# Controllability of Fractional Order Stochastic Differential Inclusions with Fractional Brownian Motion in Finite Dimensional Space

T. Sathiyaraj and P. Balasubramaniam

***Abstract*—In this paper, sufficient conditions are formulated for controllability of fractional order stochastic differential inclusions with fractional Brownian motion (fBm) via fixed point theorems, namely the Bohnenblust-Karlin fixed point theorem for the convex case and the Covitz-Nadler fixed point theorem for the nonconvex case. The controllability Grammian matrix is defined by using Mittag-Leffler matrix function. Finally, a numerical example is presented to illustrate the efficiency of the obtained theoretical results.**

***Index Terms*—Controllability, fractional Brownian motion, fractional order derivatives, Mittag-Leffler function, stochastic differential inclusions.**

## I. Introduction

MANY real dynamical systems are better characterized by using a non-integer order dynamic model based on fractional calculus or, differentiation or integration of non-integer order. The concept of fractional calculus has tremendous potential to change the model and control the nature around us. Fractional differential equations serve as an appropriate phenomenon such that it can even describe the real world problems which are impossible to describe using classical integer order differential equations. Over the past decades, the theory of fractional differential equation received more attention, and has obtained a prior position in the field of physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, electro chemistry and many other science and engineering areas, for details one may refer the books[1−5]. In recent years, fractional control techniques provide an effective way to control dynamic behaviours through the model of fractional differential equations[6]. Tuning and auto-tuning of fractional order controllers for industrial applications have been well developed see [7] and its advanced applications in various branches of physics, economics and engineering sciences, see [8−9] and references therein.

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Control theory is an interdisciplinary branch of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. The objective of control theory is to make systems to perform specific tasks using suitable control actions. Such behaviour is seen in a range of problems from mechanics, optimal control, ecology, industrial robotics, aeronautics, transportation, biotechnology, medical models, etc. Controllability of dynamical systems is one of the fundamental notions of modern control theory. Generally speaking, controllability enables one to steer the control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. This concept leads to some important conclusions regarding the behaviour of linear and nonlinear dynamical systems. Controllability of fractional order deterministic and stochastic dynamical systems in finite dimensional space has been studied in [10−11]. Controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space has been studied recently in [12]. Controllability of linear stochastic systems has been investigated in [13].

On the other hand, the theory of differential inclusions has become an active area of investigation due to its applications in various fields such as mechanics, electrical engineering, medicine biology, ecology and so on[12,14−15]. Stochastic differential equations driven by fBm have attracted great interest and potential applications in telecommunication networks, finance markets, biology and other fields[16−17].

However, to the best of our knowledge there are limited works considering the existence of solutions, and controllability results of integer order stochastic differential inclusions in finite and infinite dimensional space[12,15]. Fractional order Riemann-Liouville integral inclusions with two independent variables and multiple delays have been illustrated in [14]. Balasubramaniam[15] proposed existence of solutions of functional stochastic differential inclusions, whereas Balachandran and Kokila[10] have obtained the controllability of fractional dynamical systems, although controllability of impulsive neutral stochastic differential equations with fBm was established in [18]. In this paper, we study the controllability of fractional order stochastic differential inclusions with fBm,

$$ {}^{C}D^{q}x(t) \in Ax(t) + Bu(t) + f(t,x(t)) + \int_0^t G(s,x(s))\mathrm{d}W^H_{(s)},\ t \in J, $$

$$x(0) = x_0, \tag{1}$$

where $[0,T] := J, {}^C D^q$ denotes Caputo derivative of fractional order $q$, $A$ and $B$ are matrices of dimensions $n \times n$ and $n \times m$ respectively, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ are the state and control vectors. The nonlinear functions $f$, $G$ are appropriate functions to be defined later. $W_t^H$ is a fBm with the Hurst parameter $H \in (\frac{1}{2}, 1)$ and defined by its stochastic representation

$$W_{(t)}^H := \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \Big( \int_{-\infty}^{0} [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] \mathrm{d}W(s) + \int_0^t (t-s)^{H-\frac{1}{2}} \mathrm{d}W(s) \Big), \tag{2}$$

where $\Gamma$ represents the Gamma function $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} \exp(-x) \mathrm{d}x$ and $0 < H < 1$. The integrator $W$ is a stochastic process of ordinary Brownian motion. Note that $W$ is recovered by taking $H = \dfrac{1}{2}$ in (2). The proposed work on the controllability of fractional order stochastic differential inclusions with fBm in finite dimensional space is new to the literature.

The paper is organized as follows: In Section II, we recall some essential results on the basic definitions of fractional integral and derivatives, lemmas, propositions, notations, and some mild conditions to obtain the controllability results successfully. In Section III, we study the controllability results for the fractional system (1) under the fixed point theorems. Numerical example is illustrated in Section IV to show the effectiveness of the derived results. Finally, conclusion and future work is drawn in Section V.

## II. Preliminaries

It is well known that the fractional order integral and derivative operators, namely Riemann-Liouville, Caputo and Mittag-Leffler function play a vital role to find the solution of fractional differential equation. The following definitions and properties are well known, for a suitable function $f \in L_1(\mathbf{R}_+), \mathbf{R}_+ = [0, \infty)$. For more details, see [4, 9, 19].

Let $q > 0$, $p > 0$ with $n-1 < q < n$, $n-1 < p < n$, and $n \in \mathbf{N}$. Let $\mathbf{R}^m$ be the $m$-dimensional Euclidean space.

**Definition 1.** The fractional integral of order $q$ with the lower limit 0 for a function $f$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) \mathrm{d}s, \ t > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function. The Laplace transform of the Riemann-Liouville fractional integral is given by

$$\mathcal{L}\{I_t^q f(t)\} = \frac{1}{\lambda^q} \hat{f}(\lambda),$$

where

$$\hat{f}(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} f(t) \mathrm{d}t, \quad \mathrm{Re}(\lambda) > w.$$

**Definition 2.** Riemann-Liouville derivative of order $q$ with lower limit zero for a function $f : [0, \infty) \longrightarrow \mathbf{R}$ can be written as

$${}^L D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(s)}{(t-s)^{q+1-n}} \mathrm{d}s.$$

**Definition 3.** The Caputo fractional derivative of order $q$ for a function $f : [0, \infty) \longrightarrow \mathbf{R}$ can be written as

$${}^C D^q f(t) = I^{n-q} D^n f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q+1} f^n(s) \mathrm{d}s.$$

In particular, $I^{q\,C} D^q f(t) = f(t) - f(0)$. The following is a well-known relation

$${}^C D^q f(t) = {}^L D^q f(t) - \sum_{k=0}^{n-1} \frac{t^{k-q}}{\Gamma(k-q+1)} f^{(k)}(0^+), \quad n = \Re(q) + 1.$$

**Definition 4.** Now, consider the well-known Mittag-Leffler function:
A two parameter function of the Mittag-Leffler type function is defined by the series expansion

$$E_{q,p}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(kq+p)}, \quad q, p > 0, \quad z \in \mathbf{C}.$$

