A Collection of Optimal Control Test Problems

John T. Betts [1]

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[1]Applied Mathematical Analysis, LLC; <http://www.appliedmathematicalanalysis.com/>

This is not a book for somebody who wants to learn about optimal control. However, for the new student just learning the field it provides a set of test problems that can be used to test understanding. For the analyst doing advanced research and development of new computational algorithms it provides a comprehensive collection of problems that can be used to verify whether a new approach is efficient and robust on more than a few toy problems. Each problem in the collection is presented in a consistent format, and includes a computed solution. Every problem has been implemented in software that is available as open source code. Every problem also has an external reference with additional motivation, discussion, and formulation details.

I have spent my entire career working in a industrial environment, first at The Aerospace Corporation and then at the Boeing Company. The typical industrial application is often described as “messy,” may be poorly posed, implemented by a large team, at great expense in both manpower and budget. My activities focused on research and development of new methods that can be used improve the efficiency and/or accuracy of “real world” problems. In contrast, while the focus of an academic environment can involve new ideas and techniques, there ultimately must be a focus on teaching students. As such, good ideas developed in this setting often are untested on real world problems, and consequently the good methods are overlooked in industrial applications. A primary goal of my career was to bridge this gap and incorporate good ideas appearing in academic research into real world applications. Indeed my own approach to solving an optimal control problem reflects the transition from “old slow” methods to “new fast” algorithms. My early publications in optimal control utilized a generalized reduced gradient (GRG) algorithm for solving nonlinear programming problems, in conjuction with a “shooting method” for solving the differential equations. More recent efforts incorporate sparse nonlinear techniques — first a sparse sequential quadratic programming (SQP) algorithm, and then a sparse primal-dual interior point algorithm. When used in conjunction with a direct transcription discretization technique, these new methods demonstrate dramatically improved speed and reliability. However, in order to bridge this gap it has been imperative to collaborate with people on both sides of the fence. Dr. Wayne Hallman and his colleagues at the Aerospace Corporation have provided invaluable insight and feedback on “real world” problems for more than twenty years. Similarly Dr. Klaus Well, Mr. Andreas Wiegand and their coworkers at Astos Solutions, GmbH have gratefully shared their industrial expertise. My collaboration with Dr. Stephen L. Campbell and his doctoral students over the past twenty years has lead to significant developments in mesh refinement, optimal control theory, and more recently in the development of optimal control for delay equations. My collaboration with Dr. Raymond J. Spiteri and his students, has emphasized modern methods from computer science that can greatly enhance the software tools being used.

I would be remiss if a I failed to acknowledge the interaction and valuable discussions I have been fortunate to have with the following people:

Uri Ascher, Larry Biegler, Hans Georg Bock, Kathy Brenan, Roland Bulirsch, Christof Büskens, Kurt Chudej, Andrew Conn, John Dennis, Richard Epenoy, Roger Fletcher, Matthias Gerdts, Phillip Gill, William Hager, Matthias Heinkenschloss, Tim Kelley, Sven Leyffer, Helmut Maurer, Angelo Miele, Hans Josef Pesch, Linda Petzold, Anil V. Rao, Ekkehard Sachs, Gottfried Sachs, Roger Sargent, Michael Saunders, Oskar von Stryk, Phillipe Toint, and Margaret Wright.

The time and intellectual encouragement needed to prepare this book can be attributed to my business partner Dr. David Ferguson at Applied Mathematical Analysis, LLC (AMA) and would not be possible in a typical industrial environment. The $\mathbb{SOS}$

(Sparse Optimization Suite) software used to solve all of the test problems benefited greatly from the contributions of Dr. Paul Frank and Dr. Bill Huffman.

In spite of this effort, this collection can be improved. There are a rather small number of test problems in which the control appears linearly, leading to singular arc, and/or bang-bang solutions. There are no parameter estimation or inverse problems, which can be accommodated by the $\mathbb{SOS}$ tool. Test problems used for optimal control of delay equations are excluded from the suite. The software implementations use FORTRAN 90 as the language and an architecture suitable for $\mathbb{SOS}$ which will inevitably require some modification by scientists using different tools and computational environments. Although continuing collaborative efforts with Steve Campbell on DDE's, and Ray Spiteri in computer science may appear in a later revision of this work, at present they are absent. Nevertheless, hopefully this book will serve as a starting point, for future contributions from the entire community.

Finally, I thank my wife Jennifer for her love, patience, and support during completion of this book.

Contents

Chapter 1

Performance Testing

The development of a computational algorithm to solve a particular problem entails a number of important steps. Typically, the analyst first formulates the problem using the appropriate mathematical paradigm or framework. With a problem formulation in hand, a mathematical method or algorithm capable of solving the problem is postulated and/or selected. The method must then be implemented using a digital computer, often involving the selection of different computational environments and/or hardware. Finally, the approach is tested by solving the desired application.

Ideally, the analyst would like to select the *best* method to solve the problem. However, in practice just defining the *best* method is problematic for many reasons.

- Does *best* mean *fastest*? If so, does *fastest* mean "wall clock" time, or does it include the effort needed to formulate and implement the approach?
- Is *fastest* measured in time or is another measure, such as number of steps, more appropriate?
- If *fastest* is measured in time, what is the impact of different computer hardware?
- Does *best* mean *most accurate*? If so, what defines an accurate solution?
- What is the initial guess? When comparing methods, is the initial guess "consistent" in order to provide a "fair" comparison?

Even when an acceptable definition of *best* method can be postulated, it is challenging to demonstrate this attribute. In particular:

- Can the *best* performance be demonstrated on many problems or just one?
- Is there a standard format and/or formulation for a large suite of test problems?
- Is it possible to implement a standard format within time and budget constraints?

Historically, these diverse performance testing goals have demonstrated varying degrees of success. In the field of computational linear algebra the benchmark testing of the LINPACK [41] and LAPACK projects has been very productive. These studies have served to demonstrate both algorithmic and computational hardware performance.

The tests measure how fast a computer solves a dense n by n system of linear equations $Ax = b$, which is a common task in engineering. These benchmark testing efforts have been quite productive in part because it is relatively simple to specify the problems and ground rules.

For applications of moderate complexity useful test suites have been developed. Nonlinear programming algorithms are often tested on the collections in CUTE [25], Hock and Schittkowski [58], Hammes [54], and Moré, Garbow, and Hillstrom [71]. Similar collections have also been developed in other areas (e.g. EISPACK, ODEPACK, etc.)

Unfortunately, for many disciplines the sheer complexity of the problem precludes development of an extensive test suite. For example, it is not uncommon to spend years of time and budget to implement and execute a single practical application in computational fluid dynamics. Implementations of this type often involve different computer languages (e.g FORTRAN, C, Java, etc.) and may utilize multiple hardware platforms. Development of a test suite in this setting is simply impractical.

This document presents a collection of optimal control test problems. Given the broad applicability of optimal control problems, we hesitate to describe the formulation as “standard.” Instead, the basic elements are stated in a “common” format. Optimal control solution techniques are often classified as either “direct” or “indirect,” where the latter approach requires explicit construction of the necessary conditions for optimality. In most cases, the test problems are stated in a “direct” format, and consequently testing an indirect method will require specification of the optimality conditions by the analyst.

All test problems have been implemented using a software architecture developed for $\mathbb{SOS}$ (Sparse Optimization Suite). The test problem implementations for $\mathbb{SOS}$ are available as open source FORTRAN 90 code. Optimal solutions as computed by $\mathbb{SOS}$ are given for every problem in the test suite in addition to timing information. It is anticipated that a comprehensive comparison of results from other algorithms will require some conversion by the analyst, for example to utilize another language or environment (e.g. FORTRAN vs MATLAB). Chapter 2 presents the optimal control problem formulation format used throughout the document. A summary of the various characteristics of the test problems and procedures is given in Chapter 3. The complete suite of test problems is then defined in Chapters 4-74.

Chapter 2

Problem Formulation

2.1 The Optimal Control Problem

An optimal control problem can be formulated as a collection of N *phases*. Loosely speaking a phase describes a portion of the entire problem. In general, the independent variable t for *phase* k is defined in the region $t_I^{(k)} \leq t \leq t_F^{(k)}$. For many applications, the independent variable t is *time* and the phases are sequential, that is, $t_I^{(k+1)} = t_F^{(k)}$. However, neither of these assumptions is required. Within phase k, the dynamics of the system are described by a set of *dynamic* variables

$$\mathbf{z} = \begin{bmatrix} \mathbf{y}^{(k)}(t) \\ \mathbf{u}^{(k)}(t) \end{bmatrix} \tag{2.1}$$

made up of the $n_y^{(k)}$ *differential variables* and the $n_u^{(k)}$ *algebraic variables*, respectively. In addition, the dynamics may incorporate the $n_p^{(k)}$ *parameters* $\mathbf{p}^{(k)}$ that are independent of t.

Typically, the dynamics of the system are defined by a set of ordinary differential equations (ODEs) written in explicit form,

$$\dot{\mathbf{y}}^{(k)} = \mathbf{f}^{(k)}[\mathbf{y}^{(k)}(t), \mathbf{u}^{(k)}(t), \mathbf{p}^{(k)}, t], \tag{2.2}$$

where $\mathbf{y}^{(k)}$ is the $n_y^{(k)}$ dimension state vector. In addition, the solution must satisfy algebraic *path constraints* of the form

$$\mathbf{g}_L^{(k)} \leq \mathbf{g}^{(k)}[\mathbf{y}^{(k)}(t), \mathbf{u}^{(k)}(t), \mathbf{p}^{(k)}, t] \leq \mathbf{g}_U^{(k)}, \tag{2.3}$$

where $\mathbf{g}^{(k)}$ is a vector of size $n_g^{(k)}$, as well as simple bounds on the differential variables

$$\mathbf{y}_L^{(k)} \leq \mathbf{y}^{(k)}(t) \leq \mathbf{y}_U^{(k)} \tag{2.4}$$

and algebraic variables

$$\mathbf{u}_L^{(k)} \leq \mathbf{u}^{(k)}(t) \leq \mathbf{u}_U^{(k)}. \tag{2.5}$$

Observe that a *control variable* is an algebraic variable, whereas a *state variable* may be either differential or algebraic.

An equality constraint can be imposed if the upper and lower bounds are equal, e.g., $[g_L^{(k)}]_j = [g_U^{(k)}]_j$ for some j. In this case the dynamics are described by a set of differential-algebraic equations (DAE's). It follows that:

∘ the DAE's must be unchanged within a phase, and conversely ∘ different DAE's must be in different phases.

Using the phase structure formalism it is convenient to define quantities evaluated *over* the phase

$$\omega^{(k)} = \int_{t_I^{(k)}}^{t_F^{(k)}} w^{(k)} \left[\mathbf{y}^{(k)}(t), \mathbf{u}^{(k)}(t), \mathbf{p}^{(k)}, t\right] dt, \tag{2.6}$$

which involve the *quadrature functions* $w^{(k)}$. In contrast *point functions* can be evaluated at either end of the phase, that is

$$\psi_I^{(k)} = \psi \left[\mathbf{y}^{(k)}(t_I^{(k)}), \mathbf{u}^{(k)}(t_I^{(k)}), \mathbf{p}^{(k)}, t_I^{(k)}\right] \tag{2.7}$$

$$\psi_F^{(k)} = \psi \left[\mathbf{y}^{(k)}(t_F^{(k)}), \mathbf{u}^{(k)}(t_F^{(k)}), \mathbf{p}^{(k)}, t_F^{(k)}\right] \tag{2.8}$$

Typically the quadrature and point functions are used to impose boundary conditions of the form

$$\Psi_L \le \sum_{j=1}^{N} [a_j \psi_I^{(j)} + b_j \psi_F^{(j)} + c_j \omega^{(j)}] \le \Psi_U \tag{2.9}$$

for constants a_j, b_j, c_j. The same quantities can be used to define an *objective function*

$$J = \phi + L \tag{2.10}$$

where

$$L = \sum_{j=1}^{N} c_j \omega^{(j)} = \sum_{j=1}^{N} c_j \int_{t_I^{(j)}}^{t_F^{(j)}} w^{(j)} \left[\mathbf{y}^{(j)}(t), \mathbf{u}^{(j)}(t), \mathbf{p}^{(j)}, t\right] dt \tag{2.11}$$

$$\phi = \sum_{j=1}^{N} \left[a_j \psi_I^{(j)} + b_j \psi_F^{(j)}\right] \tag{2.12}$$

for constants a_j, b_j, c_j. As written, (2.10) is known as the *problem of Bolza*. When the function $\phi \equiv 0$ in the objective, we refer to this as the *problem of Lagrange* or, if there are no integral terms $\omega^{(j)} \equiv 0$, the optimization is termed the *problem of Mayer*.

The basic optimal control problem is to determine the $n_u^{(k)}$-dimensional control vectors $\mathbf{u}^{(k)}(t)$ and parameters $\mathbf{p}^{(k)}$ to minimize the performance index (2.10), and satisfy the differential equations (2.2), the path constraints (2.3), the simple bounds (2.4) and (2.5), in addition to the boundary conditions (2.9).

2.2 Notational Conventions

2.2.1 Problem Name

Each problem in the test suite is identified by a six character name. The first four characters are alphabetic and are derived from the problem name. The final two characters are numeric and identify the particular problem in a sequence. For example, a sequence of four problems identified as "Quadratic-Linear" are described on pages 233-234. The problems are denoted by the strings qlin01, qlin02, qlin03, and qlin04 respectively.

2.2.2 Problem Abstract

A brief abstract that describes the problem is given following the problem name. The information is displayed as in

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An early study of the dynamic maneuver of a spacecraft referred to as "aeroassisted plane change" is given in reference [4]. These examples can be considered a simplified version of the dynamics modeled in examples (5.1) and (5.2).

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which is the abstract that appears on page 33. When there are external references for the particular problem, this information is given in the abstract.

2.2.3 Phase Description

In general, the description of an optimal control problem requires information about each phase. The information needed to define a single phase is described in Sections (2.2.4) through Section (2.2.10). By convention this information is presented in the same <u>order</u> as the sections. So for example, after the phase title on phase one, the parameters are defined, followed by the independent variable on phase one, etc. Phase one information is followed by the phase two title, phase two parameters, etc. Consequently a complete problem description is of the form:

Phase 1 .... *Phase Title* .... Phase 1

⋮

Phase 2 .... *Phase Title* .... Phase 2

⋮

2.2.4 Parameters

Information about parameters is presented following the phase title. When there are no parameters on a phase, this information block is omitted. To illustrate, let us consider the second phase of problem aomp01 as it appears on page 25 which is replicated here.

Parameters: $(m_I^{(2)}, t_F^{(2)})$ ....

$1 \le m_I^{(2)}$	$1 \le t_F^{(2)} \le 4000$

In this example, two parameters are defined on the phase, namely $m_I^{(2)}$ and $t_F^{(2)}$. The first parameter which is the initial value of the variable m, i.e. $m_I^{(2)} = m[t_I^{(2)}]$, is bounded below as given by the equation $1 \le m_I^{(2)}$. The second parameter $t_F^{(2)}$ which is the final time of phase two, is bounded below and above by $1 \le t_F^{(2)} \le 4000$.

2.2.5 Independent Variable

Every phase must have an independent variable, and consequently this information is always presented. To illustrate, consider the second phase of problem aomp01 which appears on page 25 and is replicated here.

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(2)}$	$t = t_F^{(2)}$	sec

Information about the independent variable t is given for three distinct regions of the phase, namely the beginning, the interior, and the end. In this example, the initial value is fixed, i.e. $t = 0$. The final value which is free, must equal the parameter, i.e. $t = t_F^{(2)}$. The phase interior is defined when $0 < t < t_F^{(2)}$. Finally, the units for the variable t (time) are given in seconds.

2.2.6 Differential Variables

Information about the differential variables is given following the independent variable. Again consider the second phase of problem aomp01 which appears on page 25 as replicated here.

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = 60$ nm	$0 \le h \le 60$ nm	$h = 60$ nm	ft
$170^o \le \phi \le 200^o$	$170^o \le \phi \le 200^o$	$170^o \le \phi \le 200^o$	rad
$-20^o \le \theta \le 89^o$	$-20^o \le \theta \le 89^o$	$-20^o \le \theta \le 89^o$	rad
$25000 \le v \le 35000$	$25000 \le v \le 35000$	$25000 \le v \le 35000$	ft/sec
$-5^o \le \gamma \le 0^o$	$-5^o \le \gamma \le 5^o$	$0^o \le \gamma \le 5^o$	rad
$0^o \le \psi \le 40^o$	$0^o \le \psi \le 40^o$	$0^o \le \psi \le 40^o$	rad

The six differential variables are all functions of the independent variable t, that is $[h, \phi, \theta, v, \gamma, \psi] = [h(t), \phi(t), \theta(t), v(t), \gamma(t), \psi(t)]$. Information about each of the dynamic variables is given for the three distinct phase regions—beginning, interior, and end. For the first differential variable (an altitude) h, the value at the beginning of phase two is fixed, specifically $h(t_I^{(2)}) = 60$ nm. Observe, that the phase specific notation $t_I^{(2)}$ is not needed because the condition $h = 60$ nm appears in the left column. In contrast, the right column also contains the expression $h = 60$ nm, which implies that the value at the end of the phase is also fixed, i.e. $h(t_F^{(2)}) = 60$ nm. Also note that the altitude during the phase must be below the initial and final phase values, as well as above zero. This condition is reflected in the middle portion of the information by the expression $0 \le h \le 60$ nm. Observe that the internal units for h in feet (ft) are displayed on the far right, whereas the bounds are specified in nautical miles (nm). This unit conversion must be accounted for using the values in the appendix, when implementing software. Similarly the angular quantities $(\phi, \theta, \gamma, \psi)$ have bounds given in degrees and internal units of radians. Note that the bounds for the variable γ are different at the beginning, interior, and end of the phase. Finally, when a variable is unconstrained the condition is simply omitted (see for example aqua01 on page 37).

2.2.7 Algebraic Variables

Information about the algebraic variables is displayed in a format similar to the differential variables. Consider problem lnts05 as shown on page 167.

Algebraic Variables: (u) ....

$-90^o \le u \le +90^o$	$-90^o \le u \le +90^o$	$-90^o \le u \le +90^o$	rad

After presenting a list of the variables, in this case just u, conditions at the beginning, interior, and end of the phase are delineated. Internal units for the variable, (an angle in radians) are given in the far right column, and may differ from the units used to describe the variable bounds given in degrees for this example.

2.2.8 Boundary Conditions

Boundary conditions for an optimal control problem vary in complexity, and as such the presentation format must incorporate this variability. First consider problem aotv01 as shown on page 33.

Boundary Conditions ....

$$\cos\phi\cos\psi = \cos 18^o$$

This example illustrates a simple terminal condition that is imposed at the end of the phase. Display of the equation $\cos\phi\cos\psi = \cos 18^o$ in the right hand column suffices, and it is not necessary to present the information in the equivalent, albeit more explicit, format $\cos\phi_F^{(1)}\cos\psi_F^{(1)} = \cos 18^o$.

On the other hand when the boundary conditions (2.9) are more complicated it is necessary to use a more complete format. Consider the boundary conditions imposed in phase 8 of problem capt01 as they appear on page 72.

Boundary Conditions ....

$$r_I^{(8)} = r_F^{(6)} + v_F^{(6)}\left[t_F^{(7)} - t_I^{(7)}\right]$$
$$t_F^{(8)} - t_I^{(8)} \ge 1$$

These conditions involve quantities at the end of phase 6, namely $(r_F^{(6)}, v_F^{(6)})$, as well as quantities at both ends of phase 7 and 8, specifically $(t_I^{(7)}, t_F^{(7)})$ and $(t_I^{(8)}, t_F^{(8)}, r_I^{(8)})$.

2.2.9 Differential-Algebraic Equations

An ordinary differential equation stated in the explicit form (2.2) is given corresponding to each differential variable. For problem vpol01 these equations appear on page 315 and are repeated below:

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{2.13}$$

$$\dot{y}_2 = (1 - y_1^2)y_2 - y_1 + u \tag{2.14}$$

By convention any algebraic equality and/or inequality equations are defined following the ODE's. Any auxiliary information needed to complete the definition of DAE's is introduced in subsequent equations.

2.2.10 Objective Function

There is considerable flexibility possible in an objective function given by (2.10)-(2.12). Problem lnts05 as shown on page 167 provides a simple example.

Objective ....

Minimize $$J = t_F$$

$$\boxed{J^* = 5.54570879 \times 10^{-1}}$$

Here the objective function is a single parameter, namely the final time denoted t_F. The optimal objective function value computed by $\mathbb{SOS}$ is also displayed. In contrast a more complex objective function is given for problem lbrp02 on 161 repeated below.

Objective ....

Minimize $$J = \frac{1}{2} \sum_{k=1}^{k=4} \int_{t_I^{(k)}}^{t_F^{(k)}} \left(u_1^2 + u_2^2\right) dt$$

$$\boxed{J^* = 2.54291985 \times 10^{-8}}$$

In this case, the objective function requires information accumulated over four distinct phases.

Some problem formulations result in a nonlinear boundary value problem (BVP) in which case information about the objective function is either implicit in the formulation and/or does not apply. For problems of this type the objective function block is omitted.

Chapter 3

Test Suite

For the sake of reference section 3.1 presents a brief list of the problems in the current test suite. Section 3.2 tabulates a number of the important characteristics of each individual problem. The problems in the test suite are subdivided into different categories in Section 3.3.

3.1 Problem List

Problem	Description	Section	Page
robo01	*Industrial Robot; Mayer Formulation*	62.1	271
robo02	*Industrial Robot; Lagrange Formulation*	62.2	272
robo03	*Industrial Robot; Minimum Time With Regularization*	62.3	273
robo04	*Industrial Robot; Minimum Time With Switching Structure*	62.4	273
skwz01	*Andrew's Squeezer Mechanism; Initial Value Problem*	63.1	281
skwz02	*Andrew's Squeezer Mechanism; Minimum Energy*	63.2	282
skwz03	*Andrew's Squeezer Mechanism; Minimum Time*	63.3	283
skwz04	*Andrew's Squeezer Mechanism; Multiphase Minimum Energy*	63.4	283
soar01	*Dynamic Soaring; Minimum Wind Factor*	64.1	287
ssmd01	*Space Station Attitude Control; International Space Station Momentum Dumping*	65.1	289
stgl01	*Innate Immune Response*	66.1	291
tb2s01	*Two-Strain Tuberculosis Model; Minimum Infectious Strain and Cost*	67.1	293
tmpr01	*Temperature Control*	68.1	295
traj03	*Trajectory Examples; Two-Burn Orbit Transfer*	69.1	299
traj09	*Trajectory Examples; Shuttle Maximum Downrange*	69.2	303
traj21	*Trajectory Examples; Shuttle Maximum Crossrange*	69.3	304
traj22	*Trajectory Examples; Shuttle Maximum Crossrange with Control*	69.4	305
traj36	*Trajectory Examples; Shuttle Maximum Crossrange with Heat Limit*	69.5	305
tran01	*Train Problem; Minimum Fuel Cost*	70.1	307
tumr01	*Tumor Anti-angiogenesis; Minimum Tumor Size*	71.1	309
tumr02	*Tumor Anti-angiogenesis; Two Phase Formulation*	71.2	310
tumr03	*Tumor Anti-angiogenesis; Indirect Formulation*	71.3	311
vpol01	*Van der Pol Oscillator; State Bound Formulation*	72.1	315
vpol04	*Van der Pol Oscillator; Path Constraint Formulation*	72.2	316
vpol07	*Van der Pol Oscillator; Indirect Formulation*	72.3	316
wind01	*Abort Landing in the Presence of Windshear; Maximize Minimum Altitude*	73.1	319
zrml01	*Zermelo's Problem; Minimum Time*	74.1	325

3.2 Problem Characteristics

There are many factors that characterize whether a particular test problem is "hard" or "easy" for a particular computational algorithm. The following tables summarize a number of key problem characteristics that may be relevant for the selection process.

The total computation time needed to solve a problem is often a key measure of the degree of difficulty. Typically the CPU time can be computed on any computer using the appropriate utility procedure. Unfortunately the CPU time on a "fast" computer will always be less than on a "slow" computer. Furthermore the CPU time can change significantly depending on the compiler options, as well as many hardware features, such as the cache size, number of CPU's etc. Finally different computational algorithms will necessarily exhibit different solution times for the same problem. Therefore, this characteristic should be used only when comparing one problem relative to another. For the tabulated results all calculations were performed using the $\mathbb{SOS}$ algorithm with a requested accuracy of 10^{-7} or approximately eight significant figures in the differential-algebraic equations. The software was executed on a desktop computer with an Intel I7 processor (3.06 Ghz), using the SUSE Linux operating system, and GNU Fortran compiler with optimization option "O", as measured using the intrinsic function ETIME. The solution time measured in seconds is tabulated as T_s.

The total number of phases N used to model the problem is given in the third column of the tables. Since the number of differential, algebraic, and parametric variables can change from phase to phase, the table presents n_y, n_u, and n_p which are the maximum values on any phase. Finally n_ψ gives the total number of boundary conditions.

Problem Characteristic Key

T_s	Solution Time, CPU (sec)
N	Total Number of Phases
n_y	Maximum Number of Differential Variables on any Phase
n_u	Maximum Number of Algebraic Variables on any Phase
n_p	Maximum Number of Parameters on any Phase
n_ψ	Total Number of Boundary Conditions

Problem	T_s	N	n_y	n_u	n_p	n_ψ
alpr01	20.0250	1	4	2	0	0
aomp01	3.62245	3	6	2	6	36
aomp02	8.69768	9	6	2	6	93
aotv01	0.641903	1	5	2	1	1
aotv02	2.76958	1	5	2	2	1
aqua01	0.571911	1	10	4	0	0
arao01	0.355946	1	1	1	0	0
arao02	.0919876	1	2	1	0	0
ashr01	.0479927	1	2	0	0	0
ashr02	.0549927	1	2	0	0	0
ashr03	.0529900	1	2	0	0	0
ashr04	.0329933	1	2	0	0	0
ashr05	0.124981	1	2	0	0	0
ashr06	1.62775	1	2	4	0	0
asyq01	4.99924	1	6	4	1	1
asyq02	0.128979	6	6	1	2	41
bang01	.0289955	1	2	1	1	0
brac01	.0249977	1	3	1	1	0
brac02	.0699883	1	3	1	1	0
brgr01	.0829887	1	2	0	0	0
brn201	0.511921	4	7	2	2	23
brn202	0.964855	4	7	2	2	23
brn203	0.411938	4	7	0	4	23
brn204	0.996849	4	7	0	4	23
capt01	5.78612	9	5	1	2	39
capt03	8.43772	9	5	1	2	39
capt05	5.17721	9	5	1	2	39
chan01	33.5409	1	44	38	0	0
chan03	276.574	1	44	1	0	0
chmr01	.0330200	1	2	1	0	0
chmr02	.0520020	1	2	1	0	0
chmr03	.0539856	1	2	1	0	0
chmr04	.0119934	1	2	1	0	0
chmr05	.0299988	1	2	1	0	0
chmr06	.0329895	1	2	1	0	0
chmr07	.0549622	1	2	1	0	0
chmr08	.0959778	1	2	1	0	0
chmr09	0.104004	1	2	1	0	0
chmr10	0.361938	1	2	1	0	0
clym04	1.56076	1	7	1	1	0
clym13	0.823883	1	5	1	1	0
cran01	0.289948	1	6	2	0	0
cst201	0.546906	1	160	80	0	234
cstr01	0.173981	1	120	80	0	195
cstr02	2.22568	1	160	80	0	234
cstr03	0.505920	1	120	80	18	196
dlay01	.00701904	1	4	0	0	2
dlt301	1.08582	4	7	3	1	26

Problem	T_s	N	n_y	n_u	n_p	n_ψ
dock01	8.27374	1	20	6	1	6
dock02	466.894	1	20	6	1	6
dock03	80.0009	1	20	6	1	6
ffrb01	7.12390	1	6	4	0	0
fhoc01	1.43677	1	100	50	0	147
fish01	0.458923	1	200	200	0	398
gdrd02	.0429688	1	3	1	1	0
gdrd07	.0320435	3	3	1	2	11
gdrd10	.00598145	3	3	0	2	10
goll01	.0159912	1	3	3	0	4
goll02	0.174988	1	6	6	0	10
goll03	.0400391	1	6	3	0	6
gsoc01	69.9794	8	7	2	2	35
gydn01	1.44580	1	7	2	1	0
hang01	5.82520	1	4	1	1	0
hang02	6.10400	1	5	1	0	0
hang03	1.58276	1	3	1	1	0
hdae01	12.4531	1	19	2	0	0
heat01	0.795898	1	11	3	0	0
heat02	8.77673	1	50	3	0	0
jmp201	0.227905	4	7	2	3	42
jmp202	0.177979	4	7	0	4	42
jshi01	4.14929	1	2	2	0	0
jshi02	4.14734	1	3	2	0	0
kplr01	.00903320	1	0	1	0	0
lbrp01	3.59045	2	4	2	3	14
lbrp02	17.8673	4	4	2	3	26
lbrp03	2.10559	2	4	2	3	14
lbrp04	9.55847	4	4	2	3	26
lnht01	0.564941	1	5	1	0	0
lnht02	0.897827	1	4	1	0	0
lnts01	.0249023	1	8	0	1	1
lnts05	.0289307	1	4	1	1	0
lnts13	.00708008	1	4	0	3	0
lowt01	0.200928	1	4	1	0	0
lthr01	24.9033	1	7	3	2	4
ltsp01	.0319824	3	4	0	3	14
ltsp02	.0319824	3	4	0	4	16
lwbr01	66.3650	3	6	5	3	19
medi01	.0300293	1	2	1	0	0
medi02	.0159912	1	2	1	0	0
medi03	.00903320	1	2	1	0	0
medi04	.0100098	1	2	1	0	0
medi05	.00305176	1	2	1	0	0
medi06	.00195313	1	2	1	0	0
mirv01	6.77295	5	6	2	2	35

Problem	T_s	N	n_y	n_u	n_p	n_ψ
mncx01	0.608887	1	20	20	0	38
mncx01	.0899658	1	20	20	0	38
mncx02	0.135010	1	20	20	0	38
mncx02	0.135010	1	20	20	0	38
mncx03	0.158936	1	20	20	0	38
mncx03	0.160034	1	20	20	0	38
mrck01	11.5543	1	480	0	0	476
nzym01	1.06787	1	160	0	0	156
orbe01	0.267944	1	6	0	0	0
orbe02	2.35461	1	7	3	2	4
orbe05	3.58350	1	7	3	2	4
orbt01	1.37183	6	7	2	2	47
orbt02	1.65271	6	7	2	2	47
orbt03	3.90833	2	7	3	3	12
pdly01	126.117	1	160	10	0	153
plnt01	13.2080	6	7	3	2	46
pnav01	.00903320	1	2	1	1	1
pnav02	.0449219	1	2	2	1	1
pndl01	0.300903	1	4	2	0	0
pndl02	0.425049	1	5	1	0	0
putt01	0.245972	2	6	0	2	10
qlin01	.00402832	1	6	3	0	0
qlin02	.0169678	1	7	3	0	0
qlin03	.0100098	1	6	3	0	0
qlin04	.00708008	1	0	1	0	0
rayl01	0.230957	1	2	1	0	0
rayl02	.0739746	4	4	0	2	14
rayl03	0.191040	1	2	1	0	0
rayl04	0.256958	1	2	1	0	0
rayl05	.0999756	4	4	0	2	20
rbrm01	2.4877	1	6	3	1	0
rcsp01	3.33252	10	7	0	9	83
rcsp02	1.12585	8	7	0	9	69
rcsp03	1.37988	10	7	0	9	80
rcsp04	0.783813	8	7	0	9	66
rivr01	1.70776	1	2	3	1	5
rivr02	6.84399	1	2	3	1	5
robo01	0.128052	1	7	3	0	0
robo02	0.110962	1	6	3	0	0
robo03	0.745850	1	6	3	1	0
robo04	.0579834	9	6	3	2	52
skwz01	1.47473	1	14	13	0	0
skwz02	6.64490	1	14	14	0	0
skwz03	6.99792	1	14	14	1	0
skwz04	13.9249	3	14	14	0	28
soar01	2.63562	1	6	2	2	4
ssmd01	1.08179	1	9	3	0	6
stgl01	24.8481	1	40	40	0	72

Problem	T_s	N	n_y	n_u	n_p	n_ψ
tb2s01	0.787842	1	6	2	0	0
tmpr01	52.1420	1	45	1	0	0
traj03	0.422974	5	7	2	2	27
traj09	0.545898	1	4	1	1	0
traj21	0.679932	1	5	2	1	0
traj22	0.641968	1	5	2	1	0
traj36	1.86768	1	5	2	1	0
tran01	6.01709	1	2	2	0	0
tumr01	0.616943	1	3	1	1	0
tumr02	0.01599121	2	3	0	2	4
tumr03	0.03002930	2	6	0	2	9
vpol01	0.139893	1	2	1	0	0
vpol04	0.255859	1	2	2	0	0
vpol07	.00610352	3	4	0	2	9
wind01	5.36621	5	4	1	3	28
zrml01	.0310059	1	2	1	1	0

3.3 Problem Categories

The collection of test problems come from a wide variety of applications. One way to categorize the problems is by discipline or application environment. The following tables subdivide the test suite into a number of categories:

Partial Differential Equations (Method of Lines) Problems
33.1 34.1 34.2 52.1 68.1

Delay Differential Equation (Method of Steps) Problems
20.1 21.1 21.2 21.3 22.1 26.1 27.1 29.1 29.2 29.3
47.1 47.2 47.3 48.1 49.1 52.1 66.1

Multibody Systems Problems
16.1 16.2 63.1 63.2 63.3 63.4

Translational Dynamics Problems
7.1 56.1 61.1 61.2 70.1

Rotational Dynamics Problems
10.1 10.2 24.1 24.2 24.3 65.1

Orbital Trajectory Problems
5.1 5.2 6.1 6.2 14.1 14.2 14.3 14.4 35.1 35.2
39.1 39.2 39.3 39.4 42.1 43.1 50.1 50.2 50.3 51.1
51.2 53.1 60.1 60.2 60.3 60.4 69.1

Atmospheric and/or Reentry Trajectory Problems

4.1	5.1	5.2	15.1	15.2	15.3	18.1	18.2	30.1	31.1
32.1	32.2	32.3	46.1	64.1	69.2	69.3	69.4	69.5	73.1

Ascent Trajectory Problems

23.1 41.1 41.2 41.3 41.4 41.5

Biological/Medical Problems

36.1 36.2 40.1 40.2 67.1 71.1 71.2 71.3

Chemical Process Control Problems

17.1 44.1

Robotics Problems

19.1 25.1 59.1 62.1 62.2 62.3 62.4

Ill-Conditioned, Numerically Sensitive Problems

8.1 8.2 9.1 9.2 9.3 9.4 9.5 9.6 13.1 37.1

Classical Problems

11.1	12.1	12.2	28.1	28.2	28.3	41.1	41.2	41.3	41.4
41.5	45.1	45.2	45.3	45.4	45.5	45.6	54.1	54.2	55.1
55.2	57.1	57.2	57.3	57.4	58.1	58.2	58.3	58.4	58.5
72.1	72.2	72.3	74.1						

Chapter 4

alpr: **Alp Rider**

The Alp Rider example was originally proposed by Stephen Campbell to describe the path of a terrain following aircraft. The "peaks" are modeled as simple exponential spikes, and the differential equations used to model the dynamics are stiff. The example was constructed to illustrate the behavior of a specific mesh refinement algorithm described in reference [13, Sect. 4.7.6].

Example 4.1 alpr01: Stiff ODE, Terrain Following.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 \le t \le 20$	$t = 20$

Differential Variables: (y_1, y_2, y_3, y_4) ....

$y_1 = 2$	$y_1 = 2$
$y_2 = 1$	$y_2 = 3$
$y_3 = 2$	$y_3 = 1$
$y_4 = 1$	$y_4 = -2$

Algebraic Variables: (u_1, u_2) ....

Differential-Algebraic Equations ....

$$\dot{y}_1 = -10y_1 + u_1 + u_2 \tag{4.1}$$

$$\dot{y}_2 = -2y_2 + u_1 + 2u_2 \tag{4.2}$$

$$\dot{y}_3 = -3y_3 + 5y_4 + u_1 - u_2 \tag{4.3}$$

$$\dot{y}_4 = 5y_3 - 3y_4 + u_1 + 3u_2 \tag{4.4}$$

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 \ge 3p(t,3,12) + 3p(t,6,10) + 3p(t,10,6) + 8p(t,15,4) + 0.01 \tag{4.5}$$

where $p(t,a,b) = e^{-b(t-a)^2}$.

Objective ....

Minimize $$J = \int_0^{20} 10^2(y_1^2 + y_2^2 + y_3^2 + y_4^2) + 10^{-2}(u_1^2 + u_2^2)dt$$

$$\boxed{J^* = 2030.85609}$$

Chapter 5

aomp: **Multiple-Pass Aeroassisted Orbital Transfer**

A favorite summer pastime while at a seaside beach or lakefront is "stone skipping." As a flat stone hits the surface of the water a rapid change in direction takes place that alters the motion for the next "skip" and a "good throw" will result in many skips before the stone looses energy. An analogous situation occurs in orbit mechanics when a spacecraft reenters the atmosphere. An extensive study of the subject, as well as many pertinent references can be found in the paper by Rao, Tang, and Hallman [79]. A detailed presentation of the specific example problems given here can be found in reference [13, Sect. 7.2].

Example 5.1 aomp01: Maximum Final Mass, One Pass.

Phase 1 . . . . *Inbound Coast, Pass: 01* . . . . Phase 1

Parameters: $(\Delta v_x^{(1)}, \Delta v_y^{(1)}, \Delta v_z^{(1)}, m_F^{(1)}, \Delta E_F^{(1)})$ . . . .

$\Delta v_z^{(1)} \le 0$ $\quad$ $1 \le m_F^{(1)} \le 520$ $\quad$ $1^o \le \Delta E_F^{(1)} \le 180^o$

where $\boldsymbol{\Delta}\mathbf{v}^{\mathsf{T}} = (\Delta v_x, \Delta v_y, \Delta v_z)$.

Independent Variable: (ΔE) . . . .

$\Delta E = 0$ $\quad$ $0 < \Delta E < \Delta E_F^{(1)}$ $\quad$ $\Delta E = \Delta E_F^{(1)}$ $\quad$ rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ . . . .

$\mathbf{r} = \mathbf{r}_I$

where

$$\begin{aligned}\mathbf{r}^{\mathsf{T}} &= (r_x, r_y, r_z)\\ \mathbf{v}^{\mathsf{T}} &= (v_x, v_y, v_z)\\ \mathbf{r}_I^{\mathsf{T}} &= (1.38335209528 \times 10^8, 0, 0)\end{aligned}$$

$$\mathbf{v}_I^\mathsf{T} = (0, 1.00920971977 \times 10^4, 0)$$

Boundary Conditions . . . .

$$\mathbf{r} = \mathbf{h}_r(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E)$$
$$\mathbf{v} = \mathbf{h}_v(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E)$$

$$v_{xI} - \Delta v_x^{(1)} = 0$$
$$v_{yI} - \Delta v_y^{(1)} = \sqrt{\mu/r_\circ}$$
$$v_{zI} - \Delta v_z^{(1)} = 0$$
$$m_0 - m_F^{(1)} \exp\left[\frac{\|\mathbf{\Delta v}^{(1)}\|}{g_0 I_{sp}}\right] = 0$$

$$\|\mathbf{r}_F\| - R_E = 60 \text{ nm}$$

The boundary conditions are computed using values given in Table 5.1 by setting

$$(\mathbf{r}_\circ, \mathbf{v}_\circ) = (\mathbf{r}_I, \mathbf{v}_I) \tag{5.1}$$
$$\Delta E = \Delta E_F^{(1)} \tag{5.2}$$

followed by the sequence

$$r_\circ = \|\mathbf{r}_\circ\| \tag{5.3}$$
$$\sigma_\circ = \frac{1}{\sqrt{\mu}} \mathbf{r}_\circ^\mathsf{T} \mathbf{v}_\circ \tag{5.4}$$
$$v_\circ^2 = \mathbf{v}_\circ^\mathsf{T} \mathbf{v}_\circ \tag{5.5}$$
$$\frac{1}{a} = \frac{2}{r_\circ} - \left[\frac{v_\circ^2}{\mu}\right] \tag{5.6}$$
$$\rho = 1 - \frac{r_\circ}{a} \tag{5.7}$$
$$C = a(1 - \cos \Delta E) \tag{5.8}$$
$$S = \sqrt{a} \sin \Delta E \tag{5.9}$$
$$F = 1 - \frac{C}{r_\circ} \tag{5.10}$$
$$G = \frac{1}{\sqrt{\mu}} (r_\circ S + \sigma_\circ C) \tag{5.11}$$
$$r = r_\circ + \rho C + \sigma_\circ S \tag{5.12}$$
$$F_t = -\frac{\sqrt{\mu}}{r r_\circ} S \tag{5.13}$$
$$G_t = 1 - \frac{C}{r} \tag{5.14}$$
$$\mathbf{h}_r(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E) = F \mathbf{r}_\circ + G \mathbf{v}_\circ \tag{5.15}$$
$$\mathbf{h}_v(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E) = F_t \mathbf{r}_\circ + G_t \mathbf{v}_\circ \tag{5.16}$$

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{r}} = \mathbf{v} \tag{5.17}$$
$$\dot{\mathbf{v}} = -\frac{\mu}{r^3} \mathbf{r} \tag{5.18}$$

Note (5.17)-(5.18) are replaced by the analytic technique (5.3)-(5.16).

Phase 2 .... *Atmospheric Maneuver, Pass: 01* .... Phase 2

Parameters: $(m_I^{(2)}, t_F^{(2)})$ ....

$1 \leq m_I^{(2)}$ $\qquad$ $1 \leq t_F^{(2)} \leq 4000$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(2)}$	$t = t_F^{(2)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = 60$ nm	$0 \leq h \leq 60$ nm	$h = 60$ nm	ft
$170^o \leq \phi \leq 200^o$	$170^o \leq \phi \leq 200^o$	$170^o \leq \phi \leq 200^o$	rad
$-20^o \leq \theta \leq 89^o$	$-20^o \leq \theta \leq 89^o$	$-20^o \leq \theta \leq 89^o$	rad
$25000 \leq v \leq 35000$	$25000 \leq v \leq 35000$	$25000 \leq v \leq 35000$	ft/sec
$-5^o \leq \gamma \leq 0^o$	$-5^o \leq \gamma \leq 5^o$	$0^o \leq \gamma \leq 5^o$	rad
$0^o \leq \psi \leq 40^o$	$0^o \leq \psi \leq 40^o$	$0^o \leq \psi \leq 40^o$	rad

Algebraic Variables: (u_1, u_2) ....

$\underline{u} \leq u_1 \leq \overline{u}$	$\underline{u} \leq u_1 \leq \overline{u}$	$\underline{u} \leq u_1 \leq \overline{u}$
$\underline{u} \leq u_2 \leq \overline{u}$	$\underline{u} \leq u_2 \leq \overline{u}$	$\underline{u} \leq u_2 \leq \overline{u}$

where $\overline{u} = -\underline{u} = 1.1C_{LU}$.

Boundary Conditions ....

$\phi = \phi_F^{(1)}$
$\theta = \theta_F^{(1)}$
$v = v_F^{(1)}$
$\gamma = \gamma_F^{(1)}$
$\psi = \psi_F^{(1)}$
$m_F^{(1)} = m_I^{(2)}$

The quantities $(h_F^{(1)}, \phi_F^{(1)}, \theta_F^{(1)}, v_F^{(1)}, \gamma_F^{(1)}, \psi_F^{(1)})$ can be computed by setting

$$(\mathbf{r}, \mathbf{v}) = (\mathbf{r}_F^{(1)}, \mathbf{v}_F^{(1)}) \tag{5.19}$$

followed by the sequence

$$\widehat{\mathbf{z}} = -r^{-1}\mathbf{r} \tag{5.20}$$

$$\mathbf{i}_z^{\mathsf{T}} = (0, 0, 1) \tag{5.21}$$

$$\widehat{\mathbf{x}} = \|\mathbf{i}_z - \hat{z}_3\widehat{\mathbf{z}}\|^{-1}(\mathbf{i}_z - \hat{z}_3\widehat{\mathbf{z}}) \tag{5.22}$$

$$\widehat{\mathbf{y}} = \widehat{\mathbf{z}} \times \widehat{\mathbf{x}} \tag{5.23}$$
$$\mathbf{Q}_{LE}(\mathbf{r}) = \begin{bmatrix}\widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}}\end{bmatrix} \tag{5.24}$$
$$r = \|\mathbf{r}\| \tag{5.25}$$
$$\widetilde{\mathbf{v}} = \mathbf{Q}_{LE}^{\mathsf{T}}(\mathbf{r})\mathbf{v} \tag{5.26}$$
$$h = r - R_E \tag{5.27}$$
$$\phi = \tan^{-1}\left(r_y/r_x\right) \tag{5.28}$$
$$\theta = \sin^{-1}\left(r_z/r\right) \tag{5.29}$$
$$v = \|\mathbf{v}\| \tag{5.30}$$
$$\gamma = \sin^{-1}\left(-\tilde{v}_z/v\right) \tag{5.31}$$
$$\psi = \tan^{-1}\left(\tilde{v}_y/\tilde{v}_x\right) \tag{5.32}$$

Differential-Algebraic Equations . . . .

$$\dot{h} = v \sin\gamma \tag{5.33}$$
$$\dot{\phi} = \frac{v\cos\gamma\sin\psi}{r\cos\theta} \tag{5.34}$$
$$\dot{\theta} = \frac{v\cos\gamma\cos\psi}{r} \tag{5.35}$$
$$\dot{v} = -\frac{D}{m} - g\sin\gamma \tag{5.36}$$
$$\dot{\gamma} = -\frac{1}{v}\left[\frac{qS}{m}u_2 + \left(g - \frac{v^2}{r}\right)\cos\gamma\right] \tag{5.37}$$
$$\dot{\psi} = \frac{1}{v}\left[\frac{-qS}{m\cos\gamma}u_1 + \frac{v^2}{r}\cos\gamma\sin\psi\tan\theta\right] \tag{5.38}$$
$$C_{LU} \geq C_L \tag{5.39}$$
$$Q_U \geq Q \tag{5.40}$$

where $m = m_I^{(2)}$, the constants are given in Table 5.1 and

$$Q = 17600\left(\frac{\rho}{\rho_E}\right)^{\frac{1}{2}}\left(\frac{v}{v_E}\right)^{3.15} \tag{5.41}$$
$$q = \frac{1}{2}\rho v^2 \tag{5.42}$$
$$D = qSC_D \tag{5.43}$$
$$L = qSC_L \tag{5.44}$$
$$C_D = C_{D0} + KC_L^2 \tag{5.45}$$
$$r = h + R_E \tag{5.46}$$
$$g = \frac{\mu}{r^2} \tag{5.47}$$
$$\alpha = \frac{C_L}{C_{L\alpha}} \tag{5.48}$$

using ρ computed from the atmosphere model [26] with the controls given by

$$u_1 = -C_L\sin\beta \tag{5.49}$$

$$u_2 = -C_L \cos\beta \tag{5.50}$$

with the inverse transformations given by

$$C_L = \sqrt{u_1^2 + u_2^2} \tag{5.51}$$

$$\beta = \tan^{-1}(u_1/u_2) \tag{5.52}$$

Phase 3 .... *Outbound Coast, Pass: 01* .... Phase 3

Parameters: $(\Delta v_x^{(3)}, \Delta v_y^{(3)}, \Delta v_z^{(3)}, m_I^{(3)}, m_F^{(3)}, \Delta E_F^{(3)})$ ....

