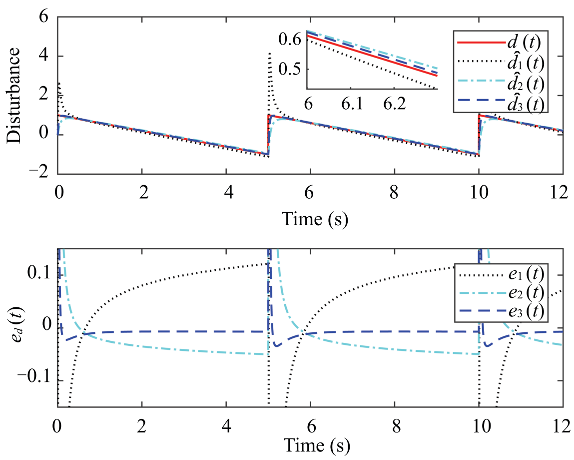

Fig. 9. Observed results for sawtooth disturbance of Example 4.

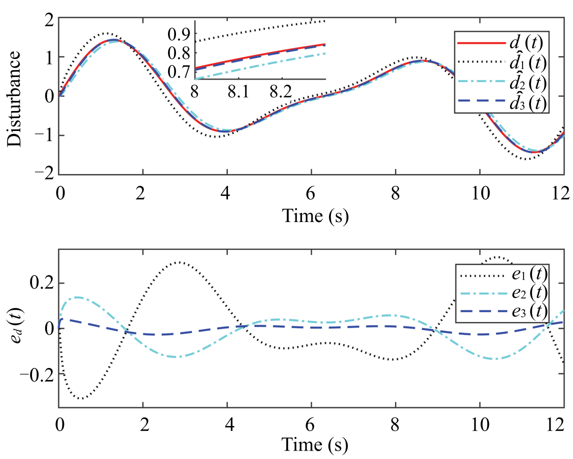

Fig. 10. Observed results for sinusoidal disturbance of Example 4.

systems has been investigated. For the case that state can be measured directly, ROESO and ROCESO are proposed. And then, ESO is developed in the case that the state cannot be measured easily. Furthermore, the CESO has been presented to observe the state and α-th order derivative of disturbance, which can be combined with ROCESO to get a more accurate observation result. In order to show the generality of the proposed observers, we consider two kinds disturbances: time series expansion and sinusoidal. And we have given concrete proofs that the time series expansion can be observed asymptotically and sinusoidal can be observed with bounded error, and in addition, the way how to reduce the observation error has been proposed. The numerical examples have shown the effectiveness of the proposed designing methods. It is believed that the approaches provide a new avenue to observe disturbance. The interesting future topics involve the following cases:

1) To utilize the designed disturbance observer to realize disturbance rejection, as well as the unmatched disturbance;

2) To study the problem of noise effect reduction where the measured output is mixed with measurement noise;

3) To investigate the observer with considering the uncertainties of the system.

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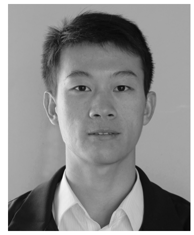

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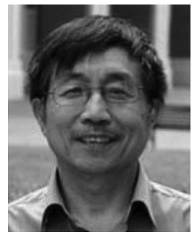

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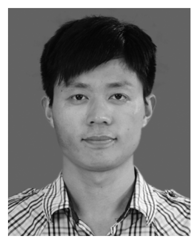

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