The general Mittag-Leffler function satisfies the following identity:

$$\int_0^\infty \mathrm{e}^{-t} t^{p-1} E_{q,p}(t^q z) \mathrm{d}t = \frac{1}{1-z} \text{ for } |z| < 1.$$

The most interesting properties of the Mittag-Leffler function are associated with their Laplace integral

$$\int_0^\infty \mathrm{e}^{-st} t^{p-1} E_{q,p}(\pm a t^q) \mathrm{d}t = \frac{s^{q-p}}{(s^q \mp a)}.$$

That is,

$$\mathcal{L}\{t^{p-1} E_{q,p}(\pm a t^q)\}(s) = \frac{s^{q-p}}{(s^q \mp a)},$$

for $\Re(s) > |a|^{\frac{1}{q}}$ and $\Re(p) > 0$. In particular, for $p = 1$,

$$E_{q,1}(az^q) = E_q(az^q) = \sum_{k=0}^\infty \frac{a^k z^{kq}}{\Gamma(qk+1)}, \ a, z \in \mathbf{C},$$

have the interesting property

$${}^C D^q E_q(az^q) = a E_q(az^q),$$

and

$$\mathcal{L}\{E_q(\pm a t^q)\}(s) = \frac{s^{q-1}}{s^q \mp a} \text{ for } p = 1.$$

Let us consider the linear fractional stochastic differential inclusions with fBm is represented in the following form

$$\begin{aligned} {}^C D^q x(t) &\in Ax(t) + Bu(t) + f(t) + \int_0^t G(s) \mathrm{d}W_{(s)}^H, \quad t \in J, \\ x(0) &= x_0, \end{aligned} \tag{3}$$

where ${}^C D^q,\ x(t),\ u(t)$ and $W^H_{(s)}$ are same as defined above, $f : J \longrightarrow \mathbf{R}^n$ and $G : J \longrightarrow \mathbf{R}^{n\times n}$. Now applying the Riemann-Liouville integral operator on both sides[20], we get

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big[Ax(s) + Bu(s) + f(s) + \int_0^s G(\theta)\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s.$$

Taking Laplace transformation on both sides, we have

$$\widehat{x}(s) = \frac{1}{s}x_0 + \frac{1}{s^q}A\widehat{x}(s) + \frac{1}{s^q}B\widehat{u}(s) + \frac{1}{s^q}\widehat{f}(s) + \frac{1}{s^q}\widehat{G}(s).$$

Taking inverse Laplace transformation on both sides, we get solution of system (3) by the expression (see [10, 19, 21] given by

$$\begin{aligned} x(t) = &\ E_q(At^q)x_0 \\ &+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q) \\ &\times \left[Bu(s) + f(s) + \int_0^s G(\theta)\mathrm{d}W^H_{(\theta)}\right]\mathrm{d}s. \end{aligned}$$

In this paper, we adopt the following notations.

Let $(\Omega, \mathcal{F}, P)$ denote the complete probability space with a right continuous and complete filtration $\{\mathcal{F}_t, t \in J\}$ ($\mathcal{F}_t$ the $\sigma-$algebra generated by the random variables $\{W^H_{(s)}, s \in [0, t]\}$ and $P-$null set) and satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $L^2(\Omega, \mathcal{F}, \mathbf{R}^n)$ be the space of all square-integrable random variables with values in $\mathbf{R}^n$. Let $\mathcal{B} = \mathcal{C}(J, \mathbf{R}^n)$ be the Banach space. Denote the class of $\mathbf{R}^n-$ valued stochastic processes $\{\xi(t) : t \in J\}$ which is $\mathcal{F}_t-$ adapted and have a finite second moments, that is

$$\|\xi\| = \sup_t \left(\mathrm{E}|\xi(t)|^2\right)^{\frac{1}{2}} < \infty.$$

**Definition 5.** A normalized fBm $W^H = \{W^H_{(t)} : 0 \le t < \infty\}$ with $0 < H < 1$ on $(\Omega, \mathcal{F}, P)$ is uniquely characterized by the following properties:

1) $W^H_{(t)}$ has stationary increments;
2) $W^H_{(0)} = 0$, and $\mathrm{E}W^H_{(t)} = 0$ for $t \ge 0$;
3) $W^H_{(t)}$ has a Gaussian distribution for $t > 0$.

From the above three properties it follows that the covariance function is given by

$$\begin{aligned} R_H(s,t) &= \mathrm{E}\left(W^H_{(s)}W^H_{(t)}\right) \\ &= \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}, \\ &\quad \text{for } 0 < s \le t. \end{aligned} \tag{4}$$

The values of $H$ determines what kind of process the fBm is:

1) if $H = \frac{1}{2}$ then the process is in fact a Brownian motion or Wiener process,

2) if $H > \frac{1}{2}$ then the increments of the process are positively correlated,

3) if $H < \frac{1}{2}$ then the increments of the process are negatively correlated.

Moreover, $W^H$ has the integral representation

$$W^H_{(t)} = \int_0^t K_H(t,s)\mathrm{d}W_{(s)}$$

where $W$ is a standard Wiener process and the kernel $K_H(t, s)$ defined as

$$K_H(t,s) = C_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\mathrm{d}u$$

and

$$\frac{\partial K}{\partial t}(t,s) = C_H\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}$$

where

$$C_H = \left[\frac{H(2H-1)}{W(2-2H, H-\frac{1}{2})}\right]^{\frac{1}{2}}\quad t > s.$$

**Remark 1.** For Gaussian process, the mean and covariance structure determine the finite dimensional distribution uniquely. Therefore, we conclude from (4) that $\{W^H_{(at)} : 0 \le t < \infty\}$ and $\{a^H W^H_{(t)} : 0 \le t < \infty\}$ have the same finite-dimensional distribution fBm. In fact, fBm is the only Gaussian process with stationary increments that is self-similar.

Let $(X, \|\cdot\|)$ be a Banach space. Denote $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}, \mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

For more details on multivalued maps, readers can refer the books[22−25].

**Definition 6.** A multivalued map $T : X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$. $T$ is bounded on bounded sets if $T(B) = \bigcup_{x\in B} T(x)$ is bounded in $X$ for all $B \in \mathcal{P}_{bd}(X)$(i.e. $\sup_{x\in B}\sup_{y\in T(x)}\|y\| < \infty$).

**Definition 7.** $T$ is called upper semi-continuous on $X$ if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $T(x_0)$, there exists an open neighborhood $N_0$ of $x_0$ such that $T(N_0) \subseteq N$.

**Definition 8.** $T$ is said to be completely continuous if $T(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. $T$ has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator $T$ will be denoted by $FixT$.

**Definition 9.** A multivalued map $T : X \longrightarrow \mathcal{P}_{cl}(\mathbf{R}^n)$ is said to be measurable if for every $v \in \mathbf{R}^n$, the function $x \mapsto d(v, T(x)) = \inf\{\|v - z\| : z \in T(x)\}$ is measurable.