$1 \le m_I^{(3)}$ $\quad$ $1 \le m_F^{(3)}$ $\quad$ $1^o \le \Delta E_F^{(3)} \le 180^o$

where $\boldsymbol{\Delta}\mathbf{v}^{\mathsf{T}} = (\Delta v_x, \Delta v_y, \Delta v_z)$.

Independent Variable: (ΔE) ....

$\Delta E = 0$ $\quad$ $0 < \Delta E < \Delta E_F^{(3)}$ $\quad$ $\Delta E = \Delta E_F^{(3)}$ $\quad$ rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

Boundary Conditions ....

$$\mathbf{r} = \mathbf{h}_r(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E)$$
$$\mathbf{v} = \mathbf{h}_v(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E)$$

$$h_I^{(3)} = 60 \text{ nm}$$
$$\phi_I^{(3)} = \phi_F^{(2)}$$
$$\theta_I^{(3)} = \theta_F^{(2)}$$
$$v_I^{(3)} = v_F^{(2)}$$
$$\gamma_I^{(3)} = \gamma_F^{(2)}$$
$$\psi_I^{(3)} = \psi_F^{(2)}$$
$$m_I^{(3)} = m_F^{(1)}$$

$$\|\mathbf{r}_F\| - R_E = 100 \text{ nm}$$
$$\|\mathbf{v}_F\| = \sqrt{\mu/r_F}$$
$$\mathbf{r}_F^{\mathsf{T}}\mathbf{v}_F = 0$$
$$i_F = 89^o$$
$$m_I^{(3)} - m_F^{(3)} \exp\left[\frac{\|\boldsymbol{\Delta}\mathbf{v}^{(3)}\|}{g_0 I_{sp}}\right] = 0$$
$$\mathbf{r}_F^{\mathsf{T}}\boldsymbol{\Delta}\mathbf{v}^{(3)} = 0$$

The boundary conditions are computed using values given in Table 5.1 by setting

$$(\mathbf{r}_\circ, \mathbf{v}_\circ) = (\mathbf{r}_I, \mathbf{v}_I) \tag{5.53}$$

$$\Delta E = \Delta E_F^{(1)} \tag{5.54}$$

followed by the sequence (5.3)-(5.16). Define

$$\mathbf{r}_F = \mathbf{r} \tag{5.55}$$

$$\mathbf{v}_F = \mathbf{v} + \mathbf{\Delta v}^{(3)}. \tag{5.56}$$

The quantities $(h_I^{(3)}, \phi_I^{(3)}, \theta_I^{(3)}, v_I^{(3)}, \gamma_I^{(3)}, \psi_I^{(3)})$ can be computed by setting

$$(\mathbf{r}, \mathbf{v}) = (\mathbf{r}_I^{(3)}, \mathbf{v}_I^{(3)}) \tag{5.57}$$

and then executing the sequence (5.20)-(5.32). Finally i_F is defined by the following:

$$\mathbf{i}_z^{\mathsf{T}} = (0, 0, 1) \tag{5.58}$$

$$i_F = \cos^{-1}\left[\mathbf{i}_z^{\mathsf{T}}\left(\frac{\mathbf{r}_F \times \mathbf{v}_F}{\|\mathbf{r}_F \times \mathbf{v}_F\|}\right)\right] \tag{5.59}$$

Differential-Algebraic Equations ....

Equations (5.17) - (5.18)

Objective ....

Maximize $J = m_F^{(3)}$

$\boxed{J^* = 212.175101}$

Example 5.2 aomp02: Maximum Final Mass, Four Passes.

Phase 1 .... *Inbound Coast, Pass: 01* .... Phase 1

Repeat phase 1 of example 5.1.

Phase 2 .... *Atmospheric Maneuver, Pass: 01* .... Phase 2

Repeat phase 2 of example 5.1.

Phase 3 .... *Intermediate Coast, Pass: 02* .... Phase 3

Parameters: $(\Delta E_F^{(3)})$ ....

$1^o \leq \Delta E_F^{(3)} \leq 360^o$

Independent Variable: (ΔE) ....

$\Delta E = 0$ $\qquad$ $0 < \Delta E < \Delta E_F^{(3)}$ $\qquad$ $\Delta E = \Delta E_F^{(3)}$ $\qquad$ rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....
Boundary Conditions ....

$$\mathbf{r} = \mathbf{h}_r(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E)$$
$$\mathbf{v} = \mathbf{h}_v(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E)$$

$h_I^{(3)} = 60$ nm
$\phi_I^{(3)} = \phi_F^{(2)}$
$\theta_I^{(3)} = \theta_F^{(2)}$
$v_I^{(3)} = v_F^{(2)}$
$\gamma_I^{(3)} = \gamma_F^{(2)}$
$\psi_I^{(3)} = \psi_F^{(2)}$

$$\|\mathbf{r}_F\| - R_E = 60 \text{ nm}$$

The boundary conditions $\mathbf{h}_r(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E)$ and $\mathbf{h}_v(\mathbf{r}_\circ, \mathbf{v}_\circ, \Delta E)$ are computed using values given in Table 5.1 by setting

$$(\mathbf{r}_\circ, \mathbf{v}_\circ) = (\mathbf{r}_I^{(3)}, \mathbf{v}_I^{(3)}) \tag{5.60}$$
$$\Delta E = \Delta E_F^{(3)} \tag{5.61}$$

followed by the sequence (5.3)-(5.16). The quantities $(h_F^{(2)}, \phi_F^{(2)}, \theta_F^{(2)}, v_F^{(2)}, \gamma_F^{(2)}, \psi_F^{(2)})$ can be computed by setting

$$(\mathbf{r}, \mathbf{v}) = (\mathbf{r}_F^{(2)}, \mathbf{v}_F^{(2)}) \tag{5.62}$$

followed by the sequence (5.20)-(5.32).

Differential-Algebraic Equations ....

Equations (5.17) - (5.18)

Phase 4 .... *Atmospheric Maneuver, Pass: 02* .... Phase 4

Repeat phase 2 of example 5.1 with the following changes:

Parameters: $(m_I^{(4)}, t_F^{(4)})$ ....

$1 \le m_I^{(4)}$ $\qquad$ $1 \le t_F^{(4)} \le 4000$

Independent Variable: (t) ....

$t = 0$ $\qquad$ $0 < t < t_F^{(4)}$ $\qquad$ $t = t_F^{(4)}$ $\qquad$ sec

Boundary Conditions . . . .

$\phi = \phi_F^{(3)}$
$\theta = \theta_F^{(3)}$
$v = v_F^{(3)}$
$\gamma = \gamma_F^{(3)}$
$\psi = \psi_F^{(3)}$
$m_F^{(1)} = m_I^{(4)}$

The quantities $(h_F^{(3)}, \phi_F^{(3)}, \theta_F^{(3)}, v_F^{(3)}, \gamma_F^{(3)}, \psi_F^{(3)})$ are given by (5.20)-(5.32) with

$$(\mathbf{r}, \mathbf{v}) = (\mathbf{r}_F^{(3)}, \mathbf{v}_F^{(3)}). \tag{5.63}$$

Phase 5 . . . . *Intermediate Coast, Pass: 03* . . . . Phase 5

Repeat the description of phase 3 with the following changes:

(a) Change quantities on phase 3 to values on phase 5, e.g. $v_I^{(3)} \to v_I^{(5)}$.
(b) Change quantities on phase 2 to values on phase 4, e.g. $\phi_F^{(2)} \to \phi_F^{(4)}$.

Phase 6 . . . . *Atmospheric Maneuver, Pass: 03* . . . . Phase 6

Repeat the description of phase 4 with the following changes:

(a) Change quantities on phase 4 to values on phase 6, e.g. $m_I^{(4)} \to m_I^{(6)}$.
(b) Change quantities on phase 3 to values on phase 5, e.g. $\phi_F^{(3)} \to \phi_F^{(5)}$.

Phase 7 . . . . *Intermediate Coast, Pass: 04* . . . . Phase 7

Repeat the description of phase 3 with the following changes:

(a) Change quantities on phase 3 to values on phase 7, e.g. $v_I^{(3)} \to v_I^{(7)}$.
(b) Change quantities on phase 2 to values on phase 6, e.g. $\phi_F^{(2)} \to \phi_F^{(6)}$.

Phase 8 . . . . *Atmospheric Maneuver, Pass: 04* . . . . Phase 8

Repeat the description of phase 4 with the following changes:

(a) Change quantities on phase 4 to values on phase 8, e.g. $m_I^{(4)} \to m_I^{(8)}$.
(b) Change quantities on phase 3 to values on phase 7, e.g. $\phi_F^{(3)} \to \phi_F^{(7)}$.

Phase 9 .... *Outbound Coast, Pass: 04* .... Phase 9

Repeat the description for phase 3 of example 5.1 with the following changes:

(a) Change quantities on phase 3 to values on phase 9, e.g. $v_I^{(3)} \to v_I^{(9)}$.
(b) Change quantities on phase 2 to values on phase 8, e.g. $\phi_F^{(2)} \to \phi_F^{(8)}$.

Objective ....

Maximize $$J = m_F^{(9)}$$

$$\boxed{J^* = 221.438830}$$

Q_U	400 BTU/(ft^2 sec)	m_0	519.5 slug
I_{sp}	310 sec	R_E	20926430 ft
μ	1.40895×10^{16} ft^3/sec^2	ρ_E	.0023769 slug/ft^3
S	125.84 ft^2	C_{D0}	.032
K	1.4	$C_{L\alpha}$	.5699
C_{LU}	0.4	v_E	$\sqrt{\mu/R_E}$ ft/sec

Table 5.1. *Dynamic Model Parameters*

Chapter 6

aotv: **Optimal Aeroassisted Plane Change**

An early study of the dynamic maneuver of a spacecraft referred to as "aeroassisted plane change" is given in reference [4]. These examples can be considered a simplified version of the dynamics modeled in examples (5.1) and (5.2).

Example 6.1 aotv01: Maximum Velocity with Heat Rate Limit.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$800 \le t_F \le 2000$

Independent Variable: (t) ....

$t = 0$	$0 \le t \le t_F$	$t = t_F$	sec

Differential Variables: $(\phi, h, v, \gamma, \psi)$ ....

$\phi = 0$	$-89^o \le \phi \le 89^o$	$-89^o \le \phi \le 89^o$	rad
$h = 365000$	$0 \le h \le 400000$	$h = 365000$	ft
$v = 25745.704$	$20000 \le v \le 28000$	$v = 25745.704$	ft/sec
$\gamma = -.55^o$	$-10^o \le \gamma \le 10^o$	$-10^o \le \gamma \le 10^o$	rad
$\psi = 0$	$-89^o \le \psi \le 89^o$	$-89^o \le \psi \le 89^o$	rad

Algebraic Variables: (C_L, β) ....

$0 \le C_L \le 2$	$0 \le C_L \le 2$	$0 \le C_L \le 2$	
$0^o \le \beta \le 180^o$	$0^o \le \beta \le 180^o$	$0^o \le \beta \le 180^o$	rad

Boundary Conditions ....

$$\cos\phi\cos\psi = \cos 18^o$$

Differential-Algebraic Equations ....

$$\dot{\phi} = \frac{v}{r}\cos\gamma\sin\psi \tag{6.1}$$
$$\dot{h} = v\sin\gamma \tag{6.2}$$
$$\dot{v} = -a_1\rho v^2\left(1 + C_L^2\right) - \frac{\mu\sin\gamma}{r^2} \tag{6.3}$$
$$\dot{\gamma} = a_0\rho v\left(C_L\cos\beta + M\cos\gamma\right) \tag{6.4}$$
$$\dot{\psi} = \frac{a_0\rho v C_L\sin\beta}{\cos\gamma} - \frac{v\cos\gamma\cos\psi\tan\phi}{r} \tag{6.5}$$
$$0 \le \dot{q} \le 800 \tag{6.6}$$

where

$$\rho = \rho_0\exp\left[-\frac{(h-h_0)}{h_r}\right] \tag{6.7}$$
$$M = \frac{1}{a_0\rho r}\left(1 - \frac{\mu}{rv^2}\right) \tag{6.8}$$
$$\dot{q} = 17600\sqrt{\frac{\rho}{\rho_s}}\left(\frac{v}{v_s}\right)^{3.15} \tag{6.9}$$
$$v_s = \sqrt{\frac{\mu}{R_e}} \tag{6.10}$$
$$r = R_e + h \tag{6.11}$$

Objective ....

Maximize $\qquad J = v(t_F)$

$$\boxed{J^* = 22043.5079; \quad t_F^* = 1005.8778}$$

Example 6.2 aotv02: Minimax Heat Rate.

Repeat example 6.1 with the following changes:

(a) Add the parameter $\dot{Q}_{max}$;

(b) Modify the bounds;

$21900 \le v \le 25745.704 \qquad 21900 \le v \le 28000 \qquad 21900 \le v \le 28000$ ft/sec

(c) Replace (6.6) with the algebraic constraint;

$$0 \le \dot{Q}_{max} - \dot{q} \tag{6.12}$$

(d) Define the objective;

Minimize $$J = \dot{Q}_{max}$$

$$\boxed{J^* = 569.650999; \quad t_F^* = 1090.8962}$$

$R_e = 2.092643 \times 10^7$	$m = 3.315 \times 10^2$
$\rho_0 = 3.3195 \times 10^{-5}$	$h_0 = 1 \times 10^5$
$h_r = 2.41388 \times 10^4$	$C_{D0} = .032$
$k = 1.4$	$S = 1.2584 \times 10^2$
$\mu = 1.40895 \times 10^{16}$	$\rho_s = \rho_0 \exp\left[h_0/h_r\right]$
$a_0 = \dfrac{S}{2m}\sqrt{\dfrac{C_{D0}}{k}}$	$a_1 = \dfrac{C_{D0} S}{2m}$

Table 6.1. *Dynamic Model Parameters*

Chapter 7

aqua: **Underwater Vehicle**

The thesis research of Christof Büskens presented in reference [35] describes an optimal control problem that models the behavior of an underwater vehicle.

Example 7.1 aqua01: MINIMUM CONTROL ENERGY.

Phase 1 . . . . Phase 1

Independent Variable: (t) . . . .

$t = 0$	$0 < t < 1$	$t = 1$

Differential Variables: $(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10})$ . . . .

$y_1 = 0$		$y_1 = 1$
$y_2 = 0$		$y_2 = .5$
$y_3 = .2$		$y_3 = 0$
$y_4 = \pi/2$	$\pi/2 - .02 \le y_4 \le \pi/2 + .02$	$y_4 = \pi/2$
$y_5 = .1$		$y_5 = 0$
$y_6 = -\pi/4$		$y_6 = 0$
$y_7 = 1$		$y_7 = 0$
$y_8 = 0$		$y_8 = 0$
$y_9 = .5$		$y_9 = 0$
$y_{10} = .1$		$y_{10} = 0$

Algebraic Variables: (u_1, u_2, u_3, u_4) . . . .

$-15 \le u_1 \le 15$	$-15 \le u_1 \le 15$	$-15 \le u_1 \le 15$
$-15 \le u_2 \le 15$	$-15 \le u_2 \le 15$	$-15 \le u_2 \le 15$
$-15 \le u_3 \le 15$	$-15 \le u_3 \le 15$	$-15 \le u_3 \le 15$
$-15 \le u_4 \le 15$	$-15 \le u_4 \le 15$	$-15 \le u_4 \le 15$

Differential-Algebraic Equations . . . .

$$\dot{y}_1 = y_7 \cos(y_6) \cos(y_5) + R_x \tag{7.1}$$
$$\dot{y}_2 = y_7 \sin(y_6) \cos(y_5) \tag{7.2}$$
$$\dot{y}_3 = -y_7 \sin(y_5) + R_z \tag{7.3}$$
$$\dot{y}_4 = y_8 + y_9 \sin(y_4) \tan(y_5) + y_{10} \cos(y_4) \tan(y_5) \tag{7.4}$$
$$\dot{y}_5 = y_9 \cos(y_4) - y_{10} \sin(y_4) \tag{7.5}$$
$$\dot{y}_6 = \frac{y_9 \sin(y_4)}{\cos(y_5)} + \frac{y_{10} \cos(y_4)}{\cos(y_5)} \tag{7.6}$$
$$\dot{y}_7 = u_1 \tag{7.7}$$
$$\dot{y}_8 = u_2 \tag{7.8}$$
$$\dot{y}_9 = u_3 \tag{7.9}$$
$$\dot{y}_{10} = u_4 \tag{7.10}$$

where

$$E = \exp\left[-\left(\frac{y_1 - c_x}{r_x}\right)^2\right] \tag{7.11}$$
$$R_x = -u_x E (y_1 - c_x) \left(\frac{y_3 - c_z}{c_z}\right)^2 \tag{7.12}$$
$$R_z = -u_z E \left(\frac{y_3 - c_z}{c_z}\right)^2 \tag{7.13}$$

with $c_x = 0.5$, $r_x = 0.1$, $u_x = 2$, $c_z = 0.1$, and $u_z = 0.1$.

Objective . . . .

Minimize $$J = \int_0^1 (u_1^2 + u_2^2 + u_3^2 + u_4^2)\, dt$$

$$\boxed{J^* = 236.527851}$$

Chapter 8

arao: **Hypersensitive Control**

In his doctoral thesis Anil Rao studied a class of "hypersensitive control" problems. Although these examples appear quite simple they can be very challenging for most computational algorithms and as such serve as excellent test problems. The original research is described by Rao and Mease in reference [78], and computational experience is given in reference [13, p. 170].

Example 8.1 arao01: Lagrange Formulation.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 \le t \le 10000$	$t = t_F = 10000$

Differential Variables: (y) ....

$y = 1$		$y = 1.5$

Algebraic Variables: (u) ....

Differential-Algebraic Equations ....

$$\dot{y} = -y^3 + u \tag{8.1}$$

Objective ....

Minimize $$J = \int_0^{t_F} \left[y^2 + u^2\right] dt$$

$$\boxed{J^* = 6.72412325}$$

Example 8.2 arao02: MAYER FORMULATION.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 \leq t \leq 10000$	$t = t_F = 10000$

Differential Variables: (y, z) ....

$y = 1$	$y = 1.5$
$z = 0$	

Algebraic Variables: (u) ....

Differential-Algebraic Equations ....

$$\dot{y} = -y^3 + u \tag{8.2}$$

$$\dot{z} = y^2 + u^2 \tag{8.3}$$

Objective ....

Minimize $J = z_F$

$\boxed{J^* = 6.72411505}$

Chapter 9

ashr: **Ill-Conditioned Boundary Value Problems**

Many optimal control problems are posed as two-point boundary value problems. Example (9.1) is presented in reference [2, Sect. 9.3.2, p. 371] and is used by Ascher, et.al. to illustrate the impact of a rapid boundary layer transition region. Example (9.3) given in reference [2, Sect. 10.1.1, p. 394] incorporates a "shock layer" transition within a boundary value setting. Examples (9.5) and (9.6) are originally described in reference [52, Sect. II.4, p. 170] to illustrate numerical integration error control strategies.

Example 9.1 ashr01: ASCHER EXAMPLE 9.2 BVP.

Phase 1 .... Phase 1

Independent Variable: (x) ....

$x = -1$	$-1 < x < 1$	$x = 1$

Differential Variables: (y_1, y_2) ....

$y_1 = -2$	$-5 \le y_1 \le 5$	$y_1 = 0$
$-2500 \le y_2 \le 2500$	$-2500 \le y_2 \le 2500$	$-2500 \le y_2 \le 2500$

Differential-Algebraic Equations ....

$$y_1' = y_2 \tag{9.1}$$

$$y_2' = -\left[xy_2 + \epsilon\pi^2 \cos \pi x + \pi x \sin \pi x\right]/\epsilon \tag{9.2}$$

where $\epsilon = 10^{-4}$.

Boundary Value Problem

$$\boxed{y_1(x) = \cos \pi x + \frac{\operatorname{erf}\left(x/\sqrt{2\epsilon}\right)}{\operatorname{erf}\left(1/\sqrt{2\epsilon}\right)}}$$

Example 9.2 ashr02: Ascher Example 9.2 IVP.

Repeat example 9.1 with the following change:

Differential Variables: (y_1, y_2) ....

$y_1 = -2$	$-5 \le y_1 \le 5$	$-5 \le y_1 \le 5$
$y_2 = 0$	$-2500 \le y_2 \le 2500$	$-2500 \le y_2 \le 2500$

Example 9.3 ashr03: Ascher Example 10.4 BVP.

Phase 1 .... Phase 1

Independent Variable: (x) ....

$x = -1$	$-1 < x < 1$	$x = 1$

Differential Variables: (y_1, y_2) ....

$y_1 = -1$	$-5 \le y_1 \le 5$	$y_1 = 1$
$-2500 \le y_2 \le 2500$	$-2500 \le y_2 \le 2500$	$-2500 \le y_2 \le 2500$

Differential-Algebraic Equations ....

$$y_1' = \frac{y_2}{\sqrt{\epsilon\pi}} \tag{9.3}$$

$$y_2' = \frac{-2xy_2}{\epsilon} \tag{9.4}$$

where $\epsilon = 10^{-4}$.

Boundary Value Problem

Example 9.4 ashr04: Stiff ODE IVP.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 5$	$t = 5$

Differential Variables: (y_1, y_2) ....

$y_1 = 0$	$-5 \leq y_1 \leq 5$	$-5 \leq y_1 \leq 5$
$y_2 = 1$	$-2500 \leq y_2 \leq 2500$	$-2500 \leq y_2 \leq 2500$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{9.5}$$
$$\dot{y}_2 = y_1 - 999.999 y_2. \tag{9.6}$$

Initial Value Problem

$$\boxed{y_1(t) = a \exp(-1000t) + b \exp\left(\frac{t}{1000}\right) \qquad \text{where } b = -a = 1/1000.001.}$$

Example 9.5 ashr05: BRUSSELATOR IVP.

Phase 1 .... Phase 1

Independent Variable: (x) ....

$x = 0$	$0 < x < 20$	$x = 20$

Differential Variables: (y_1, y_2) ....

$y_1 = 1.5$	$-10 \leq y_1 \leq 10$	$-10 \leq y_1 \leq 10$
$y_2 = 3$	$-10 \leq y_2 \leq 10$	$-10 \leq y_2 \leq 10$

Differential-Algebraic Equations ....

$$y_1' = 1 + y_2 y_1^2 - 4y_1 \tag{9.7}$$
$$y_2' = 3y_1 - y_2 y_1^2 \tag{9.8}$$

Initial Value Problem

Example 9.6 ashr06: BRUSSELATOR IVP, SLACK VARIABLE FORMULATION.

Phase 1 .... Phase 1

Independent Variable: (x) ....

$x = 0$	$0 < x < 20$	$x = 20$

Differential Variables: (y_1, y_2) ....

$y_1 = 1.5$	$-10 \le y_1 \le 10$	$-10 \le y_1 \le 10$
$y_2 = 3$	$-10 \le y_2 \le 10$	$-10 \le y_2 \le 10$

Algebraic Variables: (u_1, u_2, u_3, u_4) ....

$0 \le u_1$	$0 \le u_1$	$u_1 = 0$
$0 \le u_2$	$0 \le u_2$	$u_2 = 0$
$0 \le u_3$	$0 \le u_3$	$u_3 = 0$
$0 \le u_4$	$0 \le u_4$	$u_4 = 0$

Differential-Algebraic Equations ....

$$y_1' = 1 + y_2 y_1^2 - 4y_1 - u_1 + u_2 \tag{9.9}$$

$$y_2' = 3y_1 - y_2 y_1^2 - u_3 + u_4 \tag{9.10}$$

Objective ....

Minimize $$J = \int_0^{20} (u_1 + u_2 + u_3 + u_4)\, dx$$

$\boxed{J^* = 0}$

Chapter 10

asyq: **Reorientation of an Asymmetric Rigid Body**

The rotational motion of a spacecraft treated as a rigid body is studied in reference [47]. The computational solution of this problem leads to a bang-bang control history which is also discussed in reference [13, Sect. 6.8]. Example (10.1) formulates the problem using a single phase, whereas a multi-phase formulation is given in example (10.2). Although these examples only address rotational motion a similar application that includes translational dynamics is given by examples (24.1)-(24.3).

Example 10.1 asyq01: Minimum Time.

Phase 1 .... Phase 1

Parameters: (t_F) ....
Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: $(q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)$ ....

$q_1 = 0$	$-1.1 \le q_1 \le 1.1$	$q_1 = \sin(\phi/2)$
$q_2 = 0$	$-1.1 \le q_2 \le 1.1$	$q_2 = 0$
$q_3 = 0$	$-1.1 \le q_3 \le 1.1$	$q_3 = 0$
$\omega_1 = 0$		$\omega_1 = 0$
$\omega_2 = 0$		$\omega_2 = 0$
$\omega_3 = 0$		$\omega_3 = 0$

where $\phi = 150^o$.

Algebraic Variables: (q_4, u_1, u_2, u_3) ....

$-1.1 \le q_4 \le 1.1$	$-1.1 \le q_4 \le 1.1$	$-1.1 \le q_4 \le 1.1$
$u_1 = 50$	$-50 \le u_1 \le 50$	$-50 \le u_1 \le 50$
$u_2 = -50$	$-50 \le u_2 \le 50$	$-50 \le u_2 \le 50$

$u_3 = 50$	$-50 \leq u_3 \leq 50$	$-50 \leq u_3 \leq 50$

Boundary Conditions ....

$.01 \leq t_F \leq 50$

Differential-Algebraic Equations ....

$$\dot{q}_1 = \frac{1}{2}\left[\omega_1 q_4 - \omega_2 q_3 + \omega_3 q_2\right] \tag{10.1}$$

$$\dot{q}_2 = \frac{1}{2}\left[\omega_1 q_3 + \omega_2 q_4 - \omega_3 q_1\right] \tag{10.2}$$

$$\dot{q}_3 = \frac{1}{2}\left[-\omega_1 q_2 + \omega_2 q_1 + \omega_3 q_4\right] \tag{10.3}$$

$$\dot{\omega}_1 = \frac{u_1}{I_x} - \left(\frac{I_z - I_y}{I_x}\right)\omega_2\omega_3 \tag{10.4}$$

$$\dot{\omega}_2 = \frac{u_2}{I_y} - \left(\frac{I_x - I_z}{I_y}\right)\omega_1\omega_3 \tag{10.5}$$

$$\dot{\omega}_3 = \frac{u_3}{I_z} - \left(\frac{I_y - I_x}{I_z}\right)\omega_1\omega_2. \tag{10.6}$$

$$0 = \|\mathbf{q}\| - 1. \tag{10.7}$$

where $\mathbf{q}^\mathsf{T} = (q_1, q_2, q_3, q_4)$.

Objective ....

Minimize $\qquad J = t_F$

$$\boxed{J^* = 28.6304077}$$

Example 10.2 asyq02: Multiphase, Minimum Time.

Phase 1 .... $\mathbf{u}^T = (50, -50, 50)$ .... Phase 1

Parameters: $(t_F^{(1)})$ ....
Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$

Differential Variables: $(q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)$ ....

$q_1 = 0$	$-1.1 \leq q_1 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$
$q_2 = 0$	$-1.1 \leq q_2 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$
$q_3 = 0$	$-1.1 \leq q_3 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$

$\omega_1 = 0$
$\omega_2 = 0$
$\omega_3 = 0$

Algebraic Variables: (q_4) ....

$-1.1 \leq q_4 \leq 1.1$	$-1.1 \leq q_4 \leq 1.1$	$-1.1 \leq q_4 \leq 1.1$

Boundary Conditions ....

$.01 \leq t_F^{(1)} - t_I^{(1)} \leq 50$

Differential-Algebraic Equations ....

Equations (10.1) - (10.7)

with $(u_1, u_2, u_3) = (50, -50, 50)$.

Phase 2 .... $\mathbf{u}^T = (50, -50, -50)$ .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....
Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} \leq t \leq t_F^{(2)}$	$t_I^{(2)} \leq t \leq t_F^{(2)}$

Differential Variables: $(q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)$ ....

$q_1 = q_{1F}^{(1)}$	$-1.1 \leq q_1 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$
$q_2 = q_{2F}^{(1)}$	$-1.1 \leq q_2 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$
$q_3 = q_{3F}^{(1)}$	$-1.1 \leq q_3 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$
$\omega_1 = \omega_{1F}^{(1)}$		
$\omega_2 = \omega_{2F}^{(1)}$		
$\omega_3 = \omega_{3F}^{(1)}$		

Algebraic Variables: (q_4) ....

$-1.1 \leq q_4 \leq 1.1$	$-1.1 \leq q_4 \leq 1.1$	$-1.1 \leq q_4 \leq 1.1$

Boundary Conditions ....

$.01 \leq t_F^{(2)} - t_I^{(2)} \leq 50$

Differential-Algebraic Equations ....

Equations (10.1) - (10.7)

with $(u_1, u_2, u_3) = (50, -50, -50)$.

Phase 3 .... $\mathbf{u}^T = (50, 50, -50)$ .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....
Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} \leq t \leq t_F^{(3)}$	$t_I^{(3)} \leq t \leq t_F^{(3)}$

Differential Variables: $(q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)$ ....

$q_1 = q_{1F}^{(2)}$	$-1.1 \leq q_1 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$
$q_2 = q_{2F}^{(2)}$	$-1.1 \leq q_2 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$
$q_3 = q_{3F}^{(2)}$	$-1.1 \leq q_3 \leq 1.1$	$-1.1 \leq q_1 \leq 1.1$
$\omega_1 = \omega_{1F}^{(2)}$		
$\omega_2 = \omega_{2F}^{(2)}$		
$\omega_3 = \omega_{3F}^{(2)}$		

Algebraic Variables: (q_4) ....

$-1.1 \leq q_4 \leq 1.1$	$-1.1 \leq q_4 \leq 1.1$	$-1.1 \leq q_4 \leq 1.1$

Boundary Conditions ....

$.01 \leq t_F^{(3)} - t_I^{(3)} \leq 50$

Differential-Algebraic Equations ....

Equations (10.1) - (10.7)

with $(u_1, u_2, u_3) = (50, 50, -50)$.

Phase 4 .... $\mathbf{u}^T = (-50, 50, -50)$ .... Phase 4

Parameters: $(t_I^{(4)}, t_F^{(4)})$ ....
Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} \leq t \leq t_F^{(4)}$	$t_I^{(4)} \leq t \leq t_F^{(4)}$

Differential Variables: $(q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)$ ....

$q_1 = q_{1F}^{(3)}$	$-1.1 \le q_1 \le 1.1$	$-1.1 \le q_1 \le 1.1$
$q_2 = q_{2F}^{(3)}$	$-1.1 \le q_2 \le 1.1$	$-1.1 \le q_1 \le 1.1$
$q_3 = q_{3F}^{(3)}$	$-1.1 \le q_3 \le 1.1$	$-1.1 \le q_1 \le 1.1$
$\omega_1 = \omega_{1F}^{(3)}$		
$\omega_2 = \omega_{2F}^{(3)}$		
$\omega_3 = \omega_{3F}^{(3)}$		

Algebraic Variables: (q_4) ....

$-1.1 \le q_4 \le 1.1$	$-1.1 \le q_4 \le 1.1$	$-1.1 \le q_4 \le 1.1$

Boundary Conditions ....

$.01 \le t_F^{(4)} - t_I^{(4)} \le 50$

Differential-Algebraic Equations ....

Equations (10.1) - (10.7)

with $(u_1, u_2, u_3) = (-50, 50, -50)$.

Phase 5 .... $\mathbf{u}^T = (-50, 50, 50)$ .... **Phase 5**

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....
Independent Variable: (t) ....

$t = t_F^{(4)} = t_I^{(5)}$	$t_I^{(5)} \le t \le t_F^{(5)}$	$t_I^{(5)} \le t \le t_F^{(5)}$

Differential Variables: $(q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)$ ....

$q_1 = q_{1F}^{(4)}$	$-1.1 \le q_1 \le 1.1$	$-1.1 \le q_1 \le 1.1$
$q_2 = q_{2F}^{(4)}$	$-1.1 \le q_2 \le 1.1$	$-1.1 \le q_1 \le 1.1$
$q_3 = q_{3F}^{(4)}$	$-1.1 \le q_3 \le 1.1$	$-1.1 \le q_1 \le 1.1$
$\omega_1 = \omega_{1F}^{(4)}$		
$\omega_2 = \omega_{2F}^{(4)}$		
$\omega_3 = \omega_{3F}^{(4)}$		

Algebraic Variables: (q_4) ....

$-1.1 \le q_4 \le 1.1$	$-1.1 \le q_4 \le 1.1$	$-1.1 \le q_4 \le 1.1$

Boundary Conditions ....

$.01 \le t_F^{(5)} - t_I^{(5)} \le 50$

Differential-Algebraic Equations . . . .

Equations (10.1) - (10.7)

with $(u_1, u_2, u_3) = (-50, 50, 50)$.

Phase 6 . . . . $\mathbf{u}^T = (-50, -50, 50)$ . . . . Phase 6

Parameters: $(t_I^{(6)}, t_F^{(6)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(5)} = t_I^{(6)}$	$t_I^{(6)} \le t \le t_F^{(6)}$	$t_I^{(6)} \le t \le t_F^{(6)}$

Differential Variables: $(q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)$ . . . .

$q_1 = q_{1F}^{(5)}$	$-1.1 \le q_1 \le 1.1$	$q_1 = \sin(\phi/2)$
$q_2 = q_{2F}^{(5)}$	$-1.1 \le q_2 \le 1.1$	$q_2 = 0$
$q_3 = q_{3F}^{(5)}$	$-1.1 \le q_3 \le 1.1$	$q_3 = 0$
$\omega_1 = \omega_{1F}^{(5)}$		$\omega_1 = 0$
$\omega_2 = \omega_{2F}^{(5)}$		$\omega_2 = 0$
$\omega_3 = \omega_{3F}^{(5)}$		$\omega_3 = 0$

where $\phi = 150^o$.

Algebraic Variables: (q_4) . . . .

$-1.1 \le q_4 \le 1.1$	$-1.1 \le q_4 \le 1.1$	$-1.1 \le q_4 \le 1.1$

Boundary Conditions . . . .

$.01 \le t_F^{(6)} - t_I^{(6)} \le 50$

Differential-Algebraic Equations . . . .

Equations (10.1) - (10.7)

with $(u_1, u_2, u_3) = (-50, -50, 50)$.

$\boxed{t_F^* = 28.6304077}$

Chapter 11

bang: **Bang-Bang Control Example**

When the control variable appears linearly the differential-algebraic equations and the objective function, the optimal control value is either on a bound or defined by singular arc conditions. This simple "classical" example illustrates the phenomenon and is discussed in more detail in reference [13, Sect. 4.14.3].

Example 11.1 bang01: Minimum Time.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$0 \le t_F \le 4$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: (x, y) ....

$x = 0$		$x = 1$
$y = 0$		$y = 0$

Algebraic Variables: (u) ....

$-1 \le u \le 1$	$-1 \le u \le 1$	$-1 \le u \le 1$

Differential-Algebraic Equations ....

$$\dot{x} = y \tag{11.1}$$

$$\dot{y} = u \tag{11.2}$$

Objective ....

Minimize $J = t_F$

$$\boxed{J^* = 2.00000000}$$

Chapter 12

brac: **Brachistochrone**

Brachistochrone is the name given to a curve of fastest descent. If a body such as a bead beginning at rest, moves without friction along a wire under a constant gravitational force, the path that will carry the body from one place to another in the least amount of time is a cycloid or brachistochrone. Johann Bernoulli first studied this problem in 1697, and it is perhaps one of the oldest problems in optimal control and the calculus of variations. Example (12.1) defines the classical problem, and example (12.2) adds a constraint on one of the dynamic states. Additional discussion is found in references [13, Ex. 4.10] and [29, p. 81, p. 119]

Example 12.1 brac01: UNCONSTRAINED ANALYTIC SOLUTION.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$0 \leq t_F$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: (x, y, v) ....

$x = 0$	$0 \leq x \leq 10$	$x = 1$
$y = 0$	$0 \leq y \leq 10$	
$v = 0$	$0 \leq v \leq 10$	

Algebraic Variables: (u) ....

$0 \le u \le \pi/2$ $0 \le u \le \pi/2$ $0 \le u \le \pi/2$

Differential-Algebraic Equations . . . .

$$\dot{x} = v \cos u \tag{12.1}$$
$$\dot{y} = v \sin u \tag{12.2}$$
$$\dot{x} = g_0 \sin u \tag{12.3}$$

Objective . . . .

Minimize $J = t_F$

$$\boxed{J^* = 3.12480130 \times 10^{-1}}$$

Example 12.2 brac02: State Variable Inequality Constraint.

Repeat example 12.1 and augment the differential-algebraic equations (12.1)-(12.3) with the algebraic constraint

$$0 \ge y - x/2 - h \tag{12.4}$$

where $h = 0.1$.

$$\boxed{J^* = 3.23331161 \times 10^{-1}}$$

Chapter 13

brgr: **Burgers' Equation**

In fluid mechanics, Burgers' Equation is a fundamental partial differential equation named after Johannes Martinus Burgers. It is simplified version of the Navier-Stokes equation. The presence of a shock wave which appears in the system of ordinary differential equations derived from Burgers' equation, leads to a challenging boundary value problem. Additional discussion can be found in reference [13, Sect. 2.8.31].

Example 13.1 brgr01: BOUNDARY LAYER EXAMPLE.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 1$	$t = 1$

Differential Variables: (y_1, y_2) ....

$y_1 = 2\tanh(\epsilon^{-1})$	$0 \leq y_1$	$y_1 = 0$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{13.1}$$

$$\dot{y}_2 = \epsilon^{-1} y_1 y_2 \tag{13.2}$$

where $\epsilon = 10^{-3}$.

Objective ....

Boundary Value Problem (BVP)

Chapter 14

brn2: **Two Burn Transfer, Modified Equinoctial Elements**

When placing a satellite into orbit it is common to break the mission design into two parts. For the first portion of the mission, a launch vehicle such as the space shuttle is used to reach a low-earth orbit. After this ascent trajectory, an "upper stage" vehicle is used to transfer the spacecraft from the park orbit to the mission orbit. When the transfer vehicle utilizes a high thrust propulsion system, the most efficient trajectory involves two distinct "burn" segments with a coast between. The dynamics for this type of problem incorporate a particular form of Newtonian mechanics, that utilize *modified equinoctial elements* as described in references [9], and [86]. Four different degrees of fidelity are used to model the physics of this trajectory in examples (14.1), (14.2), (14.3), and (14.4).

Example 14.1 brn201: Variable Attitude Steering, Spherical Earth.

Phase 1 .... *Coast in Park Orbit* .... Phase 1

Parameters: $(t_F^{(1)})$ ....
Independent Variable: (t) ....

$0 \le t \le t_F^{(1)}$

Differential Variables: (p, f, g, h, k, L) ....

$p = p_1$	$\underline{p}_1 \le p \le \overline{p}_1$	$\underline{p}_1 \le p \le \overline{p}_1$	ft
$f = 0$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = 0$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_1$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = 0$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = 180^o$	$\underline{L}_1 \le L \le \overline{L}_1$	$\underline{L}_1 \le L \le \overline{L}_1$	rad

Differential-Algebraic Equations ....

Equations (53.1) and (53.5)-(53.15) where $\boldsymbol{\delta}\mathbf{g} = 0$ and $\boldsymbol{\Delta} = 0$ and Table 14.1 summarizes the problem constants.

Phase 2 . . . . *First Burn* . . . . Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} \le t \le t_F^{(2)}$

Differential Variables: (p, f, g, h, k, L, w) . . . .

$p = p_F^{(1)}$	$\underline{p}_2 \le p \le \overline{p}_2$	$\underline{p}_2 \le p \le \overline{p}_2$	ft
$f = f_F^{(1)}$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = g_F^{(1)}$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_F^{(1)}$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_F^{(1)}$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_F^{(1)}$	$\underline{L}_2 \le L \le \overline{L}_2$	$\underline{L}_2 \le L \le \overline{L}_2$	rad
$w = 1$	$.01 \le w \le 1.1$	$.01 \le w \le 1.1$	lb

Algebraic Variables: (ψ, θ) . . . .

$-20^o \le \psi \le 20^o$	$-20^o \le \psi \le 20^o$	$-20^o \le \psi \le 20^o$	rad
$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	rad

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{y}} = \mathbf{A}(\mathbf{y})\boldsymbol{\Delta} + \mathbf{b} \tag{14.1}$$

$$\dot{w} = -T_c/I_{sp} \tag{14.2}$$

where the computational sequence (53.5)-(53.15) determines $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{Q}_r$ followed by the sequence (35.10)-(35.11) to define $\mathbf{T}$ giving

$$\boldsymbol{\Delta} = \mathbf{Q}_r^{\mathsf{T}}\mathbf{T} \tag{14.3}$$

Phase 3 . . . . *Coast in Transfer Orbit* . . . . Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} \le t \le t_F^{(3)}$

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(2)}$	$\underline{p}_3 \le p \le \overline{p}_3$	$\underline{p}_3 \le p \le \overline{p}_3$	ft
$f = f_F^{(2)}$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = g_F^{(2)}$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_F^{(2)}$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_F^{(2)}$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_F^{(2)}$	$\underline{L}_3 \le L \le \overline{L}_3$	$\underline{L}_3 \le L \le \overline{L}_3$	rad

Differential-Algebraic Equations ....

Use the phase 1 differential equations.

Phase 4 .... *Second Burn* .... **Phase 4**

Parameters: $(t_I^{(4)}, t_F^{(4)})$ ....
Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} \le t \le t_F^{(4)}$

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(3)}$	$\underline{p}_4 \le p \le \overline{p}_4$	$p = p_4$	ft
$f = f_F^{(3)}$	$-1 \le f \le 1$	$f = 0$	
$g = g_F^{(3)}$	$-1 \le g \le 1$	$g = 0$	
$h = h_F^{(3)}$	$-1 \le h \le 1$	$h = 0$	
$k = k_F^{(3)}$	$-1 \le k \le 1$	$k = 0$	
$L = L_F^{(3)}$	$\underline{L}_4 \le L \le \overline{L}_4$	$\underline{L}_4 \le L \le \overline{L}_4$	rad
$w = w_F^{(2)}$	$.01 \le w \le 1.1$	$.01 \le w \le 1.1$	lb

Algebraic Variables: (ψ, θ) ....

$0^o \le \psi \le 90^o$	$0^o \le \psi \le 90^o$	$0^o \le \psi \le 90^o$	rad
$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	rad

Boundary Conditions ....

$t_F^{(4)} - t_I^{(4)} \ge 1$

Differential-Algebraic Equations ....

Use the phase 2 differential equations.

Objective ....

Maximize $J = w(t_F^{(4)})$

$$\boxed{J^* = 2.36630183 \times 10^{-1}; \quad t_F^* = 2.1703807 \times 10^4}$$

Example 14.2 brn202: Variable Attitude Steering, Oblate Earth.

References: [9], [86]

Phase 1 .... *Coast in Park Orbit* .... Phase 1

Parameters: $(t_F^{(1)})$ ....

Independent Variable: (t) ....

$0 \le t \le t_F^{(1)}$

Differential Variables: (p, f, g, h, k, L) ....

$p = p_1$	$\underline{p}_1 \le p \le \overline{p}_1$	$\underline{p}_1 \le p \le \overline{p}_1$	ft
$f = 0$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = 0$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_1$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = 0$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = 180^o$	$\underline{L}_1 \le L \le \overline{L}_1$	$\underline{L}_1 \le L \le \overline{L}_1$	rad

Differential-Algebraic Equations ....

Equations (53.1) and (53.5)-(53.15) where $\boldsymbol{\delta}\mathbf{g}$ is given by (35.12)-(35.18) with

$$\boldsymbol{\Delta} = \mathbf{Q}_r^{\mathsf{T}} \boldsymbol{\delta}\mathbf{g} \tag{14.4}$$

and Table 14.1 summarizes the problem constants.

Phase 2............................ *First Burn*............................ Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....

Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$ $\qquad t_I^{(2)} \le t \le t_F^{(2)}$

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(1)}$	$\underline{p}_2 \le p \le \overline{p}_2$	$\underline{p}_2 \le p \le \overline{p}_2$	ft
$f = f_F^{(1)}$	$-1 \le f \le 1$	$-1 \le f \le 1$	

$g = g_F^{(1)}$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_F^{(1)}$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_F^{(1)}$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_F^{(1)}$	$\underline{L}_2 \le L \le \overline{L}_2$	$\underline{L}_2 \le L \le \overline{L}_2$	rad
$w = 1$	$.01 \le w \le 1.1$	$.01 \le w \le 1.1$	lb

Algebraic Variables: (ψ, θ) ....

$-20^o \le \psi \le 20^o$	$-20^o \le \psi \le 20^o$	$-20^o \le \psi \le 20^o$	rad
$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	rad

Differential-Algebraic Equations ....

$$\dot{\mathbf{y}} = \mathbf{A}(\mathbf{y})\boldsymbol{\Delta} + \mathbf{b} \tag{14.5}$$

$$\dot{w} = -T_c/I_{sp} \tag{14.6}$$

where the computational sequence (53.5)-(53.15) determines $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{Q}_r$ followed by the sequence (35.10)-(35.18) to define $\mathbf{T}$ and $\boldsymbol{\delta}\mathbf{g}$ giving

$$\boldsymbol{\Delta} = \mathbf{Q}_r^{\mathsf{T}} [\boldsymbol{\delta}\mathbf{g} + \mathbf{T}] \tag{14.7}$$

Phase 3 .... *Coast in Transfer Orbit* .... **Phase 3**

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....
Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} \le t \le t_F^{(3)}$

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(2)}$	$\underline{p}_3 \le p \le \overline{p}_3$	$\underline{p}_3 \le p \le \overline{p}_3$	ft
$f = f_F^{(2)}$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = g_F^{(2)}$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_F^{(2)}$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_F^{(2)}$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_F^{(2)}$	$\underline{L}_3 \le L \le \overline{L}_3$	$\underline{L}_3 \le L \le \overline{L}_3$	rad

Differential-Algebraic Equations ....