For each $x \in L_2(J, \mathbf{R}^n), x(t) > 0$ defines the set of selections of $G$ by $\sigma \in N_{G,x} = \{\sigma \in L_2(J, \mathbf{R}^n) : \sigma(t) \in G(t, x(t))$ for almost everywhere (a.e.) $t \in J\}$.

**Lemma 1**[24]**.** Let $T$ be a completely continuous multivalued map with nonempty compact values, then $T$ is upper semi-continuous if and only if $T$ has a closed graph (i.e. $x_n \longrightarrow x, y_n \longrightarrow y, y_n \in T(x_n)$ imply $y \in T(x)$).

**Definition 10.** A multivalued map $T : J \times \mathbf{R}^n \longrightarrow \mathcal{P}(\mathbf{R}^n)$ is said to be $L_2$-Caratheodory if

1) $t \mapsto T(t, x)$ is measurable for each $x \in \mathbf{R}^n$,
2) $x \mapsto T(t, x)$ is upper semi-continuous for almost all $t \in J$,
3) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbf{R}^+)$ such that

$$\|T(t,x)\|^2 := \sup\{\mathrm{E}\|\sigma\|^2 : \sigma \in T(t,x)\} \le \varphi_\rho(t)$$

for all $\|x\|^2_{\mathbf{R}^n} \le \rho$ and for a.e. $t \in J$.

**Lemma 2**[26]**.** Let $X$ be a Banach space. Let $T : J \times X \longrightarrow \mathcal{P}_{cp,cv}(X)$ be a $L_2$-Caratheodory multivalued map with $N_{G,x} \neq \emptyset$ and let $\Lambda$ be a linear continuous mapping from $L^2(J, X)$ to $\mathcal{C}(J, X)$, then the operator

$$\Lambda \circ N_G : \mathcal{C}(J, X) \longrightarrow \mathcal{P}_{cp,cv}(\mathcal{C}(J, X)),$$
$$x \mapsto (\Lambda \circ N_G)(x) := \Lambda(N_{G,x})$$

is a closed graph operator in $\mathcal{C}(J, X) \times \mathcal{C}(J, X)$.

**Proposition 1**[27]**.** Let $X$ be a separable Banach space. Let $G_1, G_2 : J \longrightarrow \mathcal{P}_{cp}(X)$ be measurable multivalued maps, then the multivalued map $t \mapsto G_1(t) \cap G_2(t)$ is measurable.

**Theorem 1**[27]**.** Let $X$ be a separable metric space, $(T, \mathcal{L})$ be a measurable space, $G$ is a multivalued map from $T$ to complete nonempty subset of $X$. If for each open set $U$ in $X$, $\overline{G}(U) = \{t : G(t) \cap U \neq \emptyset\} \in \mathcal{L}$, then $G$ admits a measurable selection.

**Definition 11.** A stochastic process $x \in \mathcal{B}$ is said to be a mild solution of system (1) if $x(0) = x_0$, $u(\cdot) \in L^2_{\mathcal{F}_t}(J, \mathbf{R}^m)$ and there exists $\sigma \in N_{G,x}$ such that $\sigma(t) \in G(t, x(t))$, $t \in J$ and

$$\begin{aligned} x(t) = {} & E_q(At^q)x_0 \\ & + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)Bu(s)\mathrm{d}s \\ & + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)f(s, x(s))\mathrm{d}s \\ & + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \\ & \times \left( \int_0^s \sigma(\theta)\mathrm{d}W^H_{(\theta)} \right) \mathrm{d}s. \end{aligned}$$

**Definition 12.** The system (1) is said to be controllable on $J$ if for every $x_0, x_1 \in \mathbf{R}^n$ there exists a control $u(t)$ such that the solution $x(t)$ of (1) satisfies the conditions $x(0) = x_0$ and $x(T) = x_1$.

Let $(X, d)$ be a metric space induced from $(X, \|\cdot\|)$ be a square normed space. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbf{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where

$$d(A, b) = \inf_{a \in A} d(a, b),\ d(a, B) = \inf_{b \in B} d(a, b).$$

Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [25]).

**Definition 13.** A multivalued operator $\Phi : X \longrightarrow \mathcal{P}_{cl}(X)$ is called

1) $\gamma$-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(\Phi(x), \Phi(y)) \le \gamma d(x, y)$$

for each $x, y \in X$,

2) a contraction if and only if it is $\gamma$-Lipschitz with $\gamma < 1$.

**Lemma 3**[28] **(Bohnenblust-Karlin).** Let $X$ be a Banach space and $K \in \mathcal{P}_{cl,cv}(X)$ and suppose that the operator $\Phi : K \longrightarrow \mathcal{P}_{cl,cv}(K)$ is upper semi-continuous and the set $\Phi(K)$ is relatively compact in $X$. Then $\Phi$ has a fixed point in $K$.

**Lemma 4**[29] **(Covitz-Nadler).** Let $(X, d)$ be a complete metric space. If $\Phi : X \longrightarrow \mathcal{P}_{cl}(X)$ is a contraction, then $\Phi$ has fixed points.

In order to prove the controllability results we assume the following mild conditions.

H1) The multivalued map $G : J \times \mathbf{R}^n \longrightarrow \mathcal{P}(\mathbf{R}^n)$ be an $L_2$-Caratheodory function satisfying the following conditions
   i) for each $t \in J, x \in \mathbf{R}^n$ the function $G(t, \cdot) : \mathbf{R}^n \longrightarrow \mathcal{P}(\mathbf{R}^n)$ is upper semi-continuous. The function $G(\cdot, x) : J \longrightarrow \mathcal{P}(\mathbf{R}^n)$ is measurable and for each $x \in \mathbf{R}^n$ the set $N_{G,x} = \{\sigma \in L_2(J, \mathbf{R}^n) : \sigma(t) \in G(t, x(t))$ for a.e. $t \in J\}$ is nonempty,
   ii) There exists a positive function $\varphi_\rho : J \to \mathbf{R}_+$ such that $\sup\left\{\int_0^t \mathrm{E}\|\sigma(s)\|^2\mathrm{d}s : \sigma(t) \in G(t, x(t))\right\} \le \varphi_\rho(t)$ for a.e. $t \in J$ and the function $s \longrightarrow (t-s)^{q-1}\varphi_\rho(s) \in L^1([0, t], \mathbf{R}_+)$

$$\lim_{\rho \longrightarrow \infty} \inf \frac{\int_0^t (t-s)^{q-1}\varphi_\rho(s)\mathrm{d}s}{\rho} = \eta < \infty.$$

H2) The functions $f : J \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$, $\sigma : J \longrightarrow \mathbf{R}^{n\times n}$ are continuous and there exists a constant $M_f > 0$ such that
   i) $\mathrm{E}\|f(t, x)\|^2 \le M_f(1 + \|x\|^2)$,
   ii) $\mathrm{E}\|\int_0^t \sigma(s)\mathrm{d}W^H_{(s)}\|^2 \le 2Ht^{2H-1}\int_0^t \|\sigma(s)\|^2_{L^2}\mathrm{d}s$.