Use the phase 1 differential equations.

Phase 4	*Second Burn*	Phase 4

Parameters: $(t_I^{(4)}, t_F^{(4)})$ ....

Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} \leq t \leq t_F^{(4)}$		

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(3)}$	$\underline{p}_4 \leq p \leq \overline{p}_4$	$p = p_4$	ft
$f = f_F^{(3)}$	$-1 \leq f \leq 1$	$f = 0$	
$g = g_F^{(3)}$	$-1 \leq g \leq 1$	$g = 0$	
$h = h_F^{(3)}$	$-1 \leq h \leq 1$	$h = 0$	
$k = k_F^{(3)}$	$-1 \leq k \leq 1$	$k = 0$	
$L = L_F^{(3)}$	$\underline{L}_4 \leq L \leq \overline{L}_4$	$\underline{L}_4 \leq L \leq \overline{L}_4$	rad
$w = w_F^{(2)}$	$.01 \leq w \leq 1.1$	$.01 \leq w \leq 1.1$	lb

Algebraic Variables: (ψ, θ) ....

$0^o \leq \psi \leq 90^o$	$0^o \leq \psi \leq 90^o$	$0^o \leq \psi \leq 90^o$	rad
$-10^o \leq \theta \leq 10^o$	$-10^o \leq \theta \leq 10^o$	$-10^o \leq \theta \leq 10^o$	rad

Boundary Conditions ....

$t_F^{(4)} - t_I^{(4)} \geq 1$

Differential-Algebraic Equations ....

Use the phase 2 differential equations.

Objective ....

Maximize $\qquad J = w(t_F^{(4)})$

$$\boxed{J^* = 2.36724872 \times 10^{-1}; \quad t_F^* = 2.1683463 \times 10^4}$$

Example 14.3 brn203: Constant Attitude Steering, Spherical Earth.

Repeat example 14.1 with the following changes:

(a) In phase 2 modify the parameters as follows;

Parameters: $(\psi, \theta, t_I^{(2)}, t_F^{(2)})$ ....

$-20^o \leq \psi \leq 20^o \qquad -10^o \leq \theta \leq 10^o$

(b) In phase 2, omit the algebraic variables ψ and θ;

(c) In phase 4 modify the parameters as follows;
Parameters: $(\psi, \theta, t_I^{(4)}, t_F^{(4)})$ ....

$0^o \leq \psi \leq 90^o$ $\quad\quad$ $-20^o \leq \theta \leq 20^o$

(d) In phase 4, omit the algebraic variables ψ and θ;

$$\boxed{J^* = 2.35384459 \times 10^{-1}; \quad t_F^* = 2.1706984 \times 10^4}$$

Example 14.4 brn204: Constant Attitude Steering, Oblate Earth.

Repeat example 14.2 with the following changes:

(a) In phase 2 modify the parameters as follows;
Parameters: $(\psi, \theta, t_I^{(2)}, t_F^{(2)})$ ....

$-20^o \leq \psi \leq 20^o$ $\quad\quad$ $-10^o \leq \theta \leq 10^o$

(b) In phase 2, omit the algebraic variables ψ and θ;

(c) In phase 4 modify the parameters as follows;
Parameters: $(\psi, \theta, t_I^{(4)}, t_F^{(4)})$ ....

$0^o \leq \psi \leq 90^o$ $\quad\quad$ $-20^o \leq \theta \leq 20^o$

(d) In phase 4, omit the algebraic variables ψ and θ;

$$\boxed{J^* = 2.35477901 \times 10^{-1}; \quad t_F^* = 2.1686658 \times 10^4}$$

$T_c = 1.2$ lb	$I_{sp} = 300$ sec
$p_1 = 21837080.05283464$ ft	$p_4 = 138334442.2575590$ ft
$\mu = .1407645794 \times 10^{17}$	$h_1 = -0.2539676464749437$
$\underline{p}_1 = 2183708.005283465$ ft	$\overline{p}_1 = 109185399.2939946$ ft
$\underline{p}_2 = 2183707.985879892$ ft	$\overline{p}_2 = 188604942.2793254$ ft
$\underline{p}_3 = 3772098.845586507$ ft	$\overline{p}_3 = 188563079.4258044$ ft
$\underline{p}_4 = 3771261.588516088$ ft	$\overline{p}_4 = 691672211.2877948$ ft
$\underline{L}_1 = 90^o$	$\overline{L}_1 = 450^o$
$\underline{L}_2 = 270^o$	$\overline{L}_2 = 460^o$
$\underline{L}_3 = 280^o$	$\overline{L}_3 = 640^o$
$\underline{L}_4 = 460^o$	$\overline{L}_4 = 641^o$

Table 14.1. *Two Burn example constants.*

Chapter 15

capt: **Commercial Aircraft Trajectory Optimization**

The trajectory flown by a modern commercial aircraft is by design smooth and efficient. However, to achieve these goals using high fidelity models of the physical behavior, while also observing trajectory limitations imposed by international law and air traffic control, it is necessary to use a surprisingly complicated differentia-equation model of the dynamics. Although the vehicle parameters have been normalized, examples (15.1), (15.2), and (15.3) implement three different typical profiles, for a Boeing 767-200 ER flying from Seattle to Copenhagen. A more complete discussion can be found in reference [23], and details of the atmospheric model can be found in [26].

Example 15.1 capt01: Maximum Landing Weight.

Phase 1 .... *Climb: CAS = 250 knots* .... Phase 1

Parameters: $(t_F^{(1)})$ ....

$180 \le t_F^{(1)} \le 15$ hr	sec

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$

Differential Variables: (h, r, v, γ, w) ....

$h = 1517$	$0 \le h \le 69000$	$h = 10000$	ft
$r = 0$	$0 \le r \le 6000$	$0 \le r \le 6000$	nm
$v = v_I^{(1)}$	$1 \le v \le 1000$	$1 \le v \le 1000$	ft/sec
$0 \le \gamma \le 89^o$	$0 \le \gamma \le 89^o$	$0 \le \gamma \le 89^o$	rad
$w = 1$	$.528 \le w \le 1.06$	$.528 \le w \le 1.06$	lb

Algebraic Variables: (C_L) ....

$.1 \le C_L \le .9$ $.1 \le C_L \le .9$ $.1 \le C_L \le .9$

Differential-Algebraic Equations ....

$$\dot{h} = v \sin\gamma \tag{15.1}$$
$$\dot{r} = v \cos\gamma \tag{15.2}$$
$$\dot{v} = \frac{1}{m}(T - D) - g \sin\gamma \tag{15.3}$$
$$\dot{\gamma} = \frac{1}{v}\left(\frac{L}{m} - g\cos\gamma\right) \tag{15.4}$$
$$\dot{w} = d_w \tag{15.5}$$
$$0 = \dot{s}_C(t) \tag{15.6}$$

where $m = w/g_0$ and

$$T = d_T(M, h, \tau)\delta \tag{15.7}$$
$$d_w = d_w(d_T, M, h) \tag{15.8}$$
$$L = C_L q S \tag{15.9}$$
$$C_D = d_a(C_L, M) + d_R(w, h) \tag{15.10}$$
$$D = C_D q S \tag{15.11}$$
$$\delta = \frac{p}{p_0} \tag{15.12}$$
$$M = \frac{v}{v_c} \tag{15.13}$$
$$q = \frac{1}{2}\rho v^2. \tag{15.14}$$

The specified calibrated airspeed (CAS) in knots $\widehat{V}$ leads to the algebraic constraint (15.6) which is defined as follows:

$$s_C(t) = v - v_c\sqrt{k} \tag{15.15}$$
$$C = \sqrt{k} \tag{15.16}$$
$$\dot{s}_C(t) = \dot{v} - \dot{h}\left[v_c\left(\frac{dC}{dp}\right)\frac{dp}{dh} + C\frac{dv_c}{dh}\right] \tag{15.17}$$
$$\frac{dC}{dp} = \left(\frac{1}{2C}\right)\frac{dk}{dp} \tag{15.18}$$
$$\frac{dk}{dp} = -5\alpha\left[\frac{k_1}{p} + 1\right]^{\alpha-1} k_1 p^{-2} \tag{15.19}$$
$$\frac{dv_c}{dh} = \left(\frac{k_2}{2\sqrt{\tau}}\right)\frac{d\tau}{dh} \tag{15.20}$$
$$k = 5\left[\left(\frac{k_1}{p} + 1\right)^{\alpha} - 1\right] \tag{15.21}$$
$$k_0 = 1 + \frac{1}{5}\left(\frac{\widehat{V}\sigma_1}{v_{c0}}\right)^2 \tag{15.22}$$

$$k_1 = p_0 \left(k_0^{1/\alpha} - 1 \right) \tag{15.23}$$

The quantities ρ, p, τ, v_c, $d\rho/dh$, dp/dh, and $d\tau/dh$ are functions of h constructed as cubic spline approximations of the 1962 Standard Atmosphere [26]. Table 15.1 defines the parameters for the example. At $t = 0$, $h = 1517$, and with $\widehat{V} = 250$ from (15.15) $s_C(0) = v_I^{(1)} - v_c\sqrt{k} = 0$ which defines the initial velocity $v_I^{(1)}$. The quantities $d_T(M, h, \tau)$, $d_w(d_T, M, h)$, $d_a(C_L, M)$, and $d_R(w, h)$ are multivariate spline approximations to tabular data.

Phase 2...................... *Climb: ROC = 500 ft/min*...................... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....

$180 \le t_I^{(2)} \le 15$ hr	$180 \le t_F^{(2)} \le 15$ hr	sec

Independent Variable: (t) ....

$t = t_I^{(2)} = t_F^{(1)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$

Differential Variables: (h, r, v, γ, w) ....

$h = h_F^{(1)}$	$0 \le h \le 69000$	$0 \le h \le 69000$	ft
$r = r_F^{(1)}$	$0 \le r \le 6000$	$0 \le r \le 6000$	nm
$v = v_F^{(1)}$	$1 \le v \le 1000$	$1 \le v \le 1000$	ft/sec
$0 \le \gamma \le 89^o$	$0 \le \gamma \le 89^o$	$0 \le \gamma \le 89^o$	rad
$w = w_F^{(1)}$	$.528 \le w \le 1.06$	$.528 \le w \le 1.06$	lb

Algebraic Variables: (C_L) ....

$.1 \le C_L \le .9$	$.1 \le C_L \le .9$	$.1 \le C_L \le .9$

Boundary Conditions ....

$s_R(t_I^{(2)}) = 0$
$t_F^{(2)} - t_I^{(2)} \ge 10$

Differential-Algebraic Equations ....

Equations (15.1) - (15.14)

To achieve a specified rate of climb (ROC) the algebraic constraint (15.6) is omitted and replaced by

$$0 = \dot{s}_R(t) \tag{15.24}$$

where $\widehat{R} = 500\sigma_2$ ft/sec (500 ft/min) and

$$s_R(t) = v \sin\gamma - \widehat{R} \tag{15.25}$$

$$\dot{s}_R(t) = \dot{v} \sin\gamma + v \cos\gamma\dot{\gamma} \tag{15.26}$$

Phase 3 .... *Climb: CAS = 314 knots* .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....

$180 \le t_I^{(3)} \le 15$ hr	$180 \le t_F^{(3)} \le 15$ hr	sec

Independent Variable: (t) ....

$t = t_I^{(3)} = t_F^{(2)}$	$t_I^{(3)} < t < t_F^{(3)}$	$t = t_F^{(3)}$

Differential Variables: (h, r, v, γ, w) ....

$h = h_F^{(2)}$	$0 \le h \le 69000$	$0 \le h \le 69000$	ft
$r = r_F^{(2)}$	$0 \le r \le 6000$	$0 \le r \le 6000$	nm
$v = v_F^{(2)}$	$1 \le v \le 1000$	$1 \le v \le 1000$	ft/sec
$0 \le \gamma \le 89^o$	$0 \le \gamma \le 89^o$	$0 \le \gamma \le 89^o$	rad
$w = w_F^{(2)}$	$.528 \le w \le 1.06$	$.528 \le w \le 1.06$	lb

Algebraic Variables: (C_L) ....

$.1 \le C_L \le .9$	$.1 \le C_L \le .9$	$.1 \le C_L \le .9$

Boundary Conditions ....

$s_C(t_I^{(3)}) = 0$
$t_F^{(3)} - t_I^{(3)} \ge 10$

Differential-Algebraic Equations ....

Equations (15.1) - (15.23)

The specified CAS value is $\widehat{V} = 314$.

Phase 4 .... *Climb: M = .8* .... Phase 4

Parameters: $(t_I^{(4)}, t_F^{(4)})$ ....

$180 \leq t_I^{(4)} \leq 15$ hr	$180 \leq t_F^{(4)} \leq 15$ hr	sec

Independent Variable: (t) ....

$t = t_I^{(4)} = t_F^{(3)}$	$t_I^{(4)} < t < t_F^{(4)}$	$t = t_F^{(4)}$

Differential Variables: (h, r, v, γ, w) ....

$h = h_F^{(3)}$	$0 \leq h \leq 69000$	$h = 31000$	ft
$r = r_F^{(3)}$	$0 \leq r \leq 6000$	$0 \leq r \leq 6000$	nm
$v = v_F^{(3)}$	$1 \leq v \leq 1.125 v_F^{(4)}$	$v = v_F^{(4)}$	ft/sec
$0 \leq \gamma \leq 89^o$	$0 \leq \gamma \leq 89^o$	$0 \leq \gamma \leq 89^o$	rad
$w = w_F^{(3)}$	$.528 \leq w \leq 1.06$	$.528 \leq w \leq 1.06$	lb

Algebraic Variables: (C_L) ....

$.1 \leq C_L \leq .9$	$.1 \leq C_L \leq .9$	$.1 \leq C_L \leq .9$

Boundary Conditions ....

$t_F^{(4)} - t_I^{(4)} \geq 10$

Differential-Algebraic Equations ....

Equations (15.1) - (15.14)

In order to climb at a constant Mach number M, the algebraic constraint (15.6) is omitted and replaced by

$$0 = \dot{s}_M(t) \tag{15.27}$$

where $M = \widehat{M} = .8$ and

$$s_M(t) = v - \widehat{M} v_c \tag{15.28}$$

$$\dot{s}_M(t) = \dot{v} - \widehat{M} \frac{dv_c}{dh} \dot{h} \tag{15.29}$$

The final velocity $v_F^{(4)}$ satisfies the condition $s_M(t_F^{(4)}) = v_F^{(4)} - \widehat{M} v_c = 0$ evaluated at $h = 31000$.

Phase 5 .... *Cruise: $M = .8$, $h = 31000$ ft* .... Phase 5

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....

$180 \leq t_I^{(5)} \leq 15$ hr	$180 \leq t_F^{(5)} \leq 15$ hr	sec

Independent Variable: (t) ....

$t = t_I^{(5)} = t_F^{(4)}$	$t_I^{(5)} < t < t_F^{(5)}$	$t = t_F^{(5)}$

Differential Variables: (w) ....

$w = w_F^{(4)}$	$.528 \leq w \leq 1.06$	$.528 \leq w \leq 1.06$	lb

Boundary Conditions ....

$t_F^{(5)} - t_I^{(5)} \geq 10$

Differential-Algebraic Equations ....

$$\dot{w} = d_w(d_T, M, h) \tag{15.30}$$

where

$$d_T = \frac{T}{\delta} = \frac{D}{\delta} \tag{15.31}$$

$$C_L = \frac{L}{qS} = \frac{w}{qS} \tag{15.32}$$

where all quantities (15.10)-(15.14) are evaluated at $M = \widehat{M} = .8$ and $h = 31000$.

Phase 6 .... *Climb: $M = .8$* .... **Phase 6**

Parameters: $(t_I^{(6)}, t_F^{(6)})$ ....

$180 \leq t_I^{(6)} \leq 15$ hr	$180 \leq t_F^{(6)} \leq 15$ hr	sec

Independent Variable: (t) ....

$t = t_I^{(6)} = t_F^{(5)}$	$t_I^{(6)} < t < t_F^{(6)}$	$t = t_F^{(6)}$

Differential Variables: (h, r, v, γ, w) ....

$h = 31000$	$0 \leq h \leq 69000$	$h = 35000$	ft
$r = r_I^{(6)}$	$0 \leq r \leq 6000$	$0 \leq r \leq 6000$	nm
$v_F^{(5)} \leq v \leq v_F^{(6)}$	$1 \leq v \leq v_F^{(6)}$	$1 \leq v \leq v_F^{(6)}$	ft/sec
$0 \leq \gamma \leq 89^o$	$0 \leq \gamma \leq 89^o$	$0 \leq \gamma \leq 89^o$	rad
$w = w_F^{(5)}$	$.528 \leq w \leq 1.06$	$.528 \leq w \leq 1.06$	lb

Algebraic Variables: (C_L) ....

$.1 \le C_L \le .9$	$.1 \le C_L \le .9$	$.1 \le C_L \le .9$

Boundary Conditions ....

$r_I^{(6)} = r_F^{(4)} + v_F^{(4)} \left[t_F^{(5)} - t_I^{(5)} \right]$
$t_F^{(6)} - t_I^{(6)} \ge 10$

Differential-Algebraic Equations ....

Equations (15.1) - (15.14)

The algebraic constraint (15.6) is omitted and replaced by

$$0 = \dot{s}_M(t) \tag{15.33}$$

where $M = \widehat{M} = .8$ and

$$s_M(t) = v - \widehat{M} v_c \tag{15.34}$$

$$\dot{s}_M(t) = \dot{v} - \widehat{M} \frac{dv_c}{dh} \dot{h} \tag{15.35}$$

The velocity $v_F^{(5)}$ satisfies the condition $s_M(t_F^{(5)}) = v_F^{(5)} - \widehat{M} v_c = 0$ evaluated at $h = 31000$. The bound $v_F^{(6)} = .9 v_c$ at $h = 35000$.

Phase 7 .... *Cruise: $M = .8$, $h = 35000$ ft* .... Phase 7

Parameters: $(t_I^{(7)}, t_F^{(7)})$ ....

$180 \le t_I^{(7)} \le 15$ hr	$180 \le t_F^{(7)} \le 15$ hr	sec

Independent Variable: (t) ....

$t = t_I^{(7)} = t_F^{(6)}$	$t_I^{(7)} < t < t_F^{(7)}$	$t = t_F^{(7)}$

Differential Variables: (w) ....

$w = w_F^{(6)}$	$.528 \le w \le 1.06$	$.528 \le w \le 1.06$	lb

Boundary Conditions ....

$t_F^{(7)} - t_I^{(7)} \ge 10$

Differential-Algebraic Equations . . . .

Equations (15.30) - (15.32)

where all quantities are evaluated at $M = \widehat{M} = .8$ and $h = 35000$.

Phase 8 . . . . *Decelerate at Idle Thrust: h = 35000 ft* . . . . Phase 8

Parameters: $(t_I^{(8)}, t_F^{(8)})$ . . . .

$180 \le t_I^{(8)} \le 15$ hr	$180 \le t_F^{(8)} \le 15$ hr	sec

Independent Variable: (t) . . . .

$t = t_I^{(8)} = t_F^{(7)}$	$t_I^{(8)} < t < t_F^{(8)}$	$t = t_F^{(8)}$

Differential Variables: (r, v, w) . . . .

$r = r_I^{(8)}$	$0 \le r \le 6000$	$0 \le r \le 6000$	nm
$v = v_I^{(8)}$	$500 \le v \le 870$	$v = v_F^{(8)}$	ft/sec
$w = w_F^{(7)}$	$.528 \le w \le 1.06$	$.528 \le w \le 1.06$	lb

Boundary Conditions . . . .

$r_I^{(8)} = r_F^{(6)} + v_F^{(6)} \left[t_F^{(7)} - t_I^{(7)} \right]$
$t_F^{(8)} - t_I^{(8)} \ge 1$

Differential-Algebraic Equations . . . .

$$\dot{r} = v \tag{15.36}$$

$$\dot{v} = \frac{1}{m}(T - D) \tag{15.37}$$

$$\dot{w} = d_w \tag{15.38}$$

where

$$T = d_T(M, h, \tau)\delta \tag{15.39}$$

$$d_w = d_w(d_T, M, h) \tag{15.40}$$

$$C_L = \frac{L}{qS} = \frac{w}{qS} \tag{15.41}$$

in addition to the quantities (15.11)-(15.14) evaluated at $h = 35000$. The initial velocity $v_I^{(8)}$ satisfies the condition $s_M(t_I^{(8)}) = v_I^{(8)} - \widehat{M} v_c = 0$ given by (15.34) evaluated at

$M = \widehat{M} = .8$ and $h = 35000$. The final velocity $v_F^{(8)}$ satisfies the condition $s_c(t_F^{(8)}) = 0$ defined by (15.15) with $\widehat{V} = 250$ and $h = 35000$.

Phase 9 .... *Descent: CAS = 250 knots* .... Phase 9

Parameters: $(t_I^{(9)}, t_F^{(9)})$ ....

$180 \le t_I^{(9)} \le 15$ hr	$180 \le t_F^{(9)} \le 15$ hr	sec

Independent Variable: (t) ....

$t = t_I^{(9)} = t_F^{(8)}$	$t_I^{(9)} < t < t_F^{(9)}$	$t = t_F^{(9)}$

Differential Variables: (h, r, v, γ, w) ....

$h = 35000$	$0 \le h \le 69000$	$h = h_F^{(9)}$	ft
$r = r_F^{(8)}$	$0 \le r \le 6000$	$r = r_F^{(9)}$	nm
$v = v_F^{(8)}$	$1 \le v \le 1000$	$1 \le v \le 1000$	ft/sec
$-10^o \le \gamma \le 0$	$-10^o \le \gamma \le 0$	$-10^o \le \gamma \le 0$	rad
$w = w_F^{(8)}$	$.528 \le w \le 1.06$	$.528 \le w \le 1.06$	lb

Algebraic Variables: (C_L) ....

$.1 \le C_L \le .9$	$.1 \le C_L \le .9$	$.1 \le C_L \le .9$

Differential-Algebraic Equations ....

Equations (15.1) - (15.22)

where all quantities are evaluated with $\widehat{V} = 250$.

Objective ....

Maximize $\qquad J = w(t_F^{(9)})$

$$\boxed{J^* = .739845423 \text{ lb}; \quad t_F^* = 9.4295147 \text{ hr}}$$

Example 15.2 capt03: Maximum Range.

Repeat example 15.1 and omit the constraint in phase 9 to fix $r = r_F^{(9)}$. Replace the objective function with

Objective ....

$v_I^{(1)} = 431.04522212325520$	$v_F^{(4)} = v_F^{(5)} = 792.01573276586521$
$v_F^{(6)} = 878.32970937394043$	$v_I^{(8)} = 780.73752978474329$
$v_F^{(8)} = 722.55568194445641$	$h_F^{(9)} = 1929$
$r_F^{(9)} = 4310.9$	$S = 8.051147 \times 10^{-3}$
$\alpha = 1/3.5$	$\sigma_1 = 1.6878098571011944$ fps/knot
$k_2 = 49.02232469$	$\sigma_2 = 1/60$ sec/min

Table 15.1. *Commercial Aircraft example parameters.*

Maximize $$J = r(t_F^{(9)})$$

$$\boxed{J^* = 4327.93420 \text{ nm}; \quad t_F^* = 9.4663081 \text{ hr}}$$

Example 15.3 capt05: Minimum Takeoff Weight.

Repeat example 15.1 and add the constraint in phase 9 to fix $w = w_F^{(9)} = .739845423$ lb. Replace the objective function with

Objective ....

Minimize $$J = w(t_I^{(1)})$$

$$\boxed{J^* = .998843764 \text{ lb.}; \quad t_F^* = 9.4299183 \text{ hr}}$$

Chapter 16

chan: **Kinematic Chain**

Büskens and Gerdts [48] present an example that requires control of a *multibody system*. The problem is interesting because it can be made arbitrarily large and requires the treatment of an index 2 DAE system as described in reference [13, Sect. 6.11]. Example (16.1) defines the DAE problem formulation and the ODE formulation is given as example (16.2).

Example 16.1 chan01: Multibody System–DAE Formulation.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\quad\quad$ $0 < t < 1$ $\quad\quad$ $t = 1$

Differential Variables: $(\mathbf{p}^\mathsf{T}, \mathbf{v}^\mathsf{T})$ ....

$$\mathbf{p}_k = \begin{pmatrix} (k-1)l_k \\ 0 \\ l_k \\ 0 \end{pmatrix} \qquad k = 1, \dots, \nu$$

$$\mathbf{p}_{\nu+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{v} = \mathbf{0}$$

where $\mathbf{p}^\mathsf{T} = \left(\mathbf{p}_1^\mathsf{T}, \dots, \mathbf{p}_{\nu+1}^\mathsf{T}\right)$ and $\mathbf{v}^\mathsf{T} = \left(\mathbf{v}_1^\mathsf{T}, \dots, \mathbf{v}_{\nu+1}^\mathsf{T}\right)$.

Algebraic Variables: $(\mathbf{q}^\mathsf{T}, \boldsymbol{\lambda}^\mathsf{T}, \mathbf{u}^\mathsf{T})$ ....

Differential-Algebraic Equations ....

$$\dot{\mathbf{p}} = \mathbf{v} \tag{16.1}$$

$$\dot{\mathbf{v}} = \mathbf{q}, \tag{16.2}$$

$$\mathbf{0} = \mathbf{M}\mathbf{q} - \mathbf{f}(\mathbf{p}, \mathbf{v}, \mathbf{u}) + \mathbf{C}^{\mathsf{T}}(\mathbf{p})\boldsymbol{\lambda} - \mathbf{K}\mathbf{u}, \tag{16.3}$$

$$\mathbf{0} = \dot{\mathbf{C}}\mathbf{v} + \mathbf{C}\mathbf{q} \tag{16.4}$$

where

$$\mathbf{C}(\mathbf{p}) = \begin{bmatrix} \mathbf{C}_1(\mathbf{p}_1) & \mathbf{P}_1 & & \\ & \ddots & \ddots & \\ & & \mathbf{C}_\nu(\mathbf{p}_\nu) & \mathbf{P}_\nu \end{bmatrix} \tag{16.5}$$

$$\mathbf{C}_k(\mathbf{p}_k) = \begin{bmatrix} (0,0) & \mathbf{d}_k^{\mathsf{T}} \\ \mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix} \tag{16.6}$$

$$\mathbf{P}_k = \begin{bmatrix} \mathbf{0}_2 & \mathbf{0}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 \end{bmatrix} \tag{16.7}$$

for $k = 1, \ldots, (\nu - 1)$ and

$$\mathbf{P}_\nu = \begin{bmatrix} \mathbf{0}_2 \\ -\mathbf{I}_2 \end{bmatrix} \tag{16.8}$$

$$\mathbf{0}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{16.9}$$

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16.10}$$

In addition with $\mathbf{x}_k^{\mathsf{T}} = (x_k, y_k)$ for $k = 1, \ldots, \nu$

$$\mathbf{p}_k = \begin{pmatrix} \mathbf{x}_k \\ \mathbf{d}_k \end{pmatrix} \tag{16.11}$$

$$\mathbf{p}_{\nu+1} = \mathbf{x}_{\nu+1} \tag{16.12}$$

$$\dot{\mathbf{C}}(\mathbf{p}) = \begin{bmatrix} \dot{\mathbf{C}}_1(\mathbf{p}_1) & \dot{\mathbf{P}}_1 & & \\ & \ddots & \ddots & \\ & & \dot{\mathbf{C}}_\nu(\mathbf{p}_\nu) & \dot{\mathbf{P}}_\nu \end{bmatrix} \tag{16.13}$$

$$\dot{\mathbf{C}}_k(\mathbf{p}_k) = \begin{bmatrix} (0,0) & \dot{\mathbf{d}}_k^{\top} \\ \mathbf{0}_2 & \mathbf{0}_2 \end{bmatrix} = \begin{bmatrix} (0,0) & (v_{k,3}, v_{k,4}) \\ \mathbf{0}_2 & \mathbf{0}_2 \end{bmatrix} \tag{16.14}$$

$$\dot{\mathbf{P}}_k = \mathbf{0} \tag{16.15}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & & & & \\ & \mathbf{M}_2 & & & \\ & & \ddots & & \\ & & & \mathbf{M}_\nu & \\ & & & & \mathbf{0}_2 \end{bmatrix} \tag{16.16}$$

where

$$\mathbf{M}_1 = \left(2 + \nu^{-1}\right) \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{12} \end{bmatrix} \tag{16.17}$$

and for $k = 2, \ldots, \nu$

$$\mathbf{M}_k = \nu^{-1} \begin{bmatrix} \mathbf{I}_2 & \frac{1}{2}\mathbf{I}_2 \\ \frac{1}{2}\mathbf{I}_2 & \frac{1}{3}\mathbf{I}_2 \end{bmatrix}. \tag{16.18}$$

The matrix $\mathbf{K}$ is diagonal with

$$K_{i,i} = \begin{cases} 1 & i = 1 \\ 0 & i = 2, \ldots, 4\nu + 2. \end{cases} \tag{16.19}$$

$$\mathbf{f}^{\mathsf{T}}(\mathbf{p}, \mathbf{v}, \mathbf{u}) = (\mathbf{f}_1^{\mathsf{T}}, \mathbf{f}_2^{\mathsf{T}}, \ldots, \mathbf{f}_\nu^{\mathsf{T}}, 0, 0) \tag{16.20}$$

$$\mathbf{f}_k^{\mathsf{T}} = \begin{cases} (0, 0, 0, 0) & k = 1 \\ -g\nu^{-1}(0, 1, 0, \frac{1}{2}) & k = 2, \ldots, \nu. \end{cases} \tag{16.21}$$

where the problem constants are $g = 9.81$, $l_k = \nu^{-1}$ and $\nu = 5$.

Objective ....

Minimize $J = 1000 \int_0^1 x_1^2(t)dt + 1000 \int_0^1 y_1^2(t)dt + \frac{1}{1000} \int_0^1 u^2(t)dt$

$\boxed{J^* = 6.44798005 \times 10^{-2}}$

Example 16.2 chan03: Multibody System–ODE Formulation.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\quad 0 < t < 1$ $\quad t = 1$

Differential Variables: $(\mathbf{p}^{\mathsf{T}}, \mathbf{v}^{\mathsf{T}})$ ....

$\mathbf{p}_k = \begin{pmatrix} (k-1)l_k \\ 0 \\ l_k \\ 0 \end{pmatrix} \qquad k = 1, \ldots, \nu$

$\mathbf{p}_{\nu+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mathbf{v} = \mathbf{0}$

where $\mathbf{p}^{\mathsf{T}} = \left(\mathbf{p}_1^{\mathsf{T}}, \ldots, \mathbf{p}_{\nu+1}^{\mathsf{T}}\right)$ and $\mathbf{v}^{\mathsf{T}} = \left(\mathbf{v}_1^{\mathsf{T}}, \ldots, \mathbf{v}_{\nu+1}^{\mathsf{T}}\right)$.

Algebraic Variables: (u) ....

Differential-Algebraic Equations ....

$$\dot{\mathbf{p}} = \mathbf{v} \tag{16.22}$$

$$\dot{\mathbf{v}} = \mathbf{q}, \tag{16.23}$$

where $\mathbf{q}$ is the solution of

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}^\top \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} + \mathbf{K}\mathbf{u} \\ -\dot{\mathbf{C}}\mathbf{v} \end{bmatrix} \tag{16.24}$$

and the remaining quantities are defined in example 16.1.

$$\boxed{J^* = 6.44797578 \times 10^{-2}}$$

Chapter 17

chmr: **Chemical Reactor, Bounded Control**

Citron [37] introduces a chemical reactor problem to illustrate how the shooting method can be used to solve an optimal control boundary value problem. Ten different versions of the problem are presented here using various parameter values with bounds on the control variable. Since the differential equations cannot be evaluated for negative values of the control this presents an issue of computational concern for some solution techniques.

Example 17.1 chmr01 ... chmr10: CHEMICAL REACTOR, BOUNDED CONTROL.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 50$	$t = t_F$

Differential Variables: (x, y) ....

$x = 1$	$-0.1 \leq x \leq 1.1$	$-0.1 \leq x \leq 1.1$
$y = .01$	$-0.1 \leq y \leq 1.1$	$-0.1 \leq y \leq 1.1$

Algebraic Variables: (a) ....

$a_L \leq a \leq a_U$	$a_L \leq a \leq a_U$	$a_L \leq a \leq a_U$

Differential-Algebraic Equations ....

$$\dot{x} = -ax \tag{17.1}$$

$$\dot{y} = ax - \rho a^k y \tag{17.2}$$

where $\rho = 2.5$ and the remaining problem data for all examples is given in Table 17.1.

Objective ....

Maximize $$J = y(t_F)$$

Example	a_L	a_U	t_F	k	J^*
chmr01	.1	.5	2	1.5	.308132175
chmr02	.1	.5	4	1.5	.357577681
chmr03	.1	.5	8	1.5	.405612132
chmr04	.1	.2	2	1.5	.268290897
chmr05	.1	.3	2	1.5	.300129483
chmr06	.1	.4	2	1.5	.306107715
chmr07	.01	8	2	1.5	.310412612
chmr08	.01	8	4	1.5	.358058254
chmr09	.01	8	8	1.5	.408711527
chmr10	.1	.5	2	.5	.168229579

Table 17.1. *Chemical Reactor example constants and solution*

Chapter 18

clym: **Minimum Time to Climb**

The original minimum time to climb problem was presented by Bryson, Desai, and Hoffman [28] and has been the subject of many analyses since then. The basic problem is to choose the optimal control function, the angle of attack, such that an airplane flies from a point on a runway to a specified final altitude as quickly as possible. The problem specification includes tabular data for the aerodynamic and thrust forces. The counter intuitive optimal solution consists of a climb, followed by a dive and then a "zoom" climb to the terminal state. Since the original study was of interest to the military during the Vietnam war era, the results were very controversial when first reported. Treatment of the tabular data and a complete discussion of the results is given in reference [13, Sect. 6.2]. Two versions of the problem are defined by examples (18.1), and (18.2), respectively.

Example 18.1 clym04: MINIMUM TIME TO CLIMB.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$0 \leq t_F$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ ....

$h = 0$	$0 \leq h \leq 69000$	$h = 65617$	ft
$\phi = 0$	$-10^o \leq \phi \leq 10^o$		rad
$\theta = 0$	$-89^o \leq \theta \leq 89^o$		rad
$v = 380$	$1 \leq v \leq 2000$	$v = 986.5$	ft/sec

$\gamma = 1.7^o$	$-89^o \le \gamma \le 89^o$	$\gamma = 0$	rad
$\psi = 0$	$-90^o \le \psi \le 90^o$		rad
$w = 41955$	$0 \le w \le 45000$		lb

Algebraic Variables: (α) ....

$-20^o \le \alpha \le 20^o$	$-20^o \le \alpha \le 20^o$	$-20^o \le \alpha \le 20^o$	rad

Differential-Algebraic Equations ....

$$\dot{h} = v \sin\gamma \tag{18.1}$$

$$\dot{\phi} = 0 \tag{18.2}$$

$$\dot{\theta} = \frac{v}{r}\cos\gamma \tag{18.3}$$

$$\dot{v} = \frac{T\cos\alpha - D}{m} - g\sin\gamma \tag{18.4}$$

$$\dot{\gamma} = \frac{(T\sin\alpha + L)}{mv} + \cos\gamma\left(\frac{v}{r} - \frac{g}{v}\right) \tag{18.5}$$

$$\dot{\psi} = 0 \tag{18.6}$$

$$\dot{w} = -\frac{T}{I_{sp}} \tag{18.7}$$

where the problem parameters are given in Table 18.1. The functions $T(M,h)$, $c_{D0}(M)$, $\eta(M)$, and $c_{L\alpha}(M)$ are represented by cubic spline interpolants constructed from the data in Tables 18.2 and 18.3. A smooth approximation to the 1962 standard atmosphere [26] is used to compute $\rho = \rho(h)$ and $v_c(h)$, where $M = v/v_c(h)$.

Objective ....

Minimize $J = t_F$

$$\boxed{J^* = 325.040141}$$

Example 18.2 clym13: Minimum Time to Climb; Planar.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$0 \le t_F$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$	sec

Differential Variables: $(h, \theta, v, \gamma, w)$ ....

$h = 0$	$0 \leq h \leq 69000$	$h = 65617$	ft
$\theta = 0$	$-89^o \leq \theta \leq 89^o$		rad
$v = 380$	$1 \leq v \leq 2000$	$v = 986.5$	ft/sec
$\gamma = 1.7^o$	$-89^o \leq \gamma \leq 89^o$	$\gamma = 0$	rad
$w = 41955$	$0 \leq w \leq 45000$		lb

Algebraic Variables: (α) ....

$-20^o \leq \alpha \leq 20^o$	$-20^o \leq \alpha \leq 20^o$	$-20^o \leq \alpha \leq 20^o$	rad

Differential-Algebraic Equations ....

$$\dot{h} = v \sin\gamma \tag{18.8}$$

$$\dot{\theta} = \frac{v}{r}\cos\gamma \tag{18.9}$$

$$\dot{v} = \frac{T\cos\alpha - D}{m} - g\sin\gamma \tag{18.10}$$

$$\dot{\gamma} = \frac{(T\sin\alpha + L)}{mv} + \cos\gamma\left(\frac{v}{r} - \frac{g}{v}\right) \tag{18.11}$$

$$\dot{w} = -\frac{T}{I_{sp}} \tag{18.12}$$

where the problem parameters are given in Table 18.1. The functions $T(M,h)$, $c_{D0}(M)$, $\eta(M)$, and $c_{L\alpha}(M)$ are represented by cubic spline interpolants constructed from the data in Tables 18.2 and 18.3. A smooth approximation to the 1962 standard atmosphere [26] is used to compute $\rho = \rho(h)$ and $v_c(h)$, where $M = v/v_c(h)$.

Objective ....

Minimize $\qquad J = t_F$

$$\boxed{J^* = 325.040141}$$

$D = \frac{1}{2}c_D S\rho v^2$	$c_D = c_{D0}(M) + \eta(M)c_{L\alpha}(M)\alpha^2$
$L = \frac{1}{2}c_L S\rho v^2$	$c_L = c_{L\alpha}(M)\alpha$
$g = \mu/r^2$	$r = R_e + h$
$\mu = 0.14076539 \times 10^{17}$	$R_e = 20902900$
$S = 530$	$I_{sp} = 1600$
$m = w/g_0$	$g_0 = 32.174$

Table 18.1. *Minimum Time to Climb parameters.*

Thrust $T(M, h)$ (thousands of lb)

	Altitude h (thousands of ft)									
M	0	5	10	15	20	25	30	40	50	70
0.0	24.2									
0.2	28.0	24.6	21.1	18.1	15.2	12.8	10.7			
0.4	28.3	25.2	21.9	18.7	15.9	13.4	11.2	7.3	4.4	
0.6	30.8	27.2	23.8	20.5	17.3	14.7	12.3	8.1	4.9	
0.8	34.5	30.3	26.6	23.2	19.8	16.8	14.1	9.4	5.6	1.1
1.0	37.9	34.3	30.4	26.8	23.3	19.8	16.8	11.2	6.8	1.4
1.2	36.1	38.0	34.9	31.3	27.3	23.6	20.1	13.4	8.3	1.7
1.4		36.6	38.5	36.1	31.6	28.1	24.2	16.2	10.0	2.2
1.6				38.7	35.7	32.0	28.1	19.3	11.9	2.9
1.8						34.6	31.1	21.7	13.3	3.1

Table 18.2. *Propulsion data.*

M	0	0.4	0.8	0.9	1.0	1.2	1.4	1.6	1.8
$c_{L\alpha}$	3.44	3.44	3.44	3.58	4.44	3.44	3.01	2.86	2.44
c_{D0}	0.013	0.013	0.013	0.014	0.031	0.041	0.039	0.036	0.035
η	0.54	0.54	0.54	0.75	0.79	0.78	0.89	0.93	0.93

Table 18.3. *Aerodynamic data.*

Chapter 19

cran: Container Crane Problem

Augustin and Maurer [3] describe a modified version of a model originally developed by Sakawa and Shindo. The problem requires control of a container crane subject to state constraints on the vertical velocity. Augustin and Maurer demonstrate second order sufficient conditions for this rather complex application.

Example 19.1 cran01: MINIMUM CONTROL ENERGY.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 9$	$t = 9$

Differential Variables: $(x_1, x_2, x_3, x_4, x_5, x_6)$ ....

$x_1 = 0$		$x_1 = 10$
$x_2 = 22$		$x_2 = 14$
$x_3 = 0$		$x_3 = 0$
$x_4 = 0$	$-2.5 \leq x_4 \leq 2.5$	$x_4 = 2.5$
$x_5 = -1$	$-1 \leq x_5 \leq 1$	$x_5 = 0$
$x_6 = 0$		$x_6 = 0$

Algebraic Variables: (u_1, u_2) ....

$-c_1 \leq u_1 \leq c_1$	$-c_1 \leq u_1 \leq c_1$	$-c_1 \leq u_1 \leq c_1$
$c_2 \leq u_2 \leq c_3$	$c_2 \leq u_2 \leq c_3$	$c_2 \leq u_2 \leq c_3$

Differential-Algebraic Equations ....

$$\dot{x}_1 = x_4 \tag{19.1}$$

$$\dot{x}_2 = x_5 \tag{19.2}$$
$$\dot{x}_3 = x_6 \tag{19.3}$$
$$\dot{x}_4 = u_1 + c_4 x_3 \tag{19.4}$$
$$\dot{x}_5 = u_2 \tag{19.5}$$
$$\dot{x}_6 = -\left[u_1 + c_5 x_3 + 2 x_5 x_6\right] / x_2 \tag{19.6}$$

where $\rho = .01$, $c_1 = 2.83374$, $c_2 = -.80865$, $c_3 = .71265$, $c_4 = 17.2656$, and $c_5 = 27.0756$.

Objective ....

Minimize $$J = \frac{1}{2} \int_0^9 \left[x_3^2 + x_6^2 + \rho(u_1^2 + u_2^2)\right] \; dt$$

$$\boxed{J^* = 3.75194596 \times 10^{-2}}$$

Chapter 20

cst2: **Two Stage Stirred Tank Reactor**

Büskens, Göllmann, and Maurer [34] describe a problem that requires control of a chemical process in a two stage stirred tank reactor with time delay in the dynamic model. The formulation extends work originally presented in reference [51]. Because the delay terms in the model are constant, the delay-differential equations can be recast as a system of ordinary differential equations using the *method of steps*. In this example there are 80 control variables that appear in 160 nonlinear differential equations.

Example 20.1 cst201: Two Stage CSTR Optimal Control.

Phase 1 . . . . *DDE: Method of Steps* . . . . Phase 1

Independent Variable: (t) . . . .

$t = 0$ $\qquad$ $0 < t < \delta$ $\qquad$ $t = \delta = .05$

Differential Variables: $(y_1, \ldots, y_{LN} \qquad LN = 160)$ . . . .

$y_1 = 0.15$
$y_2 = -0.03$
$y_3 = 0.1$
$y_4 = 0.0$

where $N = 2/\delta = 40$, $L = 4$ and $M = 2$.

Algebraic Variables: $(v_1, \ldots, v_{MN} \qquad MN = 80)$ . . . .

Boundary Conditions . . . .

$y_{j+kL}(\delta) = y_{j+L+kL}(0) \qquad j = 1, 2, 3, 4$
$v_{j+kM}(\delta) = v_{j+M+kM}(0) \qquad j = 1, 2$

for $k = 0, 1, \ldots, N - 2$.

Differential-Algebraic Equations . . . .

$$\dot{y}_{1+kL} = 0.5 - y_{1+kL} - R_1(y_{1+kL}, y_{2+kL}) \tag{20.1}$$
$$\dot{y}_{2+kL} = -2\left[y_{2+kL} + 0.25\right] - v_{1+kM}\left[y_{2+kL} + 0.25\right] + R_1(y_{1+kL}, y_{2+kL}) \tag{20.2}$$
$$\dot{y}_{3+kL} = y_{1+kL-\sigma L} - y_{3+kL} - R_2(y_{3+kL}, y_{4+kL}) + 0.25 \tag{20.3}$$
$$\dot{y}_{4+kL} = y_{2+kL-\sigma L} - 2y_{4+kL} - v_{2+kM}\left[y_{4+kL} + 0.25\right] + R_2(y_{3+kL}, y_{4+kL}) - 0.25 \tag{20.4}$$

where

$$R_1 = [x_1 + 0.5] \exp\left[\frac{25x_2}{x_2 + 2}\right] \tag{20.5}$$
$$R_2 = [x_3 + 0.25] \exp\left[\frac{25x_4}{x_4 + 2}\right] \tag{20.6}$$

for $k = 0, 1, \ldots, N-1$, where $L = 4$ and $M = 2$. In addition for $r = .4$, $\sigma = r/\delta = 8$ and when $k < \sigma$ and $0 \le t \le \delta$

$$y_{1+kL-\sigma L}(t) = 0.15 \tag{20.7}$$
$$y_{2+kL-\sigma L}(t) = -.03 \tag{20.8}$$

for $k = 0, 1, 2, 3$.

Objective . . . .

Minimize

$$J = \int_0^\delta \sum_{k=0}^{N-1} \left[\sum_{j=1}^{4} y_{j+kL}^2(t) + .1 \sum_{j=1}^{2} v_{j+kM}^2(t) \right] dt \tag{20.9}$$

$$\boxed{J^* = .0246128799}$$

Chapter 21

cstr: Continuous Stirred Tank Reactor

In reference [50, Sect 7] Göllmann, Kern, and Maurer discuss a different version of the stirred tank reactor problem than given in example (20.1). Using the method of steps to convert the delay-differential equations to an ODE system results in a problem in Lagrange form with 120 state, and 80 control variables which is given as example (21.1). The problem is recast in Mayer form in example (21.2) leading to a system with 160 states. In example (21.3) the prehistory is represented using a piecewise cubic polynomial and the coefficients are chosen in addition to the control variables to optimize the solution.

Example 21.1 cstr01: CSTR DDE OPTIMAL CONTROL.

Phase 1 .... *DDE: Method of Steps* .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < \delta$	$t = \delta = .005$

Differential Variables: $(y_1, \ldots, y_{LN} \qquad LN = 120)$ ....

$y_1 = 0.49$
$y_2 = -0.0002$
$y_3 = -0.02$

where $N = 0.2/\delta = 40$, $L = 3$ and $M = 2$.

Algebraic Variables: $(v_1, \ldots, v_{MN} \qquad MN = 80)$ ....

$-500 \le v_k \le 500$	$-500 \le v_k \le 500$	$-500 \le v_k \le 500$

for $k = 1, 3, 5, \ldots, MN$.

Boundary Conditions ....

$y_{1+kL}(\delta) = y_{1+L+kL}(0)$
$y_{2+kL}(\delta) = y_{2+L+kL}(0)$
$y_{3+kL}(\delta) = y_{3+L+kL}(0)$
$v_{1+kM}(\delta) = v_{1+M+kM}(0)$
$v_{2+kM}(\delta) = v_{2+M+kM}(0)$

for $k = 0, 1, \ldots, N-2$.

Differential-Algebraic Equations ....

$$\dot{y}_{1+kL} = -y_{1+kL} - R(y_{1+kL}, y_{2+kL}, y_{3+kL}) \tag{21.1}$$
$$\dot{y}_{2+kL} = -y_{2+kL} + 0.9v_{2+kM-4M} + 0.1v_{2+kM} \tag{21.2}$$
$$\dot{y}_{3+kL} = -2y_{3+kL} + 0.25R(y_{1+kL}, y_{2+kL}, y_{3+kL}) - 1.05v_{1+kM}y_{3+kL-3L} \tag{21.3}$$

where

$$R(x_1, x_2, x_3) = [1 + x_1]\,[1 + x_2]\exp\left[\frac{25x_3}{1 + x_3}\right] \tag{21.4}$$

for $k = 0, 1, \ldots, N-1$. In addition

$$y_{3+kL-3L} = -0.02 \tag{21.5}$$

for $k = 0, 1, 2$ and

$$v_{2+kM-4M} = 1 \tag{21.6}$$

for $k = 0, 1, 2, 3$.

Objective ....

Minimize

$$J = \int_0^{\delta} \sum_{k=0}^{N-1} \left[y_{1+kL}^2(t) + y_{2+kL}^2(t) + y_{3+kL}^2(t) + .01v_{1+kM}^2(t) + .01v_{2+kM}^2(t)\right] dt \tag{21.7}$$

$$\boxed{J^* = .0213328235}$$

Example 21.2 cstr02: CSTR DDE Optimal Control, Mayer Form.

Phase 1 .... *DDE: Method of Steps* .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < \delta$	$t = \delta = .005$

Differential Variables: $(y_1, \ldots, y_{LN} \quad LN = 160)$ ....

$y_1 = 0.49$
$y_2 = -0.0002$
$y_3 = -0.02$
$y_4 = 0$

where $N = 0.2/\delta = 40$, $L = 4$ and $M = 2$.

Algebraic Variables: $(v_1, \ldots, v_{MN} \qquad MN = 80)$ ....

$-500 \leq v_k \leq 500$ $\qquad$ $-500 \leq v_k \leq 500$ $\qquad$ $-500 \leq v_k \leq 500$

for $k = 1, 3, 5, \ldots, MN$.

Boundary Conditions ....

$y_{1+kL}(\delta) = y_{1+L+kL}(0)$
$y_{2+kL}(\delta) = y_{2+L+kL}(0)$
$y_{3+kL}(\delta) = y_{3+L+kL}(0)$
$y_{4+kL}(\delta) = y_{4+L+kL}(0)$
$v_{1+kM}(\delta) = v_{1+M+kM}(0)$
$v_{2+kM}(\delta) = v_{2+M+kM}(0)$

for $k = 0, 1, \ldots, N-2$.

Differential-Algebraic Equations ....

$$\dot{y}_{1+kL} = -y_{1+kL} - R(y_{1+kL}, y_{2+kL}, y_{3+kL}) \tag{21.8}$$
$$\dot{y}_{2+kL} = -y_{2+kL} + 0.9v_{2+kM-4M} + 0.1v_{2+kM} \tag{21.9}$$
$$\dot{y}_{3+kL} = -2y_{3+kL} + 0.25R(y_{1+kL}, y_{2+kL}, y_{3+kL}) - 1.05v_{1+kM}y_{3+kL-3L} \tag{21.10}$$
$$\dot{y}_{4+kL} = y_{1+kL}^2 + y_{2+kL}^2 + y_{3+kL}^2 + .01v_{1+kM}^2 + .01v_{2+kM}^2 \tag{21.11}$$

where

$$R(x_1, x_2, x_3) = [1 + x_1]\,[1 + x_2]\exp\left[\frac{25x_3}{1 + x_3}\right] \tag{21.12}$$

for $k = 0, 1, \ldots, N-1$. In addition

$$y_{3+kL-3L} = -0.02 \tag{21.13}$$

for $k = 0, 1, 2$ and

$$v_{2+kM-4M} = 1 \tag{21.14}$$

for $k = 0, 1, 2, 3$.

Objective ....

Minimize

$$J = y_{LN}(\delta) \tag{21.15}$$

$\boxed{J^* = .0213328232}$

Example 21.3 cstr03: CSTR, Optimal Spline Prehistory.

Phase 1 .... *DDE: Method of Steps* .... Phase 1

Parameters: $(r_0, r_1, r_2, r_3, r'_0, r'_1, r'_2, r'_3, s_0, s_1, s_2, s_3, s_4, s'_0, s'_1, s'_2, s'_3, s'_4)$ ....

$r_3 = -0.02$ $\quad$ $s_4 = 1$

Independent Variable: (t) ....

$t = 0$ $\quad$ $0 < t < \delta$ $\quad$ $t = \delta = .005$

Differential Variables: $(y_1, \ldots, y_{LN} \qquad LN = 120)$ ....

$y_1 = 0.49$
$y_2 = -0.0002$
$y_3 = -0.02$

where $N = 0.2/\delta = 40$, $L = 3$ and $M = 2$.

Algebraic Variables: $(v_1, \ldots, v_{MN} \qquad MN = 80)$ ....

$-500 \leq v_k \leq 500$ $\quad$ $-500 \leq v_k \leq 500$ $\quad$ $-500 \leq v_k \leq 500$

for $k = 1, 3, 5, \ldots, MN$.

Boundary Conditions ....

$y_{1+kL}(\delta) = y_{1+L+kL}(0)$
$y_{2+kL}(\delta) = y_{2+L+kL}(0)$
$y_{3+kL}(\delta) = y_{3+L+kL}(0)$
$v_{1+kM}(\delta) = v_{1+M+kM}(0)$
$v_{2+kM}(\delta) = v_{2+M+kM}(0)$

for $k = 0, 1, \ldots, N-2$.

Differential-Algebraic Equations ....

$$\dot{y}_{1+kL} = -y_{1+kL} - R(y_{1+kL}, y_{2+kL}, y_{3+kL}) \tag{21.16}$$
$$\dot{y}_{2+kL} = -y_{2+kL} + 0.9v_{2+kM-4M} + 0.1v_{2+kM} \tag{21.17}$$
$$\dot{y}_{3+kL} = -2y_{3+kL} + 0.25R(y_{1+kL}, y_{2+kL}, y_{3+kL}) - 1.05v_{1+kM}y_{3+kL-3L} \tag{21.18}$$

where

$$R(x_1, x_2, x_3) = [1 + x_1]\,[1 + x_2]\exp\left[\frac{25x_3}{1 + x_3}\right] \tag{21.19}$$

for $k = 0, 1, \ldots, N-1$. In addition for $k = 0, 1, 2$ define

$$\tau_L = (k-3)\delta \tag{21.20}$$
$$\tau_U = \tau_L + \delta \tag{21.21}$$
$$\tau = \tau_L + t \tag{21.22}$$
$$y_{3+kL-3L} = H(\tau, \tau_L, \tau_U, r_k, r'_k, r_{k+1}, r'_{k+1}) \tag{21.23}$$

Similarly for $k = 0, 1, 2, 3$ define

$$\rho_L = (k-4)\delta \tag{21.24}$$
$$\rho_U = \rho_L + \delta \tag{21.25}$$
$$\rho = \rho_L + t \tag{21.26}$$
$$v_{2+kM-4M} = H(\rho, \rho_L, \rho_U, s_k, s'_k, s_{k+1}, s'_{k+1}) \tag{21.27}$$

The Hermite function is defined by the following sequence:

$$h = t_U - t_L \tag{21.28}$$
$$\delta = \frac{t - t_L}{h} \tag{21.29}$$
$$d = 2(f_L - f_U) + h(f'_L + f'_U) \tag{21.30}$$
$$c = -h(f'_U + 2f'_L) - 3(f_L - f_U) \tag{21.31}$$
$$b = hf'_L \tag{21.32}$$
$$H(t, t_L, t_U, f_L, f'_L, f_U, f'_U) = f_L + (b + (c + d\delta)\delta)\delta \tag{21.33}$$

Objective . . . .

Minimize

$$J = \int_0^\delta \sum_{k=0}^{N-1} \left[y_{1+kL}^2(t) + y_{2+kL}^2(t) + y_{3+kL}^2(t) + .01v_{1+kM}^2(t) + .01v_{2+kM}^2(t) \right] dt \tag{21.34}$$

$$\boxed{J^* = .0213308582}$$

Chapter 22

dlay: **Delay Differential Equation**

Ascher, Mattheij, and Russell [2, Ex. 11.12, p 506] use this very simple delay differential equation (DDE) example to illustrate the *method of steps* (MOS). Using this technique which is applicable for problems with a constant delay, the DDE is replaced by a system of ordinary differential equations.

Example 22.1 dlay01: Delay Example, MOS.

Phase 1 .... *Method of Steps (MOS)* .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 \le t \le 1$	$t = t_F = 1$

Differential Variables: (y_1, y_2, y_3, y_4) ....

$y_1 = -0.5$		
		$y_3 = -0.5$

Boundary Conditions ....

$y_{3I} = y_{1F}$
$y_{4I} = y_{2F}$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{22.1}$$
$$\dot{y}_2 = -(1/16)\sin y_1 - (t+1)(t-1.5) + t \tag{22.2}$$
$$\dot{y}_3 = y_4 \tag{22.3}$$
$$\dot{y}_4 = -(1/16)\sin y_3 - (t+2)y_1 + t + 1 \tag{22.4}$$

Chapter 23

dlt3: **Delta III Ascent Trajectory Optimization**

The design of an ascent trajectory is one of the most common applications for modern optimal control methods. Although the trajectory dynamics are usually rather benign, a realistic simulation of translational motion (a so-called 3D trajectory) must incorporate accurate approximations to real data representing the aerodynamic, propulsion and gravitational forces. When both translational and rotational motion are incorporated (a 6D trajectory), the simulation becomes significantly more complex. In his Ph.D. thesis Benson [6], presents a typical 3D ascent trajectory for the Delta III vehicle from the launch pad to a common low-earth park orbit. Fortunately, vehicle data that is often proprietary, is publicly available for the Delta III, presumably because it is no longer produced after two launch failures, and a partially successful third launch. Despite the lackluster history of this launch vehicle, it serves as an reasonable representation for this problem class. Rao [77] uses this application to illustrate the GPOCS tool, and a complete discussion is found in reference [13, Sect. 6.15].

Example 23.1 dlt301: Maximum Final Mass.

Phase 1 . . . . *6 Solid Rocket Boosters + Stage 1 Engine* . . . . Phase 1

Independent Variable: (t) . . . .

$t = 0$	$0 < t < 75.2$	$t = 75.2$

Differential Variables: $(\mathbf{r}^\mathsf{T}, \mathbf{v}^\mathsf{T}, m)$ . . . .

$r_1 = R_E \cos\psi_L$
$r_2 = 0$
$r_3 = R_E \sin\psi_L$
$v_1 = 0$
$v_2 = \|\boldsymbol{\omega} \times \mathbf{r}_0\|$
$v_3 = 0$

$m = m_I^{(1)}$	$\underline{m} \le m \le \overline{m}$	$\underline{m} \le m \le \overline{m}$

where $\mathbf{r}^\mathsf{T} = (r_1, r_2, r_3)$ and $\mathbf{v}^\mathsf{T} = (v_1, v_2, v_3)$ and the values in Table 23.2 define the sequence

$$m_I^{(1)} = 9\varpi_s + \varpi_1 + \varpi_2 + \varpi_p \tag{23.1}$$
$$m_F^{(1)} = m_I^{(1)} - 6\varrho_s - \frac{\tau_s}{\tau_1}\varrho_1 \tag{23.2}$$
$$m_I^{(2)} = m_F^{(1)} - 6\varphi_s \tag{23.3}$$
$$m_F^{(2)} = m_I^{(2)} - 3\varrho_s - \frac{\tau_s}{\tau_1}\varrho_1 \tag{23.4}$$
$$m_I^{(3)} = m_F^{(2)} - 3\varphi_s \tag{23.5}$$
$$m_F^{(3)} = m_I^{(3)} - \left(1 - 2\frac{\tau_s}{\tau_1}\right)\varrho_1 \tag{23.6}$$
$$m_I^{(4)} = m_F^{(3)} - \varphi_1 \tag{23.7}$$

and

$$\underline{m} = m_F^{(1)} - 10 \tag{23.8}$$
$$\overline{m} = m_I^{(1)} + 10. \tag{23.9}$$

Algebraic Variables: $(\mathbf{u}^\mathsf{T})$ ....

$-1.1 \le u_1 \le 1.1$	$-1.1 \le u_1 \le 1.1$	$-1.1 \le u_1 \le 1.1$
$-1.1 \le u_2 \le 1.1$	$-1.1 \le u_2 \le 1.1$	$-1.1 \le u_2 \le 1.1$
$-1.1 \le u_3 \le 1.1$	$-1.1 \le u_3 \le 1.1$	$-1.1 \le u_3 \le 1.1$

where $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_3)$.

Differential-Algebraic Equations ....

$$\dot{\mathbf{r}} = \mathbf{v} \tag{23.10}$$
$$\dot{\mathbf{v}} = -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \frac{T}{m}\mathbf{u} + \frac{1}{m}\mathbf{D} \tag{23.11}$$
$$\dot{m} = -\xi \tag{23.12}$$
$$1 = \|\mathbf{u}\| \tag{23.13}$$
$$R_E \le \|\mathbf{r}\| \tag{23.14}$$
$$q \le q_{max} \tag{23.15}$$

where the model and vehicle parameters are given in Tables 23.1 and 23.2 respectively and

$$h = \|\mathbf{r}\| - R_E \tag{23.16}$$
$$\rho = \rho_0 e^{(-h/h_0)} \tag{23.17}$$
$$\boldsymbol{\omega}^\mathsf{T} = (0, 0, \omega_E) \tag{23.18}$$
$$\mathbf{v}_r = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} \tag{23.19}$$

$$v_r = \|\mathbf{v}_r\| \tag{23.20}$$

$$\widehat{\mathbf{d}} = \begin{cases} \mathbf{r}/\|\mathbf{r}\| & \text{if } v_r < .001 \\ \mathbf{v}_r/v_r & \text{if } v_r \geq .001 \end{cases} \tag{23.21}$$

$$q = \frac{1}{2}\rho v_r^2 \tag{23.22}$$

$$\mathbf{D} = -C_D S q \widehat{\mathbf{d}} \tag{23.23}$$

$$T = 6T_s + T_1 \tag{23.24}$$

$$\xi = \frac{6T_s}{g_0 \mathcal{I}_s} + \frac{T_1}{g_0 \mathcal{I}_1} \tag{23.25}$$

Phase 2 .... *3 Solid Rocket Boosters + Stage 1 Engine* .... Phase 2

Independent Variable: (t) ....

$t = 75.2$	$75.2 < t < 150.4$	$t = 150.4$

Differential Variables: $(\mathbf{r}^\mathsf{T}, \mathbf{v}^\mathsf{T}, m)$ ....

$r_1 = r_{1F}^{(1)}$		
$r_2 = r_{2F}^{(1)}$		
$r_3 = r_{3F}^{(1)}$		
$v_1 = v_{1F}^{(1)}$		
$v_2 = v_{2F}^{(1)}$		
$v_3 = v_{3F}^{(1)}$		
$m = m_I^{(2)}$	$\underline{m} \leq m \leq \overline{m}$	$\underline{m} \leq m \leq \overline{m}$

where (23.1)-(23.7) are used with

$$\underline{m} = m_F^{(2)} - 10 \tag{23.26}$$

$$\overline{m} = m_I^{(2)} + 10. \tag{23.27}$$

Algebraic Variables: $(\mathbf{u}^\mathsf{T})$ ....

$-1.1 \leq u_1 \leq 1.1$	$-1.1 \leq u_1 \leq 1.1$	$-1.1 \leq u_1 \leq 1.1$
$-1.1 \leq u_2 \leq 1.1$	$-1.1 \leq u_2 \leq 1.1$	$-1.1 \leq u_2 \leq 1.1$
$-1.1 \leq u_3 \leq 1.1$	$-1.1 \leq u_3 \leq 1.1$	$-1.1 \leq u_3 \leq 1.1$

where $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_3)$.

Differential-Algebraic Equations ....

Equations (23.10) - (23.23)

Omit (23.15) and replace (23.24)-(23.25) with

$$T = 3T_s + T_1 \tag{23.28}$$

$$\xi = \frac{3T_s}{g_0\mathcal{I}_s} + \frac{T_1}{g_0\mathcal{I}_1} \tag{23.29}$$

Phase 3 .... *Stage 1 Engine* .... Phase 3

Independent Variable: (t) ....

$t = 150.4$	$150.4 < t < 261$	$t = 261$

Differential Variables: $(\mathbf{r}^\mathsf{T}, \mathbf{v}^\mathsf{T}, m)$ ....

$r_1 = r_{1F}^{(2)}$		
$r_2 = r_{2F}^{(2)}$		
$r_3 = r_{3F}^{(2)}$		
$v_1 = v_{1F}^{(2)}$		
$v_2 = v_{2F}^{(2)}$		
$v_3 = v_{3F}^{(2)}$		
$m = m_I^{(3)}$	$\underline{m} \le m \le \overline{m}$	$\underline{m} \le m \le \overline{m}$

where (23.1)-(23.7) are used with

$$\underline{m} = m_F^{(3)} - 10 \tag{23.30}$$

$$\overline{m} = m_I^{(3)} + 10. \tag{23.31}$$

Algebraic Variables: $(\mathbf{u}^\mathsf{T})$ ....

$-1.1 \le u_1 \le 1.1$	$-1.1 \le u_1 \le 1.1$	$-1.1 \le u_1 \le 1.1$
$-1.1 \le u_2 \le 1.1$	$-1.1 \le u_2 \le 1.1$	$-1.1 \le u_2 \le 1.1$
$-1.1 \le u_3 \le 1.1$	$-1.1 \le u_3 \le 1.1$	$-1.1 \le u_3 \le 1.1$

where $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_3)$.

Differential-Algebraic Equations ....

Equations (23.10) - (23.23)

Omit (23.15) and replace (23.24)-(23.25) with

$$T = T_1 \tag{23.32}$$

$$\xi = \frac{T_1}{g_0\mathcal{I}_1} \tag{23.33}$$

Phase 4 .... *Stage 2 Engine* .... Phase 4

Parameters: (t_F) ....

$611 \le t_F$

Independent Variable: (t) ....

$t = 261$	$261 < t < t_F$	$t = t_F$

Differential Variables: $(\mathbf{r}^\mathsf{T}, \mathbf{v}^\mathsf{T}, m)$ ....

$r_1 = r_{1F}^{(3)}$		
$r_2 = r_{2F}^{(3)}$		
$r_3 = r_{3F}^{(3)}$		
$v_1 = v_{1F}^{(3)}$		
$v_2 = v_{2F}^{(3)}$		
$v_3 = v_{3F}^{(3)}$		
$m = m_I^{(4)}$	$\underline{m} \le m \le \overline{m}$	$\underline{m} \le m \le \overline{m}$

where (23.1)-(23.7) are used with

$$\underline{m} = 10 \tag{23.34}$$

$$\overline{m} = m_I^{(4)} + 10. \tag{23.35}$$

Algebraic Variables: $(\mathbf{u}^\mathsf{T})$ ....

$-1.1 \le u_1 \le 1.1$	$-1.1 \le u_1 \le 1.1$	$-1.1 \le u_1 \le 1.1$
$-1.1 \le u_2 \le 1.1$	$-1.1 \le u_2 \le 1.1$	$-1.1 \le u_2 \le 1.1$
$-1.1 \le u_3 \le 1.1$	$-1.1 \le u_3 \le 1.1$	$-1.1 \le u_3 \le 1.1$

where $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_3)$.

Boundary Conditions ....

$$a_f = 24361140$$
$$e_f = .7308$$
$$i_f = 28.5^o$$
$$\Omega_f = 269.8^o$$
$$\omega_f = 130.5^o$$

where the classical elements $(a, e, i, \Omega, \omega)$ can be computed from $\mathbf{r}$ and $\mathbf{v}$.

Differential-Algebraic Equations ....

Equations (23.10) - (23.23)

Omit (23.15) and replace (23.24)-(23.25) with

$$T = T_2 \tag{23.36}$$

$$\xi = \frac{T_2}{g_0 \mathcal{I}_2}. \tag{23.37}$$

Objective ....

Maximize $$J = m(t_F)$$

$$\boxed{J^* = 7529.50823; \quad t_F^* = 924.14770}$$

$\mu = 3.986012 \times 10^{14}$	$R_E = 6378145.$
$g_0 = 9.80665$	$h_0 = 7200.$
$\rho_0 = 1.225$	$\omega_E = 7.29211585 \times 10^{-5}$
$C_D = .5$	$S = 4\pi$
$\psi_L = 28.5^o$	$q_{max} = 60000$

Table 23.1. *Dynamic Model Parameters*

$\varpi_s = 19290$	$\varpi_1 = 104380$	$\varpi_2 = 19300$
$\varrho_s = 17010$	$\varrho_1 = 95550$	$\varrho_2 = 16820$
$\varphi_s = 2280$	$\varphi_1 = 8830$	$\varphi_2 = 2480$
$T_s = 628500$	$T_1 = 1083100$	$T_2 = 110094$
$\mathcal{I}_s = 283.33364$	$\mathcal{I}_1 = 301.68776$	$\mathcal{I}_2 = 467.21311$
$\tau_s = 75.2$	$\tau_1 = 261$	$\tau_2 = 700$

Table 23.2. *Vehicle Parameters*

Chapter 24

dock: **Optimal Spacecraft Docking Maneuver**

A formulation of the optimal docking maneuver of a service vehicle and a target vehicle is described in a paper by Michael, Chudej, Gerdts and Pannek [69]. The relative motion of two vehicles in the local-vertical local-horizontal (LVLH) system is modeled using the *Hill-Clohessy-Wilshire* equations which in conjunction with the rotational dynamics yields a six-degree of freedom trajectory. The rotational dynamics are specified using quaternions [60, pp. 18–31]. Example (24.1) yields the minimum control energy solution, example (24.2) the minimum time solution, and (24.3) describes a composite objective function. All three examples are discussed in reference [14].

Example 24.1 dock01: Minimum Control Energy.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$t_F \leq 420$

Independent Variable: (t) ....

$t = t_I = 0$	$t_I < t < t_F$	$t = t_F$	sec

Differential Variables: $(\mathbf{x}^\top, \mathbf{v}^\top, \mathbf{q}^\top, \boldsymbol{\omega}^\top, \mathbf{p}^\top, \boldsymbol{\phi}^\top)$ ....

$x = 0$	m
$y = -10$	m
$z = 0$	m
$v_x = 0$	m/sec
$v_y = 0$	m/sec
$v_z = 0$	m/sec
$q_1 = 0$	

$q_2 = 0$
$q_3 = 0$
$q_4 = 1$
$\omega_1 = 0$ rad
$\omega_2 = 0$ rad
$\omega_3 = 0$ rad
$p_1 = -.05$
$p_2 = 0$
$p_3 = 0$
$p_4 = \sqrt{1-(.05)^2}$
$\phi_1 = 0$ rad
$\phi_2 = .0349$ rad
$\phi_3 = .017453$ rad

Algebraic Variables: $(\boldsymbol{\alpha}^{\mathsf{T}}, \boldsymbol{\tau}^{\mathsf{T}})$ ....

$\|\boldsymbol{\tau}\|_\infty \le \tau_{max}$ $\|\boldsymbol{\tau}\|_\infty \le \tau_{max}$ $\|\boldsymbol{\tau}\|_\infty \le \tau_{max}$

Boundary Conditions ....

$$\begin{aligned} \mathbf{0} &= \mathbf{x} + \mathbf{S}\mathbf{a} - \mathbf{T}\mathbf{b} \\ \mathbf{0} &= \mathbf{v} + \mathbf{S}\boldsymbol{\omega} \times \mathbf{S}\mathbf{a} - \mathbf{T}\boldsymbol{\phi} \times \mathbf{T}\mathbf{b} \end{aligned}$$

Differential-Algebraic Equations ....

$$\dot{x} = v_x \tag{24.1}$$
$$\dot{y} = v_y \tag{24.2}$$
$$\dot{z} = v_z \tag{24.3}$$
$$\dot{v}_x = 2nv_y + 3n^2 x + \frac{\alpha_x}{m} \tag{24.4}$$
$$\dot{v}_y = -2nv_x + \frac{\alpha_y}{m} \tag{24.5}$$
$$\dot{v}_z = -n^2 z + \frac{\alpha_z}{m} \tag{24.6}$$
$$\dot{\mathbf{q}} = \frac{1}{2}\boldsymbol{\Omega}\,\mathbf{q} \tag{24.7}$$
$$\dot{\omega}_1 = J_{11}^{-1}\left[\omega_2\omega_3(J_{22} - J_{33}) + \tau_1\right] \tag{24.8}$$
$$\dot{\omega}_2 = J_{22}^{-1}\left[\omega_1\omega_3(J_{33} - J_{11}) + \tau_2\right] \tag{24.9}$$
$$\dot{\omega}_3 = J_{33}^{-1}\left[\omega_1\omega_2(J_{11} - J_{22}) + \tau_3\right] \tag{24.10}$$
$$\dot{\mathbf{p}} = \frac{1}{2}\boldsymbol{\Phi}\,\mathbf{p} \tag{24.11}$$
$$\dot{\phi}_1 = K_{11}^{-1}\left[\phi_2\phi_3(K_{22} - K_{33})\right] \tag{24.12}$$
$$\dot{\phi}_2 = K_{22}^{-1}\left[\phi_1\phi_3(K_{33} - K_{11})\right] \tag{24.13}$$
$$\dot{\phi}_3 = K_{33}^{-1}\left[\phi_1\phi_2(K_{11} - K_{22})\right]. \tag{24.14}$$
$$-\alpha_{max} \le Q_{11}\alpha_x + Q_{12}\alpha_y + Q_{13}\alpha_z \le \alpha_{max} \tag{24.15}$$
$$-\alpha_{max} \le Q_{21}\alpha_x + Q_{22}\alpha_y + Q_{23}\alpha_z \le \alpha_{max} \tag{24.16}$$

$$-\alpha_{max} \leq Q_{31}\alpha_x + Q_{32}\alpha_y + Q_{33}\alpha_z \leq \alpha_{max} \tag{24.17}$$

$$\|\mathbf{x}\| \geq 2. \tag{24.18}$$

using the parameter definitions given in Table 24.1. The relative position vector of the vehicles is $\mathbf{x}^\mathsf{T} = (x, y, z)$ with relative velocity $\mathbf{v}^\mathsf{T} = (v_x, v_y, v_z)$. The spacecraft orientation is defined by $\mathbf{q}^\mathsf{T} = (q_1, q_2, q_3, q_4)$ called quaternions [60, pp. 18–31], where $\|\mathbf{q}\| = 1$ with angular velocities $\boldsymbol{\omega}^\mathsf{T} = (\omega_1, \omega_2, \omega_3)$, and diagonal moment of inertia matrix $\mathbf{J}$. Define

$$\mathbf{Q} = \begin{bmatrix} q_1^2 + q_4^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_4q_3) & 2(q_1q_3 - q_4q_2) \\ 2(q_1q_2 - q_4q_3) & q_2^2 + q_4^2 - q_1^2 - q_3^2 & 2(q_2q_3 + q_4q_1) \\ 2(q_1q_3 + q_4q_2) & 2(q_2q_3 - q_4q_1) & q_3^2 + q_4^2 - q_1^2 - q_2^2 \end{bmatrix} \tag{24.19}$$

$$= \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \mathbf{S}^\mathsf{T}. \tag{24.20}$$

and

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}. \tag{24.21}$$

The target vehicle orientation is $\mathbf{p}^\mathsf{T} = (p_1, p_2, p_3, p_4)$, with corresponding angular velocities $\boldsymbol{\phi}^\mathsf{T} = (\phi_1, \phi_2, \phi_3)$, and diagonal moment of inertia matrix $\mathbf{K}$. For the target vehicle we define

$$\mathbf{T} = \begin{bmatrix} p_1^2 + p_4^2 - p_2^2 - p_3^2 & 2(p_1p_2 - p_4p_3) & 2(p_1p_3 + p_4p_2) \\ 2(p_1p_2 + p_4p_3) & p_2^2 + p_4^2 - p_1^2 - p_3^2 & 2(p_2p_3 - p_4p_1) \\ 2(p_1p_3 - p_4p_2) & 2(p_2p_3 + p_4p_1) & p_3^2 + p_4^2 - p_1^2 - p_2^2 \end{bmatrix}. \tag{24.22}$$

Also define the matrix

$$\boldsymbol{\Phi} = \begin{bmatrix} 0 & \phi_3 & -\phi_2 & \phi_1 \\ -\phi_3 & 0 & \phi_1 & \phi_2 \\ \phi_2 & -\phi_1 & 0 & \phi_3 \\ -\phi_1 & -\phi_2 & -\phi_3 & 0 \end{bmatrix}. \tag{24.23}$$

Objective ....

Minimize $$J = w_0 t_F + \int_0^{t_F} \left[w_1 \boldsymbol{\alpha}^\mathsf{T}\boldsymbol{\alpha} + w_2 \boldsymbol{\tau}^\mathsf{T}\boldsymbol{\tau} \right] dt$$

with $(w_0, w_1, w_2) = (0, 1, 1)$.

$$\boxed{J^* = 5.27584533 \times 10^{-1}; \quad t_F^* = 4.0322676 \times 10^2}$$

Example 24.2 dock02: Minimum Time.
Repeat example 24.1 with $(w_0, w_1, w_2) = (1, 0, 0)$.

parameter	value	definition
a	7071000	orbit radius [m]
μ	398×10^{12}	gravitational constant $[N(m/kg)^2]$
n	$\sqrt{\mu/a^3}$	mean motion [1/sec]
m	100	satellite mass [kg]
α_{max}	0.1	maximum thrust [N]
τ_{max}	1	maximum torque [Nm]
J_{11}, K_{11}	1000	moment of inertia around x [kg/m^2]
J_{22}, K_{22}	2000	moment of inertia around y [kg/m^2]
J_{33}, K_{33}	1000	moment of inertia around z [kg/m^2]
$\mathbf{a}, \mathbf{b}$	$(0, 1.01, 0)^\mathsf{T}$	docking point for servicer, target [m]

Table 24.1. *Parameter Definitions*

$$\boxed{J^* = 1.72214926 \times 10^2; \quad t_F^* = 1.72214926 \times 10^2}$$

Example 24.3 dock03: Bolza Composite Objective.
Repeat example 24.1 with $(w_0, w_1, w_2) = (1, 0.1, 0.1)$.

$$\boxed{J^* = 1.81054716 \times 10^2; \quad t_F^* = 1.7600356 \times 10^2}$$

Chapter 25

ffrb: Free-Flying Robot

Sakawa [81] presents an example that describes the motion of a free-flying robot equipped with a propulsion system. Unfortunately, the objective function as written by Sakawa has discontinuous derivatives because it involves the absolute value function. An approach for treating absolute values motivated by a dynamic MPEC (mathematical program with equilibrium constraints) formulation is presented here and fully described in reference [13, pp 326-330].

Example 25.1 ffrb01: ABSOLUTE VALUE ELIMINATION BY SLACKS.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 12$	$t = 12$

Differential Variables: $(y_1, y_2, y_3, y_4, y_5, y_6)$ ....

$y_1 = -10$		$y_1 = 0$
$y_2 = -10$		$y_2 = 0$
$y_3 = \pi/2$		$y_3 = 0$
$y_4 = 0$		$y_4 = 0$
$y_5 = 0$		$y_5 = 0$
$y_6 = 0$		$y_6 = 0$

Algebraic Variables: (u_1, u_2, u_3, u_4) ....

$0 \le u_1 \le 1$	$0 \le u_1 \le 1$	$0 \le u_1 \le 1$
$0 \le u_2 \le 1$	$0 \le u_2 \le 1$	$0 \le u_2 \le 1$
$0 \le u_3 \le 1$	$0 \le u_3 \le 1$	$0 \le u_3 \le 1$
$0 \le u_4 \le 1$	$0 \le u_4 \le 1$	$0 \le u_4 \le 1$

Differential-Algebraic Equations . . . .

$$\dot{y}_1 = y_4 \tag{25.1}$$
$$\dot{y}_2 = y_5 \tag{25.2}$$
$$\dot{y}_3 = y_6 \tag{25.3}$$
$$\dot{y}_4 = [u_1 - u_2 + u_3 - u_4] \cos y_3 \tag{25.4}$$
$$\dot{y}_5 = [u_1 - u_2 + u_3 - u_4] \sin y_3 \tag{25.5}$$
$$\dot{y}_6 = \alpha(u_1 - u_2) - \beta(u_3 - u_4) \tag{25.6}$$
$$1 \geq u_1 + u_2 \tag{25.7}$$
$$1 \geq u_3 + u_4 \tag{25.8}$$

where $\alpha = \beta = .2$.

Objective . . . .

Minimize $$J = \int_0^{12} (u_1 + u_2 + u_3 + u_4)\, dt$$

$$\boxed{J^* = 7.91014874}$$

Chapter 26

fhoc: **Finite Horizon Optimal Control**

Deshmukh, Ma, and Butcher [38] present an example they describe as follows

> The mathematical models of certain engineering processes and systems are represented by delay differential equations with time periodic coefficients. Such processes and systems include the machine tool dynamics in metal cutting operations such as milling and turning with periodically varying cutting speed or impedance and parametric control of robots, etc. Delay differential equations have been used to model nonlinear systems where finite delay in feedback control can have adverse effects on closed loop stability.

The example defined here is obtained when the method of steps is used to transform the delay system into a system of ODE's as described in reference [13, Sect. 7.3]. The resulting problem has 100 states, 50 controls, and 147 boundary conditions.

Example 26.1 fhoc01: Delay Equation; Fifty Intervals.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\quad\quad$ $0 \le t \le \tau$ $\quad\quad$ $t = \tau$

Differential Variables: $(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N)$ ....

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Algebraic Variables: $(u_1, u_2, \ldots, u_N)$ ....

Boundary Conditions ....

$$\mathbf{y}_k(0) = \mathbf{y}_{k-1}(\tau)$$

$u_k(0) = u_{k-1}(\tau)$

for $k = 2, \ldots, N$.

Differential-Algebraic Equations ....

$$\dot{\mathbf{y}}_k = \mathbf{A}_1(\alpha)\mathbf{y}_k + \mathbf{A}_2(\alpha)\mathbf{y}_{k-1} + \mathbf{B}(\alpha)u_k \tag{26.1}$$

for $k = 1, \ldots, N$ where

$$\alpha = t + (k-1)\tau \tag{26.2}$$

$$\mathbf{A}_1(\alpha) = \begin{bmatrix} 0 & 1 \\ -4\pi^2 \{a + c\cos(2\pi\alpha)\} & 0 \end{bmatrix} \tag{26.3}$$

$$\mathbf{A}_2(\alpha) = \begin{bmatrix} 0 & 0 \\ 4\pi^2 b\cos(2\pi\alpha) & 0 \end{bmatrix} \tag{26.4}$$

$$\mathbf{B}(\alpha) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{26.5}$$

and for $-1 \leq \alpha \leq 0$ define

$$\mathbf{y}_0(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{26.6}$$

When $N = 50$ with $\tau = 1$ the model parameters are $a = 0.2$, $b = 0.5$, and $c = 0.2$.

Objective ....

Minimize

$$J = \frac{10^4}{2}\mathbf{y}_N^{\mathsf{T}}(\tau)\mathbf{y}_N(\tau) + \int_0^{\tau} \sum_{k=1}^{N} \left[\mathbf{y}_k^{\mathsf{T}}\mathbf{y}_k + u_k^2\right] dt$$

$$\boxed{J^* = 45.6775203}$$

Chapter 27

fish: **Optimal Renewable Resource**

In reference [50, Sect 8] Göllmann, Kern, and Maurer present a delay system used to model the optimal harvest of a renewable resource, i.e. fish. This DDE example can be transformed into an equivalent set of ODE's with appropriate boundary conditions using the *method of steps*. The resulting problem has 200 states and controls, in addition to 398 boundary conditions.

Example 27.1 fish01: Optimal Fish Harvesting.

Phase 1 .... *DDE: Method of Steps* .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < \tau$	$t = \tau = 0.1$

Differential Variables: $(x_1, \ldots, x_N)$ ....

$x_1 = 2$	$x_1 \geq 2$	$x_1 \geq 2$
$x_k \geq 2$	$x_k \geq 2$	$x_k \geq 2$

for $k = 2, \ldots, N$ and $N = 200$.

Algebraic Variables: $(u_1, \ldots, u_N)$ ....

$u_k \geq 0$	$u_k \geq 0$	$u_k \geq 0$

for $k = 1, \ldots, N$.

Boundary Conditions ....

$$x_j(0) = x_{j-1}(\tau)$$
$$u_j(0) = u_{j-1}(\tau)$$

for $j = 2, \ldots, N$.

Differential-Algebraic Equations ....

$$\dot{x}_k = a_1 x_k \left[1 - a_2 x_{k-\sigma}\right] - u_k \tag{27.1}$$

for $k = 1, \ldots, N$. The model coefficients are

$$(a_1, a_2, a_3, a_4, a_5) = (3, 0.2, 0.2, 0.05, 2)$$

and $t_F = 20$. Thus $\tau = t_F/N = 0.1$ and with $r = .3$, $\sigma = r/\tau = 3$.

Objective ....

Maximize

$$J = \int_0^\tau \sum_{k=1}^{N} e^{-a_4[t+(k-1)\tau]} \left[a_5 u_k(t) - a_3 x_k^{-1}(t) u_k^3(t)\right] dt \tag{27.2}$$

$$\boxed{J^* = 56.6620647}$$

Chapter 28

gdrd: **Goddard Rocket Problem**

Robert H. Goddard first posed the problem that bears his name in 1919. By making assumptions about the atmospheric density it is possible to derive expressions that define the optimal trajectory which contains a *singular arc* [29]. The appearance of a singular arc also introduces a number of computational issues as discussed in reference [13, Sect. 4.14.1]. Three versions of the problem are given here. In example (28.1) the objective is to maximize the terminal velocity. In example (28.2) the goal is to maximize the final altitude, which is formulated using a separate phase for the singular arc expressed using a differential-algebraic equation. In example (28.3) a feedback control law is used for the singular arc phase.

Example 28.1 gdrd02: Maximum Terminal Velocity.

Phase 1 . . . . Phase 1

Parameters: (t_F) . . . .
Independent Variable: (t) . . . .

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: (h, v, m) . . . .

$h = 0$		
$v = 0$		
$m = 3$	$.1 \leq m$	$m = 1$

Algebraic Variables: (T) . . . .

$0 \leq T \leq T_m$	$0 \leq T \leq T_m$	$0 \leq T \leq T_m$

Differential-Algebraic Equations ....

$$\dot{h} = v \tag{28.1}$$

$$\dot{v} = \frac{1}{m}\left[T - \sigma v^2 \exp[-h/h_0]\right] - g \tag{28.2}$$

$$\dot{m} = -T/c. \tag{28.3}$$

The problem definition is completed by the following parameters: $T_m = 193.044$, $g = 32.174$, $\sigma = 5.49153484923381010 \times 10^{-5}$, $c = 1580.9425279876559$, $h_0 = 23800$.

Objective ....

Maximize $J = v_F$

$$\boxed{J^* = 1.06029900 \times 10^3; \quad t_F^* = 16.379090}$$

Example 28.2 gdrd07: Singular Arc Problem.

Phase 1 .... *Maximum Thrust* .... Phase 1

Parameters: $(t_F^{(1)})$ ....

$1 \le t_F^{(1)} \le 45$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$

Differential Variables: (h, v, m) ....

$h = 0$
$v = 0$
$m = 3$

Differential-Algebraic Equations ....

$$\dot{h} = v \tag{28.4}$$

$$\dot{v} = \frac{1}{m}\left[T_m - \sigma v^2 \exp[-h/h_0]\right] - g \tag{28.5}$$

$$\dot{m} = -T_m/c. \tag{28.6}$$

where $T_m = 193.044$, $g = 32.174$, $\sigma = 5.49153484923381010 \times 10^{-5}$, $c = 1580.9425279876559$, $h_0 = 23800$.

Phase 2 . . . . *Singular Arc* . . . . Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$

Differential Variables: (h, v, m) . . . .

$h = h_F^{(1)}$
$v = v_F^{(1)}$
$m = m_F^{(1)}$

Algebraic Variables: (T) . . . .

$0 \le T \le T_m$	$0 \le T \le T_m$	$0 \le T \le T_m$

Boundary Conditions . . . .

$0 = mg - \left(1 + \frac{v}{c}\right) \sigma v^2 \exp[-h/h_0]$
$t_F^{(2)} - t_I^{(2)} \ge 1$

Differential-Algebraic Equations . . . .

$$\dot{h} = v \tag{28.7}$$

$$\dot{v} = \frac{1}{m}\left[T - \sigma v^2 \exp[-h/h_0]\right] - g \tag{28.8}$$

$$\dot{m} = -T/c. \tag{28.9}$$

$$\begin{aligned} 0 &= T - \sigma v^2 \exp[-h/h_0] - mg \\ &\quad - \frac{mg}{1 + 4(c/v) + 2(c^2/v^2)} \left[\frac{c^2}{h_0 g}\left(1 + \frac{v}{c}\right) - 1 - 2\frac{c}{v}\right] \end{aligned} \tag{28.10}$$

Phase 3 . . . . *No Thrust* . . . . Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} < t < t_F^{(3)}$	$t = t_F^{(3)}$

Differential Variables: (h, v, m) . . . .

$h = h_F^{(2)}$		
$v = v_F^{(2)}$		$v = 0$
$m = m_F^{(2)}$	$.1 \le m$	$m = 1$

Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} \geq 1$

Differential-Algebraic Equations ....

$$\dot{h} = v \tag{28.11}$$

$$\dot{v} = -\left(\frac{\sigma v^2}{m}\right)\exp[-h/h_0] - g \tag{28.12}$$

$$\dot{m} = 0 \tag{28.13}$$

Objective ....

Maximize $J = h_F$

$$\boxed{J^* = 18550.872; \quad t_F^{(1)} = 13.751270; \quad t_F^{(2)} = 21.987363; \quad t_F^{(3)} = 42.887912}$$

Example 28.3 gdrd10: Singular Arc Feedback Control.

Phase 1 .... *Maximum Thrust* .... Phase 1

Parameters: $(t_F^{(1)})$ ....
Independent Variable: (t) ....

$t = 0$ $\quad$ $0 < t < t_F^{(1)}$ $\quad$ $t = t_F^{(1)}$

Differential Variables: (h, v, m) ....

$h = 0$
$v = 0$
$m = 3$

Differential-Algebraic Equations ....

Equations (28.4) - (28.6)

Phase 2 .... *Singular Arc* .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....
Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$ $\qquad$ $t_I^{(2)} < t < t_F^{(2)}$ $\qquad$ $t = t_F^{(2)}$

Differential Variables: (h, v, m) ....

$h = h_F^{(1)}$
$v = v_F^{(1)}$
$m = m_F^{(1)}$ $\qquad$ $m = 1$

Boundary Conditions ....

$0 = mg - \left(1 + \frac{v}{c}\right) \sigma v^2 \exp[-h/h_0]$
$t_F^{(2)} - t_I^{(2)} \geq 1$

Differential-Algebraic Equations ....

$$\dot{h} = v \tag{28.14}$$

$$\dot{v} = \frac{1}{m}\left[T_s - \sigma v^2 \exp[-h/h_0]\right] - g \tag{28.15}$$

$$\dot{m} = -T_s/c. \tag{28.16}$$

where

$$\begin{aligned} T_s &= \sigma v^2 \exp[-h/h_0] + mg \\ &\quad + \frac{mg}{1 + 4(c/v) + 2(c^2/v^2)}\left[\frac{c^2}{h_0 g}\left(1 + \frac{v}{c}\right) - 1 - 2\frac{c}{v}\right] \end{aligned} \tag{28.17}$$

Phase 3 .... *No Thrust* .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....
Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$ $\qquad$ $t_I^{(3)} < t < t_F^{(3)}$ $\qquad$ $t = t_F^{(3)}$

Differential Variables: (h, v) ....

$h = h_F^{(2)}$
$v = v_F^{(2)}$ $\qquad$ $v = 0$

Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} \geq 1$

Differential-Algebraic Equations . . . .

$$\dot{h} = v \tag{28.18}$$

$$\dot{v} = -\sigma v^2 \exp[-h/h_0] - g \tag{28.19}$$

Objective . . . .

Maximize $J = h_F$

$$\boxed{J^* = 18550.872; \quad t_F^{(1)} = 13.751270; \quad t_F^{(2)} = 21.987363; \quad t_F^{(3)} = 42.887912}$$

Chapter 29

goll: **Delay Equation, Göllmann, Kern, Maurer**

In reference [50, Sect 6] Göllmann, Kern, and Maurer present an optimal control problem with fixed delays in the state and control. The authors also derive the optimality conditions for the example, which permits an analytic solution. The method of steps is used to convert the original delay problems into ordinary optimal control cases. Examples (29.1) and (29.2) are fully described in the reference and example (29.3) corresponds to the Mayer formulation of example (29.1).

Example 29.1 goll01: DDE Optimal Control, Analytical Example.

Phase 1 .... *DDE: Method of Steps* .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\quad$ $0 < t < 1$ $\quad$ $t = 1$

Differential Variables: (x_1, x_2, x_3) ....

$x_1 = 1$

Algebraic Variables: (u_1, u_2, u_3) ....
Boundary Conditions ....