H3) The linear stochastic differential inclusions (3) are controllable on $J$ if and only if the controllability Grammian matrix

$$\begin{aligned} W = {} & \int_0^T (T-s)^{q-1}[E_{q,q}(A(T-s)^q)B] \\ & \times [E_{q,q}(A(T-s)^q)B]^*\mathrm{d}s \end{aligned}$$

is positive definite, for some $T > 0$ (see [20]).

H4) The multifunction $G : J \times \mathbf{R}^n \longrightarrow \mathcal{P}_{cp}(\mathbf{R}^n)$ has the property that $G(\cdot, x) : J \longrightarrow \mathcal{P}_{cp}(\mathbf{R}^n)$ is measurable for each $x \in \mathbf{R}^n$.

H5) There exists a non-negative function $m \in L^2(J)$ such that

$$H_d(G(t, x), G(t, y)) \le m(t)\|x - y\|^2$$

for every $x, y \in \mathbf{R}^n$, and

$$d(0, G(t, 0)) \le m(t)$$

a.e. $t \in J$.

For convenience, let us introduce the following constants $a_1 = \sup\|E_{q,q}(A(T-s)^q)\|^2$, $a_2 = \sup\|E_q(AT^q)x_0\|^2$, $l = \|W^{-1}\|^2$.

## III. MAIN RESULTS

In this section, we discuss the controllability criteria of fractional order stochastic differential inclusions with fBm. The fixed point technique is effectively used to study the controllability of nonlinear systems. The essential part of this method is to guarantee the existence of an invariant subset for an appropriate nonlinear operator. Due to their importance, several researchers have used different kinds of fixed point theorems. Here the controllability results are obtained by adopting Bohnenblust-Karlin fixed point theorem for the convex case and the Covitz-Nadler for the nonconvex case.

**Theorem 2 (Convex Case).** Suppose that the hypotheses H1)-H3) are satisfied, then the system (1) is controllable on $J$, provided that the following holds

$$\begin{aligned}1 > {} & 4a_2\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2\|B^*\|^2l\right)+16\frac{T^{2q}}{q^2}\\ & \times\ a_1^2\|B\|^2l\|B^*\|^2\|x_1\|^2+4\frac{T^{2q}}{q^2}a_1M_f\\ & \times\ (1+\mathrm{E}\|x\|^2)\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2l\|B^*\|^2\right)\\ & +\ 4\frac{T^q}{q}a_1 2HT^{2H-1}\eta\\ & \times\ \left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2l\|B^*\|^2\right).\end{aligned} \tag{5}$$

**Proof.** For any arbitrary function $x \in \mathbf{R}^n$, we can define the control function $u_x(t)$

$$\begin{aligned}u_x(t) = {} & B^*E_{q,q}(A^*(T-t)^q)W^{-1}\Big\{x_1-E_q(AT^q)x_0\\ & -\int_0^T(T-s)^{q-1}E_{q,q}(A(T-s)^q)\\ & \times\left[f(s,x(s))+\int_0^s\sigma(\theta)\mathrm{d}W^H_{(\theta)}\right]\mathrm{d}s\Big\}\end{aligned}$$

where $t \in J,\ \sigma \in N_{G,x}$. Using the above control, we show that the operator $\Phi : \mathcal{B} \longrightarrow \mathcal{P}(\mathcal{B})$, defined as

$$\begin{aligned}\Phi(x) = {} & \Big\{\Psi \in \mathcal{B} : \Psi(t) = E_q(At^q)x_0\\ & +\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\\ & \times\left[Bu(s)+f(s,x(s))+\int_0^s\sigma(\theta)\mathrm{d}W^H_{(\theta)}\right]\mathrm{d}s,\\ & \qquad t \in J,\ \sigma \in N_{G,x}\Big\}\end{aligned}$$

has a fixed point $x$, which is a solution of the system (1). We observe that $x_1 \in (\Phi x)(T)$ which means that $u_x$ steers the system (1) from $x_0$ to $x_1$ in finite time $T$. This implies that system (1) is controllable on $J$.

We now show that $\Phi$ satisfies all the conditions of Lemma 3. For the sake of convenience, we subdivide the proof into four steps.

**Step 1.** $\Phi$ is convex, for each $x \in \mathcal{B}$.

In fact, if $\Psi_1, \Psi_2 \in \Phi(x)$, then there exists $\sigma_1, \sigma_2 \in N_{G,x}$ such that for each $t \in J$, we have

$$\begin{aligned}\Psi_i(t) = {} & E_q(At^q)x_0+\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\\ & \times BB^*E_{q,q}(A^*(T-s)^q)W^{-1}\Big\{x_1\\ & -E_q(AT^q)x_0-\int_0^T(T-s)^{q-1}\\ & \times E_{q,q}(A(T-s)^q)\Big[f(s,x(s))\\ & +\int_0^s\sigma_i(\theta)\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s\Big\}(s)\mathrm{d}s\\ & +\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\Big[f(s,x(s))\\ & +\int_0^s\sigma_i(\theta)\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s,\ i=1,2.\end{aligned}$$

Let $0 \le \lambda \le 1$, then for each $t \in J$, we have

$$\begin{aligned}& [\lambda\Psi_1+(1-\lambda)\Psi_2](t)\\ & = E_q(At^q)x_0+\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\\ & \times BB^*E_{q,q}(A^*(T-s)^q)W^{-1}\Big\{x_1\\ & -E_q(AT^q)x_0-\int_0^T(T-s)^{q-1}\\ & \times E_{q,q}(A(T-s)^q)\Big[f(s,x(s))\\ & +\int_0^s[\lambda\sigma_1(\theta)+(1-\lambda)\sigma_2(\theta)]\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s\Big\}(s)\mathrm{d}s\\ & +\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\Big[f(s,x(s))\\ & +\int_0^s[\lambda\sigma_1(\theta)+(1-\lambda)\sigma_2(\theta)]\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s.\end{aligned}$$

It is easy to see that $N_{G,x}$ is convex since $G$ has convex values. So, $\lambda\sigma_1+(1-\lambda)\sigma_2 \in N_{G,x}$. Thus, $\lambda\Psi_1+(1-\lambda)\Psi_2 \in \Phi(x)$.

**Step 2.** For each positive number $\rho > 0$, let $\mathcal{B}_\rho = \{x \in \mathcal{B} : \|x\|^2_{\mathcal{B}} \le \rho\}$. Obviously, $\mathcal{B}_\rho$ is a bounded, closed and convex set of $\mathcal{B}$. We claim that there exists a positive number $\rho$ such that $\Phi(\mathcal{B}_\rho) \subset \mathcal{B}_\rho$.

If this is not true, then for each positive number $\rho$, there exists a function $x^\rho \in \mathcal{B}_\rho$, but $\Phi(x^\rho) \notin \mathcal{B}_\rho$ i.e. $\|\Phi(x^\rho)\|^2_{\mathcal{B}} \equiv \sup\{\|\Psi\|^2_{\mathcal{B}} : \Psi^\rho \in (\Phi x^\rho)\} > \rho$ and

$$\begin{aligned}\Psi^\rho(t) = {} & E_q(At^q)x_0+\int_0^t(t-s)^{q-1}E_{q,q}(A(T-s)^q)\\ & \times\left[Bu^\rho_x(s)+f(s,x^\rho(s))+\int_0^s\sigma^\rho(\theta)\mathrm{d}W^H_{(\theta)}\right]\mathrm{d}s\end{aligned}$$

for some $\sigma^\rho \in N_{G,x^\rho}$. Using H2) we have

$$\begin{aligned}& \mathrm{E}\|u_x(t)\|^2\\ & \le 4\|B^*\|^2\|E_{q,q}(A^*(T-t)^q)\|^2\left\|W^{-1}\right\|^2\\ & \times\Big\{\|x_1\|^2+\|E_q(AT^q)x_0\|^2\\ & +\mathrm{E}\Big\|\int_0^T(T-s)^{q-1}\end{aligned}$$

$$
\begin{aligned}
&\times E_{q,q}(A(T-s)^q)f(s,x(s))\mathrm{d}s\Big\|^2 \\
&+\mathrm{E}\Big\|\int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q) \\
&\times\left(\int_0^s \sigma(\theta)\mathrm{d}W^H_{(\theta)}\right)\mathrm{d}s\Big\|^2\Big\} \\
&\leq 4\left\|B^*\right\|^2 a_1 l\|x_1\|^2 \\
&+4\left\|B^*\right\|^2 a_1 l a_2 + 4\left\|B^*\right\|^2 a_1^2 l\frac{T^{2q}}{q^2}M_f \\
&\times(1+\mathrm{E}\|x\|^2)+4\left\|B^*\right\|^2 a_1^2 l 2HT^{2H-1} \\
&\times\frac{T^q}{q}\int_0^T (T-s)^{q-1}\varphi_\rho(s)\mathrm{d}s,
\end{aligned}
$$

and, we find that

$$
\begin{aligned}
\rho &< \mathrm{E}\|(\Phi x^\rho)(t)\|^2 \\
&\leq 4\mathrm{E}\left\|E_q(At^q)x_0\right\|^2 \\
&+4\mathrm{E}\Big\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q) \\
&\times Bu^\rho_x(s)\mathrm{d}s\Big\|^2 \\
&+4\mathrm{E}\Big\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q) \\
&\times f(s,x^\rho(s))\mathrm{d}s\Big\|^2 \\
&+4\mathrm{E}\Big\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q) \\
&\times\left(\int_0^s \sigma^\rho(\theta)\mathrm{d}W^H_{(\theta)}\right)\mathrm{d}s\Big\|^2 \\
&\leq 4a_2+16\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\|x_1\|^2 \\
&+16\frac{T^{2q}}{q^2}a_1^2 a_2\|B\|^2 l\|B^*\|^2+4\frac{T^{2q}}{q^2}a_1\|B\|^2 \\
&\times\left(4\|B^*\|^2 a_1^2 l\frac{T^{2q}}{q^2}M_f(1+\mathrm{E}\|x\|^2)\right) \\
&+4\frac{T^{2q}}{q^2}a_1\|B\|^2 \\
&\times\Big(4\|B^*\|^2 a_1^2 l\frac{T^q}{q}2HT^{2H-1} \\
&\times\int_0^T (T-s)^{q-1}\varphi_\rho(s)\mathrm{d}s\Big) \\
&+4\frac{T^{2q}}{q^2}a_1 M_f(1+\mathrm{E}\|x\|^2) \\
&+4\frac{T^q}{q}a_1 2HT^{2H-1}\int_0^T (T-s)^{q-1}\varphi_\rho(s)\mathrm{d}s \\
&\leq 4a_2\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right)+16\frac{T^{2q}}{q^2}a_1^2 \\
&\times\|B\|^2 l\|B^*\|^2\|x_1\|^2+4\frac{T^{2q}}{q^2}a_1 M_f \\
&\times(1+\mathrm{E}\|x\|^2)\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right) \\
&+4\frac{T^q}{q}a_1 2HT^{2H-1}\int_0^T (T-s)^{q-1}\varphi_\rho(s)\mathrm{d}s \\
&\times\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right).
\end{aligned}
$$

Dividing both sides of the above inequality by $\rho$ and taking limit as $\rho \longrightarrow \infty$, using H1) we get

$$
\begin{aligned}
1 \leq\ & 4a_2\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right) \\
&+16\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\|x_1\|^2+4\frac{T^{2q}}{q^2}a_1 M_f \\
&\times(1+\mathrm{E}\|x\|^2)\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right) \\
&+4\frac{T^q}{q}a_1 2HT^{2H-1}\eta \\
&\times\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right),
\end{aligned}
$$

which is a contradiction to (5). Hence, for some $\rho > 0$, $\Phi(\mathcal{B}_\rho)\subset\mathcal{B}_\rho$.

**Step 3.** Compactness of $\Phi$.

To prove this, we first prove that the set $\Phi(\mathcal{B}_\rho)$ is relatively compact in $\mathcal{B}_\rho$. Subsequently, we show that $\Phi(\mathcal{B}_\rho)$ is uniformly bounded. Note that by using the same method as in Step 2, it can be manifested that the operator $\Phi$ is uniformly bounded that is

$$
\begin{aligned}
&4a_2\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right)+16\frac{T^{2q}}{q^2}a_1^2 \\
&\quad\times\|B\|^2 l\|B^*\|^2\|x_1\|^2+4\frac{T^{2q}}{q^2}a_1 M_f \\
&\quad\times(1+\mathrm{E}\|x\|^2)\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right) \\
&\quad+4\frac{T^q}{q}a_1 2HT^{2H-1}\eta \\
&\quad\times\left(1+4\frac{T^{2q}}{q^2}a_1^2\|B\|^2 l\|B^*\|^2\right)<\infty,
\end{aligned}
$$

the set $\Phi(\mathcal{B}_\rho)$ is relatively compact. Finally, we prove that $\Phi(\mathcal{B}_\rho)$ is equicontinuous. For any $x\in\mathcal{B}_\rho$ and $t_1,t_2\in J$ with $0<t_1<t_2\leq T$, we get

$$
\begin{aligned}
&\mathrm{E}\|\Psi(t_1)-\Psi(t_2)\|^2 \\
&\quad\leq 7\mathrm{E}\left\|[E_q(At_1^q)-E_q(At_2^q)]x_0\right\|^2 \\
&\quad+7\mathrm{E}\Big\|\int_{t_1}^{t_2}(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)Bu_x(s)\mathrm{d}s\Big\|^2 \\
&\quad+7\mathrm{E}\Big\|\int_0^{t_1}\Big[(t_1-s)^{q-1}E_{q,q}(A(t_1-s)^q) \\
&\quad-(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)\Big]Bu_x(s)\mathrm{d}s\Big\|^2 \\
&\quad+7\mathrm{E}\Big\|\int_{t_1}^{t_2}(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)
\end{aligned}
$$