$x_1(1) = x_2(0)$
$x_2(1) = x_3(0)$
$u_1(1) = u_2(0)$
$u_2(1) = u_3(0)$

Differential-Algebraic Equations ....

$$\dot{x}_1 = x_0 u_{-1} \tag{29.1}$$
$$\dot{x}_2 = x_1 u_0 \tag{29.2}$$

$$\dot{x}_3 = x_2 u_1 \tag{29.3}$$

where

$$x_0 = 1 \tag{29.4}$$
$$u_{-1} = u_0 = 0 \tag{29.5}$$

Objective ....

Minimize $$J = \sum_{j=1}^{3} \int_0^1 x_j^2(t)dt + \sum_{j=1}^{3} \int_0^1 u_j^2(t)dt$$

$$\boxed{J^* = 2.76159451}$$

Example 29.2 goll02: DDE Optimal Control, Mixed State-Control Constraint.

Phase 1........................*DDE: Method of Steps*........................Phase 1

Independent Variable: (t) ....

$t = 0$ $\quad 0 < t < 1 \quad$ $t = 1$

Differential Variables: $(x_1, x_2, x_3, x_4, x_5, x_6)$ ....

$x_1 = 1$

Algebraic Variables: $(u_1, u_2, u_3, u_4, u_5, u_6)$ ....
Boundary Conditions ....

$x_1(1) = x_2(0)$
$x_2(1) = x_3(0)$
$x_3(1) = x_4(0)$
$x_4(1) = x_5(0)$
$x_5(1) = x_6(0)$
$u_1(1) = u_2(0)$
$u_2(1) = u_3(0)$
$u_3(1) = u_4(0)$
$u_4(1) = u_5(0)$
$u_5(1) = u_6(0)$

Differential-Algebraic Equations ....

$$\dot{x}_1 = x_0 u_{-1} \tag{29.6}$$
$$\dot{x}_2 = x_1 u_0 \tag{29.7}$$

$$\dot{x}_3 = x_2 u_1 \tag{29.8}$$
$$\dot{x}_4 = x_3 u_2 \tag{29.9}$$
$$\dot{x}_5 = x_4 u_3 \tag{29.10}$$
$$\dot{x}_6 = x_5 u_4 \tag{29.11}$$
$$.3 \le u_1 + x_1 \tag{29.12}$$
$$.3 \le u_2 + x_2 \tag{29.13}$$
$$.3 \le u_3 + x_3 \tag{29.14}$$
$$.3 \le u_4 + x_4 \tag{29.15}$$
$$.3 \le u_5 + x_5 \tag{29.16}$$
$$.3 \le u_6 + x_6 \tag{29.17}$$

where

$$x_0 = 1 \tag{29.18}$$
$$u_{-1} = u_0 = 0 \tag{29.19}$$

Objective ....

Minimize $$J = \sum_{j=1}^{6} \int_0^1 x_j^2(t)dt + \sum_{j=1}^{6} \int_0^1 u_j^2(t)dt$$

$$\boxed{J^* = 3.10812214}$$

Example 29.3 goll03: DDE Optimal Control, Mayer Form.

Phase 1 .... *DDE: Method of Steps* .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\qquad$ $0 < t < 1$ $\qquad$ $t = 1$

Differential Variables: $(x_1, y_1, x_2, y_2, x_3, y_3)$ ....

$x_1 = 1$
$y_1 = 0$

Algebraic Variables: (u_1, u_2, u_3) ....
Boundary Conditions ....

$x_1(1) = x_2(0)$
$y_1(1) = y_2(0)$
$x_2(1) = x_3(0)$

$y_2(1) = y_3(0)$
$u_1(1) = u_2(0)$
$u_2(1) = u_3(0)$

Differential-Algebraic Equations ....

$$\dot{x}_1 = x_0 u_{-1} \tag{29.20}$$
$$\dot{y}_1 = x_1^2 + u_1^2 \tag{29.21}$$
$$\dot{x}_2 = x_1 u_0 \tag{29.22}$$
$$\dot{y}_2 = x_2^2 + u_2^2 \tag{29.23}$$
$$\dot{x}_3 = x_2 u_1 \tag{29.24}$$
$$\dot{y}_3 = x_3^2 + u_3^2 \tag{29.25}$$

where

$$x_0 = 1 \tag{29.26}$$
$$u_{-1} = u_0 = 0 \tag{29.27}$$

Objective ....

Minimize $J = y_3(1)$

$\boxed{J^* = 2.76159420}$

Chapter 30

gsoc: **Multi-path Multi-phase Optimization**

This example illustrates an application with many features that are typical of a mission design for a military aircraft. The problem definition requires multiple phases and multiple paths that are optimized simultaneously. The basic path is specified by a collection of "way-points," through which the aircraft must fly. A second trajectory branch is introduced to model the dynamics of an un-powered "glide bomb" that is launched during the mission, and must hit a specified target. The overall goal of the mission is to fly the aircraft as fast as possible, and also hit the target with maximum velocity. The mission is modeled using eight phases, where the first seven phases define the aircraft trajectory between way-points, and phase eight defines the "glide bomb" trajectory to the target. Boundary conditions at the phase boundaries ensure state continuity for the aircraft. Continuity between the aircraft state at the end of phase three and the "glide bomb" state at the beginning of phase eight, defines the trajectory branch point. Different dynamic variables and constraints are used to reflect different flight conditions in the various phases.

Example 30.1 gsoc01: Branched Trajectory Optimization.

Phase 1 .... *Waypoint* $1 \Rightarrow 2$ .... Phase 1

Parameters: $(t_F^{(1)})$ ....
Independent Variable: (t) ....

$t = 0$	$0 \le t \le t_F^{(1)}$	$0 \le t \le t_F^{(1)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ ....

$h = 300$	$0 \le h \le 70000$	$h = 60000$	ft
$\phi = 0^o$	$-.5^o \le \phi \le 1.5^o$	$\phi = (1/6)^o$	rad
$\theta = 0^o$	$-1.5^o \le \theta \le .5^o$	$\theta = -(2/3)^o$	rad
$v = 948.0148985067440$	$200 \le v \le 3000$	$200 \le v \le 3000$	ft/sec

$\gamma = 0^o$	$-89^o \le \gamma \le 89^o$	$-89^o \le \gamma \le 89^o$	rad
$\psi = 165.9643839443566^o$	$\underline{\psi}_1 \le \psi \le \overline{\psi}_1$	$\underline{\psi}_1 \le \psi \le \overline{\psi}_1$	rad
$w = 41955$	$\underline{w} \le w \le \overline{w}$	$\underline{w} \le w \le \overline{w}$	lb

Algebraic Variables: (α, β) ....

$0 \le \alpha \le 45^o$	$0 \le \alpha \le 45^o$	$0 \le \alpha \le 45^o$	rad
$-180^o \le \beta \le 180^o$	$-180^o \le \beta \le 180^o$	$-180^o \le \beta \le 180^o$	rad

Differential-Algebraic Equations ....

$$\dot{h} = v \sin\gamma \tag{30.1}$$

$$\dot{\phi} = \frac{v \cos\gamma \sin\psi}{r \cos\theta} \tag{30.2}$$

$$\dot{\theta} = \frac{v}{r} \cos\gamma \cos\psi \tag{30.3}$$

$$\dot{v} = \frac{1}{m}(T\cos\alpha - D) - g\sin\gamma \tag{30.4}$$

$$\dot{\gamma} = \frac{\cos\beta}{mv}(T\sin\alpha + L) + \cos\gamma\left[\frac{v}{r} - \frac{g}{v}\right] \tag{30.5}$$

$$\dot{\psi} = \frac{(T\sin\alpha + L)\sin\beta}{mv\cos\gamma} + \frac{v\cos\gamma\sin\psi\sin\theta}{r\cos\theta} \tag{30.6}$$

$$\dot{w} = \frac{-T}{I_{sp}} \tag{30.7}$$

where the problem parameters are given in Table 18.1 and Table 30.1. The functions $T(M,h)$, $c_{D0}(M)$, $\eta(M)$, and $c_{L\alpha}(M)$ are represented by cubic spline interpolants constructed from the data in Tables 18.2 and 18.3. A smooth approximation to the 1962 standard atmosphere [26] is used to compute $\rho = \rho(h)$ and $v_c(h)$, where $M = v/v_c(h)$.

Phase 2 .... *Waypoint* $2 \Rightarrow 3$ .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....
Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} \le t \le t_F^{(2)}$	$t_I^{(2)} \le t \le t_F^{(2)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ ....

$h = 60000$	$0 \le h \le 70000$	$h = 3000$	ft
$\phi = (1/6)^o$	$-.5^o \le \phi \le 1.5^o$	$\phi = .5^o$	rad
$\theta = -(2/3)^o$	$-1.5^o \le \theta \le .5^o$	$\theta = -(5/6)^o$	rad
$v = v_F^{(1)}$	$200 \le v \le 3000$	$200 \le v \le 3000$	ft/sec
$\gamma = \gamma_F^{(1)}$	$-89^o \le \gamma \le 89^o$	$\gamma = 0^o$	rad

$\psi = \psi_F^{(1)}$	$\underline{\psi}_2 \le \psi \le \overline{\psi}_2$	$\underline{\psi}_2 \le \psi \le \overline{\psi}_2$	rad
$w = w_F^{(1)}$	$\underline{w} \le w \le \overline{w}$	$\underline{w} \le w \le \overline{w}$	lb

Algebraic Variables: (α, β) ....

$0 \le \alpha \le 45^o$	$0 \le \alpha \le 45^o$	$0 \le \alpha \le 45^o$	rad
$-180^o \le \beta \le 180^o$	$-180^o \le \beta \le 180^o$	$-180^o \le \beta \le 180^o$	rad

Differential-Algebraic Equations ....

Equations (30.1) - (30.7)

Phase 3 .... *Waypoint* $3 \Rightarrow 4$ .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....
Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} \le t \le t_F^{(3)}$	$t_I^{(3)} \le t \le t_F^{(3)}$	sec

Differential Variables: $(\phi, \theta, v, \psi, w)$ ....

$\phi = .5^o$	$-.5^o \le \phi \le 1.5^o$	$-.5^o \le \phi \le 1.5^o$	rad
$\theta = -(5/6)^o$	$-1.5^o \le \theta \le .5^o$	$-1.5^o \le \theta \le .5^o$	rad
$v = v_F^{(2)}$	$200 \le v \le 3000$	$200 \le v \le 3000$	ft/sec
$\psi = \psi_F^{(2)}$	$\underline{\psi}_3 \le \psi \le \overline{\psi}_3$	$\underline{\psi}_3 \le \psi \le \overline{\psi}_3$	rad
$w = w_F^{(2)}$	$\underline{w} \le w \le \overline{w}$	$\underline{w} \le w \le \overline{w}$	lb

Algebraic Variables: (α, β) ....

$0 \le \alpha \le 45^o$	$0 \le \alpha \le 45^o$	$0 \le \alpha \le 45^o$	rad
$-180^o \le \beta \le 180^o$	$-180^o \le \beta \le 180^o$	$-180^o \le \beta \le 180^o$	rad

Differential-Algebraic Equations ....
Equations (30.2), (30.3), (30.4), (30.6), and (30.7).

Phase 4 .... *Waypoint* $4 \Rightarrow 5$ .... Phase 4

Parameters: $(t_I^{(4)}, t_F^{(4)})$ ....
Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} \le t \le t_F^{(4)}$	$t_I^{(4)} \le t \le t_F^{(4)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ ....

$h = 3000$	$0 \leq h \leq 70000$	$h = 25000$	ft
$\phi = \phi_F^{(3)}$	$-.5^o \leq \phi \leq 1.5^o$	$\phi = (5/6)^o$	rad
$\theta = \theta_F^{(3)}$	$-1.5^o \leq \theta \leq .5^o$	$\theta = -(1/3)^o$	rad
$v = v_F^{(3)}$	$200 \leq v \leq 3000$	$200 \leq v \leq 3000$	ft/sec
$\gamma = 0^o$	$-89^o \leq \gamma \leq 89^o$	$-89^o \leq \gamma \leq 89^o$	rad
$\psi = \psi_F^{(3)}$	$-180^o \leq \psi \leq 180^o$	$-180^o \leq \psi \leq 180^o$	rad
$w = w_F^{(3)}$	$\underline{w} \leq w \leq \overline{w}$	$\underline{w} \leq w \leq \overline{w}$	lb

Algebraic Variables: (α, β) ....

$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	rad
$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	rad

Differential-Algebraic Equations ....

Equations (30.1) - (30.7)

Phase 5 .... *Waypoint* $5 \Rightarrow 6$ .... Phase 5

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....
Independent Variable: (t) ....

$t = t_F^{(4)} = t_I^{(5)}$	$t_I^{(5)} \leq t \leq t_F^{(5)}$	$t_I^{(5)} \leq t \leq t_F^{(5)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ ....

$h = 25000$	$0 \leq h \leq 70000$	$h = 40000$	ft
$\phi = (5/6)^o$	$-.5^o \leq \phi \leq 1.5^o$	$\phi = .5^o$	rad
$\theta = -(1/3)^o$	$-1.5^o \leq \theta \leq .5^o$	$\theta = -(1/6)^o$	rad
$v = v_F^{(4)}$	$200 \leq v \leq 3000$	$200 \leq v \leq 3000$	ft/sec
$\gamma = \gamma_F^{(4)}$	$-89^o \leq \gamma \leq 89^o$	$-89^o \leq \gamma \leq 89^o$	rad
$\psi = \psi_F^{(4)}$	$\underline{\psi}_5 \leq \psi \leq \overline{\psi}_5$	$\underline{\psi}_5 \leq \psi \leq \overline{\psi}_5$	rad
$w = w_F^{(4)}$	$\underline{w} \leq w \leq \overline{w}$	$\underline{w} \leq w \leq \overline{w}$	lb

Algebraic Variables: (α, β) ....

$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	rad
$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	rad

Differential-Algebraic Equations ....

Equations (30.1) - (30.7)

Phase 6 .... *Waypoint* $6 \Rightarrow 7$ .... Phase 6

Parameters: $(t_I^{(6)}, t_F^{(6)})$ ....
Independent Variable: (t) ....

$t = t_F^{(5)} = t_I^{(6)}$	$t_I^{(6)} \leq t \leq t_F^{(6)}$	$t_I^{(6)} \leq t \leq t_F^{(6)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ ....

$h = 40000$	$0 \leq h \leq 70000$	$h = 20000$	ft
$\phi = .5^o$	$-.5^o \leq \phi \leq 1.5^o$	$\phi = (1/6)^o$	rad
$\theta = -(1/6)^o$	$-1.5^o \leq \theta \leq .5^o$	$\theta = -.5^o$	rad
$v = v_F^{(5)}$	$200 \leq v \leq 3000$	$200 \leq v \leq 3000$	ft/sec
$\gamma = \gamma_F^{(5)}$	$-89^o \leq \gamma \leq 89^o$	$-89^o \leq \gamma \leq 89^o$	rad
$\psi = \psi_F^{(5)}$	$\underline{\psi}_6 \leq \psi \leq \overline{\psi}_6$	$\underline{\psi}_6 \leq \psi \leq \overline{\psi}_6$	rad
$w = w_F^{(5)}$	$\underline{w} \leq w \leq \overline{w}$	$\underline{w} \leq w \leq \overline{w}$	lb

Algebraic Variables: (α, β) ....

$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	rad
$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	rad

Differential-Algebraic Equations ....

Equations (30.1) - (30.7)

Phase 7 .... *Waypoint* $7 \Rightarrow 8$ .... Phase 7

Parameters: $(t_I^{(7)}, t_F^{(7)})$ ....
Independent Variable: (t) ....

$t = t_F^{(6)} = t_I^{(7)}$	$t_I^{(7)} \leq t \leq t_F^{(7)}$	$t_I^{(7)} \leq t \leq t_F^{(7)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ ....

$h = 20000$	$0 \leq h \leq 70000$	$h = 1000$	ft
$\phi = (1/6)^o$	$-.5^o \leq \phi \leq 1.5^o$	$\phi = 0^o$	rad

$\theta = -.5^o$	$-1.5^o \leq \theta \leq .5^o$	$\theta = -1^o$	rad
$v = v_F^{(6)}$	$200 \leq v \leq 3000$	$200 \leq v \leq 3000$	ft/sec
$\gamma = \gamma_F^{(6)}$	$-89^o \leq \gamma \leq 89^o$	$-89^o \leq \gamma \leq 89^o$	rad
$\psi = \psi_F^{(6)}$	$\underline{\psi}_7 \leq \psi \leq \overline{\psi}_7$	$\underline{\psi}_7 \leq \psi \leq \overline{\psi}_7$	rad
$w = w_F^{(6)}$	$\underline{w} \leq w \leq \overline{w}$	$\underline{w} \leq w \leq \overline{w}$	lb

Algebraic Variables: (α, β) ....

$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	rad
$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	rad

Differential-Algebraic Equations ....

Equations (30.1) - (30.7)

Phase 8 .... *Waypoint* 4 $\Rightarrow$ *Target* .... Phase 8

Parameters: $(t_I^{(8)}, t_F^{(8)})$ ....
Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(8)}$	$t_I^{(8)} \leq t \leq t_F^{(8)}$	$t_I^{(8)} \leq t \leq t_F^{(8)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = 30000$	$0 \leq h \leq 70000$	$h = 0$	ft
$\phi = \phi_F^{(3)}$	$-.5^o \leq \phi \leq 1.5^o$	$\phi = 1^o$	rad
$\theta = -.5^o$	$-1.5^o \leq \theta \leq .5^o$	$\theta = 0^o$	rad
$v = v_F^{(3)}$	$200 \leq v \leq 3000$	$200 \leq v \leq 3000$	ft/sec
$\gamma = 0^o$	$-89^o \leq \gamma \leq 89^o$	$-89^o \leq \gamma \leq 89^o$	rad
$\psi = \psi_F^{(3)}$	$-180^o \leq \psi \leq 180^o$	$-180^o \leq \psi \leq 180^o$	rad

Algebraic Variables: (α, β) ....

$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	$0 \leq \alpha \leq 45^o$	rad
$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	rad

Differential-Algebraic Equations ....
Equations (30.1)-(30.6), with $T = 0$ and $w = 50000$.

Objective ....

Minimize $J = t_F^{(7)} - v_F^{(8)}$

$$J^* = 233.120824; \quad t_F^{(7)} = 863.01096; \quad v_F^{(8)} = 629.89014$$

$\underline{w} = 4195.5$	$\overline{w} = 46150.5$
$\underline{\psi}_1 = -14.03561605564343^o$	$\overline{\psi}_1 = 345.9643839443565^o$
$\underline{\psi}_2 = -63.43087633909731^o$	$\overline{\psi}_2 = 296.5691236609027^o$
$\underline{\psi}_3 = -116.5647604454595^o$	$\overline{\psi}_3 = 243.4352395545405^o$
$\underline{\psi}_5 = -243.4355305990111^o$	$\overline{\psi}_5 = 116.5644694009889^o$
$\underline{\psi}_6 = -315.0013332415543^o$	$\overline{\psi}_6 = 44.99866675844564^o$
$\underline{\psi}_7 = -341.5675479435504^o$	$\overline{\psi}_7 = 18.43245205644953^o$

Table 30.1. *Multi-phase Multi-path example constants.*

Chapter 31

gydn: **Reentry Guidance Problem**

For a reentry vehicle such as the space shuttle a common goal is to determine a trajectory that can reach a specified point on the ground, while minimizing the acceleration normal to the trajectory path. In this example the goal is to steer the trajectory by choosing the angle of attack and bank angle to minimize the lateral acceleration over the duration of the reentry trajectory.

Example 31.1 gydn01: MINIMUM LATERAL ACCELERATION GUIDANCE.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$.005 \le t_F \le 1000$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, a)$ ....

$h = h_I$	$0 \le h \le 100000$	$h = h_F$	ft
$\phi = \phi_I$	$-114^o \le \phi \le -112^o$	$\phi = \phi_F$	rad
$\theta = \theta_I$	$36^o \le \theta \le 38^o$	$\theta = \theta_F$	rad
$v = v_I$	$1 \le v \le 1000$		ft/sec
$\gamma = 0$	$-89^o \le \gamma \le +89^o$	$\gamma = \gamma_F$	rad
$\psi = \psi_I$	$-180^o \le \gamma \le +180^o$	$\psi = \psi_F$	rad
$a = 0$	$0 \le a$	$0 \le a$	

Algebraic Variables: (α, β) ....

$-6.5^o \le \alpha \le +13.5^o$	$-6.5^o \le \alpha \le +13.5^o$	$-6.5^o \le \alpha \le +13.5^o$	rad

$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	$-180^o \leq \beta \leq 180^o$	rad

Differential-Algebraic Equations . . . .

$$\dot{h} = v \sin\gamma \tag{31.1}$$

$$\dot{\phi} = \frac{v}{r\cos\theta}\cos\gamma\sin\psi \tag{31.2}$$

$$\dot{\theta} = \frac{v}{r}\cos\gamma\cos\psi \tag{31.3}$$

$$\dot{v} = -\frac{D}{m} - g\sin\gamma \tag{31.4}$$

$$\dot{\gamma} = \frac{L}{mv}\cos\beta + \cos\gamma\left(\frac{v}{r} - \frac{g}{v}\right) \tag{31.5}$$

$$\dot{\psi} = \frac{1}{mv\cos\gamma}L\sin\beta + \frac{v}{r\cos\theta}\cos\gamma\sin\psi\sin\theta \tag{31.6}$$

$$\dot{a} = \dot{\gamma}^2 + \dot{\psi}^2 \tag{31.7}$$

$$0 \leq M \leq .93 \tag{31.8}$$

for the parameter definitions given in Table 31.1.

Objective . . . .

Minimize $\qquad J = a(t_F)$

$$\boxed{J^* = 1.8511591 \times 10^{-1}; \quad t_F^* = 202.04126}$$

$h_I = 5075$	$h_F = 5749.3$
$\phi_I = -113.2205667^o$	$\phi_F = -113.2200639^o$
$\theta_I = 37.23631389^o$	$\theta_F = 37.27560603^o$
$v_I = 877.3894136$	$\gamma_F = -3^o$
$\psi_I = 4.42788880^o$	$\psi_F = 160^o$
$D = \frac{1}{2}c_D S\rho v^2$	$L = \frac{1}{2}c_L S\rho v^2$
$\rho = \rho(h)$ (Ref. [26])	$v_c = v_c(h)$ (Ref. [26])
$c_L = d_L(\alpha, M)$	$c_D = d_D(\alpha, M)$
$M = v/v_c$	$S = 13$
$g = \mu/r^2$	$\mu = 0.14076539 \times 10^{17}$
$r = R_e + h$	$R_e = 20902900$
$w = 1650$	$m = w/g_0$

Table 31.1. *Guidance reentry example parameters.*

Chapter 32

hang: **Maximum Range of a Hang Glider**

Originally posed by Bulirsch, Nerz, Pesch, and von Stryk [33], this problem describes the optimal control of a hang glider in the presence of a specified thermal updraft. It is particularly sensitive to the accuracy of the dynamics in the updraft region, a difficulty resolved by Oskar von Stryk in his thesis by exploiting a combination of direct and indirect methods. A detailed discussion of the problem is given in reference [13, Sect. 6.5]. The original problem formulation is given in example (32.1). Example (32.2) introduces a modified formulation of fixed duration, and example (32.3) recasts the dynamics using horizontal distance and the independent variable.

Example 32.1 hang01: Original Formulation.

Phase 1 . . . . Phase 1

Parameters: (t_F) . . . .

$0 \le t_F \le 110$

Independent Variable: (t) . . . .

$t = 0$	$0 < t < t_F$	$t = t_F$	sec

Differential Variables: (x, y, v_x, v_y) . . . .

$x = 0$		m
$y = 1000$	$y = 900$	m
$v_x = \bar{v}_x$	$v_x = \bar{v}_x$	m/sec
$v_y = \bar{v}_y$	$v_y = \bar{v}_y$	m/sec

Algebraic Variables: (C_L) . . . .

$0 \le C_L \le 1.4$ $0 \le C_L \le 1.4$ $0 \le C_L \le 1.4$

Differential-Algebraic Equations ....

$$\dot{x} = v_x \tag{32.1}$$
$$\dot{y} = v_y \tag{32.2}$$
$$\dot{v}_x = \frac{1}{m}\left(-L\sin\eta - D\cos\eta\right) \tag{32.3}$$
$$\dot{v}_y = \frac{1}{m}\left(L\cos\eta - D\sin\eta - mg\right) \tag{32.4}$$

where

$$C_D(C_L) = C_0 + kC_L^2 \tag{32.5}$$
$$D = \frac{1}{2}C_D\rho S v_r^2 \tag{32.6}$$
$$L = \frac{1}{2}C_L\rho S v_r^2 \tag{32.7}$$
$$X = \left(\frac{x}{R} - 2.5\right)^2 \tag{32.8}$$
$$V_y = v_y - u_M\left(1 - X\right)\exp[-X] \tag{32.9}$$
$$v_r = \sqrt{v_x^2 + V_y^2} \tag{32.10}$$
$$v_r = \sqrt{v_x^2 + V_y^2} \tag{32.11}$$
$$\sin\eta = \frac{V_y}{v_r} \tag{32.12}$$
$$\cos\eta = \frac{v_x}{v_r} \tag{32.13}$$

and model constants are given in Table 32.1.

Objective ....

Maximize $J = x(t_F)$

$\boxed{J^* = 1248.03103; \quad t_F^* = 98.436940}$

Example 32.2 hang02: Augmented Formulation.

Phase 1 .... Phase 1

Independent Variable: (τ) ....

$\tau = 0$ $0 < \tau < 1$ $\tau = 1$

Differential Variables: (x, y, v_x, v_y, t_F) ....

$x = 0$			m
$y = 1000$		$y = 900$	m
$v_x = \bar{v}_x$		$v_x = \bar{v}_x$	m/sec
$v_y = \bar{v}_y$		$v_y = \bar{v}_y$	m/sec
		$0 \le t_F \le 110$	sec

Algebraic Variables: (C_L) ....

$0 \le C_L \le 1.4$	$0 \le C_L \le 1.4$	$0 \le C_L \le 1.4$

Differential-Algebraic Equations ....

$$x' = t_F v_x \tag{32.14}$$
$$y' = t_F v_y \tag{32.15}$$
$$v_x' = \frac{t_F}{m}\left(-L\sin\eta - D\cos\eta\right) \tag{32.16}$$
$$v_y' = \frac{t_F}{m}\left(L\cos\eta - D\sin\eta - mg\right) \tag{32.17}$$
$$t_F' = 0 \tag{32.18}$$

where (32.5)-(32.13) are used with the model constants given in Table 32.1.

Objective ....

Maximize $\qquad J = x(1)$

$$\boxed{J^* = 1248.03102; \quad t_F^* = 98.436735}$$

Example 32.3 hang03: Compressed Formulation.

Phase 1 .... Phase 1

Parameters: (x_F) ....

$0 \le x_F \le 1500$

Independent Variable: (x) ....

$x = 0$	$0 < x < x_F$	$x = x_F$	m

Differential Variables: (y, v_x, v_y) ....

$y = 1000$		$y = 900$	m
$v_x = \bar{v}_x$		$v_x = \bar{v}_x$	m/sec
$v_y = \bar{v}_y$		$v_y = \bar{v}_y$	m/sec

Algebraic Variables: (C_L) . . . .

$0 \leq C_L \leq 1.4$ $\quad$ $0 \leq C_L \leq 1.4$ $\quad$ $0 \leq C_L \leq 1.4$

Differential-Algebraic Equations . . . .

$$y' = \frac{v_y}{v_x} \tag{32.19}$$

$$v_x' = \frac{1}{mv_x}\left(-L\sin\eta - D\cos\eta\right) \tag{32.20}$$

$$v_y' = \frac{1}{mv_x}\left(L\cos\eta - D\sin\eta - mg\right) \tag{32.21}$$

where (32.5)-(32.13) are used with the model constants given in Table 32.1.

Objective . . . .

Maximize $\quad J = x_F$

$$\boxed{J^* = 1248.03103}$$

u_M	2.5	m	100. (kg)
R	100.	S	14. (m^2)
C_0	.034	ρ	1.13 ($\mathrm{kg/m}^3$)
k	.069662	g	9.80665 ($\mathrm{m/sec}^2$)
$\bar{v}_x$	13.227567500 (m/sec)	$\bar{v}_y$	-1.2875005200 (m/sec)

Table 32.1. *Dynamic Model Constants*

Chapter 33

hdae: **Heat Diffusion Process with Inequality**

The method of lines is a technique for constructing a system of ordinary differential equations that approximate the solution of a partial differential equation. When state constraints are imposed, it is expected that a differential-algebraic equation will describe the dynamics in regions where the state constraints are binding. However, when the control variable is introduced on the boundary of the region, this approach suggests that the *index* of the DAE can be arbitrarily high when the state constraint is active. This example was first introduced by Stephen Campbell and studied in references [18, 19, 20, 63, 72, 73]. It is also described extensively in reference [13, Sect. 4.12].

Example 33.1 hdae01: High Index DAE from Method of Lines.

Phase 1 . . . . Phase 1

Independent Variable: (t) . . . .

$t = 0$ $\qquad$ $0 < t < 5$ $\qquad$ $t = 5$

Differential Variables: $(y_1, y_2, \ldots, y_{n-1})$ . . . .

$y_k = 0$

Algebraic Variables: (u_0, u_π) . . . .

$u_0 = 0$
$u_\pi = 0$

Differential-Algebraic Equations . . . .

$$\dot{y}_1 = \frac{1}{\delta^2}\left(y_2 - 2y_1 + u_0\right) \tag{33.1}$$

$$\dot{y}_k = \frac{1}{\delta^2}(y_{k+1} - 2y_k + y_{k-1}) \qquad k = 2, \ldots, n-2 \tag{33.2}$$

$$\dot{y}_{n-1} = \frac{1}{\delta^2}(u_\pi - 2y_{n-1} + y_{n-2}) \tag{33.3}$$

$$0 \geq g(x_k, t) - y_k \qquad k = 0, \ldots, n \tag{33.4}$$

where $x_k = k\delta = k\frac{\pi}{n}$ for $k = 0, \ldots, n$ and

$$g(x,t) = c\left[\sin x \sin\left(\frac{\pi t}{5}\right) - a\right] - b \tag{33.5}$$

To complete the problem definition set $n = 20$ with constants $q_1 = q_2 = 10^{-3}$, $a = .5$, $b = .2$, and $c = 1$.

Objective ....

Minimize

$$J = \int_0^5 \left[\frac{1}{2}\delta + q_1\right] u_0^2(t)dt + \delta \sum_{k=1}^{n-1} \int_0^5 y_k^2(t)dt + \int_0^5 \left[\frac{1}{2}\delta + q_2\right] u_\pi^2(t)dt$$

$$\boxed{J^* = 4.68159793 \times 10^{-1}}$$

Chapter 34

heat: **Heat Equation**

The optimal control of a distributed parameter system, that is a system defined by partial differential equations can be transformed to a system of ordinary differential equations using the method of lines. Two different problems that demonstrate this technique are given here. Example (34.1) first appeared in reference [22]. A more complex process given here as example (34.2) was first discussed by Heinkenschloss in reference [56] and is also addressed in reference [13, Sect. 4.6.10]

Example 34.1 heat01: Minimum Deviation Heating, Boundary Control.

Phase 1 .... *PDE using Method of Lines* .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\qquad$ $0 < t < 0.2$ $\qquad$ $t = 0.2$

Differential Variables: $(q_1, \ldots, q_{10}, w)$ ....

$q_1 = 0$
$\vdots$
$q_{10} = 0$
$w = 0$

Algebraic Variables: (v, q_0, q_{11}) ....

$0 \le v \le 1$ $\qquad$ $0 \le v \le 1$ $\qquad$ $0 \le v \le 1$

Differential-Algebraic Equations ....

$$\dot{q}_1 = \frac{1}{\delta^2}\left(q_2 - 2q_1 + q_0\right) \tag{34.1}$$

$$\dot{q}_k = \frac{1}{\delta^2}\left(q_{k+1} - 2q_k + q_{k-1}\right) \qquad k = 2,\ldots,9 \tag{34.2}$$

$$\dot{q}_{10} = \frac{1}{\delta^2}\left(q_{11} - 2q_{10} + q_9\right) \tag{34.3}$$

$$\dot{w} = \frac{1}{\gamma}(v - w) \tag{34.4}$$

$$0 = h(q_1 - w) - \frac{1}{2\delta}(q_2 - q_0) \tag{34.5}$$

$$0 = \frac{1}{2\delta}(q_{11} - q_9) \tag{34.6}$$

where $q_a = .2$, $\gamma = .04$, $h = 10$, and $\delta = 1/9$.

Objective ....

Minimize

$$J = \left[\frac{1}{2\delta}(q_a - q_1)^2 + \frac{1}{\delta}\sum_{k=2}^{9}(q_a - q_k)^2 + \frac{1}{2\delta}(q_a - q_{10})^2\right]\Bigg|_{t=0.2}$$

$$\boxed{J^* = 2.45476113 \times 10^{-3}}$$

Example 34.2 heat02: Optimal Kiln Heating Process.

Phase 1 .... *PDE using Method of Lines* .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < T$	$t = T = 0.5$

Differential Variables: $(y_1, \ldots, y_N$ for $N = 50)$ ....

$y_i = y_I(x_i) = 2 + \cos(\pi x_i)$

where $x_i = (i-1)/(N-1)$ and $i = 1, \ldots, N$.

Algebraic Variables: (v_1, v_2, v_3) ....

$v_1 \le 0.1$	$v_1 \le 0.1$	$v_1 \le 0.1$

Differential-Algebraic Equations ....

$$\dot{y}_1 = \frac{1}{(a_1 + a_2 y_1)}\left[q_1 + \frac{1}{\delta^2}(a_3 + a_4 y_1)(y_2 - 2y_1 + v_2) + a_4\left(\frac{y_2 - v_2}{2\delta}\right)^2\right] \tag{34.7}$$

$$\dot{y}_i = \frac{1}{(a_1 + a_2 y_i)}\left[q_i + \frac{1}{\delta^2}(a_3 + a_4 y_i)(y_{i+1} - 2y_i + y_{i-1}) + a_4\left(\frac{y_{i+1} - y_{i-1}}{2\delta}\right)^2\right] \tag{34.8}$$

for $i = 2, \ldots, N-1$

$$\dot{y}_N = \frac{1}{(a_1 + a_2 y_N)} \left[q_N + \frac{1}{\delta^2}(a_3 + a_4 y_N)(v_3 - 2y_N + y_{N-1}) + a_4 \left(\frac{v_3 - y_{N-1}}{2\delta} \right)^2 \right] \tag{34.9}$$

$$0 = y_1 - v_1 - \frac{1}{2\delta}(a_3 + a_4 y_1)(y_2 - v_2), \tag{34.10}$$

$$0 = \frac{1}{2\delta}(a_3 + a_4 y_N)(v_3 - y_{N-1}) \tag{34.11}$$

and for $i = 1, \ldots, N$

$$x_i = \frac{i-1}{N-1} \tag{34.12}$$

$$\begin{aligned} q_i = {} & \left[\rho(a_1 + 2a_2) + \pi^2(a_3 + 2a_4) \right] e^{\rho t} \cos(\pi x_i) \\ & - a_4 \pi^2 e^{2\rho t} + (2a_4 \pi^2 + \rho a_2) e^{2\rho t} \cos^2(\pi x_i) \end{aligned} \tag{34.13}$$

where $\delta = 1/(N-1)$ and the constants are

$$a_1 = 4 \qquad a_2 = 1 \qquad a_3 = 4 \qquad a_4 = -1 \qquad \rho = -1 \qquad \gamma = 10^{-3}.$$

Objective ....

Minimize $$J = \frac{1}{2} \int_0^T \left\{ \left[y_N - y_d \right]^2 + \gamma v_1^2 \right\} dt$$

where $y_d(t) = 2 - e^{\rho t}$.

$$\boxed{J^* = 3.87868446 \times 10^{-5}}$$

Chapter 35

jmp2: **Analytic Propagation Two Burn Transfer**

The two burn orbit transfer is an important aerospace problem, and consequently the physics of the problem is often treated using mathematical models of different fidelity. Preliminary mission studies can utilize lower fidelity models, whereas higher accuracy is needed for final mission design. Examples (14.1)-(14.4) provide implementations of moderate accuracy. In reference [59] Huffman develops closed form approximations to the orbit dynamics. This technique permits specification of the coast phases using a few parameters to replace the system of differential equations. Example (35.1) incorporates the analytic orbit propagation technique when the thrust direction varies during the burn phases and example (35.2) uses steering that is constant during the burn.

Example 35.1 jmp201: Optimal Time Varying Steering.

Phase 1 .... *Coast in Park Orbit* .... Phase 1

Parameters: $(\alpha^{(1)}, t_F^{(1)})$ ....

$90^o \le \alpha^{(1)} \le 270^o$

Independent Variable: (t) ....

$0 \le t \le t_F^{(1)}$

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = r_0$	$-10r_0 \le r_x \le 10r_0$	$-10r_0 \le r_x \le 10r_0$	ft
$r_y = 0$	$-10r_0 \le r_y \le 10r_0$	$-10r_0 \le r_y \le 10r_0$	ft
$r_z = 0$	$-10r_0 \le r_z \le 10r_0$	$-10r_0 \le r_z \le 10r_0$	ft
$v_x = 0$	$-10v_0 \le v_x \le 10v_0$	$-10v_0 \le v_x \le 10v_0$	ft/sec
$v_y = -v_o \cos i_0$	$-10v_0 \le v_y \le 10v_0$	$-10v_0 \le v_y \le 10v_0$	ft/sec
$v_z = v_o \sin i_0$	$-10v_0 \le v_z \le 10v_0$	$-10v_0 \le v_z \le 10v_0$	ft/sec

Boundary Conditions . . . .

$$\mathbf{z}_F = \boldsymbol{\xi}[\mathbf{z}_I^{(1)}, \alpha^{(1)}]$$

where $\boldsymbol{\xi}[\mathbf{z}_I^{(1)}, \alpha^{(1)}]$ is computed using the propagation algorithm [59] and

$$\mathbf{z}^\mathsf{T} = (\mathbf{r}^\mathsf{T}, \mathbf{v}^\mathsf{T}, t) \tag{35.1}$$

$$r_0 = h_0 + R_e \tag{35.2}$$

$$\mathbf{r}^\mathsf{T} = (r_x, r_y, r_z) \tag{35.3}$$

$$r = \|\mathbf{r}\| \tag{35.4}$$

$$v_o = \sqrt{\frac{\mu}{r}} \tag{35.5}$$

$$\mathbf{v}^\mathsf{T} = (v_x, v_y, v_z) \tag{35.6}$$

Phase 2 . . . . *First Burn* . . . . Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} \le t \le t_F^{(2)}$

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ . . . .

$r_x = r_{xF}^{(1)}$	$-10r_0 \le r_x \le 10r_0$	$-10r_0 \le r_x \le 10r_0$	ft
$r_y = r_{yF}^{(1)}$	$-10r_0 \le r_y \le 10r_0$	$-10r_0 \le r_y \le 10r_0$	ft
$r_z = r_{zF}^{(1)}$	$-10r_0 \le r_z \le 10r_0$	$-10r_0 \le r_z \le 10r_0$	ft
$v_x = v_{xF}^{(1)}$	$-10v_0 \le v_x \le 10v_0$	$-10v_0 \le v_x \le 10v_0$	ft/sec
$v_y = v_{yF}^{(1)}$	$-10v_0 \le v_y \le 10v_0$	$-10v_0 \le v_y \le 10v_0$	ft/sec
$v_z = v_{zF}^{(1)}$	$-10v_0 \le v_z \le 10v_0$	$-10v_0 \le v_z \le 10v_0$	ft/sec
$w = 1$			lb

Algebraic Variables: (ψ, θ) . . . .

$-20^o \le \psi \le 20^o$	$-20^o \le \psi \le 20^o$	$-20^o \le \psi \le 20^o$	rad
$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	rad

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{r}} = \mathbf{v} \tag{35.7}$$

$$\dot{\mathbf{v}} = \mathbf{g} + \mathbf{T} \tag{35.8}$$

$$\dot{w} = -T_c/I_{sp} \tag{35.9}$$

using the definitions in (35.1)-(35.6) and

$$\mathbf{Q}_v = \left[\begin{array}{ccc} \frac{\mathbf{v}}{\|\mathbf{v}\|} & \frac{\mathbf{v}\times\mathbf{r}}{\|\mathbf{v}\times\mathbf{r}\|} & \frac{\mathbf{v}}{\|\mathbf{v}\|} \times \left(\frac{\mathbf{v}\times\mathbf{r}}{\|\mathbf{v}\times\mathbf{r}\|}\right) \end{array} \right] \tag{35.10}$$

$$\mathbf{T} = \frac{T_c g_0}{w}\mathbf{Q}_v \begin{pmatrix} \cos\theta\cos\psi \\ \cos\theta\sin\psi \\ \sin\theta \end{pmatrix} \tag{35.11}$$

$$\mathbf{i}_r = \frac{\mathbf{r}}{\|\mathbf{r}\|} \tag{35.12}$$

$$\boldsymbol{\delta}\mathbf{g} = \delta g_n \mathbf{i}_n - \delta g_r \mathbf{i}_r \tag{35.13}$$

$$\mathbf{i}_n = \frac{\mathbf{e}_n - (\mathbf{e}_n^\top \mathbf{i}_r)\mathbf{i}_r}{\|\mathbf{e}_n - (\mathbf{e}_n^\top \mathbf{i}_r)\mathbf{i}_r\|} \tag{35.14}$$

$$\mathbf{e}_n^\mathsf{T} = (0, 0, 1) \tag{35.15}$$

$$\cos\phi = \sqrt{1 - (r_3/r)^2} \tag{35.16}$$

$$\delta g_n = -\frac{\mu\cos\phi}{r^2}\sum_{k=2}^{4}\left(\frac{R_e}{r}\right)^k P_k' J_k \tag{35.17}$$

$$\delta g_r = -\frac{\mu}{r^2}\sum_{k=2}^{4}(k+1)\left(\frac{R_e}{r}\right)^k P_k J_k \tag{35.18}$$

$$\mathbf{g} = -\frac{\mu}{r^2}\mathbf{i}_r + \boldsymbol{\delta}\mathbf{g} \tag{35.19}$$

where P_k are Legendre polynomials.

Phase 3 .... *Coast in Transfer Orbit* .... Phase 3

Parameters: $(\alpha^{(3)}, t_I^{(3)}, t_F^{(3)})$ ....

$90^o \leq \alpha^{(3)} \leq 270^o$

Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$ $\qquad$ $t_I^{(3)} \leq t \leq t_F^{(3)}$

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = r_{xF}^{(2)}$	$-10r_0 \leq r_x \leq 10r_0$	$-10r_0 \leq r_x \leq 10r_0$	ft
$r_y = r_{yF}^{(2)}$	$-10r_0 \leq r_y \leq 10r_0$	$-10r_0 \leq r_y \leq 10r_0$	ft
$r_z = r_{zF}^{(2)}$	$-10r_0 \leq r_z \leq 10r_0$	$-10r_0 \leq r_z \leq 10r_0$	ft
$v_x = v_{xF}^{(2)}$	$-10v_0 \leq v_x \leq 10v_0$	$-10v_0 \leq v_x \leq 10v_0$	ft/sec
$v_y = v_{yF}^{(2)}$	$-10v_0 \leq v_y \leq 10v_0$	$-10v_0 \leq v_y \leq 10v_0$	ft/sec
$v_z = v_{zF}^{(2)}$	$-10v_0 \leq v_z \leq 10v_0$	$-10v_0 \leq v_z \leq 10v_0$	ft/sec

Boundary Conditions ....

$$\mathbf{z}_F = \boldsymbol{\xi}[\mathbf{z}_I^{(3)}, \alpha^{(3)}]$$

where $\boldsymbol{\xi}[\mathbf{z}_I^{(3)}, \alpha^{(3)}]$ is computed using the propagation algorithm [59] and the definitions (35.1)-(35.6).

Phase 4 .... *Second Burn* .... Phase 4

Parameters: $(t_I^{(4)}, t_F^{(4)})$ ....
Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} \le t \le t_F^{(4)}$

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(3)}$	$-10r_0 \le r_x \le 10r_0$	$-10r_0 \le r_x \le 10r_0$	ft
$r_y = r_{yF}^{(3)}$	$-10r_0 \le r_y \le 10r_0$	$-10r_0 \le r_y \le 10r_0$	ft
$r_z = r_{zF}^{(3)}$	$-10r_0 \le r_z \le 10r_0$	$-10r_0 \le r_z \le 10r_0$	ft
$v_x = v_{xF}^{(3)}$	$-10v_0 \le v_x \le 10v_0$	$-10v_0 \le v_x \le 10v_0$	ft/sec
$v_y = v_{yF}^{(3)}$	$-10v_0 \le v_y \le 10v_0$	$-10v_0 \le v_y \le 10v_0$	ft/sec
$v_z = v_{zF}^{(3)}$	$-10v_0 \le v_z \le 10v_0$	$-10v_0 \le v_z \le 10v_0$	ft/sec
$w = w_F^{(2)}$			lb

Algebraic Variables: (ψ, θ) ....

$0^o \le \psi \le 90^o$	$0^o \le \psi \le 90^o$	$0^o \le \psi \le 90^o$	rad
$-20^o \le \theta \le 20^o$	$-20^o \le \theta \le 20^o$	$-20^o \le \theta \le 20^o$	rad

Boundary Conditions ....

$$t_F^{(4)} - t_I^{(4)} \ge 1$$

$$\boldsymbol{\psi}_1[\mathbf{r}, \mathbf{v}] = 19323.\ \text{nm}$$
$$\boldsymbol{\psi}_2[\mathbf{r}, \mathbf{v}] = \sqrt{\mu/r}$$
$$\boldsymbol{\psi}_3[\mathbf{r}, \mathbf{v}] = 0^o$$
$$\boldsymbol{\psi}_4[\mathbf{r}, \mathbf{v}] = 90^o$$
$$\boldsymbol{\psi}_5[\mathbf{r}, \mathbf{v}] = 0^o$$

where the terminal boundary conditions are computed as follows:

$$r = \|\mathbf{r}\| \tag{35.20}$$
$$v = \|\mathbf{v}\| \tag{35.21}$$
$$\mathbf{k} = -\mathbf{r}/r \tag{35.22}$$

$$\widetilde{\mathbf{i}} = \begin{pmatrix} -k_3k_1 \\ -k_3k_2 \\ 1 - k_3k_3 \end{pmatrix} \tag{35.23}$$

$$\mathbf{i} = \widetilde{\mathbf{i}}/\|\widetilde{\mathbf{i}}\| \tag{35.24}$$

$$\mathbf{j} = \mathbf{k} \times \mathbf{i} \tag{35.25}$$

$$\mathbf{Q}_L = \begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\end{bmatrix} \tag{35.26}$$

$$\boldsymbol{\eta} = \mathbf{Q}_L^{\mathsf{T}}\mathbf{v} \tag{35.27}$$

$$\boldsymbol{\psi}_1 = r - R_e \tag{35.28}$$

$$\boldsymbol{\psi}_2 = v \tag{35.29}$$

$$\boldsymbol{\psi}_3 = \sin^{-1}(\eta_3/v) \tag{35.30}$$

$$\boldsymbol{\psi}_4 = \tan^{-1}(\eta_2/\eta_1) \tag{35.31}$$

$$\boldsymbol{\psi}_5 = \sin^{-1}(r_3/r) \tag{35.32}$$

Objective ....