$$
\begin{aligned}
&\times f(s,x(s))\mathrm{d}s\Big\|^2\\
&+7\mathrm{E}\Big\|\int_0^{t_1}\Big[(t_1-s)^{q-1}E_{q,q}(A(t_1-s)^q)\\
&-(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)\Big]\\
&\times f(s,x(s))\mathrm{d}s\Big\|^2+7\mathrm{E}\Big\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\\
&\times E_{q,q}(A(t_2-s)^q)\left(\int_0^s\sigma(\theta)\mathrm{d}W^H_{(\theta)}\right)\mathrm{d}s\Big\|^2\\
&+7\mathrm{E}\Big\|\int_0^{t_1}\Big[(t_1-s)^{q-1}E_{q,q}(A(t_1-s)^q)\\
&-(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)\Big]\\
&\times\left(\int_0^s\sigma(\theta)\mathrm{d}W^H_{(\theta)}\right)\mathrm{d}s\Big\|^2\\
&\le 7\mathrm{E}\,\|[E_q(At_1^q)-E_q(At_2^q)]x_0\|^2\\
&+\frac{7(t_2-t_1)^q}{q}\int_{t_1}^{t_2}(t_2-s)^{q-1}\\
&\times\|E_{q,q}(A(t_2-s)^q)\|^2\|B\|^2\mathrm{E}\|u_x(s)\|^2\mathrm{d}s\\
&+7t_1\int_0^{t_1}\Big\|[(t_1-s)^{q-1}E_{q,q}(A(t_1-s)^q)\\
&-(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)]\Big\|^2\|B\|^2\\
&\times\mathrm{E}\|u_x(s)\|^2\mathrm{d}s+\frac{7(t_2-t_1)^q}{q}\int_{t_1}^{t_2}(t_2-s)^{q-1}\\
&\times\|E_{q,q}(A(t_2-s)^q)\|^2M_f(1+\mathrm{E}\|x\|^2)\mathrm{d}s\\
&+7t_1\int_0^{t_1}\Big\|[(t_1-s)^{q-1}E_{q,q}(A(t_1-s)^q)\\
&-(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)]\Big\|^2\\
&\times M_f(1+\mathrm{E}\|x\|^2)\mathrm{d}s+\frac{7(t_2-t_1)^q}{q}\\
&\times\int_{t_1}^{t_2}(t_2-s)^{q-1}\|E_{q,q}(A(t_2-s)^q)\|^2\mathrm{d}s\\
&\times 2HT^{2H-1}\int_0^s\mathrm{E}\|\sigma(\theta)\|^2\mathrm{d}\theta\\
&+7t_1\int_0^{t_1}\Big\|[(t_1-s)^{q-1}E_{q,q}(A(t_1-s)^q)\\
&-(t_2-s)^{q-1}E_{q,q}(A(t_2-s)^q)]\Big\|^2\mathrm{d}s\\
&\times 2HT^{2H-1}\int_0^s\mathrm{E}\|\sigma(\theta)\|^2\mathrm{d}\theta.
\end{aligned}
$$

As $t_1\longrightarrow t_2$, the right-hand side of the above inequality tends to zero. An application of the Arzela-Ascoli theorem yields that $\Phi$ maps $\mathcal{B}_\rho$ into $\mathcal{B}$, that is $\Phi:\mathcal{B}_\rho\longrightarrow\mathcal{P}(\mathcal{B})$ is a compact operator. Thus $\Phi(\mathcal{B}_\rho)$ is relatively compact.

**Step 4.** $\Phi$ is upper semi-continuous on $\mathcal{B}_\rho$.

Let $x^n\longrightarrow x^*$, as $n\longrightarrow\infty$ and $\Psi^n\longrightarrow\Psi^*$ as $n\longrightarrow\infty$. We need to show that $\Psi^*\in\Phi(x^*)$. Since $\Psi^n\in\Phi(x^n)$ means that there exists $\sigma^n\in N_{G,x^n}$ such that

$$
\begin{aligned}
\Psi^n(t)=\ &E_q(At^q)x_0+\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\\
&\times BB^*E_{q,q}(A^*(T-s)^q)W^{-1}\Big\{x_1\\
&-E_q(AT^q)x_0-\int_0^T(T-s)^{q-1}\\
&\times E_{q,q}(A(T-s)^q)\Big[f(s,x^n(s))\\
&+\int_0^s\sigma^n(\theta)\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s\Big\}(s)\mathrm{d}s\\
&+\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\\
&\times\left[f(s,x^n(s))+\int_0^s\sigma^n(\theta)\mathrm{d}W^H_{(\theta)}\right]\mathrm{d}s. \qquad (6)
\end{aligned}
$$

We must show that there exists $\sigma^*\in N_{G,x^*}$ such that

$$
\begin{aligned}
\Psi^*(t)=\ &E_q(At^q)x_0+\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\\
&\times BB^*E_{q,q}(A^*(T-s)^q)W^{-1}\Big\{x_1\\
&-E_q(AT^q)x_0-\int_0^T(T-s)^{q-1}\\
&\times E_{q,q}(A(T-s)^q)\Big[f(s,x^*(s))\\
&+\int_0^s\sigma^*(\theta)\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s\Big\}(s)\mathrm{d}s\\
&+\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\\
&\times\left[f(s,x^*(s))+\int_0^s\sigma^*(\theta)\mathrm{d}W^H_{(\theta)}\right]\mathrm{d}s.
\end{aligned}
$$

Now, we consider the continuous operator

$$\Lambda:L_2(J,\mathbf{R}^n)\longrightarrow\mathcal{B},\sigma\mapsto\Lambda(\sigma)(t)$$

such that,

$$
\begin{aligned}
\Lambda(\sigma)(t)=&\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)\\
&\times\Big[\int_0^s\sigma(\theta)\mathrm{d}W^H_{(\theta)}-BB^*E_{q,q}(A^*(T-s)^q)\\
&\times W^{-1}\int_0^T(T-s)^{q-1}E_{q,q}(A(T-s)^q)\\
&\times\left(\int_0^s\sigma(\theta)\mathrm{d}W^H_{(\theta)}\right)\mathrm{d}s\Big](s)\mathrm{d}s.
\end{aligned}
$$

From Lemma 2, it follows that $\Lambda\circ N_G$ is a closed graph operator. Clearly, for each $t\in J$, we have

$$
\begin{aligned}
&\Big(\Psi^n(t)-E_q(At^q)x_0-\int_0^t(t-s)^{q-1}\\
&\quad\times E_{q,q}(A(t-s)^q)\Big[Bu^n(s)+f(s,x^n(s))\Big]\mathrm{d}s\Big)\\
&\quad\in\Lambda(N_{G,x^n}).
\end{aligned}
$$

Since $y^n\longrightarrow y^*$, it follows from Lemma 2 that, for some $y^*\in N_{G,x^*}$, we have

$$
\begin{aligned}
&\Big(\Psi^*(t)-E_q(At^q)x_0-\int_0^t(t-s)^{q-1}\\
&\quad\times E_{q,q}(A(t-s)^q)\Big[Bu^*(s)+f(s,x^*(s))\Big]\mathrm{d}s\Big)\\
&\quad\in\Lambda(N_{G,x^*}).
\end{aligned}
$$

Clearly, for each $t \in J$, we have

$$\begin{aligned}&\Big\|\Big(\Psi^n(t) - E_q(At^q)x_0 - \int_0^t (t-s)^{q-1} \\ &\quad\times E_{q,q}(A(t-s)^q)\Big[Bu^n(s) + f(s, x^n(s))\Big]\mathrm{d}s\Big) \\ &\quad- \Big(\Psi^*(t) - E_q(At^q)x_0 - \int_0^t (t-s)^{q-1} \\ &\quad\times E_{q,q}(A(t-s)^q)\Big[Bu^*(s) + f(s, x^*(s))\Big]\mathrm{d}s\Big)\Big\|^2_{\mathcal{B}} \\ &\quad\longrightarrow 0,\end{aligned}$$

as $n \longrightarrow \infty$. From Lemma 1 we can conclude that $\Phi$ is upper semi-continuous. As a consequence of Lemma 3, we deduce that $\Phi$ has a fixed point which is the solution of the system (1), and it is easy to verify that $x(T) = x_1$. Hence the system (1) is controllable on $J$. □

**Theorem 3 (Non-Convex Case).** Assume that conditions H3)-H5) are satisfied, then the system (1) has atleast one solution in $J$, provided that

$$8\frac{T^{2q}}{q^2}a_1 m(t)\left(1 + T2HT^{2H-1}\right) < 1. \tag{7}$$

**Proof.** Under the assumption H5) it is easy to see that for each $x \in \mathcal{B}$, the set $N_{G,x}$ is nonempty. Therefore, $G$ has a nonempty measurable selection (by Theorem 1). We shall show that $\Phi$ defined in Theorem 2 satisfies the assumption of Lemma 4. The proof will be given in two steps.

**Step 1.** $\Phi(x) \in \mathcal{P}_{cl}(\mathcal{B})$ for each $x \in \mathcal{B}$.

Indeed, let $(\Psi^n)_{n\geq 0} \in \Phi(x)$ such that $\Psi^n \longrightarrow \Psi$. Then, $\Psi \in \mathcal{B}$ and there exists $\sigma^n \in N_{G,x}$ such that, for each $t \in J$, $\Psi^n(t)$ is defined in (6). Using H5) we have for a.e. $t \in J$

$$|\sigma^n(t)| \leq m(t)\|x\|^2 + m(t),\ n \in \mathbf{N}.$$

The Lebesgue dominated convergence theorem implies that

$$\|\sigma^n - \sigma\|_{L^2} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence $\sigma \in N_{G,x}$. Then, for each $t \in J$, $\Psi^n(t) \longrightarrow \Psi(t)$, where

$$\begin{aligned}\Psi(t) &= E_q(At^q)x_0 \\ &\quad+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\Big[Bu_x(s) \\ &\quad+ f(s, x(s)) + \int_0^s \sigma(\theta)\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s.\end{aligned} \tag{8}$$

So, $\Psi \in \Phi(x)$.

**Step 2.** There exists $\gamma < 1$ such that $H_d(\Phi(x), \Phi(y)) \leq \gamma\|x - y\|_{\mathcal{B}}$ for each $x, y \in \mathcal{B}$. Let $x, y \in \mathcal{B}$ and $\Psi \in \Phi(x)$. Then, there exists $\sigma \in N_{G,x}$ such that $\Psi(t)$ is defined in (8). From H5), it follows that

$$H_d(G(t, x(t)), G(t, y(t))) \leq m(t)\|x(t) - y(t)\|^2.$$

Hence, there exists $\omega \in N_{G,y}$ such that

$$\|\sigma(t) - \omega(t)\|^2 \leq m(t)\|x(t) - y(t)\|^2,\ t \in J.$$

Consider $U : J \longrightarrow \mathcal{P}(\mathbf{R}^n)$ given by $U(t) = \{\omega(t)|\omega : J \longrightarrow \mathbf{R}^n$ is Lebesgue integrable and $\|\sigma(t) - \omega(t)\|^2 \leq m(t)\|x(t) - y(t)\|^2\}$. Since the multivalued operator $U(t) \cap G(t, y(t))$ is measurable (Proposition 1), there exists a functions $\overline{\sigma}(t)$ which is a measurable selection for $U$. So, $\overline{\sigma}(t) \in N_{G,y}$, and for each $t \in J$,

$$\|\sigma(t) - \overline{\sigma}(t)\|^2 \leq m(t)\|x(t) - y(t)\|^2.$$

Let us define

$$\begin{aligned}\overline{\Psi}(t) &= E_q(At^q)x_0 \\ &\quad+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\Big[Bu_y(s) \\ &\quad+ f(s, y(s)) + \int_0^s \overline{\sigma}(\theta)\mathrm{d}W^H_{(\theta)}\Big]\mathrm{d}s.\end{aligned}$$

Then, for each $t \in J$, we get

$$\begin{aligned}&\mathrm{E}\|\Psi(t) - \overline{\Psi}(t)\|^2 \\ &\leq 4\mathrm{E}\Big\|\int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q) \\ &\quad\times[f(s, y(s)) - f(s, x(s))]\mathrm{d}s\Big\|^2 \\ &\quad+ 4\mathrm{E}\Big\|\int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q) \\ &\quad\times\left(\int_0^s\Big[\overline{\sigma}(\theta) - \sigma(\theta)\Big]\mathrm{d}W^H_{(\theta)}\right)\mathrm{d}s\Big\|^2 \\ &\quad+ 4\mathrm{E}\Big\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q) \\ &\quad\times[f(s, x(s)) - f(s, y(s))]\mathrm{d}s\Big\|^2 \\ &\quad+ 4\mathrm{E}\Big\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q) \\ &\quad\times\left(\int_0^s[\sigma(\theta) - \overline{\sigma}(\theta)]\mathrm{d}W^H_{(\theta)}\right)\mathrm{d}s\Big\|^2 \\ &\leq 4\frac{T^{2q}}{q^2}a_1\mathrm{E}\|f(t, y(t)) - f(t, x(t))\|^2 \\ &\quad+ 4\frac{T^{2q}}{q^2}Ta_1 2HT^{2H-1}\mathrm{E}\|\sigma(t) - \overline{\sigma}(t)\|^2 \\ &\quad+ 4\frac{T^{2q}}{q^2}a_1\mathrm{E}\|f(t, x(t)) - f(t, y(t))\|^2 \\ &\quad+ 4\frac{T^{2q}}{q^2}Ta_1 2HT^{2H-1}\mathrm{E}\|\overline{\sigma}(t) - \sigma(t)\|^2 \\ &\leq 4\frac{T^{2q}}{q^2}a_1 m(t)\|x - y\|^2 \\ &\quad+ 4\frac{T^{2q}}{q^2}Ta_1 2HT^{2H-1}m(t)\|x - y\|^2 \\ &\quad+ 4\frac{T^{2q}}{q^2}a_1 m(t)\|x - y\|^2 \\ &\quad+ 4\frac{T^{2q}}{q^2}Ta_1 2HT^{2H-1}m(t)\|x - y\|^2 \\ &\leq 8\frac{T^{2q}}{q^2}a_1 m(t)\left(1 + T2HT^{2H-1}\right)\|x - y\|^2.\end{aligned}$$