Maximize $\qquad J = w(t_F^{(4)})$

$$\boxed{J^* = 2.36724612 \times 10^{-1}; \quad t_F^* = 2.1682950 \times 10^4}$$

Example 35.2 jmp202: Optimal Constant Attitude Steering.

Repeat example 35.1 with the following changes:

(a) In phase 2 modify the parameters as follows;
Parameters: $(\psi, \theta, t_I^{(2)}, t_F^{(2)})$ ....

$-20^o \le \psi \le 20^o \qquad -10^o \le \theta \le 10^o$

(b) In phase 2, omit the algebraic variables ψ and θ;

(c) In phase 4 modify the parameters as follows;
Parameters: $(\psi, \theta, t_I^{(4)}, t_F^{(4)})$ ....

$0^o \le \psi \le 90^o \qquad -20^o \le \theta \le 20^o$

(d) In phase 4, omit the algebraic variables ψ and θ;

$$\boxed{J^* = 2.35477657 \times 10^{-1}; \quad t_F^* = 2.1686144 \times 10^4}$$

$h_0 = 150$ nm $= 911417.32283464505$	$R_e = 20925662.73$
$\mu = .1407645794 \times 10^{17}$	$i_0 = 28.5^o$
$T_c = 1.2$ lb	$I_{sp} = 300$ sec

Table 35.1. *Analytic Propagation example constants.*

Chapter 36

jshi: **HIV Immunology Model**

In reference [61], Hem Raj Joshi describes an application of modern optimal control techniques to design a drug treatment schedule for the treatment of HIV. Example (36.1) poses the problem in Lagrange form and a Mayer form is used in example (36.2). An alternate formulation for a similar application is given as examples (40.1) and (40.2).

Example 36.1 jshi01: Optimal Drug Treatment Strategy.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 50$	$t = 50$

Differential Variables: (T, V) ....

$T = 400$	$0 \leq T \leq 1200$	$0 \leq T \leq 1200$
$V = 3$	$.05 \leq V \leq 5$	$.05 \leq V \leq 5$

Algebraic Variables: (u_1, u_2) ....

$0 \leq u_1 \leq .02$	$0 \leq u_1 \leq .02$	$0 \leq u_1 \leq .02$
$0 \leq u_2 \leq .9$	$0 \leq u_2 \leq .9$	$0 \leq u_2 \leq .9$

Differential-Algebraic Equations ....

$$\dot{T} = s_1 - \frac{s_2 V}{b_1 + V} - \mu T - kVT + u_1 T \tag{36.1}$$

$$\dot{V} = \frac{g(1 - u_2)V}{b_2 + V} - cVT \tag{36.2}$$

where the problem constants are defined in Table 36.1.

Objective . . . .

Maximize $$J = \int_0^{50} \left[T - \left(A_1 u_1^2 + A_2 u_2^2\right)\right] dt$$

$$\boxed{J^* = 29514.4477}$$

Example 36.2 jshi02: Optimal Drug Treatment Strategy.

Repeat example 36.1 with the following changes:

(a) Add the differential variable z with $z(0) = 0$

(b) Add the differential equation

$$\dot{z} = T - \left(A_1 u_1^2 + A_2 u_2^2\right) \tag{36.3}$$

(c) Define

Objective . . . .

Maximize $$J = z(50)$$

$$\boxed{J^* = 29514.4477}$$

$s_1 = 2$	$s_2 = 1.5$
$\mu = .002$	$k = 2.5 \times 10^{-4}$
$c = .007$	$g = 30$
$b_1 = 14$	$b_2 = 1$
$A_1 = 2.5 \times 10^5$	$A_2 = 75$

Table 36.1. *Immunology example constants.*

Chapter 37

kplr: **Kepler's Equation**

One of the simplest transcendental equations is Kepler's equation. This trivial example poses a problem in which a single algebraic variable, the eccentric anomaly, is treated as a function of the eccentricity as the independent variable. The resulting problem serves as a test for software, when there are no differential equations and/or objective function.

Example 37.1 kplr01: TRANSCENDENTAL EQUATION.

Phase 1 . . . . Phase 1

Independent Variable: (e) . . . .

$e = 0$ $\qquad\qquad\qquad$ $e = .9$

Algebraic Variables: (E) . . . .
Differential-Algebraic Equations . . . .

$$0 = E - e \sin E - 1 \tag{37.1}$$

Objective . . . .

Root of Nonlinear Algebraic Equation

Chapter 38

lbri: Optimal Libration Point Transfer, Indirect Collocation

A formulation of an optimal low thrust transfer between libration point orbits is presented by Epenoy [45]. A direct formulation of this example is given in examples (39.1)-(39.2). In contrast reference [15] describes an indirect collocation formulation, given here as examples (38.1) and (38.2).

Example 38.1 lbri01: Indirect Formulation; Short Transfer Duration.

Phase 1 .... Phase 1

Parameters: (τ_0, τ_f) ....

Independent Variable: (t) ....

$t = t_I = 0$ $\qquad$ $t_I < t < t_F$ $\qquad$ $t = t_F = 2.7596586$

Differential Variables: $(x, y, v_x, v_y, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ ....

Boundary Conditions ....

$\mathbf{z} = \boldsymbol{\xi}_1(\tau_0)$

$\mathbf{z} = \boldsymbol{\xi}_2(\tau_f)$

Differential-Algebraic Equations ....

$$\dot{x} = v_x \tag{38.1}$$

$$\dot{y} = v_y \tag{38.2}$$

$$\dot{v}_x = x + 2v_y - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} + u_1 \tag{38.3}$$

$$\dot{v}_y = y - 2v_x - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} + u_2 \tag{38.4}$$

$$\dot{\lambda}_1 = -\lambda_3 \frac{\partial f_3}{\partial x} - \lambda_4 \frac{\partial f_4}{\partial x} \tag{38.5}$$

$$\dot{\lambda}_2 = -\lambda_3 \frac{\partial f_3}{\partial y} - \lambda_4 \frac{\partial f_4}{\partial y} \tag{38.6}$$

$$\dot{\lambda}_3 = -\lambda_1 + 2\lambda_4 \tag{38.7}$$

$$\dot{\lambda}_4 = -\lambda_2 - 2\lambda_3 \tag{38.8}$$

where

$$u_1 = -\lambda_3 \tag{38.9}$$

$$u_2 = -\lambda_4 \tag{38.10}$$

$$r_1 = \sqrt{(x+\mu)^2 + y^2} \tag{38.11}$$

$$r_2 = \sqrt{(x+\mu-1)^2 + y^2} \tag{38.12}$$

$$\frac{\partial f_3}{\partial x} = 1 - d_1 - d_2 \tag{38.13}$$

$$\frac{\partial f_4}{\partial x} = -d_3 - d_4 \tag{38.14}$$

$$\frac{\partial f_3}{\partial y} = -d_5 - d_6 \tag{38.15}$$

$$\frac{\partial f_4}{\partial y} = 1 - d_7 - d_8 \tag{38.16}$$

$$d_1 = (1-\mu)(x+\mu)\frac{\partial}{\partial x}\left\{r_1^{-3}\right\} + (1-\mu)r_1^{-3} \tag{38.17}$$

$$d_2 = \mu(x+\mu-1)\frac{\partial}{\partial x}\left\{r_2^{-3}\right\} + \mu r_2^{-3} \tag{38.18}$$

$$d_3 = (1-\mu)y\frac{\partial}{\partial x}\left\{r_1^{-3}\right\} \tag{38.19}$$

$$d_4 = \mu y\frac{\partial}{\partial x}\left\{r_2^{-3}\right\} \tag{38.20}$$

$$d_5 = (1-\mu)(x+\mu)\frac{\partial}{\partial y}\left\{r_1^{-3}\right\} \tag{38.21}$$

$$d_6 = \mu(x+\mu-1)\frac{\partial}{\partial y}\left\{r_2^{-3}\right\} \tag{38.22}$$

$$d_7 = (1-\mu)y\frac{\partial}{\partial y}\left\{r_1^{-3}\right\} + (1-\mu)r_1^{-3} \tag{38.23}$$

$$d_8 = \mu y\frac{\partial}{\partial y}\left\{r_2^{-3}\right\} + \mu r_2^{-3} \tag{38.24}$$

$$\frac{\partial}{\partial x}\left\{r_1^{-3}\right\} = -\frac{3(x+\mu)}{r_1^5} \tag{38.25}$$

$$\frac{\partial}{\partial y}\left\{r_1^{-3}\right\} = -\frac{3y}{r_1^5} \tag{38.26}$$

$$\frac{\partial}{\partial x}\left\{r_2^{-3}\right\} = -\frac{3(x+\mu-1)}{r_2^5} \tag{38.27}$$

$$\frac{\partial}{\partial y}\left\{r_2^{-3}\right\} = -\frac{3y}{r_2^5} \tag{38.28}$$

with $\mu = 0.0121506683$, $\mathbf{z}^\mathsf{T} = (x, y, v_x, v_y)$ and the Lyapunov orbits are denoted by

$\boldsymbol{\xi}_1(\tau_0)$ and $\boldsymbol{\xi}_2(\tau_f)$. The functions $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are computed as described in Ref. [45].

Objective ....

Minimize $$J = \frac{1}{2}\int_{t_I}^{t_F} \left(u_1^2 + u_2^2\right) dt$$

$$\boxed{J^* = 3.6513908 \times 10^{-3}}$$

Example 38.2 lbri02: Indirect Formulation; Long Transfer Duration.

References: [15, 45]
Repeat example (38.1) with $t_F = 10.11874803$.

$$\boxed{J^* = 2.54291985 \times 10^{-8}}$$

Chapter 39

lbrp: **Optimal Low-Thrust Transfer Between Libration Points**

A formulation of an optimal low thrust transfer between libration point orbits is presented by Epenoy [45]. The dynamic model is based on the Planar Circular Restricted Three Body Problem (PCR3BP) with Earth as one primary and the Moon as the second. The equations of motion are constructed in a rotating reference frame, in which the x-axis extends from the barycenter of the Earth-Moon system to the Moon, and the y-axis completes the right hand coordinate frame. A set of non-dimensional units is chosen such that the unit of distance is the distance between the two primaries, the unit of mass is the sum of the primaries' masses, and the unit of time is such that the angular velocity of the primaries around their barycenter is one. The initial and final states must lie on a manifold referred to as the Lyapunov orbit. The Lyapunov states are computed by means of Lindstedt-Poincare approximation as functions of non-dimensional parameters that determine the departure and arrival locations. A single phase formulation is used by Epenoy to construct both short and long duration transfers. In contrast reference [15] describes a formulation with multiple phases, given here as examples (39.1) and (39.2). Examples (39.3) and (39.4) implement short and long transfers when the boundary manifolds are approximated using splines.

Example 39.1 lbrp01: Short Transfer Duration.

Phase 1 .... *Departure Leg* .... Phase 1

Parameters: $(\tau_0, t_F^{(1)})$ ....

$t_F^{(1)} \geq .001$

Independent Variable: (t) ....

$t = t_I = 0$ $\qquad$ $t_I < t < t_F^{(1)}$ $\qquad$ $t = t_F^{(1)}$

Differential Variables: (x, y, v_x, v_y) ....

$$x = 1 - \mu$$
$$y \le y_{min} = -.04$$
$$v_x \ge 0$$

Algebraic Variables: (u_1, u_2) ....

Boundary Conditions ....

$\mathbf{z} = \boldsymbol{\xi}_1(\tau_0)$

Differential-Algebraic Equations ....

$$\dot{x} = v_x \tag{39.1}$$
$$\dot{y} = v_y \tag{39.2}$$
$$\dot{v}_x = x + 2v_y - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} + u_1 \tag{39.3}$$
$$\dot{v}_y = y - 2v_x - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} + u_2 \tag{39.4}$$

where

$$r_1 = \sqrt{(x+\mu)^2 + y^2} \tag{39.5}$$
$$r_2 = \sqrt{(x+\mu-1)^2 + y^2} \tag{39.6}$$

with $\mu = 0.0121506683$, $\mathbf{z}^\mathsf{T} = (x, y, v_x, v_y)$ and the Lyapunov orbits are denoted by $\boldsymbol{\xi}_1(\tau_0)$ and $\boldsymbol{\xi}_2(\tau_f)$. The functions $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are computed as described in Ref. [45].

Phase 2 .... *Arrival Leg* .... Phase 2

Parameters: $(\tau_f, t_I^{(2)}, t_F^{(2)})$ ....

$t_F^{(2)} \le 2.7596586$

Independent Variable: (t) ....

$t = t_I^{(2)} = t_F^{(1)}$ $\qquad$ $t_I^{(2)} < t < t_F^{(2)}$ $\qquad$ $t = t_F^{(2)}$

Differential Variables: (x, y, v_x, v_y) ....

$x = x_F^{(1)}$
$y = y_F^{(1)}$
$v_x = v_{xF}^{(1)}$
$v_y = v_{yF}^{(1)}$

Algebraic Variables: (u_1, u_2) ....

Boundary Conditions ....

$$\mathbf{z} = \boldsymbol{\xi}_2(\tau_f)$$

Differential-Algebraic Equations ....

Equations (39.1) - (39.4)

Objective ....

Minimize $$J = \frac{1}{2}\sum_{k=1}^{k=2}\int_{t_I^{(k)}}^{t_F^{(k)}} \left(u_1^2 + u_2^2\right) dt$$

$$\boxed{J^* = 3.6513908 \times 10^{-3}}$$

Example 39.2 lbrp02: LONG TRANSFER DURATION.

References: [15, 45]

Phase 1 .... *Departure Leg* .... Phase 1

Parameters: $(\tau_0, t_F^{(1)})$ ....

$t_F^{(1)} \geq .001$

Independent Variable: (t) ....

$t = t_I = 0$ $\qquad t_I < t < t_F^{(1)}$ $\qquad t = t_F^{(1)}$

Differential Variables: (x, y, v_x, v_y) ....

$$x = 1 - \mu$$
$$y \leq y_{min} = -.04$$
$$v_x \geq 0$$

Algebraic Variables: (u_1, u_2) ....

Boundary Conditions ....

$\mathbf{z} = \boldsymbol{\xi}_1(\tau_0)$

Differential-Algebraic Equations ....

Equations (39.1) - (39.4)

Phase 2 .... *First Lunar Revolution* .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....
Independent Variable: (t) ....

$t = t_I^{(2)} = t_F^{(1)}$ $\quad$ $t_I^{(2)} < t < t_F^{(2)}$ $\quad$ $t = t_F^{(2)}$

Differential Variables: (x, y, v_x, v_y) ....

$x = x_F^{(1)}$ $\quad$ $x = 1 - \mu$
$y = y_F^{(1)}$ $\quad$ $y \le y_{min} = -.04$
$v_x = v_{xF}^{(1)}$ $\quad$ $v_x \ge 0$
$v_y = v_{yF}^{(1)}$

Algebraic Variables: (u_1, u_2) ....
Boundary Conditions ....

$t_F^{(2)} - t_I^{(2)} \ge .001$

Differential-Algebraic Equations ....

Equations (39.1) - (39.4)

Phase 3 .... *Second Lunar Revolution* .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....
Independent Variable: (t) ....

$t = t_I^{(3)} = t_F^{(2)}$ $\quad$ $t_I^{(3)} < t < t_F^{(3)}$ $\quad$ $t = t_F^{(3)}$

Differential Variables: (x, y, v_x, v_y) ....

$x = x_F^{(2)}$ $\quad$ $x = 1 - \mu$
$y = y_F^{(2)}$ $\quad$ $y \le y_{min} = -.04$
$v_x = v_{xF}^{(2)}$ $\quad$ $v_x \ge 0$
$v_y = v_{yF}^{(2)}$

Algebraic Variables: (u_1, u_2) ....
Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} \geq .001$

Differential-Algebraic Equations ....

Equations (39.1) - (39.4)

Phase 4 .... *Arrival Leg* .... Phase 4

Parameters: $(\tau_f, t_I^{(4)}, t_F^{(4)})$ ....

$t_F^{(4)} \leq 10.11874803$

Independent Variable: (t) ....

$t = t_I^{(4)} = t_F^{(3)}$ $\quad$ $t_I^{(4)} < t < t_F^{(4)}$ $\quad$ $t = t_F^{(4)}$

Differential Variables: (x, y, v_x, v_y) ....

$x = x_F^{(3)}$
$y = y_F^{(3)}$
$v_x = v_{xF}^{(3)}$
$v_y = v_{yF}^{(3)}$

Algebraic Variables: (u_1, u_2) ....
Boundary Conditions ....

$\mathbf{z} = \boldsymbol{\xi}_2(\tau_f)$

Differential-Algebraic Equations ....

Equations (39.1) - (39.4)

Objective ....

Minimize $$J = \frac{1}{2}\sum_{k=1}^{k=4}\int_{t_I^{(k)}}^{t_F^{(k)}} \left(u_1^2 + u_2^2\right) dt$$

$$\boxed{J^* = 2.54291985 \times 10^{-8}}$$

Example 39.3 lbrp03: Short Transfer Duration; Spline BC.
Repeat example 39.1 with a cubic B-spline approximation to the boundary functions $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.

$$\boxed{J^* = 3.65139078 \times 10^{-3}}$$

Example 39.4 lbrp04: Long Transfer Duration; Spline BC.
Repeat example 39.2 with a cubic B-spline approximation to the boundary functions $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.

$$\boxed{J^* = 2.57838882 \times 10^{-8}}$$

Chapter 40

lnht: **Chemotherapy of HIV**

Kirschner, Lenhart, and Serbin [64] describe the formulation of a biological system that can be used to construct a chemotherapy treatment strategy for HIV. Example (40.1) poses a Mayer formulation and (40.2) recasts the problem in Lagrange form. An alternate formulation for a similar application is given as examples (36.1) and (36.2).

Example 40.1 lnht01: OPTIMAL TREATMENT STRATEGY.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 500$	$t = 500$

Differential Variables: $(y_1, y_2, y_3, y_4, y_5)$ ....

$y_1 = 982$
$y_2 = .05$
$y_3 = 6.2 \times 10^{-4}$
$y_4 = .02$
$y_5 = 0$

Algebraic Variables: (u) ....

$0 \le u \le 1$	$0 \le u \le 1$	$0 \le u \le 1$

Differential-Algebraic Equations ....

$$\dot{y}_1 = \frac{c_8}{1 + y_4} - c_1 y_1 + c_6 y_1 \left[1 - \frac{1}{c_7}(y_1 + y_2 + y_3)\right] - c_4 y_4 y_1 \tag{40.1}$$

$$\dot{y}_2 = c_4 y_4 y_1 - c_1 y_2 - c_5 y_2 \tag{40.2}$$

$$\dot{y}_3 = c_5 y_2 - c_2 y_3 \tag{40.3}$$
$$\dot{y}_4 = c_9 c_2 y_3 u - c_4 y_4 y_1 - c_3 y_4 \tag{40.4}$$
$$\dot{y}_5 = 10^{-5}\left[-y_1 + 50(1-u)^2\right] \tag{40.5}$$

where the problem constants are defined in Table 40.1.

Objective ....

Minimize $J = y_5(500)$

$$\boxed{J^* = -4.92803496}$$

Example 40.2 Inht02: Optimal Treatment Strategy.

Repeat example 40.1 with the following changes:

(a) Eliminate the differential variable y_5

(b) Eliminate differential equation (40.5)

(c) Define
Objective ....

Minimize $$J = 10^{-5}\int_0^{500}\left[-y_1 + 50(1-u)^2\right] dt$$

$$\boxed{J^* = -4.92803496}$$

$c_1 = 2.0 \times 10^{-2}$	$c_2 = 2.4 \times 10^{-1}$	$c_3 = 2.4$
$c_4 = 2.4 \times 10^{-5}$	$c_5 = 3 \times 10^{-3}$	$c_6 = 3 \times 10^{-2}$
$c_7 = 1500$	$c_8 = 10$	$c_9 = 1200$

Table 40.1. *Chemotherapy example constants.*

Chapter 41

lnts: **Linear Tangent Steering**

When the goal is to minimize the time required for a vehicle to move from a fixed initial state to a terminal position in a constant gravity field, by choosing the steering angle, the problem has an analytic solution referred to as "linear tangent steering" [29]. There are many different versions of this problem as discussed in reference [13, Sect. 4.11.4, Sect. 5.6]. This problem also is of considerable practical interest since it is a simplified version of the steering algorithm used by many launch vehicles, including the space shuttle. Five different versions of this problem are given as examples (41.1)- (41.5).

Example 41.1 lnts01: Indirect Formulation.

Phase 1 . . . . Phase 1

Parameters: (t_F) . . . .

$.001 \le t_F$

Independent Variable: (t) . . . .

$t = 0$	$t = t_F$

Differential Variables: $(x_1, x_2, x_3, x_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ . . . .

$x_1 = 0$	
$x_2 = 0$	$x_2 = 5$
$x_3 = 0$	$x_3 = 45$
$x_4 = 0$	$x_4 = 0$
	$\lambda_1 = 0$

Boundary Conditions . . . .

$$0 = 1 + \lambda_1 x_3 + \lambda_2 x_4 + a\lambda_3 \cos u + a\lambda_4 \sin u$$

Differential-Algebraic Equations ....

$$\dot{x}_1 = x_3 \tag{41.1}$$
$$\dot{x}_2 = x_4 \tag{41.2}$$
$$\dot{x}_3 = a \cos u \tag{41.3}$$
$$\dot{x}_4 = a \sin u \tag{41.4}$$
$$\dot{\lambda}_1 = 0 \tag{41.5}$$
$$\dot{\lambda}_2 = 0 \tag{41.6}$$
$$\dot{\lambda}_3 = -\lambda_1 \tag{41.7}$$
$$\dot{\lambda}_4 = -\lambda_2 \tag{41.8}$$

where $a = 100$ and

$$\cos u = \frac{-\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}} \tag{41.9}$$
$$\sin u = \frac{-\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}}. \tag{41.10}$$

Objective ....

Minimize (TPBVP) $J = t_F$

$$\boxed{J^* = 5.5457088 \times 10^{-1}}$$

Example 41.2 Ints05: Direct Formulation.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$0 \le t_F$

Independent Variable: (t) ....

$t = 0$	$t = t_F$

Differential Variables: (x_1, x_2, x_3, x_4) ....

$x_1 = 0$	
$x_2 = 0$	$x_2 = 5$
$x_3 = 0$	$x_3 = 45$
$x_4 = 0$	$x_4 = 0$

Algebraic Variables: (u) ....

$-90^o \leq u \leq +90^o$ $\quad$ $-90^o \leq u \leq +90^o$ $\quad$ $-90^o \leq u \leq +90^o$ $\quad$ rad

Differential-Algebraic Equations ....

Equations (41.1) - (41.4)

Objective ....

Minimize $\quad J = t_F$

$$\boxed{J^* = 5.54570879 \times 10^{-1}}$$

Example 41.3 lnts13: Explicit Parameterization.

Phase 1 .... Phase 1

Parameters: (p_1, p_2, t_F) ....

$0 \leq p_1 \leq 10$ $\quad$ $0 \leq p_2 \leq 10$ $\quad$ $.001 \leq t_F$

Independent Variable: (t) ....

$t = 0$ $\quad$ $t = t_F$

Differential Variables: (x_1, x_2, x_3, x_4) ....

$x_1 = 0$	
$x_2 = 0$	$x_2 = 5$
$x_3 = 0$	$x_3 = 45$
$x_4 = 0$	$x_4 = 0$

Differential-Algebraic Equations ....

Equations (41.1) - (41.4)

with

$$u = \tan^{-1}\left[p_1 - p_2 t\right]. \tag{41.11}$$

Objective ....

Minimize $\quad J = t_F$

$$\boxed{J^* = 5.5457088 \times 10^{-1}; \quad p_1^* = 1.4085084; \quad p_2^* = 5.0796333}$$

Example 41.4 ltsp01: MULTIPHASE, NORMALIZED DOMAIN.

Phase 1 . . . . Phase 1

Parameters: $(p_1^{(1)}, p_2^{(1)}, T^{(1)})$ . . . .

$0 \le p_1^{(1)}$	$0 \le p_2^{(1)}$	$0 \le T^{(1)}$

Independent Variable: (τ) . . . .

$\tau = 0$	$0 < \tau < 1/3$	$\tau = 1/3$

Differential Variables: (x_1, x_2, x_3, x_4) . . . .

$x_1 = 0$
$x_2 = 0$
$x_3 = 0$
$x_4 = 0$

Differential-Algebraic Equations . . . .

$$\dot{x}_1 = T^{(1)} x_3 \tag{41.12}$$
$$\dot{x}_2 = T^{(1)} x_4 \tag{41.13}$$
$$\dot{x}_3 = T^{(1)} a \cos u \tag{41.14}$$
$$\dot{x}_4 = T^{(1)} a \sin u \tag{41.15}$$

where $a = 100$ and

$$t = \tau T^{(1)} \tag{41.16}$$
$$\tan u = p_1^{(1)} - p_2^{(1)} t \tag{41.17}$$
$$D = (1 + \tan^2 u)^{-1/2} \tag{41.18}$$
$$\sin u = D \tan u \tag{41.19}$$
$$\cos u = D \tag{41.20}$$

Phase 2 . . . . Phase 2

Parameters: $(p_1^{(2)}, p_2^{(2)}, T^{(2)})$ . . . .

$0 \leq p_1^{(2)}$ $\quad$ $0 \leq p_2^{(2)}$ $\quad$ $0 \leq T^{(2)}$

Independent Variable: (τ) ....

$\tau = 1/3$ $\quad$ $1/3 < \tau < 2/3$ $\quad$ $\tau = 2/3$

Differential Variables: (x_1, x_2, x_3, x_4) ....

$x_1 = x_1[\tau_F^{(1)}]$
$x_2 = x_2[\tau_F^{(1)}]$
$x_3 = x_3[\tau_F^{(1)}]$
$x_4 = x_4[\tau_F^{(1)}]$

Boundary Conditions ....

$p_1^{(1)} = p_1^{(2)}$
$p_2^{(1)} = p_2^{(2)}$
$T^{(1)} = T^{(2)}$

Differential-Algebraic Equations ....

$$\dot{x}_1 = T^{(2)} x_3 \tag{41.21}$$

$$\dot{x}_2 = T^{(2)} x_4 \tag{41.22}$$

$$\dot{x}_3 = T^{(2)} a \cos u \tag{41.23}$$

$$\dot{x}_4 = T^{(2)} a \sin u \tag{41.24}$$

using (41.18)-(41.20) with $a = 100$ and

$$t = \tau T^{(2)} \tag{41.25}$$

$$\tan u = p_1^{(2)} - p_2^{(2)} t \tag{41.26}$$

Phase 3 .... Phase 3

Parameters: $(p_1^{(3)}, p_2^{(3)}, T^{(3)})$ ....

$0 \leq p_1^{(3)}$ $\quad$ $0 \leq p_2^{(3)}$ $\quad$ $0 \leq T^{(3)}$

Independent Variable: (τ) ....

$\tau = 2/3$ $\quad$ $2/3 < \tau < 1$ $\quad$ $\tau = 1$

Differential Variables: (x_1, x_2, x_3, x_4) ....

$x_1 = x_1[\tau_F^{(2)}]$
$x_2 = x_2[\tau_F^{(2)}]$ $\qquad x_2 = 5$
$x_3 = x_3[\tau_F^{(2)}]$ $\qquad x_3 = 45$
$x_4 = x_4[\tau_F^{(2)}]$ $\qquad x_4 = 0$

Boundary Conditions ....

$p_1^{(2)} = p_1^{(3)}$
$p_2^{(2)} = p_2^{(3)}$
$T^{(2)} = T^{(3)}$

Differential-Algebraic Equations ....

$$\dot{x}_1 = T^{(3)} x_3 \tag{41.27}$$

$$\dot{x}_2 = T^{(3)} x_4 \tag{41.28}$$

$$\dot{x}_3 = T^{(3)} a \cos u \tag{41.29}$$

$$\dot{x}_4 = T^{(3)} a \sin u \tag{41.30}$$

using (41.18)-(41.20) with $a = 100$ and

$$t = \tau T^{(3)} \tag{41.31}$$

$$\tan u = p_1^{(3)} - p_2^{(3)} t \tag{41.32}$$

Objective ....

Minimize (BVP) $\qquad J = T$

where $T^* = T^{(k)}$, $p_1^* = p_1^{(k)}$, and $p_2^* = p_2^{(k)}$ for $k = 1, 2, 3$.

$$\boxed{T^* = 5.5457088 \times 10^{-1}; \quad p_1^* = 1.4085084; \quad p_2^* = 5.0796333}$$

Example 41.5 ltsp02: MULTIPHASE, VARIABLE TIME.

Phase 1 .... Phase 1

Parameters: $(p_1^{(1)}, p_2^{(1)}, t_F^{(1)})$ ....

$0 \le p_1^{(1)}$ $\qquad 0 \le p_2^{(1)}$

Independent Variable: (t) ....

$t = 0$ $\qquad 0 < t < t_F^{(1)}$ $\qquad t = t_F^{(1)}$

Differential Variables: (x_1, x_2, x_3, x_4) ....

$x_1 = 0$
$x_2 = 0$
$x_3 = 0$
$x_4 = 0$

Differential-Algebraic Equations ....

Equations (41.1) - (41.4)

using (41.18)-(41.20) with $a = 100$ and $\tan u = p_1^{(1)} - p_2^{(1)} t$.

Phase 2 .... Phase 2

Parameters: $(p_1^{(2)}, p_2^{(2)}, t_I^{(2)}, t_F^{(2)})$ ....

$0 \le p_1^{(2)}$ $\quad$ $0 \le p_2^{(2)}$

Independent Variable: (t) ....

$t = t_F^{(1)}$ $\quad$ $t_I^{(2)} < t < t_F^{(2)}$ $\quad$ $t = t_F^{(2)}$

Differential Variables: (x_1, x_2, x_3, x_4) ....

$x_1 = x_1[t_F^{(1)}]$
$x_2 = x_2[t_F^{(1)}]$
$x_3 = x_3[t_F^{(1)}]$
$x_4 = x_4[t_F^{(1)}]$

Boundary Conditions ....

$p_1^{(1)} = p_1^{(2)}$
$p_2^{(1)} = p_2^{(2)}$
$t_F^{(2)} - 2t_I^{(2)} = 0$

Differential-Algebraic Equations ....

Equations (41.1) - (41.4)

using (41.18)-(41.20) with $a = 100$ and $\tan u = p_1^{(2)} - p_2^{(2)} t$.

Phase 3 .... Phase 3

Parameters: $(p_1^{(3)}, p_2^{(3)}, t_I^{(3)}, t_F^{(3)})$ ....

$0 \le p_1^{(3)}$ $0 \le p_2^{(3)}$

Independent Variable: (t) ....

$t = t_F^{(2)}$	$t_I^{(3)} < t < t_F^{(3)}$	$t = t_F^{(3)}$

Differential Variables: (x_1, x_2, x_3, x_4) ....

$x_1 = x_1[t_F^{(2)}]$	
$x_2 = x_2[t_F^{(2)}]$	$x_2 = 5$
$x_3 = x_3[t_F^{(2)}]$	$x_3 = 45$
$x_4 = x_4[t_F^{(2)}]$	$x_4 = 0$

Boundary Conditions ....

$p_1^{(2)} = p_1^{(3)}$
$p_2^{(2)} = p_2^{(3)}$
$t_F^{(3)} - 2t_I^{(3)} + t_I^{(2)} = 0$

Differential-Algebraic Equations ....

Equations (41.1) - (41.4)

using (41.18)-(41.20) with $a = 100$ and $\tan u = p_1^{(3)} - p_2^{(3)} t$.

Objective ....

Minimize (BVP) $J = t_F$

where $t_F^* = t_F^{(3)}$, $p_1^* = p_1^{(k)}$, and $p_2^* = p_2^{(k)}$ for $k = 1, 2, 3$.

$$\boxed{t_F^* = 5.5457088 \times 10^{-1}; \quad p_1^* = 1.4085084; \quad p_2^* = 5.0796333}$$

Chapter 42

lowt: **Planar Thrust Orbit Transfer**

Albert Herman and Bruce Conway define a planar orbit transfer problem in reference [57], extending earlier work in references [43] and [44]. In this example the kinetic plus potential energy is minimized for a fixed duration transfer departing from a circular orbit.

Example 42.1 lowt01: PLANAR THRUST ORBIT TRANSFER.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 50$	$t = 50$

Differential Variables: $(r, \theta, v_r, v_\theta)$ ....

$r = 1.1$	$.5 \le r \le 5$	$.5 \le r \le 5$
$\theta = 0$	$0 \le \theta \le 8\pi$	$0 \le \theta \le 8\pi$
$v_r = 0$	$-10 \le v_r \le 10$	$-10 \le v_r \le 10$
$v_\theta = 1/\sqrt{1.1}$	$0 \le v_\theta \le 10$	$0 \le v_\theta \le 10$

Algebraic Variables: (β) ....

$-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$

Differential-Algebraic Equations ....

$$\dot{r} = v_r \tag{42.1}$$

$$\dot{\theta} = \frac{v_\theta}{r} \tag{42.2}$$

$$\dot{v}_r = \frac{v_\theta^2}{r} - \frac{1}{r^2} + .01 \sin\beta \tag{42.3}$$

$$\dot{v}_\theta = -\frac{v_\theta v_r}{r} + .01 \cos\beta \tag{42.4}$$

Objective ....

Minimize $$J = \left[\frac{1}{r} - \frac{1}{2}\left(v_r^2 + v_\theta^2\right)\right]\Bigg|_{t=50}$$

$$\boxed{J^* = 9.51233834 \times 10^{-2}}$$

Chapter 43

lthr: **Low Thrust Orbit Transfer**

Constructing the trajectory for a spacecraft as it transfers from a low earth orbit to a mission orbit leads to a class of challenging optimal control examples. The dynamics are very nonlinear and because the thrust applied to the vehicle is small in comparison to the weight of the spacecraft, the duration of the trajectory can be very long. Problems of this type have been of considerable interest in the aerospace industry [8, 9, 10, 11, 24, 42, 43, 80, 89]. Typically, the goal is to construct the optimal steering during the transfer such that the final weight is maximized (i.e., minimum fuel consumed). The specific example given here is described in reference [13, Sect. 6.3] and represents the trajectory from a low-earth circular orbit to a highly inclined, eccentric mission orbit.

Example 43.1 lthr01: Low Thrust Transfer to Molniya Orbit.

Phase 1 .... Phase 1

Parameters: (τ, t_F) ....

$-99 \leq \tau \leq 0$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$	sec

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_I$	$.1p_I \leq p \leq 5p_F$	$p = p_F$	ft
$f = 0$	$-1 \leq f \leq 1$	$-1 \leq f \leq 1$	
$g = 0$	$-1 \leq g \leq 1$	$-1 \leq g \leq 1$	
$h = h_I$	$-1 \leq h \leq 1$	$-1 \leq h \leq 1$	
$k = 0$	$-1 \leq k \leq 1$	$-1 \leq k \leq 1$	
$L = \pi$	$\pi \leq L \leq 34\pi$	$\pi \leq L \leq 34\pi$	rad

$w = w_I$	$.001 \le w \le 1.01$	$.001 \le w \le 1.01$	lb

Algebraic Variables: (u_r, u_θ, u_h) . . . .

$-1.1 \le u_r \le 1.1$	$-1.1 \le u_r \le 1.1$	$-1.1 \le u_r \le 1.1$
$-1.1 \le u_\theta \le 1.1$	$-1.1 \le u_\theta \le 1.1$	$-1.1 \le u_\theta \le 1.1$
$-1.1 \le u_h \le 1.1$	$-1.1 \le u_h \le 1.1$	$-1.1 \le u_h \le 1.1$

Boundary Conditions . . . .

$$\begin{aligned} \sqrt{f^2+g^2} &= e_F \\ \sqrt{h^2+k^2} &= \tan(i_F/2) \\ fh + gk &= 0 \\ gh - kf &\le 0 \end{aligned}$$

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{y}} = \mathbf{A}(\mathbf{y})\boldsymbol{\Delta} + \mathbf{b}, \tag{43.1}$$

$$\dot{w} = -T\left[1 + 0.01\tau\right]/I_{sp}, \tag{43.2}$$

$$0 = \|\mathbf{u}\| - 1, \tag{43.3}$$

using the parameter definitions given in Table 43.1 where $\mathbf{y}^{\mathsf{T}} = [p, f, g, h, k, L]$, $\mathbf{u}^{\mathsf{T}} = [u_r, u_\theta, u_h]$. The formulation utilizes the following quantities:

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{2p}{q}\sqrt{\frac{p}{\mu}} & 0 \\ \sqrt{\frac{p}{\mu}}\sin L & \sqrt{\frac{p}{\mu}}\frac{1}{q}\{(q+1)\cos L + f\} & -\sqrt{\frac{p}{\mu}}\frac{g}{q}\{h\sin L - k\cos L\} \\ -\sqrt{\frac{p}{\mu}}\cos L & \sqrt{\frac{p}{\mu}}\frac{1}{q}\{(q+1)\sin L + g\} & \sqrt{\frac{p}{\mu}}\frac{f}{q}\{h\sin L - k\cos L\} \\ 0 & 0 & \sqrt{\frac{p}{\mu}}\frac{s^2\cos L}{2q} \\ 0 & 0 & \sqrt{\frac{p}{\mu}}\frac{s^2\sin L}{2q} \\ 0 & 0 & \sqrt{\frac{p}{\mu}}\frac{1}{q}\{h\sin L - k\cos L\} \end{bmatrix} \tag{43.4}$$

$$\mathbf{b}^{\mathsf{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{\mu p}\left(\frac{q}{p}\right)^2 \end{bmatrix} \tag{43.5}$$

$$q = 1 + f\cos L + g\sin L \tag{43.6}$$

$$r = \frac{p}{q} \tag{43.7}$$

$$\alpha^2 = h^2 - k^2 \tag{43.8}$$

$$\chi = \sqrt{h^2+k^2} \tag{43.9}$$

$$s^2 = 1 + \chi^2 \tag{43.10}$$

$$\mathbf{r} = \begin{bmatrix} \frac{r}{s^2}\left(\cos L + \alpha^2\cos L + 2hk\sin L\right) \\ \frac{r}{s^2}\left(\sin L - \alpha^2\sin L + 2hk\cos L\right) \\ \frac{2r}{s^2}\left(h\sin L - k\cos L\right) \end{bmatrix} \tag{43.11}$$

$$\mathbf{v} = \begin{bmatrix} -\frac{1}{s^2}\sqrt{\frac{\mu}{p}}\left(\sin L + \alpha^2 \sin L - 2hk\cos L + g - 2fhk + \alpha^2 g\right) \\ -\frac{1}{s^2}\sqrt{\frac{\mu}{p}}\left(-\cos L + \alpha^2 \cos L + 2hk\sin L - f + 2ghk + \alpha^2 f\right) \\ \frac{2}{s^2}\sqrt{\frac{\mu}{p}}\left(h\cos L + k\sin L + fh + gk\right) \end{bmatrix} \tag{43.12}$$

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}_g + \boldsymbol{\Delta}_T \tag{43.13}$$

$$\mathbf{Q}_r = \begin{bmatrix}\mathbf{i}_r & \mathbf{i}_\theta & \mathbf{i}_h\end{bmatrix} = \begin{bmatrix}\frac{\mathbf{r}}{\|\mathbf{r}\|} & \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{\|\mathbf{r}\times\mathbf{v}\|\|\mathbf{r}\|} & \frac{\mathbf{r}\times\mathbf{v}}{\|\mathbf{r}\times\mathbf{v}\|}\end{bmatrix} \tag{43.14}$$

$$\boldsymbol{\delta}\mathbf{g} = \delta g_n \mathbf{i}_n - \delta g_r \mathbf{i}_r \tag{43.15}$$

$$\mathbf{i}_n = \frac{\mathbf{e}_n - (\mathbf{e}_n^\top \mathbf{i}_r)\mathbf{i}_r}{\|\mathbf{e}_n - (\mathbf{e}_n^\top \mathbf{i}_r)\mathbf{i}_r\|} \tag{43.16}$$

$$\mathbf{e}_n^\mathsf{T} = (0, 0, 1) \tag{43.17}$$

$$\delta g_n = -\frac{\mu\cos\phi}{r^2}\sum_{k=2}^{4}\left(\frac{R_e}{r}\right)^k P_k' J_k \tag{43.18}$$

$$\delta g_r = -\frac{\mu}{r^2}\sum_{k=2}^{4}(k+1)\left(\frac{R_e}{r}\right)^k P_k J_k \tag{43.19}$$

$$\boldsymbol{\Delta}_g = \mathbf{Q}_r^\mathsf{T}\boldsymbol{\delta}\mathbf{g} \tag{43.20}$$

$$\boldsymbol{\Delta}_T = \frac{g_o T\left[1 + .01\tau\right]}{w}\mathbf{u} \tag{43.21}$$

where P_k are Legendre polynomials.

Objective ....

Maximize $\qquad J = w(t_F)$

$J^* = 2.20179127 \times 10^{-1}; \quad t_F^* = 8.6810014 \times 10^4$

$p_I = 21837080.052835$	$p_F = 40007346.015232$
$e_F = 0.73550320568829$	$\tan(i_F/2) = 0.61761258786099$
$w_I = 1$	$g_0 = 32.174$
$I_{sp} = 450$	$T = 4.446618 \times 10^{-3}$
$\mu = 1.407645794 \times 10^{16}$	$R_e = 20925662.73$
$J_2 = 1082.639 \times 10^{-6}$	$J_3 = -2.565 \times 10^{-6}$
$J_4 = -1.608 \times 10^{-6}$	$h_I = -0.25396764647494$

Table 43.1. *Low Thrust Transfer Parameters.*

Chapter 44

lwbr: **Kinetic Batch Reactor**

In his doctoral thesis Daniel Leineweber [66] presents a problem originally given by Caracotsios and Stewart [36] that describes

> an optimal control problem which has several interesting features: stiff nonlinear DAE's, two model stages, a nonlinear inequality path constraint, equality and inequality boundary conditions, and unspecified terminal time. The example in its original form was given by the Dow Chemical Company as a challenging test problem for parameter estimation software ...

Leineweber presents a kinetic model of the batch reactor system in terms of both differential and algebraic states, and the three phase formulation given here is described in reference [13, Sect. 6.14].

Example 44.1 lwbr01: CHEMICAL PROCESS CONTROL.

Phase 1 .... *Transient Stage 1* .... Phase 1

Parameters: $(p^{(1)})$ ....

$0 \le p^{(1)} \le .0262$

Independent Variable: (t) ....

$t = t_I^{(1)} = 0$	$0 \le t \le t_F^{(1)}$	$t = t_F^{(1)} = .01$

Differential Variables: $(y_1, y_2, y_3, y_4, y_5, y_6)$ ....

$y_1 = 1.5776$	$y_1 \le 2$	$y_1 \le 2$
$y_2 = 8.32$	$5 \le y_2 \le 10$	$5 \le y_2 \le 10$
$y_3 = 0$	$y_3 \le 2$	$y_3 \le 2$
$y_4 = 0$	$y_4 \le 2$	$y_4 \le 2$

$y_5 = 0$	$y_5 \le 2$	$y_5 \le 2$
$y_6 = y_{6I}^{(1)} \le 0.1$	$y_6 \le 0.1$	$y_6 \le 0.1$

Algebraic Variables: $(u_1, u_2, u_3, u_4, u_5)$ . . . .

$0 \le u_1 \le 15$	$0 \le u_1 \le 15$	$0 \le u_1 \le 15$
$u_2 \le .02$	$u_2 \le .02$	$u_2 \le .02$
$u_3 \le 5 \times 10^{-5}$	$u_3 \le 5 \times 10^{-5}$	$u_3 \le 5 \times 10^{-5}$
$u_4 \le 5 \times 10^{-5}$	$u_4 \le 5 \times 10^{-5}$	$u_4 \le 5 \times 10^{-5}$
$293.15 \le u_5 \le 393.15$	$293.15 \le u_5 \le 393.15$	$293.15 \le u_5 \le 393.15$

Boundary Conditions . . . .

$y_{6I}^{(1)} = p^{(1)}$

Differential-Algebraic Equations . . . .

$$\dot{y}_1 = -k_2 y_2 u_2 \tag{44.1}$$
$$\dot{y}_2 = -k_1 y_2 y_6 + k_{-1} u_4 - k_2 y_2 u_2 \tag{44.2}$$
$$\dot{y}_3 = k_2 y_2 u_2 + k_3 y_4 y_6 - k_{-3} u_3 \tag{44.3}$$
$$\dot{y}_4 = -k_3 y_4 y_6 + k_{-3} u_3 \tag{44.4}$$
$$\dot{y}_5 = k_1 y_2 y_6 - k_{-1} u_4 \tag{44.5}$$
$$\dot{y}_6 = -k_1 y_2 y_6 + k_{-1} u_4 - k_3 y_4 y_6 + k_{-3} u_3 \tag{44.6}$$
$$0 = p^{(1)} - y_6 + 10^{-u_1} - u_2 - u_3 - u_4 \tag{44.7}$$
$$0 = u_2 - K_2 y_1/(K_2 + 10^{-u_1}) \tag{44.8}$$
$$0 = u_3 - K_3 y_3/(K_3 + 10^{-u_1}) \tag{44.9}$$
$$0 = u_4 - K_1 y_5/(K_1 + 10^{-u_1}) \tag{44.10}$$
$$0 \ge y_4 - 2t^2 \tag{44.11}$$

where

$$k_1 = \hat{k}_1 \exp(-\beta_1/u_5)$$
$$k_{-1} = \hat{k}_{-1} \exp(-\beta_{-1}/u_5)$$
$$k_2 = \hat{k}_2 \exp(-\beta_2/u_5)$$
$$k_3 = k_1$$
$$k_{-3} = (k_{-1})/2$$

The values for the model constants are:

$\hat{k}_1 = 1.3708 \times 10^{12}$,	$\beta_1 = 9.2984 \times 10^3$,	$K_1 = 2.575 \times 10^{-16}$
$\hat{k}_{-1} = 1.6215 \times 10^{20}$	$\beta_{-1} = 1.3108 \times 10^4$,	$K_2 = 4.876 \times 10^{-14}$
$\hat{k}_2 = 5.2282 \times 10^{12}$,	$\beta_2 = 9.5999 \times 10^3$,	$K_3 = 1.7884 \times 10^{-16}$.