Thus, for each $t \in J$, we get

$$\mathrm{E}\|\Psi - \overline{\Psi}\|_{\mathcal{B}}^2 \le 8\frac{T^{2q}}{q^2}a_1 m(t) \times \left(1 + T2HT^{2H-1}\right)\|x-y\|_{\mathcal{B}}^2.$$

By an analogous relation obtained by interchanging the roles of $x$ and $y$, it follows that

$$H_d(\Phi(x), \Phi(y)) \le 8\frac{T^{2q}}{q^2}a_1 m(t) \times \left(1 + T2HT^{2H-1}\right)\|x-y\|_{\mathcal{B}}^2.$$

By (7), $\Phi$ is a contraction and thus, by Lemma 4, $\Phi$ has a fixed point $x$ which is the solution of (1) on $J$, and it is easy to verify that $x(T) = x_1$. Hence the fractional order system (1) is controllable on $J$. □

**Remark 2.** Existence of solutions for integer order stochastic differential inclusions without control vector and fractional Brownian motion have been investigated in [15]. Also the controllability problem for fractional dynamical systems without stochastic differential equations has been studied in [10]. Since fBm has dependent increments, it is an interesting generalization of ordinary Brownian motion to model the noise process in many applications such as finance, network simulations and environmental processes. So, it is significant to study the controllability of fractional order stochastic differential inclusions with fBm due to the potential applications. It should be mentioned that different from literature, this paper makes use of stochastic analysis technique with fBm and the controllability Grammian martrix which is formulated using Mittag-Leffler matrix function. The main advantage of the proposed technique is the utilization of fixed point theorem for both cases of the multivalued map. Moreover, Covitz-Nadler fixed point theorem is utilized for the nonconvex case of the multivalued map to establish controllability of fractional stochastic systems with fBm.

## IV. EXAMPLE

In this section an example is illustrated to show the effectiveness of the proposed technique.

As an application of the derived results, we consider the fractional Harmonic Oscillator equation[30]

$$\left(m^C D^{2q} + k\right)x(t) = 0,$$

where $k$ and $m$ are appropriate constants. Introducing a control variable and a nonlinear forcing term, we get the following controlled fractional Harmonic Oscillator equation with Brownian motion:

$$^C D^{2q}x(t) + x(t) = u(t) + \frac{x(t)}{1+3x(t)} + \frac{5x(t)}{1+x(t)}, \quad t \in J,$$

where $x(t)$ specifies the position of the particle or Oscillator at time $t$, $u(t)$ is a control term, $\frac{x(t)}{1+3x(t)}$ is a nonlinear forcing term and $\frac{5x(t)}{1+x(t)}$ describes a Brownian motion in an external quadratic potential. Introduce the auxiliary variables $x_1(t) = x(t)$ and $x_2(t) = {}^C D^q x_1(t)$. Then

$$\begin{aligned} {}^C D^q x_1(t) &= {}^C D^q x(t) = x_2(t), \\ {}^C D^q x_2(t) &= {}^C D^{2q} x(t) \\ &= -x_1(t) + u(t) + \frac{x_1(t)}{1+3x_1(t)} \\ &\quad + \frac{5x_1(t)}{1+x_1(t)}, \ t \in J. \end{aligned}$$

The above system can be rewritten as follows

$$\begin{aligned} {}^C D^q x(t) = &\ Ax(t) + Bu(t) + f(t, x(t)) \\ &+ \int_0^t G(s, x(s))\mathrm{d}W_{(s)}^H, \\ &t \in J \end{aligned} \tag{9}$$

with

$$q = \frac{1}{2},\ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ f(t, x(t)) = \begin{pmatrix} 0 \\ \frac{x_1(t)}{1+3x_1(t)} \end{pmatrix}$$

$$\text{and} \quad G(t, x(t)) = \begin{pmatrix} \frac{5x_1(t)}{1+x_1(t)} \\ 0 \end{pmatrix}.$$

The Mittag-Leffler matrix function of the system is given by (see [8])

$$E_q(At^q) = \begin{pmatrix} \sum\limits_{j=0}^{\infty} \frac{(-1)^j t^{2jq}}{\Gamma[1+2jq]} & \sum\limits_{j=0}^{\infty} \frac{(-1)^j t^{(2j+1)q}}{\Gamma[1+(2j+1)q]} \\ -\sum\limits_{j=0}^{\infty} \frac{(-1)^j t^{(2j+1)q}}{\Gamma[1+(2j+1)q]} & \sum\limits_{j=0}^{\infty} \frac{(-1)^j t^{2jq}}{\Gamma[1+2jq]} \end{pmatrix}.$$

By simple matrix calculations, one can see that the controllability matrix

$$\begin{aligned} W &= \int_0^T (T-s)^{q-1}[E_{q,q}(A(T-s)^q)B] \\ &\quad \times [E_{q,q}(A(T-s)^q)B]^* \mathrm{d}s \\ &= \int_0^T (T-s)^{q-1} \\ &\quad \times \begin{pmatrix} S_1^2 + S_2^2 & S_1S_3 + S_2S_4 \\ S_1S_3 + S_2S_4 & S_1^2 + S_2^2 \end{pmatrix} \mathrm{d}s \\ &= \begin{pmatrix} 3.5363 & 0 \\ 0 & 3.5363 \end{pmatrix} > 0 \end{aligned}$$

is positive definite. Here

$$S_1 = S_4 = \sum_{j=0}^{\infty} \frac{(-1)^j (T-s)^{2jq}}{\Gamma[(1+2j)q]},$$

$$S_2 = -S_3 = -\sum_{j=0}^{\infty} \frac{(-1)^j (T-s)^{(2j+1)q}}{\Gamma[2q(j+1)]}.$$

Moreover, it is easy to show that for all $x \in \mathbf{R}^2$, $\|f(t, x(t))\|^2 \le \frac{\|x_1\|^2}{1+9\|x_1(t)\|^2}$ and $\|G(t, x(t))\|^2 \le \frac{25\|x_1(t)\|^2}{1+\|x_1(t)\|^2}$. One can see that the inequalities (5) and (7) hold and all other conditions stated in Theorems 2 and 3 are satisfied. Hence, the fractional order stochastic differential inclusions with fBm (9) are completely controllable on $J$.

## V. Conclusion and Future Work

This paper has advanced the controllability result of fractional order stochastic differential inclusions with fBm in finite dimensional space. The results have been obtained upon suitable fixed point theorems, namely the Bohnenblust-Karlin fixed point theorem for the convex case and the Covitz-Nadler for the nonconvex case. Finally, a numerical example has been given to validate the efficiency of the proposed theoretical results.

In recent years, the applications of an integro-differential equations model play an important role in many areas from science and engineering, particularly in the analysis of electrical circuit. Inspired by the applications of fractional order system and integro-differential equation, solving the fractional stochastic integro-differential equations with nonlocal condition deserves our future concern.

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