Phase 2 .... *Transient Stage 2* .... Phase 2

Parameters: $(p^{(2)}, t_F^{(2)})$ ....

$0 \le p^{(2)} \le .0262$

Independent Variable: (t) ....

$t = .01$	$.01 \le t \le t_F^{(2)}$	$t = t_F^{(2)}$

Differential Variables: $(y_1, y_2, y_3, y_4, y_5, y_6)$ ....

$y_1 = y_{1F}^{(1)}$	$y_1 \le 2$	$y_1 \le 2$
$y_2 = y_{2F}^{(1)}$	$5 \le y_2 \le 10$	$5 \le y_2 \le 10$
$y_3 = y_{3F}^{(1)}$	$y_3 \le 2$	$y_3 \le 2$
$y_4 = y_{4F}^{(1)}$	$y_4 \le 2$	$y_4 \le 2$
$y_5 = y_{5F}^{(1)}$	$y_5 \le 2$	$y_5 \le 2$
$y_6 = y_{6F}^{(1)}$	$y_6 \le 0.1$	$y_6 \le 0.1$

Algebraic Variables: $(u_1, u_2, u_3, u_4, u_5)$ ....

$0 \le u_1 \le 15$	$0 \le u_1 \le 15$	$0 \le u_1 \le 15$
$u_2 \le .02$	$u_2 \le .02$	$u_2 \le .02$
$u_3 \le 5 \times 10^{-5}$	$u_3 \le 5 \times 10^{-5}$	$u_3 \le 5 \times 10^{-5}$
$u_4 \le 5 \times 10^{-5}$	$u_4 \le 5 \times 10^{-5}$	$u_4 \le 5 \times 10^{-5}$
$u_5 = u_{5F}^{(1)}$	$293.15 \le u_5 \le 393.15$	$293.15 \le u_5 \le 393.15$

Boundary Conditions ....

$p^{(2)} = p^{(1)}$

Differential-Algebraic Equations ....

Equations (44.1) - (44.11)

Phase 3 .... *Steady State* .... Phase 3

Parameters: $(p^{(3)}, t_I^{(3)}, t_F^{(3)})$ ....

$0 \leq p^{(3)} \leq .0262$
$1.5 \leq t_F^{(3)}$

Independent Variable: (t) ....

$t = t_I^{(3)} = t_F^{(2)}$	$t_I^{(3)} \leq t \leq t_F^{(3)}$	$t = t_F^{(3)}$

Differential Variables: $(y_1, y_2, y_3, y_4, y_5, y_6)$ ....

$y_1 = y_{1F}^{(2)}$	$y_1 \leq 2$	$y_1 \leq 2$
$y_2 = y_{2F}^{(2)}$	$5 \leq y_2 \leq 10$	$5 \leq y_2 \leq 10$
$y_3 = y_{3F}^{(2)}$	$y_3 \leq 2$	$y_3 \leq 2$
$y_4 = y_{4F}^{(2)}$	$y_4 \leq 2$	$y_4 \leq 2$
$y_5 = y_{5F}^{(2)}$	$y_5 \leq 2$	$y_5 \leq 2$
$y_6 = y_{6F}^{(2)}$	$y_6 \leq 0.1$	$y_6 \leq 0.1$

Algebraic Variables: $(u_1, u_2, u_3, u_4, u_5)$ ....

$0 \leq u_1 \leq 15$	$0 \leq u_1 \leq 15$	$0 \leq u_1 \leq 15$
$u_2 \leq .02$	$u_2 \leq .02$	$u_2 \leq .02$
$u_3 \leq 5 \times 10^{-5}$	$u_3 \leq 5 \times 10^{-5}$	$u_3 \leq 5 \times 10^{-5}$
$u_4 \leq 5 \times 10^{-5}$	$u_4 \leq 5 \times 10^{-5}$	$u_4 \leq 5 \times 10^{-5}$
$u_5 = u_{5F}^{(2)}$	$293.15 \leq u_5 \leq 393.15$	$293.15 \leq u_5 \leq 393.15$

Boundary Conditions ....

$p^{(3)} = p^{(2)}$
$4t_I^{(3)} = t_F^{(3)}$

Differential-Algebraic Equations ....

Equations (44.1) - (44.10)

Objective ....

Minimize $\quad J = t_F^{(3)} + 100p^{(3)}$

$$\boxed{J^* = 3.16466910; \quad t_F^{(3)} = 1.7468208}$$

Chapter 45

medi: **Minimum Energy Double Integrator**

Bryson and Ho [29, pp 120-123] present an example they label *A minimum energy problem with a second-order state variable inequality constraint*. The problem is simple enough that analytic solutions are available for all values of the state bound. The examples given here correspond to solutions over all regions of the problem.

Example 45.1 medi01: Minimum Control Energy ($\ell = 0.1$).

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 1$	$t = 1$

Differential Variables: (x, v) ....

$x = 0$		$x = 0$
$v = 1$		$v = -1$

Algebraic Variables: (u) ....

Differential-Algebraic Equations ....

$$\dot{x} = v \tag{45.1}$$

$$\dot{v} = u \tag{45.2}$$

$$x \le \ell \tag{45.3}$$

where $\ell = 0.1$ and $\hat{J} = 4/(9\ell) = 4.444444444$.

Objective ....

Minimize
$$J = \frac{1}{2}\int_0^1 u^2 dt$$

$\boxed{J^* = 4.44444433}$

Example 45.2 medi02: MINIMUM CONTROL ENERGY ($\ell = 0.1$).

Phase 1 . . . . Phase 1

Independent Variable: (t) . . . .

$t = 0$	$0 < t < 1$	$t = 1$

Differential Variables: (x, v) . . . .

$x = 0$	$x \le \ell$	$x = 0$
$v = 1$		$v = -1$

Algebraic Variables: (u) . . . .
Differential-Algebraic Equations . . . .

$$\dot{x} = v \tag{45.4}$$

$$\dot{v} = u \tag{45.5}$$

where $\ell = 0.1$ and $\hat{J} = 4/(9\ell) = 4.444444444$.
Objective . . . .

Minimize
$$J = \frac{1}{2}\int_0^1 u^2 dt$$

$\boxed{J^* = 4.44439748}$

Example 45.3 medi03: MINIMUM CONTROL ENERGY ($\ell = 0.2$).

Repeat example 45.1 with $\ell = 0.2$ and $\hat{J} = 2 + 6(1 - 4\ell)^2 = 2.24$.

$\boxed{J^* = 2.24000000}$

Example 45.4 medi04: MINIMUM CONTROL ENERGY ($\ell = 0.2$).

Repeat example 45.2 with $\ell = 0.2$ and $\hat{J} = 2 + 6(1 - 4\ell)^2 = 2.24$.

$\boxed{J^* = 2.24000000}$

Example 45.5 medi05: MINIMUM CONTROL ENERGY ($\ell = 0.5$).

Repeat example 45.1 with $\ell = 0.5$ and $\hat{J} = 2$.

$$\boxed{J^* = 2.00000000}$$

Example 45.6 medi06: MINIMUM CONTROL ENERGY ($\ell = 0.5$).

Repeat example 45.2 with $\ell = 0.5$ and $\hat{J} = 2$.

$$\boxed{J^* = 2.00000000}$$

Chapter 46

mirv: **Multiple Independent Reentry Vehicles**

Anti-ballistic missile (ABM) systems were designed during the cold war to defend against the threat of attack by ballistic missiles. The ABM missiles were designed to intercept an incoming missile assuming it follows a *ballistic* trajectory. However, if the incoming missile maneuvers away from the ballistic trajectory the ABM is not effective. This scenario requires a model with two distinct trajectory branches. First, a ballistic trajectory must be defined such that it reenters the atmosphere and impacts a given target location. Second, an aerodynamically controlled maneuver must be computed, such that the reentry vehicle begins and ends on the ballistic path, but deviates as far as possible from the ballistic path during the maneuver. This scenario is implemented using five phases, with the first three covering portions of the ballistic path, and the final two modeling the maneuver branch of the trajectory. Boundary conditions ensure that the end of phase one coincides with the beginning of phase four, and the end of phase three, coincides with the end of phase five. The goal of the optimization is to maximize the distance between the ballistic trajectory at the end of phase two, and the maneuvering vehicle at the end of phase four.

Example 46.1 mirv01: Maximum Deviation From Ballistic.

Phase 1 . . . . *Ballistic Reentry Segment 1* . . . . Phase 1

Parameters: $(t_F^{(1)})$ . . . .

$0 \le t_F^{(1)} \le 300$

Independent Variable: (t) . . . .

$t = 0$	$0 \le t \le t_F^{(1)}$	$0 \le t \le t_F^{(1)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ . . . .

$h = 200000$	$-10 \leq h \leq 300000$	$h = 36000$	ft
$-10^o \leq \phi \leq 20^o$	$-10^o \leq \phi \leq 20^o$	$\phi = 0^o$	rad
$\theta = 0^o$	$-10^o \leq \theta \leq 10^o$	$-10^o \leq \theta \leq 10^o$	rad
$v = 20000$	$0 \leq v \leq 21000$	$0 \leq v \leq 21000$	ft/sec
$\gamma = -1^o$	$-89^o \leq \gamma \leq 89^o$	$-89^o \leq \gamma \leq 89^o$	rad
$\psi = 90^o$	$-180^o \leq \psi \leq 180^o$	$-180^o \leq \psi \leq 180^o$	rad

Differential-Algebraic Equations ....

$$\dot{h} = v \sin\gamma \tag{46.1}$$

$$\dot{\phi} = \frac{v\cos\gamma\sin\psi}{r\cos\theta} \tag{46.2}$$

$$\dot{\theta} = \frac{v\cos\gamma\cos\psi}{r} \tag{46.3}$$

$$\dot{v} = -\frac{D}{m} - g\sin\gamma \tag{46.4}$$

$$\dot{\gamma} = \cos\gamma\left(\frac{v}{r} - \frac{g}{v}\right) \tag{46.5}$$

$$\dot{\psi} = \frac{v\cos\gamma\sin\psi\sin\theta}{r\cos\theta} \tag{46.6}$$

where the problem constants are given in Table 46.1 and

$$D = \frac{1}{2}\hat{c}_D\rho v^2 \tag{46.7}$$

$$g = \mu/r^2 \tag{46.8}$$

$$r = R_e + h \tag{46.9}$$

$$\rho = \rho_0\exp[-h/h_r] \tag{46.10}$$

Phase 2 .... *Ballistic Reentry Segment 2* .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....

$0 \leq t_F^{(2)} \leq 300$

Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} \leq t \leq t_F^{(2)}$	$t_I^{(2)} \leq t \leq t_F^{(2)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = h_F^{(1)}$	$-10 \leq h \leq 300000$	$h = 17500$	ft
$\phi = \phi_F^{(1)}$	$-10^o \leq \phi \leq 20^o$	$-10^o \leq \phi \leq 20^o$	rad

$\theta = \theta_F^{(1)}$	$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	rad
$v = v_F^{(1)}$	$0 \le v \le 21000$	$0 \le v \le 21000$	ft/sec
$\gamma = \gamma_F^{(1)}$	$-89^o \le \gamma \le 89^o$	$-89^o \le \gamma \le 89^o$	rad
$\psi = \psi_F^{(1)}$	$-180^o \le \psi \le 180^o$	$-180^o \le \psi \le 180^o$	rad

Boundary Conditions ....

$t_F^{(2)} - t_I^{(2)} \ge 1$

Differential-Algebraic Equations ....

Equations (46.1) - (46.10)

Phase 3 .... *Ballistic Reentry Segment 3* .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....

$0 \le t_F^{(3)} \le 300$

Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} \le t \le t_F^{(3)}$	$t_I^{(3)} \le t \le t_F^{(3)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = h_F^{(2)}$	$-10 \le h \le 300000$	$h = 0$	ft
$\phi = \phi_F^{(2)}$	$-10^o \le \phi \le 20^o$	$-10^o \le \phi \le 20^o$	rad
$\theta = \theta_F^{(2)}$	$-10^o \le \theta \le 10^o$	$-10^o \le \theta \le 10^o$	rad
$v = v_F^{(2)}$	$0 \le v \le 21000$	$0 \le v \le 21000$	ft/sec
$\gamma = \gamma_F^{(2)}$	$-89^o \le \gamma \le 89^o$	$-89^o \le \gamma \le 89^o$	rad
$\psi = \psi_F^{(2)}$	$-180^o \le \psi \le 180^o$	$-180^o \le \psi \le 180^o$	rad

Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} \ge 1$

Differential-Algebraic Equations ....

Equations (46.1) - (46.10)

Phase 4 . . . . *Manuevering Reentry Segment 1* . . . . Phase 4

Parameters: $(t_I^{(4)}, t_F^{(4)})$ . . . .

$0 \leq t_F^{(4)} \leq 300$

Independent Variable: (t) . . . .

$t = t_F^{(1)} = t_I^{(4)}$	$t_I^{(4)} \leq t \leq t_F^{(4)}$	$t = t_F^{(4)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ . . . .

$h = h_F^{(2)}$	$-10 \leq h \leq 300000$	$-10 \leq h \leq 300000$	ft
$\phi = \phi_F^{(2)}$	$-10^o \leq \phi \leq 20^o$	$-10^o \leq \phi \leq 20^o$	rad
$\theta = \theta_F^{(2)}$	$-10^o \leq \theta \leq 10^o$	$-10^o \leq \theta \leq 10^o$	rad
$v = v_F^{(2)}$	$0 \leq v \leq 21000$	$0 \leq v \leq 21000$	ft/sec
$\gamma = \gamma_F^{(2)}$	$-89^o \leq \gamma \leq 89^o$	$-89^o \leq \gamma \leq 89^o$	rad
$\psi = \psi_F^{(2)}$	$-180^o \leq \psi \leq 180^o$	$-180^o \leq \psi \leq 180^o$	rad

Algebraic Variables: (c_β, s_β) . . . .

$-1.1 \leq c_\beta \leq 1.1$	$-1.1 \leq c_\beta \leq 1.1$	$-1.1 \leq c_\beta \leq 1.1$
$-1.1 \leq s_\beta \leq 1.1$	$-1.1 \leq s_\beta \leq 1.1$	$-1.1 \leq s_\beta \leq 1.1$

Boundary Conditions . . . .

$t_F^{(4)} - t_I^{(4)} \geq 1$

$t_F^{(2)} = t_F^{(4)}$

$\xi = 0$

where $\sigma = .5$ and

$$\xi = \sigma \left[v_F^{(4)} \sin \gamma_F^{(4)} \right] + (1 - \sigma) R_e \left[\frac{v_F^{(4)} \cos \gamma_F^{(4)} \cos \psi_F^{(4)}}{r_F^{(4)}} \right] - \sigma \left[v_F^{(2)} \sin \gamma_F^{(2)} \right] \tag{46.11}$$

Differential-Algebraic Equations . . . .

$$\dot{h} = v \sin \gamma \tag{46.12}$$

$$\dot{\phi} = \frac{v \cos \gamma \sin \psi}{r \cos \theta} \tag{46.13}$$

$$\dot{\theta} = \frac{v \cos \gamma \cos \psi}{r} \tag{46.14}$$

$$\dot{v} = -\frac{D}{m} - g \sin \gamma \tag{46.15}$$

$$\dot{\gamma} = \frac{Lc_\beta}{mv} + \cos\gamma\left(\frac{v}{r} - \frac{g}{v}\right) \tag{46.16}$$

$$\dot{\psi} = \frac{Ls_\beta}{mv\cos\gamma} + \frac{v\cos\gamma\sin\psi\sin\theta}{r\cos\theta} \tag{46.17}$$

$$1 = \sqrt{s_\beta^2 + c_\beta^2} \tag{46.18}$$

where the problem constants are given in Table 46.1 and

$$D = \frac{1}{2}c_D\rho v^2 \tag{46.19}$$

$$L = \frac{1}{2}c_L\rho v^2 \tag{46.20}$$

Objective ....

Maximize $\qquad J = \sigma h_F^{(4)} + (1-\sigma)R_e\phi_F^{(4)} - \sigma h_F^{(2)}$

$\boxed{J^* = 2392.06937; \quad t_F^{(4)} = 167.60889}$

Phase 5 .... *Manuevering Reentry Segment 2* .... Phase 5

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....

$0 \le t_F^{(5)} \le 300$

Independent Variable: (t) ....

$t = t_F^{(4)} = t_I^{(5)}$	$t_I^{(5)} \le t \le t_F^{(5)}$	$t = t_F^{(5)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = h_F^{(4)}$	$-10 \le h \le 300000$	$h = 0$	ft
$\phi = \phi_F^{(4)}$	$-10^o \le \phi \le 20^o$	$\phi = \phi_F^{(3)}$	rad
$\theta = \theta_F^{(4)}$	$-10^o \le \theta \le 10^o$	$\theta = 0^o$	rad
$v = v_F^{(4)}$	$0 \le v \le 21000$	$0 \le v \le 21000$	ft/sec
$\gamma = \gamma_F^{(4)}$	$-89^o \le \gamma \le 89^o$	$-89^o \le \gamma \le 89^o$	rad
$\psi = \psi_F^{(4)}$	$-180^o \le \psi \le 180^o$	$-180^o \le \psi \le 180^o$	rad

Algebraic Variables: (c_β, s_β) ....

$-1.1 \le c_\beta \le 1.1$	$-1.1 \le c_\beta \le 1.1$	$-1.1 \le c_\beta \le 1.1$
$-1.1 \le s_\beta \le 1.1$	$-1.1 \le s_\beta \le 1.1$	$-1.1 \le s_\beta \le 1.1$

Boundary Conditions ....

$t_F^{(5)} - t_I^{(5)} \geq 1$

Differential-Algebraic Equations ....
Equations (46.12) - (46.20)

$h_r = 23800$ ft	$R_e = 20902900$ ft
$\mu = 0.14076539 \times 10^{17}$	$\rho_0 = 0.002378$
$m = 2.4411015267444376$	$c_L = .029244$
$\hat{c}_D = .07854$	$c_D = .073002208$

Table 46.1. *Multiple Independent Reentry Vehicles example constants.*

Chapter 47

mncx: Non-Convex Delay

A delay equation example given by Maurer [67] is posed here using the method of steps. Three different versions corresponding to different delay times are stated.

Example 47.1 mncx01: NON-CONVEX DELAY, $r = 0$.

Phase 1 .... *DDE: Method of Steps* .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < \delta$	$t = \tau = 0.1$

Differential Variables: $(x_1, \ldots, x_N)$ ....

$x_1 = x_0 = 1$	$0.7 \le x_1$	$0.7 \le x_1$
$0.7 \le x_j$	$0.7 \le x_j$	$0.7 \le x_j$

where $j = 2, \ldots, N$. For $N = 20$ and $t_F = 2$, $\tau = t_F/N = 0.1$.

Algebraic Variables: $(u_1, \ldots, u_N)$ ....

Boundary Conditions ....

$x_j(0) = x_{j-1}(\tau)$
$u_j(0) = u_{j-1}(\tau)$

for $j = 2, \ldots, N$.

Differential-Algebraic Equations ....

$$\dot{x}_k = x_{k-\sigma}^2 - u_k \tag{47.1}$$

for $k = 1, \ldots, N$, with $\sigma = r/\tau$. When $r = 0$, $\sigma = 0$ and

$$x_{k-\sigma} = x_0 \tag{47.2}$$

for $k - \sigma \leq 0$.

Objective . . . .

Minimize

$$J = \int_0^\tau \sum_{k=1}^{N} \left[x_k^2(t) + u_k^2(t)\right] dt \tag{47.3}$$

$\boxed{J^* = 2.26991831}$

Example 47.2 mncx02: Non-Convex Delay, $r = 0.1$.

Repeat example 47.1 with $r = 0.1$, $\sigma = 1$.

$\boxed{J^* = 2.40054167}$

Example 47.3 mncx03: Non-Convex Delay, $r = 0.5$.

Repeat example 47.1 with $r = 0.5$, $\sigma = 5$.

$\boxed{J^* = 2.79685764}$

Chapter 48

mrck: **Immunology DDE**

A example originally published in Russian by G. I. Marchuk is also cited by Hairer, Norsett, and Wanner [52, pp. 349–351]. The example is used to illustrate solution techniques for a challenging delay differential equation and is posed here as an initial value problem as discussed in reference [13, pp. 389-393]. This formulation leads to a system with 480 states, and 476 boundary conditions.

Example 48.1 mrck01: MARCHUK DDE; 120 DELAY INTERVALS.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\qquad$ $0 \le t \le \tau$ $\qquad$ $t = \tau$

Differential Variables: $(y_{1+kL}, y_{2+kL}, y_{3+kL}, y_{4+kL})$ ....

$y_1 = 10^{-6}$
$y_2 = 1$
$y_3 = 1$
$y_4 = 0$

for $j = 1, \ldots, L$, and $k = 0, \ldots, N-1$ with $N = 120$ and $L = 4$.

Boundary Conditions ....

$y_{j+(k+1)L}(0) = y_{j+kL}(\tau)$

for $j = 1, \ldots, L$, and $k = 0, \ldots, N-2$.

Differential-Algebraic Equations ....

$$\dot{y}_{1+kL} = [h_1 - h_2 y_{3+kL}]\, y_{1+kL} \tag{48.1}$$

$$\dot{y}_{2+kL} = \xi(y_{4+kL}) h_3 y_{3+(k-1)L} y_{1+(k-1)L} - h_5 [y_{2+kL} - 1] \tag{48.2}$$

$$\dot{y}_{3+kL} = h_4 \left[y_{2+kL} - y_{3+kL} \right] - h_8 y_{3+kL} y_{1+kL} \tag{48.3}$$
$$\dot{y}_{4+kL} = h_6 y_{1+kL} - h_7 y_{4+kL} \tag{48.4}$$

where

$$\xi(m) = \begin{cases} 1 & \text{if } m \leq 0.1, \\ (1-m)\frac{10}{9} & \text{if } 0.1 \leq m \leq 1. \end{cases} \tag{48.5}$$

for $k = 0, \ldots, N-1$ where $N = 120$ and $L = 4$. When $-\tau \leq t \leq 0$ define

$$y_{1-L}(t) = \max\left(0, 10^{-6} + t\right) \tag{48.6}$$
$$y_{2-L}(t) = 1 \tag{48.7}$$
$$y_{3-L}(t) = 1 \tag{48.8}$$
$$y_{4-L}(t) = 0 \tag{48.9}$$

The model parameters are $\tau = 0.5$, $h_1 = 2$, $h_2 = 0.8$, $h_3 = 10^4$, $h_4 = 0.17$, $h_5 = 0.5$, $h_6 = 300.$, $h_7 = 0.12$, and $h_8 = 8$.

Boundary Value Problem

Chapter 49

nzym: **Enzyme Kinetics**

A particular example that was originally published by Okamoto and Hayashi [74] and cited by Hairer, Norsett, and Wanner [52, pp. 348–349], describes enzyme kinetics. Formulation using the method of steps is described in reference [13, p 386-389]. Using this approach simulation for a period of 160 with a delay of 4, leads to a system with 160 state variables subject to 156 boundary conditions.

Example 49.1 nzym01: ENZYME KINETICS, MOS.

Phase 1 .... *Method of Steps (MOS)* .... Phase 1

Independent Variable: (t) ....

$t = t_I = 0$ $\qquad 0 \leq t \leq t_F$ $\qquad t = t_F = 4$

Differential Variables: $(y_1, \dots, y_{160})$ ....

$y_1 = 60$
$y_2 = 10$
$y_3 = 10$
$y_4 = 20$

Boundary Conditions ....

$y_{j+4k}(t_F) = y_{j+4(k+1)}(t_I)$

for $j = 1, \dots, 4$ and $k = 0, 1, \dots, (40 - 1)$.

Differential-Algebraic Equations ....

$$\dot{y}_{1+4k} = I - z y_{1+4k} \tag{49.1}$$

$$\dot{y}_{2+4k} = z y_{1+4k} - c_2 y_{2+4k} \tag{49.2}$$

$$\dot{y}_{3+4k} = c_2 y_{2+4k} - c_3 y_{3+4k} \tag{49.3}$$
$$\dot{y}_{4+4k} = c_3 y_{3+4k} - c_4 y_{4+4k} \tag{49.4}$$

for $k = 0, 1, \ldots, 39$ where

$$z = \frac{c_1}{1 + \alpha [y_{4+4(k-1)}]^3}. \tag{49.5}$$

The problem constants are given by $I = 10.5$, $c_1 = c_2 = c_3 = 1$, $c_4 = 0.5$, and $\alpha = 0.0005$ in addition to the values

$$y_{-3} = 60 \tag{49.6}$$
$$y_{-2} = 10 \tag{49.7}$$
$$y_{-1} = 10 \tag{49.8}$$
$$y_0 = 20 \tag{49.9}$$

Chapter 50

orbe: **Low Thrust Orbit Transfer using Equinoctial Elements**

This low thrust orbit transfer was first described in reference [8]. The physical application is similar to that represented in example (43.1). However, the different dynamics used here are referred to as *equinoctial elements*, and the three examples (50.1)-(50.3) require multiple revolutions about the earth.

Example 50.1 orbe01: Coast in Molniya Orbit.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$	sec

Differential Variables: (a, h, k, p, q, F) ....

$a = a_1$	$c_1 \le a \le c_2$	$c_1 \le a \le c_2$	ft
$h = h_1$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = 0$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$p = 0$	$-1 \le p \le 1$	$-1 \le p \le 1$	
$q = q_1$	$-1 \le q \le 1$	$-1 \le q \le 1$	
$F = \pi$	$\pi \le F \le 6\pi$	$\pi \le F \le 6\pi$	rad

Differential-Algebraic Equations ....

$$\dot{\mathbf{z}} = \mathbf{M}\boldsymbol{\Delta} + \mathbf{m} \tag{50.1}$$

where $\mathbf{z}^\mathsf{T} = (a, h, k, p, q, F)$. The right hand side is computed by sequentially executing the following expressions:

$$n = \sqrt{\frac{\mu}{a^3}} \tag{50.2}$$

$$G = \sqrt{1 - h^2 - k^2} \tag{50.3}$$

$$\beta = \frac{1}{(1+G)} \tag{50.4}$$

$$s_F = \sin F \tag{50.5}$$

$$c_F = \cos F \tag{50.6}$$

$$r = a(1 - kc_F - hs_F) \tag{50.7}$$

$$K = (1 + p^2 + q^2) \tag{50.8}$$

$$m_6 = \frac{na}{r} \tag{50.9}$$

$$X = a\left[(1 - h^2\beta)c_F + hk\beta s_F - k\right] \tag{50.10}$$

$$Y = a\left[hk\beta c_F + (1 - k^2\beta)s_F - h\right] \tag{50.11}$$

$$\dot{X} = a^2 n r^{-1}\left[hk\beta c_F - (1 - h^2\beta)s_F\right] \tag{50.12}$$

$$\dot{Y} = a^2 n r^{-1}\left[(1 - k^2\beta)c_F - hk\beta s_F\right] \tag{50.13}$$

$$\frac{\partial X}{\partial h} = a\left[-(hc_F - ks_F)\left\{\beta + \frac{\beta^3}{(1-\beta)}h^2\right\} + \frac{a}{r}c_F(s_F - h\beta)\right] \tag{50.14}$$

$$\frac{\partial X}{\partial k} = -a\left[(hc_F - ks_F)hk\frac{\beta^3}{(1-\beta)} + 1 + \frac{a}{r}s_F(s_F - h\beta)\right] \tag{50.15}$$

$$\frac{\partial Y}{\partial h} = a\left[(hc_F - ks_F)hk\frac{\beta^3}{(1-\beta)} - 1 - \frac{a}{r}c_F(c_F - k\beta)\right] \tag{50.16}$$

$$\frac{\partial Y}{\partial k} = a\left[(hc_F - ks_F)\left\{\beta + \frac{\beta^3}{(1-\beta)}k^2\right\} + \frac{a}{r}s_F(c_F - k\beta)\right] \tag{50.17}$$

$$M_{11} = 2a^{-1}n^{-2}\dot{X} \tag{50.18}$$

$$M_{12} = 2a^{-1}n^{-2}\dot{Y} \tag{50.19}$$

$$M_{13} = 0 \tag{50.20}$$

$$M_{21} = Gn^{-1}a^{-2}\left(\frac{\partial X}{\partial k} - \dot{X}\frac{h\beta}{n}\right) \tag{50.21}$$

$$M_{22} = Gn^{-1}a^{-2}\left(\frac{\partial Y}{\partial k} - \dot{Y}\frac{h\beta}{n}\right) \tag{50.22}$$

$$M_{23} = G^{-1}n^{-1}a^{-2}k\,(qY - pX) \tag{50.23}$$

$$M_{31} = -Gn^{-1}a^{-2}\left(\frac{\partial X}{\partial h} + \dot{X}\frac{k\beta}{n}\right) \tag{50.24}$$

$$M_{32} = -Gn^{-1}a^{-2}\left(\frac{\partial Y}{\partial h} + \dot{Y}\frac{k\beta}{n}\right) \tag{50.25}$$

$$M_{33} = -G^{-1}n^{-1}a^{-2}h\,(qY - pX) \tag{50.26}$$

$$M_{41} = 0 \tag{50.27}$$

$$M_{42} = 0 \tag{50.28}$$

$$M_{43} = \frac{G^{-1}n^{-1}a^{-2}KY}{2} \tag{50.29}$$

$$M_{51} = 0 \tag{50.30}$$

$$M_{52} = 0 \tag{50.31}$$

$$M_{53} = \frac{G^{-1}n^{-1}a^{-2}KX}{2} \tag{50.32}$$

$$\tilde{M}_{61} = n^{-1}a^{-2}\left[-2X + G\left(h\beta\frac{\partial X}{\partial h} + k\beta\frac{\partial X}{\partial k}\right)\right] \tag{50.33}$$

$$\tilde{M}_{62} = n^{-1}a^{-2}\left[-2Y + G\left(h\beta\frac{\partial Y}{\partial h} + k\beta\frac{\partial Y}{\partial k}\right)\right] \tag{50.34}$$

$$\tilde{M}_{63} = G^{-1}n^{-1}a^{-2}\left(qY - pX\right) \tag{50.35}$$

$$M_{61} = \frac{a}{r}\left(\tilde{M}_{61} + s_F M_{31} - c_F M_{21}\right) \tag{50.36}$$

$$M_{62} = \frac{a}{r}\left(\tilde{M}_{62} + s_F M_{32} - c_F M_{22}\right) \tag{50.37}$$

$$M_{63} = \frac{a}{r}\left(\tilde{M}_{63} + s_F M_{33} - c_F M_{23}\right) \tag{50.38}$$

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} \\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} \\ M_{51} & M_{52} & M_{53} & M_{54} & M_{55} & M_{56} \\ M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66} \end{bmatrix} \tag{50.39}$$

$$\mathbf{m}^{\mathsf{T}} = (0, 0, 0, 0, 0, m_6). \tag{50.40}$$

The perturbing force $\boldsymbol{\Delta}$ is computed by executing the following expressions in sequence:

$$\widehat{\mathbf{f}} = K^{-1}\begin{pmatrix} 1 - p^2 + q^2 \\ 2pq \\ -2p \end{pmatrix} \tag{50.41}$$

$$\widehat{\mathbf{g}} = K^{-1}\begin{pmatrix} 2pq \\ 1 + p^2 - q^2 \\ 2q \end{pmatrix} \tag{50.42}$$

$$\widehat{\mathbf{w}} = K^{-1}\begin{pmatrix} 2p \\ -2q \\ 1 - p^2 - q^2 \end{pmatrix} \tag{50.43}$$

$$\mathbf{r} = X\widehat{\mathbf{f}} + Y\widehat{\mathbf{g}} \tag{50.44}$$

$$\mathbf{v} = \dot{X}\widehat{\mathbf{f}} + \dot{Y}\widehat{\mathbf{g}} \tag{50.45}$$

$$\widetilde{\mathbf{k}} = \frac{-\mathbf{r}}{\|\mathbf{r}\|} \tag{50.46}$$

$$\breve{\mathbf{i}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \tilde{k}_3\widetilde{\mathbf{k}} \tag{50.47}$$

$$\widetilde{\mathbf{i}} = \frac{\breve{\mathbf{i}}}{\|\breve{\mathbf{i}}\|} \tag{50.48}$$

$$\sin\phi = \frac{r_3}{r} \tag{50.49}$$

$$\cos\phi = \sqrt{1 - \sin^2\phi} \tag{50.50}$$

$$g_x = -\frac{\mu \cos\phi}{r^2} \sum_{k=2}^{4} \left(\frac{R_e}{r}\right)^k P_k' J_k \tag{50.51}$$

$$g_z = -\frac{\mu}{r^2} \sum_{k=2}^{4} (k+1) \left(\frac{R_e}{r}\right)^k P_k J_k \tag{50.52}$$

$$\alpha_1 = g_x \widetilde{\mathbf{i}}^{\mathsf{T}} \widehat{\mathbf{f}} + g_z \widetilde{\mathbf{k}}^{\mathsf{T}} \widehat{\mathbf{f}} \tag{50.53}$$

$$\alpha_2 = g_x \widetilde{\mathbf{i}}^{\mathsf{T}} \widehat{\mathbf{g}} + g_z \widetilde{\mathbf{k}}^{\mathsf{T}} \widehat{\mathbf{g}} \tag{50.54}$$

$$\alpha_3 = g_x \widetilde{\mathbf{i}}^{\mathsf{T}} \widehat{\mathbf{w}} + g_z \widetilde{\mathbf{k}}^{\mathsf{T}} \widehat{\mathbf{w}} \tag{50.55}$$

$$\boldsymbol{\Delta}_g = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{50.56}$$

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}_g \tag{50.57}$$

where $P_k(\sin\phi)$ is the k-th order Legendre polynomial with corresponding derivative P_k'. Table 50.1 summarizes the remaining problem constants.

Objective ....

Initial Value Problem

$\boxed{a_F^* = 8.7155322 \times 10^7; \quad F_F^* = 539.91847^o}$

Example 50.2 orbe02: LOW-THRUST, MAX PAYLOAD, TWO REV.

Phase 1 .... Phase 1

Parameters: (T, t_F) ....

$1 \times 10^{-5} \le T \le 1$	$1 \le t_F$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F = t_F$	sec

Differential Variables: (a, h, k, p, q, F, w) ....

$a = a_2$	$0.1a_2 \le a \le 5a_1$	$a = a_1$	ft
$h = 0$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = 0$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$p = 0$	$-1 \le p \le 1$	$-1 \le p \le 1$	
$q = q_2$	$-1 \le q \le 1$	$-1 \le q \le 1$	
$F = \pi$	$\pi \le F \le 10\pi$	$4.5\pi \le F \le 5.5\pi$	rad
$w = 1$	$1 \times 10^{-4} \le w \le 1$	$1 \times 10^{-4} \le w \le 1$	lb

Algebraic Variables: (u_x, u_y, u_z) ....

$-1.1 \le u_x \le 1.1$	$-1.1 \le u_x \le 1.1$	$-1.1 \le u_x \le 1.1$
$-1.1 \le u_y \le 1.1$	$-1.1 \le u_y \le 1.1$	$-1.1 \le u_y \le 1.1$
$-1.1 \le u_z \le 1.1$	$-1.1 \le u_z \le 1.1$	$-1.1 \le u_z \le 1.1$

Boundary Conditions ....

$$\begin{aligned} e_F &= \sqrt{h_F^2 + k_F^2} \\ \tan\frac{i_F}{2} &= \sqrt{p_F^2 + q_F^2} \\ 0 &= k_F q_F + h_F p_F \\ 0 &\ge h_F q_F - p_F k_F \end{aligned}$$

Differential-Algebraic Equations ....

$$\dot{\mathbf{z}} = \mathbf{M}\boldsymbol{\Delta} + \mathbf{m} \tag{50.58}$$

$$\dot{w} = \frac{-T}{I_{sp}} \tag{50.59}$$

$$0 = \|\mathbf{u}\| - 1. \tag{50.60}$$

where $\mathbf{M}$, $\mathbf{m}$, and $\boldsymbol{\Delta}_g$ are computed by executing the sequence (50.2)-(50.56) and $\mathbf{u}^{\mathsf{T}} = (u_x, u_y, u_z)$. The definition of the right hand side is completed by computing the following sequence:

$$b_1 = \dot{X}(\dot{X}^2 + \dot{Y}^2)^{-\frac{1}{2}} \tag{50.61}$$

$$b_2 = \dot{Y}(\dot{X}^2 + \dot{Y}^2)^{-\frac{1}{2}} \tag{50.62}$$

$$\boldsymbol{\Delta}_T = \frac{g_0 T}{w}\begin{pmatrix} u_x b_1 + u_z b_2 \\ u_x b_2 - u_z b_1 \\ u_y \end{pmatrix} \tag{50.63}$$

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}_T + \boldsymbol{\Delta}_g \tag{50.64}$$

Objective ....

Maximize $\qquad J = w(t_F)$

$$\boxed{J^* = .244318271; \quad t_F^* = 19330.329; \quad T^* = .017591878}$$

Example 50.3 orbe05: LOW-THRUST, MAX PAYLOAD, FOUR REV.

Repeat example 50.2 with the following change

$F = \pi$ $\qquad$ $\pi \leq F \leq 18\pi$ $\qquad$ $8.5\pi \leq F \leq 9.5\pi$ $\qquad$ rad

$$\boxed{J^* = .230052256; \quad t_F^* = 41388.706; \quad T^* = .0083712810}$$

$\mu = 1.407645794 \times 10^{16}$	$R_e = 20925662.73$
$J_2 = 1082.3 \times 10^{-6}$	$J_3 = -2.3 \times 10^{-6}$
$J_4 = -1.8 \times 10^{-6}$	$I_{sp} = 450$
$e_F = .73550320568829042$	$i_F = 63.4^o$
$a_1 = 87155321.522650868$	$h_1 = .73550320568829042$
$q_1 = -0.61761258786098949$	$t_F = 43089.756402388135$
$c_1 = 2183708.0052834647$	$c_2 = 435776607.61325431$
$a_2 = 21837080.052834645$	$q_2 = -0.25396764647494369$

Table 50.1. *Equinoctial Orbit example constants.*

Chapter 51

orbt: **Elliptic Mission Orbit Transfer**

This collection of orbit transfer problems is stated using the more common Cartesian coordinates. However, the independent variable in this set of examples is a "range angle" instead of the usual time. Consequently the boundary conditions appearing here also differ when compared with examples (50.1)-(50.3) as well as example (43.1).

Example 51.1 orbt01: Three Burn Transfer.

References: [46, pp 50-51]

Phase 1 .... *First Coast* .... Phase 1

Parameters: $(\phi_F^{(1)})$ ....

$1 \times 10^{-8} \leq \phi_F^{(1)} \leq 4\pi$

Independent Variable: (ϕ) ....

$\phi = 0$	$0 \leq \phi \leq \phi_F^{(1)}$	$\phi = \phi_F^{(1)}$	rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = c_1$	$-c_4 \leq r_x \leq c_4$	$-c_4 \leq r_x \leq c_4$	ft
$r_y = 0$	$-c_4 \leq r_y \leq c_4$	$-c_4 \leq r_y \leq c_4$	ft
$r_z = 0$	$-c_4 \leq r_z \leq c_4$	$-c_4 \leq r_z \leq c_4$	ft
$v_x = 0$	$-c_5 \leq v_x \leq c_5$	$-c_5 \leq v_x \leq c_5$	ft/sec
$v_y = c_2$	$-c_5 \leq v_y \leq c_5$	$-c_5 \leq v_y \leq c_5$	ft/sec
$v_z = c_3$	$-c_5 \leq v_z \leq c_5$	$-c_5 \leq v_z \leq c_5$	ft/sec

Differential-Algebraic Equations ....

$$\mathbf{r}' = \left(\frac{dt}{d\phi}\right)\dot{\mathbf{r}} = \left(\frac{dt}{d\phi}\right)\mathbf{v} \tag{51.1}$$

$$\mathbf{v}' = \left(\frac{dt}{d\phi}\right)\dot{\mathbf{v}} = \left(\frac{dt}{d\phi}\right)\mathbf{g}(\mathbf{r}) \tag{51.2}$$

where $\mathbf{r}^\mathsf{T} = (r_x, r_y, r_z)$, $\mathbf{v}^\mathsf{T} = (v_x, v_y, v_z)$

$$\frac{d\phi}{dt} = \frac{v}{r}\sqrt{1 - \left(\frac{\mathbf{r}^\mathsf{T}\mathbf{v}}{rv}\right)^2} \tag{51.3}$$

$$r = \|\mathbf{r}\| = \sqrt{r_x^2 + r_y^2 + r_z^2} \tag{51.4}$$

$$v = \|\mathbf{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{51.5}$$

and $\mathbf{g}(\mathbf{r})$ is defined in [46, pp 50-51]. The additional problem parameters are given in Table 51.1.

Phase 2 .... *First Burn* .... Phase 2

Parameters: $(\phi_I^{(2)}, \phi_F^{(2)})$ ....

$0 \le \phi_I^{(2)} \le 4\pi$	$0 \le \phi_F^{(2)} \le 4\pi$

Independent Variable: (ϕ) ....

$\phi = \phi_F^{(1)} = \phi_I^{(2)}$	$\phi_I^{(2)} \le \phi \le \phi_F^{(2)}$	$\phi = \phi_F^{(2)}$	rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(1)}$	$-c_8 \le r_x \le c_8$	$-c_8 \le r_x \le c_8$	ft
$r_y = r_{yF}^{(1)}$	$-c_8 \le r_y \le c_8$	$-c_8 \le r_y \le c_8$	ft
$r_z = r_{zF}^{(1)}$	$-c_8 \le r_z \le c_8$	$-c_8 \le r_z \le c_8$	ft
$v_x = v_{xF}^{(1)}$	$-c_5 \le v_x \le c_5$	$-c_5 \le v_x \le c_5$	ft/sec
$v_y = v_{yF}^{(1)}$	$-c_5 \le v_y \le c_5$	$-c_5 \le v_y \le c_5$	ft/sec
$v_z = v_{zF}^{(1)}$	$-c_5 \le v_z \le c_5$	$-c_5 \le v_z \le c_5$	ft/sec
$w = 1$	$1 \times 10^{-4} \le w \le 1$	$1 \times 10^{-4} \le w \le 1$	lb

Algebraic Variables: (θ, ψ) ....

$-180^o \le \theta \le +180^o$	$-180^o \le \theta \le +180^o$	$-180^o \le \theta \le +180^o$	rad
$-89^o \le \psi \le 89^o$	$-89^o \le \psi \le 89^o$	$-89^o \le \psi \le 89^o$	rad

Boundary Conditions ....

$0 \le \phi_F^{(2)} - \phi_I^{(2)} \le 10^o$

Differential-Algebraic Equations ....

$$\mathbf{r}' = \left(\frac{dt}{d\phi}\right)\dot{\mathbf{r}} = \left(\frac{dt}{d\phi}\right)\mathbf{v} \tag{51.6}$$

$$\mathbf{v}' = \left(\frac{dt}{d\phi}\right)\dot{\mathbf{v}} = \left(\frac{dt}{d\phi}\right)\left[\mathbf{g}(\mathbf{r}) + \frac{g_0}{w}\mathbf{T}\right] \tag{51.7}$$

$$w' = -\left(\frac{dt}{d\phi}\right)\frac{T}{I_{sp}} \tag{51.8}$$

$$100 \text{ nm} \leq h \leq 50000 \text{ nm} \tag{51.9}$$

using (51.3)-(51.5) and

$$\mathbf{T} = \mathbf{Q}_v \begin{bmatrix} T\cos\theta\cos\psi \\ T\cos\theta\sin\psi \\ T\sin\theta \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \tag{51.10}$$

$$\mathbf{Q}_v = \left[\frac{\mathbf{v}}{\|\mathbf{v}\|}, \quad \frac{\mathbf{v}\times\mathbf{r}}{\|\mathbf{v}\times\mathbf{r}\|}, \quad \frac{\mathbf{v}}{\|\mathbf{v}\|} \times \left(\frac{\mathbf{v}\times\mathbf{r}}{\|\mathbf{v}\times\mathbf{r}\|}\right)\right] \tag{51.11}$$

$$T = \|\mathbf{T}\| \tag{51.12}$$

$$h = r - R_e \tag{51.13}$$

where $T = 2$.

Phase 3 .... *Second Coast* .... Phase 3

Parameters: $(\phi_I^{(3)}, \phi_F^{(3)})$ ....

$0 \leq \phi_I^{(3)} \leq 4\pi$ $\qquad$ $0 \leq \phi_F^{(3)} \leq 4\pi$

Independent Variable: (ϕ) ....

$\phi = \phi_F^{(2)} = \phi_I^{(3)}$	$\phi_I^{(3)} \leq \phi \leq \phi_F^{(3)}$	$\phi = \phi_F^{(3)}$	rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = r_{xF}^{(2)}$	$-c_7 \leq r_x \leq c_7$	$-c_7 \leq r_x \leq c_7$	ft
$r_y = r_{yF}^{(2)}$	$-c_7 \leq r_y \leq c_7$	$-c_7 \leq r_y \leq c_7$	ft
$r_z = r_{zF}^{(2)}$	$-c_7 \leq r_z \leq c_7$	$-c_7 \leq r_z \leq c_7$	ft
$v_x = v_{xF}^{(2)}$	$-c_5 \leq v_x \leq c_5$	$-c_5 \leq v_x \leq c_5$	ft/sec
$v_y = v_{yF}^{(2)}$	$-c_5 \leq v_y \leq c_5$	$-c_5 \leq v_y \leq c_5$	ft/sec
$v_z = v_{zF}^{(2)}$	$-c_5 \leq v_z \leq c_5$	$-c_5 \leq v_z \leq c_5$	ft/sec

Boundary Conditions ....

$0 \leq \phi_F^{(3)} - \phi_I^{(3)}$

Differential-Algebraic Equations ....

Equations (51.1) - (51.5)

Phase 4 .... *Second Burn* .... Phase 4

Parameters: $(\phi_I^{(4)}, \phi_F^{(4)})$ ....

$0 \leq \phi_I^{(4)} \leq 4\pi$ $\qquad$ $0 \leq \phi_F^{(4)} \leq 4\pi$

Independent Variable: (ϕ) ....

$\phi = \phi_F^{(3)} = \phi_I^{(4)}$	$\phi_I^{(4)} \leq \phi \leq \phi_F^{(4)}$	$\phi = \phi_F^{(4)}$	rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(3)}$	$-c_6 \leq r_x \leq c_6$	$-c_6 \leq r_x \leq c_6$	ft
$r_y = r_{yF}^{(3)}$	$-c_6 \leq r_y \leq c_6$	$-c_6 \leq r_y \leq c_6$	ft
$r_z = r_{zF}^{(3)}$	$-c_6 \leq r_z \leq c_6$	$-c_6 \leq r_z \leq c_6$	ft
$v_x = v_{xF}^{(3)}$	$-c_5 \leq v_x \leq c_5$	$-c_5 \leq v_x \leq c_5$	ft/sec
$v_y = v_{yF}^{(3)}$	$-c_5 \leq v_y \leq c_5$	$-c_5 \leq v_y \leq c_5$	ft/sec
$v_z = v_{zF}^{(3)}$	$-c_5 \leq v_z \leq c_5$	$-c_5 \leq v_z \leq c_5$	ft/sec
$w = w_F^{(2)}$	$1 \times 10^{-4} \leq w \leq 1$	$1 \times 10^{-4} \leq w \leq 1$	lb

Algebraic Variables: (θ, ψ) ....

$-180^o \leq \theta \leq +180^o$	$-180^o \leq \theta \leq +180^o$	$-180^o \leq \theta \leq +180^o$	rad
$-89^o \leq \psi \leq 89^o$	$-89^o \leq \psi \leq 89^o$	$-89^o \leq \psi \leq 89^o$	rad

Boundary Conditions ....

$0 \leq \phi_F^{(4)} - \phi_I^{(4)} \leq 10^o$

Differential-Algebraic Equations ....

Equations (51.6) - (51.9)

Phase 5 .... *Third Coast* .... Phase 5

Parameters: $(\phi_I^{(5)}, \phi_F^{(5)})$ ....

$0 \le \phi_I^{(5)} \le 4\pi$ $\qquad$ $0 \le \phi_F^{(5)} \le 4\pi$

Independent Variable: (ϕ) ....

$\phi = \phi_F^{(4)} = \phi_I^{(5)}$	$\phi_I^{(5)} \le \phi \le \phi_F^{(5)}$	$\phi = \phi_F^{(5)}$	rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = r_{xF}^{(4)}$	$-c_9 \le r_x \le c_9$	$-c_9 \le r_x \le c_9$	ft
$r_y = r_{yF}^{(4)}$	$-c_9 \le r_y \le c_9$	$-c_9 \le r_y \le c_9$	ft
$r_z = r_{zF}^{(4)}$	$-c_9 \le r_z \le c_9$	$-c_9 \le r_z \le c_9$	ft
$v_x = v_{xF}^{(4)}$	$-c_5 \le v_x \le c_5$	$-c_5 \le v_x \le c_5$	ft/sec
$v_y = v_{yF}^{(4)}$	$-c_5 \le v_y \le c_5$	$-c_5 \le v_y \le c_5$	ft/sec
$v_z = v_{zF}^{(4)}$	$-c_5 \le v_z \le c_5$	$-c_5 \le v_z \le c_5$	ft/sec

Boundary Conditions ....

$0 \le \phi_F^{(5)} - \phi_I^{(5)}$

Differential-Algebraic Equations ....

Equations (51.1) - (51.5)

Phase 6 .... *Third Burn* .... Phase 6

Parameters: $(\phi_I^{(6)}, \phi_F^{(6)})$ ....

$0 \le \phi_I^{(6)} \le 4\pi$ $\qquad$ $0 \le \phi_F^{(6)} \le 4\pi$

Independent Variable: (ϕ) ....

$\phi = \phi_F^{(5)} = \phi_I^{(6)}$	$\phi_I^{(6)} \le \phi \le \phi_F^{(6)}$	$\phi = \phi_F^{(6)}$	rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(5)}$	$-c_9 \le r_x \le c_9$	$-c_9 \le r_x \le c_9$	ft
$r_y = r_{yF}^{(5)}$	$-c_9 \le r_y \le c_9$	$-c_9 \le r_y \le c_9$	ft
$r_z = r_{zF}^{(5)}$	$-c_9 \le r_z \le c_9$	$-c_9 \le r_z \le c_9$	ft
$v_x = v_{xF}^{(5)}$	$-c_5 \le v_x \le c_5$	$-c_5 \le v_x \le c_5$	ft/sec
$v_y = v_{yF}^{(5)}$	$-c_5 \le v_y \le c_5$	$-c_5 \le v_y \le c_5$	ft/sec
$v_z = v_{zF}^{(5)}$	$-c_5 \le v_z \le c_5$	$-c_5 \le v_z \le c_5$	ft/sec
$w = w_F^{(4)}$	$1 \times 10^{-4} \le w \le 1$	$1 \times 10^{-4} \le w \le 1$	lb

Algebraic Variables: (θ, ψ) ....

$-180^o \leq \theta \leq +180^o$	$-180^o \leq \theta \leq +180^o$	$-180^o \leq \theta \leq +180^o$	rad
$-89^o \leq \psi \leq 89^o$	$-89^o \leq \psi \leq 89^o$	$-89^o \leq \psi \leq 89^o$	rad

Boundary Conditions ....

$0 \leq \phi_F^{(6)} - \phi_I^{(6)} \leq 10^o$

$$
\begin{aligned}
0 &= \Psi_1(\mathbf{r}_F, \mathbf{v}_F) \\
0 &= \Psi_2(\mathbf{r}_F, \mathbf{v}_F) \\
0 &= \Psi_3(\mathbf{r}_F, \mathbf{v}_F) \\
-1 \leq \Psi_4(\mathbf{r}_F, \mathbf{v}_F) &\leq 0 \\
0 &= \Psi_5(\mathbf{r}_F, \mathbf{v}_F)
\end{aligned}
$$

The boundary conditions are computed using $\mathbf{r} = \mathbf{r}_F$ and $\mathbf{v} = \mathbf{v}_F$ with $a_F = a_1$ and $e_F = e_1$ using the following sequence of expressions:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \tag{51.14}$$

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{\|\mathbf{r}\|} \tag{51.15}$$

$$a = \left[\frac{2}{\|\mathbf{r}\|} - \left(\frac{\mathbf{v}^\mathsf{T}\mathbf{v}}{\mu}\right)\right]^{-1} \tag{51.16}$$

$$\cos i = \frac{\mathbf{h}_3}{\|\mathbf{h}\|} \tag{51.17}$$

$$\mathbf{k}^\mathsf{T} = (0, 0, 1) \tag{51.18}$$

$$\mathbf{n} = \mathbf{k} \times \mathbf{h} \tag{51.19}$$

$$\cos\omega = \frac{\mathbf{n}^\mathsf{T}\mathbf{e}}{\|\mathbf{n}\|\|\mathbf{e}\|} \tag{51.20}$$

$$\Psi_1 = a_F - a \tag{51.21}$$

$$\Psi_2 = e_F - \|\mathbf{e}\| \tag{51.22}$$

$$\Psi_3 = \cos\omega_F - \cos\omega \tag{51.23}$$

$$\Psi_4 = \mathbf{e}_3 \tag{51.24}$$

$$\Psi_5 = \cos i_F - \cos i \tag{51.25}$$

Differential-Algebraic Equations ....

Equations (51.6) - (51.9)

Objective ....

Maximize $\qquad J = w(t_F^{(6)})$

$J^* = .411558794; \quad \phi_F^* = 506.39484^o$

Example 51.2 orbt02: Three Burn Transfer.
Repeat example 51.1 and replace the problem constants $(c_1, c_2, c_3, c_4, a_1, e_1)$ with the values $(c_{11}, c_{12}, c_{13}, c_{14}, a_2, e_2)$ given in Table 51.1.

$J^* = .356868150; \quad \phi_F^* = 500.22783^o$

Example 51.3 orbt03: Variable Thrust Transfer.

Phase 1 .... *Park Orbit Coast* .... Phase 1

Parameters: $(\phi_F^{(1)})$ ....

$1^o \le \phi_F^{(1)} \le 3\pi$

Independent Variable: (ϕ) ....

$\phi = 0$	$0 \le \phi \le \phi_F^{(1)}$	$\phi = \phi_F^{(1)}$	rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = c_{11}$	$-c_{10} \le r_x \le c_{10}$	$-c_{10} \le r_x \le c_{10}$	ft
$r_y = 0$	$-c_{10} \le r_y \le c_{10}$	$-c_{10} \le r_y \le c_{10}$	ft
$r_z = 0$	$-c_{10} \le r_z \le c_{10}$	$-c_{10} \le r_z \le c_{10}$	ft
$v_x = 0$	$-c_5 \le v_x \le c_5$	$-c_5 \le v_x \le c_5$	ft/sec
$v_y = c_{12}$	$-c_5 \le v_y \le c_5$	$-c_5 \le v_y \le c_5$	ft/sec
$v_z = c_{13}$	$-c_5 \le v_z \le c_5$	$-c_5 \le v_z \le c_5$	ft/sec

Differential-Algebraic Equations ....

Equations (51.1) - (51.5)

Phase 2 .... *Variable Magnitude Burn* .... Phase 2

Parameters: $(\phi_I^{(2)}, \phi_F^{(2)}, T)$ ....

$0 \le \phi_I^{(2)} \le 4\pi$	$1^o \le \phi_F^{(2)} \le 4\pi$	$0 \le T \le 2$

Independent Variable: (ϕ) ....

$\phi = \phi_F^{(1)} = \phi_I^{(2)}$	$\phi_I^{(2)} \le \phi \le \phi_F^{(2)}$	$\phi = \phi_F^{(2)}$	rad

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(1)}$	$-c_{10} \le r_x \le c_{10}$	$-c_{10} \le r_x \le c_{10}$	ft
$r_y = r_{yF}^{(1)}$	$-c_{10} \le r_y \le c_{10}$	$-c_{10} \le r_y \le c_{10}$	ft
$r_z = r_{zF}^{(1)}$	$-c_{10} \le r_z \le c_{10}$	$-c_{10} \le r_z \le c_{10}$	ft
$v_x = v_{xF}^{(1)}$	$-c_5 \le v_x \le c_5$	$-c_5 \le v_x \le c_5$	ft/sec
$v_y = v_{yF}^{(1)}$	$-c_5 \le v_y \le c_5$	$-c_5 \le v_y \le c_5$	ft/sec
$v_z = v_{zF}^{(1)}$	$-c_5 \le v_z \le c_5$	$-c_5 \le v_z \le c_5$	ft/sec
$w = 1$	$1 \times 10^{-4} \le w \le 1$	$1 \times 10^{-4} \le w \le 1$	lb

Algebraic Variables: (T_x, T_y, T_z) ....

$-2 \le T_x \le 2$	$-2 \le T_x \le 2$	$-2 \le T_x \le 2$	ft/sec^2
$-2 \le T_y \le 2$	$-2 \le T_y \le 2$	$-2 \le T_y \le 2$	ft/sec^2
$-2 \le T_z \le 2$	$-2 \le T_z \le 2$	$-2 \le T_z \le 2$	ft/sec^2

Boundary Conditions ....

$$
\begin{aligned}
0 &= \Psi_1(\mathbf{r}_F, \mathbf{v}_F) \\
0 &= \Psi_2(\mathbf{r}_F, \mathbf{v}_F) \\
0 &= \Psi_3(\mathbf{r}_F, \mathbf{v}_F) \\
-1 \le \Psi_4(\mathbf{r}_F, \mathbf{v}_F) &\le 0 \\
0 &= \Psi_5(\mathbf{r}_F, \mathbf{v}_F)
\end{aligned}
$$

The boundary conditions are computed using (51.14)-(51.25) with $a_F = a_2$ and $e_F = e_2$.

Differential-Algebraic Equations ....

$$\mathbf{r}' = \left(\frac{dt}{d\phi}\right)\dot{\mathbf{r}} = \left(\frac{dt}{d\phi}\right)\mathbf{v} \tag{51.26}$$

$$\mathbf{v}' = \left(\frac{dt}{d\phi}\right)\dot{\mathbf{v}} = \left(\frac{dt}{d\phi}\right)\left[\mathbf{g}(\mathbf{r}) + \frac{g_0}{w}\mathbf{T}\right] \tag{51.27}$$

$$w' = -\left(\frac{dt}{d\phi}\right)\frac{T}{I_{sp}} \tag{51.28}$$

$$100 \text{ nm} \le h \le 50000 \text{ nm} \tag{51.29}$$

$$0 = T - \|\mathbf{T}\| = T - \sqrt{T_x^2 + T_y^2 + T_z^2}. \tag{51.30}$$

where h is given by (51.13).

Objective ....

Maximize $J = w(t_F^{(2)})$

$$\boxed{J^* = .20850003; \quad \phi_F^* = 613.54471^o; \quad T^* = .022890463}$$

$a_1 = 138312691.$	$a_2 = 87155321.522650868.$
$e_1 = .67$	$e_2 = .73550320568829042$
$i_F = 63.4^o$	$\omega_F = 270^o$
$c_1 = -21715557.743123360$	$c_{11} = -21837080.052834645$
$c_2 = -19215.029798030402$	$c_{12} = -22312.483663879691$
$c_3 = 16703.370570171435$	$c_{13} = 12114.690178392992$
$c_4 = 43431115.486246720$	$c_{14} = 43674160.105669290$
$c_5 = 36679.387990635936$	$c_6 = 591957486.55575049$
$c_7 = 236782994.62230018$	$c_8 = 43422593.607642516$
$c_9 = 750706894.87775517$	$c_{10} = 104628313.65$
$I_{sp} = 450$	$R_e = 20925662.73$

Table 51.1. *Elliptic Orbit example parameters.*

Chapter 52

pdly: **Delay Partial Differential Equation**

Reference [21, Sect. 10.6.1] presents an optimal control problem, in which the dynamic model is given by a partial differential equation with a time delay. First, by introducing a spatial discretization the method of lines is used to approximate the PDE by a system of ordinary differential equations with a delay. Although spatial dependent delays are considered in the reference, for the case given here the delay is constant, with no spatial dependence. Using the method of steps, the delay ODE system is recast as a larger system of ODEs with no delay. Using sixteen spatial discretization lines, and ten delay steps, the final problem has 160 state variables, 10 control variables, and 153 boundary conditions.

Example 52.1 pdly01: Delay Partial Differential Equation.

Phase 1 .... *DPDE: Method of Lines and Method of Steps* .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\qquad$ $0 < t < 0.5$ $\qquad$ $t = r = 0.5$

Differential Variables: $(S_{k,j} : \quad k = 0, \ldots, n; \quad j = 1, \ldots, N)$ ....

$S_{k,1}(0) = \alpha_k \qquad k = 0, \ldots, n$

where $n = 15$ and for $T = 5$, $N = T/r = 10$.

$$x_k = k\delta = k\frac{\pi}{n} \qquad k = 0, \ldots, n \tag{52.1}$$

$$\alpha_k = \alpha(x_k) = 1 + \sin(2x_k - \frac{\pi}{2}) \qquad k = 0, \ldots, n. \tag{52.2}$$

Algebraic Variables: $(u_j : \quad j = 1, \ldots, N)$ ....

$0 \le u_j$ $\qquad$ $0 \le u_j$ $\qquad$ $0 \le u_j$

Boundary Conditions . . . .

$S_{k,j}(0) = S_{k,j-1}(r)$ $\qquad$ $k = 0, \dots, n$ $\qquad$ $j = 2, \dots, N$
$u_j(0) = u_{j-1}(r)$ $\qquad$ $j = 2, \dots, N$

Differential-Algebraic Equations . . . .

For $j = 1, \dots, N$

$$\dot{S}_{0,j} = \frac{2c_1}{\delta^2}\left(S_{1,j} - S_{0,j}\right) - c_2 S_{0,j-1}\left[1 + S_{0,j}\right] + u_j \tag{52.3}$$

$$\begin{aligned}\dot{S}_{k,j} = {} & \frac{c_1}{\delta^2}\left(S_{k+1,j} - 2S_{k,j} + S_{k-1,j}\right) \\ & - c_2 S_{k,j-1}\left[1 + S_{k,j}\right] + u_j \qquad k = 1, \dots, n-1\end{aligned} \tag{52.4}$$

$$\dot{S}_{n,j} = \frac{2c_1}{\delta^2}\left(S_{n-1,j} - S_{n,j}\right) - c_2 S_{n,j-1}\left[1 + S_{n,j}\right] + u_j \tag{52.5}$$

where $c_1 = 1$, $c_2 = .5$, and when $0 \le t \le r$

$$S_{k,0}(t) = \alpha_k \qquad k = 0, \dots, n \tag{52.6}$$

Objective . . . .

Minimize

$$J = \sum_{j=1}^{N} \int_0^r c_3 u_j^2(t)dt + \frac{1}{2}\delta \cdot f_0 + \delta \sum_{k=1}^{n-1} f_k + \frac{1}{2}\delta \cdot f_n, \tag{52.7}$$

with $c_3 = 0.1$ and $h(x) = 5$

$$f_k = \left[S_{k,N}(r) - h(x_k)\right]^2. \tag{52.8}$$

$\boxed{J^* = 3.80079537}$

Chapter 53

plnt: **Earth to Mars with Venus Swingby**

This example describes the design of an interplanetary trajectory between Earth and Mars, with a *swingby* of the planet Venus. The problem described in reference [9], is implemented using six distinct phases. All phases incorporate cubic spline approximations to the gravitational attraction of the planetary ephemerides given in reference [86]. The sun is treated as the primary body of attraction during phases one, two, five, and six. During phase three and four, Venus is considered the primary body. Nonlinear boundary conditions are introduced to ensure continuity at the interface between Venus centered and Sun centered gravitational fields. The goal is to minimize fuel consumption during the mission, by optimally steering the burns during phase one and six.

Example 53.1 plnt01: EARTH TO MARS WITH VENUS SWINGBY.

Phase 1 . . . . . . *First Heliocentric Burn* . . . . . . Phase 1

Parameters: $(t_F^{(1)})$ . . . .

$0 \le t_F^{(1)} \le 1095$

Independent Variable: (t) . . . .

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$	days

Differential Variables: (p, f, g, h, k, L, m) . . . .

$p = p_1$	$\underline{p}_1 \le p \le \overline{p}_1$	$\underline{p}_1 \le p \le \overline{p}_1$	km
$f = f_1$	$-10 \le f \le 10$	$-10 \le f \le 10$	
$g = g_1$	$-10 \le g \le 10$	$-10 \le g \le 10$	
$h = h_1$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_1$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_1$	$\underline{L}_1 \le L \le \overline{L}_1$	$\underline{L}_1 \le L \le \overline{L}_1$	rad

$m = m_1$	$10^{-5}m_1 \le m \le 1.1m_1$	$10^{-5}m_1 \le m \le 1.1m_1$	kg

Algebraic Variables: (u_r, u_θ, u_h) ....

$-2 \le u_r \le 2$	$-2 \le u_r \le 2$	$-2 \le u_r \le 2$
$-2 \le u_\theta \le 2$	$-2 \le u_\theta \le 2$	$-2 \le u_\theta \le 2$
$-2 \le u_h \le 2$	$-2 \le u_h \le 2$	$-2 \le u_h \le 2$

Differential-Algebraic Equations ....

$$\dot{\mathbf{y}} = \mathbf{A}(\mathbf{y})\boldsymbol{\Delta} + \mathbf{b} \tag{53.1}$$

$$\dot{m} = \frac{T}{g_0 I_{sp}} \tag{53.2}$$

$$0 = \|\mathbf{u}\| - 1 \tag{53.3}$$

$$R_m \le r \tag{53.4}$$

where the problem constants are given in Table 53.1. Denoting $\mu \doteq \mu_{\sigma_1}$ define the following:

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{2p}{q}\sqrt{\frac{p}{\mu}} & 0 \\ \sqrt{\frac{p}{\mu}}\sin L & \sqrt{\frac{p}{\mu}}\frac{1}{q}\{(q+1)\cos L + f\} & -\sqrt{\frac{p}{\mu}}\frac{g}{q}\{h\sin L - k\cos L\} \\ -\sqrt{\frac{p}{\mu}}\cos L & \sqrt{\frac{p}{\mu}}\frac{1}{q}\{(q+1)\sin L + g\} & \sqrt{\frac{p}{\mu}}\frac{f}{q}\{h\sin L - k\cos L\} \\ 0 & 0 & \sqrt{\frac{p}{\mu}}\frac{s^2\cos L}{2q} \\ 0 & 0 & \sqrt{\frac{p}{\mu}}\frac{s^2\sin L}{2q} \\ 0 & 0 & \sqrt{\frac{p}{\mu}}\frac{1}{q}\{h\sin L - k\cos L\} \end{bmatrix} \tag{53.5}$$

$$\mathbf{b}^\mathsf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{\mu p}\left(\frac{q}{p}\right)^2 \end{bmatrix} \tag{53.6}$$

$$q = 1 + f\cos L + g\sin L \tag{53.7}$$

$$r = \frac{p}{q}, \tag{53.8}$$

$$\alpha^2 = h^2 - k^2 \tag{53.9}$$

$$\chi = \sqrt{h^2 + k^2} \tag{53.10}$$

$$s^2 = 1 + \chi^2 \tag{53.11}$$

$$\mathbf{r} = \begin{bmatrix} \frac{r}{s^2}\left(\cos L + \alpha^2\cos L + 2hk\sin L\right) \\ \frac{r}{s^2}\left(\sin L - \alpha^2\sin L + 2hk\cos L\right) \\ \frac{2r}{s^2}\left(h\sin L - k\cos L\right) \end{bmatrix} \tag{53.12}$$

$$\mathbf{v} = \begin{bmatrix} -\frac{1}{s^2}\sqrt{\frac{\mu}{p}}\left(\sin L + \alpha^2\sin L - 2hk\cos L + g - 2fhk + \alpha^2 g\right) \\ -\frac{1}{s^2}\sqrt{\frac{\mu}{p}}\left(-\cos L + \alpha^2\cos L + 2hk\sin L - f + 2ghk + \alpha^2 f\right) \\ \frac{2}{s^2}\sqrt{\frac{\mu}{p}}\left(h\cos L + k\sin L + fh + gk\right) \end{bmatrix} \tag{53.13}$$

$$v = \|\mathbf{v}\| \tag{53.14}$$

$$\mathbf{Q}_r = \begin{bmatrix}\mathbf{i}_r & \mathbf{i}_\theta & \mathbf{i}_h\end{bmatrix} = \begin{bmatrix}\frac{\mathbf{r}}{\|\mathbf{r}\|} & \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{\|\mathbf{r}\times\mathbf{v}\|\|\mathbf{r}\|} & \frac{\mathbf{r}\times\mathbf{v}}{\|\mathbf{r}\times\mathbf{v}\|}\end{bmatrix} \tag{53.15}$$

and the following quantities are computed

$$k = \sigma_{\jmath+1} \tag{53.16}$$
$$\mathbf{s}_k = \overline{\mathbf{r}}_k(t) - \overline{\mathbf{r}}_{\sigma_1}(t) \tag{53.17}$$
$$\mathbf{d}_k = \mathbf{r} - \mathbf{s}_k \tag{53.18}$$
$$d_k = \|\mathbf{d}_k\| \tag{53.19}$$
$$q_k = \frac{\mathbf{r}^\mathsf{T}(\mathbf{r} - 2\mathbf{s}_k)}{\mathbf{s}_k^\mathsf{T}\mathbf{s}_k} \tag{53.20}$$
$$F(q_k) = q_k\left[\frac{3 + 3q_k + q_k^2}{1 + (\sqrt{1 + q_k})^3}\right] \tag{53.21}$$

for $\jmath = 1, \ldots, 5$ followed by

$$\mathbf{a}_d = -\sum_k \frac{\mu_k}{d_k^3}\left[\mathbf{r} + F(q_k)\mathbf{s}_k\right] \tag{53.22}$$
$$\boldsymbol{\Delta}_g = \mathbf{Q}_r^\mathsf{T}\mathbf{a}_d \tag{53.23}$$

and with $\mathbf{u}^\mathsf{T} = (u_r, u_\theta, u_h)$

$$\boldsymbol{\Delta}_T = \frac{T}{m}\mathbf{u} \tag{53.24}$$
$$\boldsymbol{\Delta} = \boldsymbol{\Delta}_g + \boldsymbol{\Delta}_T \tag{53.25}$$

Phase 2 .... *First Heliocentric Coast* .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....

$0 \le t_I^{(2)} \le 1095$
$0 \le t_F^{(2)} \le 1095$

Independent Variable: (t) ....

$t = t_I^{(2)} = t_F^{(1)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$	days

Differential Variables: (p, f, g, h, k, L) ....

$p = p_F^{(1)}$	$\underline{p}_2 \le p \le \overline{p}_2$	$\underline{p}_2 \le p \le \overline{p}_2$	km
$f = f_F^{(1)}$	$-10 \le f \le 10$	$-10 \le f \le 10$	

$g = g_F^{(1)}$	$-10 \le g \le 10$	$-10 \le g \le 10$	
$h = h_F^{(1)}$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_F^{(1)}$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_F^{(1)}$	$\underline{L}_2 \le L \le \overline{L}_2$	$\underline{L}_2 \le L \le \overline{L}_2$	rad

Boundary Conditions ....

$t_F^{(2)} - t_I^{(2)} \ge 10$ min

Differential-Algebraic Equations ....

Equation (53.1) and (53.5) - (53.23), with $\mathbf{\Delta} = \mathbf{\Delta}_g$.

Phase 3 .... *Venus Arrival Coast* .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....

174 days $\le t_I^{(3)} \le$ 379 days
174 days $\le t_F^{(3)} \le$ 379 days

Independent Variable: (t) ....

$t = t_I^{(3)} = t_F^{(2)}$ days	$t_I^{(3)} < t < t_F^{(3)}$	$t = t_F^{(3)}$	sec

Differential Variables: (p, f, g, h, k, L) ....

$\underline{p}_3 \le p \le \overline{p}_3$	$\underline{p}_3 \le p \le \overline{p}_3$	$\underline{p}_3 \le p \le \overline{p}_3$	km
$-10 \le f \le 10$	$-10 \le f \le 10$	$-10 \le f \le 10$	
$-10 \le g \le 10$	$-10 \le g \le 10$	$-10 \le g \le 10$	
$-1 \le h \le 1$	$-1 \le h \le 1$	$h = 0$	
$-1 \le k \le 1$	$-1 \le k \le 1$	$k = 0$	
$\underline{L}_3 \le L \le \overline{L}_3$	$\underline{L}_3 \le L \le \overline{L}_3$	$\underline{L}_3 \le L \le \overline{L}_3$	rad

Boundary Conditions ....

$\mathbf{r}_F^{(2)} - [\overline{\mathbf{r}}_{\sigma_1}(t_F^{(2)}) - \mathbf{r}_0] = \mathbf{r}$	
$\mathbf{v}_F^{(2)} - \overline{\mathbf{v}}_{\sigma_1}(t_F^{(2)}) = \mathbf{v}$	
$r = \rho_{\circledast}$	
$\mathbf{r}^{\mathsf{T}}\mathbf{v}/(rv) \le 0$	
	$r/r_{\circledast} = 2$
	$\mathbf{r}^{\mathsf{T}}\mathbf{v}/(rv) = 0$
$t_F^{(3)} - t_I^{(3)} \ge 600$	

Differential-Algebraic Equations ....

Equation (53.1) and (53.5) - (53.23), with the following changes:

- replace (53.16) with $k = \varrho_{\jmath+1}$;
- replace (53.17) with $\mathbf{s}_k = \overline{\mathbf{r}}_k(t) - \overline{\mathbf{r}}_{\varrho_1}(t)$;
- define $\mu \doteq \mu_{\varrho_1}$, and;
- $\boldsymbol{\Delta} = \boldsymbol{\Delta}_g$.

Phase 4 .... *Venus Departure Coast* .... Phase 4

Parameters: $(t_I^{(4)}, t_F^{(4)})$ ....

174 days $\le t_I^{(4)} \le$ 379 days
174 days $\le t_F^{(4)} \le$ 379 days

Independent Variable: (t) ....

$t = t_I^{(4)} = t_F^{(3)}$	$t_I^{(4)} < t < t_F^{(4)}$	$t = t_F^{(4)}$	sec

Differential Variables: (p, f, g, h, k, L) ....

$p = p_F^{(3)}$	$\underline{p}_4 \le p \le \overline{p}_4$	$\underline{p}_4 \le p \le \overline{p}_4$	km
$f = f_F^{(3)}$	$-10 \le f \le 10$	$-10 \le f \le 10$	
$g = g_F^{(3)}$	$-10 \le g \le 10$	$-10 \le g \le 10$	
$h = h_F^{(3)}$	$-1 \le h \le 1$	$h = 0$	
$k = k_F^{(3)}$	$-1 \le k \le 1$	$k = 0$	
$L = L_F^{(3)}$	$\underline{L}_4 \le L \le \overline{L}_4$	$\underline{L}_4 \le L \le \overline{L}_4$	rad

Boundary Conditions ....

		$r = \rho_{\circledast}$
		$\mathbf{r}^{\mathsf{T}}\mathbf{v}/(rv) \ge 0$
$t_F^{(4)} - t_I^{(4)} \ge 600$		

Differential-Algebraic Equations ....

Equation (53.1) and (53.5) - (53.23), with the following changes:

- replace (53.16) with $k = \varrho_{\jmath+1}$;
- replace (53.17) with $\mathbf{s}_k = \overline{\mathbf{r}}_k(t) - \overline{\mathbf{r}}_{\varrho_1}(t)$;
- define $\mu \doteq \mu_{\varrho_1}$, and;

- $\boldsymbol{\Delta} = \boldsymbol{\Delta}_g$.

Phase 5 .... *Second Heliocentric Coast* .... Phase 5

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....

$0 \le t_I^{(5)} \le 1095$
$0 \le t_F^{(5)} \le 1095$

Independent Variable: (t) ....

$t = t_I^{(5)} = t_F^{(4)}$ sec	$t_I^{(5)} < t < t_F^{(5)}$	$t = t_F^{(5)}$	days

Differential Variables: (p, f, g, h, k, L) ....

$\underline{p}_5 \le p \le \overline{p}_5$	$\underline{p}_5 \le p \le \overline{p}_5$	$\underline{p}_5 \le p \le \overline{p}_5$	km
$-10 \le f \le 10$	$-10 \le f \le 10$	$-10 \le f \le 10$	
$-10 \le g \le 10$	$-10 \le g \le 10$	$-10 \le g \le 10$	
$-1 \le h \le 1$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$-1 \le k \le 1$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$\underline{L}_5 \le L \le \overline{L}_5$	$\underline{L}_5 \le L \le \overline{L}_5$	$\underline{L}_5 \le L \le \overline{L}_5$	rad

Boundary Conditions ....

$\mathbf{r}_I^{(5)} - [\overline{\mathbf{r}}_{\sigma_1}(t_I^{(5)}) - \mathbf{r}_0] = \mathbf{r}_F^{(4)}$
$\mathbf{v}_I^{(5)} - \overline{\mathbf{v}}_{\sigma_1}(t_I^{(5)}) = \mathbf{v}_F^{(4)}$
$t_F^{(5)} - t_I^{(5)} \ge 10$ min

Differential-Algebraic Equations ....

Equation (53.1) and (53.5) - (53.23), with $\boldsymbol{\Delta} = \boldsymbol{\Delta}_g$.

Phase 6 .... *Second Heliocentric Burn* .... Phase 6

Parameters: $(t_I^{(6)})$ ....

$0 \le t_I^{(6)} \le 1095$

Independent Variable: (t) ....

$t = t_I^{(6)} = t_F^{(5)}$	$t_I^{(6)} < t < 675$	$t = t_6 = 675$	days

Differential Variables: (p, f, g, h, k, L, m) ....

$p = p_F^{(5)}$	$\underline{p}_6 \leq p \leq \overline{p}_6$	$p = p_6$	km
$f = f_F^{(5)}$	$-10 \leq f \leq 10$	$f = f_6$	
$g = g_F^{(5)}$	$-10 \leq g \leq 10$	$g = g_6$	
$h = h_F^{(5)}$	$-1 \leq h \leq 1$	$h = h_6$	
$k = k_F^{(5)}$	$-1 \leq k \leq 1$	$k = k_6$	
$L = L_F^{(5)}$	$\underline{L}_6 \leq L \leq \overline{L}_6$	$L = L_6$	rad
$m = m_F^{(1)}$	$10^{-5} m_1 \leq m \leq 1.1 m_1$	$10^{-5} m_1 \leq m \leq 1.1 m_1$	kg

Algebraic Variables: (u_r, u_θ, u_h) ....

$-2 \leq u_r \leq 2$	$-2 \leq u_r \leq 2$	$-2 \leq u_r \leq 2$
$-2 \leq u_\theta \leq 2$	$-2 \leq u_\theta \leq 2$	$-2 \leq u_\theta \leq 2$
$-2 \leq u_h \leq 2$	$-2 \leq u_h \leq 2$	$-2 \leq u_h \leq 2$

Differential-Algebraic Equations ....

Equations (53.1) - (53.25)

Objective ....

Maximize $\quad J = m(t_6)$

$$\boxed{J^* = 2.97400307 \times 10^5}$$

$\mu_0 = 1.327124 \times 10^{11}$ km^3/sec^2	$\mu_1 = 22034$ km^3/sec^2
$\mu_2 = 324888$ km^3/sec^2	$\mu_3 = 398634$ km^3/sec^2
$\mu_4 = 42832$ km^3/sec^2	$\mu_5 = 1.2670 \times 10^8$ km^3/sec^2
$T = .306$ kg-km/sec^2	$I_{sp} = 10000$ sec
$p_1 = 149556812.03600001$ km	$f_1 = -4.03253858617000013 \times 10^{-3}$
$g_1 = 1.62135319770000015 \times 10^{-2}$	$h_1 = -6.93223616339000019 \times 10^{-5}$
$k_1 = -7.49214107310999997 \times 10^{-6}$	$L_1 = 70.346635323223751^o$
$p_6 = 193497106.77643296$ km	$f_6 = -4.92530906533987373 \times 10^{-2}$
$g_6 = 0.22127102921358094$	$h_6 = -2.54326301299256366 \times 10^{-3}$
$k_6 = 1.60487978920904849 \times 10^{-2}$	$L_6 = 1006.7133109199491^o$
$\underline{p}_1 = 3.740 \times 10^7$ km	$\overline{p}_1 = 2.990 \times 10^8$ km
$\underline{p}_2 = 3.740 \times 10^7$ km	$\overline{p}_2 = 1.950 \times 10^8$ km
$\underline{p}_3 = 3.030 \times 10^3$ km	$\overline{p}_3 = 5.290 \times 10^4$ km
$\underline{p}_4 = 3.030 \times 10^3$ km	$\overline{p}_4 = 5.080 \times 10^4$ km
$\underline{p}_5 = 3.740 \times 10^7$ km	$\overline{p}_5 = 2.240 \times 10^8$ km
$\underline{p}_6 = 3.740 \times 10^7$ km	$\overline{p}_6 = 3.870 \times 10^8$ km
$\underline{L}_1 = 35.18^o$	$\overline{L}_1 = 534.6^o$
$\underline{L}_2 = 133.5^o$	$\overline{L}_2 = 1020^o$
$\underline{L}_3 = 111.7^o$	$\overline{L}_3 = 744.8^o$
$\underline{L}_4 = 186.8^o$	$\overline{L}_4 = 1049^o$
$\underline{L}_5 = 258.4^o$	$\overline{L}_5 = 1335^o$
$\underline{L}_6 = 334^o$	$\overline{L}_6 = 2011^o$
$m_1 = 400000$ kg	$R_m = .5$ au
$\rho_{\circledast} = 536540.11739530240$ km	$r_{\circledast} = 6052$ km
$\boldsymbol{\sigma}^{\mathsf{T}} = (0, 1, 2, 3, 4, 5)$	$\boldsymbol{\varrho}^{\mathsf{T}} = (2, 0, 1, 3, 4, 5)$
The functions $\overline{\mathbf{r}}_\jmath(t), \overline{\mathbf{v}}_\jmath(t)$ for $\jmath = 0, \ldots, 5$ are represented as spline approximations to the ephemerides in [86] for a period of 675 days beginning on 12/10/2010, (Julian date = 2455532.0)	

Table 53.1. *Interplanetary example constants.*

Chapter 54

pnav: **Proportional Navigation**

Bryson and Ho [29, pp 154-155] describe a popular guidance scheme referred to as *proportional navigation*. Example (54.1) poses the open loop control problem, and in example (54.2) the optimal coefficients of the closed loop control law are computed. In addition an integral boundary condition is used to fix the final time.

Example 54.1 pnav01: FEEDBACK CONTROL–(OPEN LOOP).

Phase 1 .... Phase 1

Parameters: (t_F) ....
Independent Variable: (t) ....

$t = 0$ $\qquad$ $0 < t < t_F$ $\qquad$ $t = t_F$

Differential Variables: (v, y) ....

$v = 1$
$y = 1$

Algebraic Variables: (a) ....
Boundary Conditions ....

$$\int_0^{t_F} dt = 1$$

Differential-Algebraic Equations ....

$$\dot{v} = a \tag{54.1}$$

$$\dot{y} = v \tag{54.2}$$

Objective ....

Minimize $\qquad J = \frac{1}{2} \begin{bmatrix} v & y \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix} \Bigg|_{t=t_F} + \frac{1}{2} \int_0^{t_F} a^2 \, dt$

where $c_1 = 1$ and $c_2 = 2$.

$$\boxed{J^* = 2.41176471}$$

Example 54.2 pnav02: FEEDBACK CONTROL–(CLOSED LOOP).

Phase 1 .... Phase 1

Parameters: (t_F) ....
Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: (v, y) ....

$v = 1$
$y = 1$

Algebraic Variables: (Λ_v, Λ_y) ....
Boundary Conditions ....

$$\int_0^{t_F} dt = 1$$

Differential-Algebraic Equations ....

$$\dot{v} = a \tag{54.3}$$
$$\dot{y} = v \tag{54.4}$$

where

$$a = -\Lambda_v v - \Lambda_y y \tag{54.5}$$

Objective ....

Minimize $$J = \frac{1}{2} \begin{bmatrix} v & y \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix} \Bigg|_{t=t_F} + \frac{1}{2} \int_0^{t_F} a^2 \, dt$$

where $c_1 = 1$ and $c_2 = 2$.

$$\boxed{J^* = 2.41176471}$$

Chapter 55

pndl: **Pendulum Problem**

Two versions of the mathematical pendulum problem are given. Example (55.1) formulates the problem as an index one differential-algebraic system, and in example (55.2) further index reduction yields and ODE problem statement.

Example 55.1 pndl01: Index 1 DAE Formulation.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 3$	$t = 3$

Differential Variables: (y_1, y_2, y_3, y_4) ....

$y_1 = 1$	$-5 \leq y_1 \leq 5$	$y_1 = 0$
$y_2 = 0$	$-5 \leq y_2 \leq 5$	$-5 \leq y_2 \leq 5$
$y_3 = 0$	$-5 \leq y_3 \leq 5$	$y_3 = 0$
$y_4 = 0$	$-5 \leq y_4 \leq 5$	$-5 \leq y_4 \leq 5$

Algebraic Variables: (y_5, u) ....

$-1 \leq y_5 \leq 15$	$-1 \leq y_5 \leq 15$	$-1 \leq y_5 \leq 15$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_3 \tag{55.1}$$
$$\dot{y}_2 = y_4 \tag{55.2}$$
$$\dot{y}_3 = -2y_5y_1 + uy_2 \tag{55.3}$$
$$\dot{y}_4 = -g - 2y_5y_2 - uy_1 \tag{55.4}$$
$$0 = y_3^2 + y_4^2 - 2y_5 - gy_2 \tag{55.5}$$

where $g = 9.81$.

Objective ....

Maximize
$$J = \int_0^3 u^2 \, dt$$

$$\boxed{J^* = 12.8738850}$$

Example 55.2 pndl02: ODE Formulation.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 3$	$t = 3$

Differential Variables: $(y_1, y_2, y_3, y_4, y_5)$ ....

$y_1 = 1$	$-5 \le y_1 \le 5$	$y_1 = 0$
$y_2 = 0$	$-5 \le y_2 \le 5$	$-5 \le y_2 \le 5$
$y_3 = 0$	$-5 \le y_3 \le 5$	$y_3 = 0$
$y_4 = 0$	$-5 \le y_4 \le 5$	$-5 \le y_4 \le 5$
$y_5 = 0$	$-1 \le y_5 \le 15$	$-1 \le y_5 \le 15$

Algebraic Variables: (u) ....

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_3 \tag{55.6}$$
$$\dot{y}_2 = y_4 \tag{55.7}$$
$$\dot{y}_3 = -2y_5y_1 + uy_2 \tag{55.8}$$
$$\dot{y}_4 = -g - 2y_5y_2 - uy_1 \tag{55.9}$$
$$\dot{y}_5 = y_3\dot{y}_3 + y_4\dot{y}_4 - g\dot{y}_2/2 \tag{55.10}$$

where $g = 9.81$.

Objective ....

Maximize
$$J = \int_0^3 u^2 \, dt$$

$$\boxed{J^* = 12.8738861}$$

Chapter 56

putt: **Golf Putting On Parabaloid Green**

To motivate the boundary value problem, Alessandrini [1] describes a problem as follows:

> Suppose that Arnold Palmer is on the 18th green at Pebble Beach. He needs to sink this putt to beat Jack Nicklaus and walk away with the $1,000,000 grand prize. What should he do? Solve a BVP! By modeling the surface of the green, Arnie sets up the equations of motion of his golf ball.

A more accurate formulation of the example as an optimal control problem is discussed in reference [13, Sect. 3.6].

Example 56.1 putt01: Minimum horizontal terminal velocity.

Phase 1 .... *Rolling On the Green* .... Phase 1

Parameters: $(t_F^{(1)})$ ....

$0 \le t_F^{(1)}$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(1)}$		sec

Differential Variables: (y_1, y_2, y_3, y_4) ....

$y_1 = 0$	$-25 \le y_1 \le 25$	$\underline{y}_1 \le y_1 \le \overline{y}_1$	ft
$y_2 = 0$	$-25 \le y_2 \le 25$	$\underline{y}_2 \le y_2 \le \overline{y}_2$	ft
$-100 \le y_3 \le 100$	$-100 \le y_3 \le 100$	$-100 \le y_3 \le 100$	ft/sec
$-100 \le y_4 \le 100$	$-100 \le y_4 \le 100$	$-100 \le y_4 \le 100$	ft/sec

Boundary Conditions ....

$$r_H = \|\mathbf{x} - \mathbf{x}_H\|$$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_3 \tag{56.1}$$
$$\dot{y}_2 = y_4 \tag{56.2}$$
$$\dot{y}_3 = g_0 n_1 n_3 - \mu_k g_0 n_3 \frac{y_3}{s} \tag{56.3}$$
$$\dot{y}_4 = g_0 n_2 n_3 - \mu_k g_0 n_3 \frac{y_4}{s} \tag{56.4}$$

where $\mathbf{x}^\mathsf{T} = (y_1, y_2)$, $\mathbf{x}_H^\mathsf{T} = (20, 0)$, $\mu_k = .2$ and

$$S = \frac{(y_1 - 10)^2}{125} + \frac{(y_2 - 5)^2}{125} - 1 + r_b \tag{56.5}$$
$$\dot{S} = \frac{2}{125}(y_1 - 10)y_3 + \frac{2}{125}(y_2 - 5)y_4 \tag{56.6}$$
$$s = \sqrt{y_3^2 + y_4^2 + \dot{S}^2} \tag{56.7}$$
$$\mathbf{N}^\mathsf{T} = \left[-\frac{\partial S}{\partial y_1}, -\frac{\partial S}{\partial y_2}, 1\right] = \left[-\frac{2}{125}(y_1 - 10), -\frac{2}{125}(y_2 - 5), 1\right] \tag{56.8}$$
$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} \tag{56.9}$$

Phase 2 .... *Dropping In the Hole* .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....
Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} < t < t_F^{(2)}$	sec

Differential Variables: $(y_1, y_2, y_3, y_4, y_5, y_6)$ ....

$y_1 = y_{1F}^{(1)}$	$\underline{y}_1 \le y_1 \le \overline{y}_1$	$\underline{y}_1 \le y_1 \le \overline{y}_1$	ft
$y_2 = y_{2F}^{(1)}$	$\underline{y}_2 \le y_2 \le \overline{y}_2$	$\underline{y}_2 \le y_2 \le \overline{y}_2$	ft
$\underline{y}_3 \le y_3 \le \overline{y}_3$	$\underline{y}_3 \le y_3 \le \overline{y}_3$	$\underline{y}_3 \le y_3 \le 0$	ft
$y_4 = y_{3F}^{(1)}$	$-100 \le y_4 \le 100$	$-100 \le y_4 \le 100$	ft/sec
$y_5 = y_{4F}^{(1)}$	$-100 \le y_5 \le 100$	$-100 \le y_5 \le 100$	ft/sec
$-100 \le y_6 \le 100$	$-100 \le y_6 \le 100$	$-100 \le y_6 \le 100$	ft/sec

Boundary Conditions ....

$S(\mathbf{y}) = y_3$
$\dot{S}(\mathbf{y}) = y_6$
$t_F^{(2)} - t_I^{(2)} \geq 10^{-5}$

$$\sqrt{(y_1 - 20)^2 + y_2^2} \leq r_H - r_b$$

where

$$S(\mathbf{y}) = \frac{(y_1 - 10)^2}{125} + \frac{(y_2 - 5)^2}{125} - 1 + r_b \tag{56.10}$$

$$\dot{S}(\mathbf{y}) = \frac{2}{125}(y_1 - 10)y_4 + \frac{2}{125}(y_2 - 5)y_5 \tag{56.11}$$

Differential-Algebraic Equations . . . .

$$\dot{y}_1 = y_4, \tag{56.12}$$
$$\dot{y}_2 = y_5, \tag{56.13}$$
$$\dot{y}_3 = y_6, \tag{56.14}$$
$$\dot{y}_4 = 0, \tag{56.15}$$
$$\dot{y}_5 = 0, \tag{56.16}$$
$$\dot{y}_6 = -g_0. \tag{56.17}$$

Objective . . . .

Minimize $$J = \left(y_4^2 + y_5^2\right)\Big|_{t=t_F^{(2)}}$$

$$\boxed{J^* = 1.8655284 \times 10^{-1}; \quad t_F^* = 2.9361307}$$

$\underline{y}_1 = x_{1H} - 2r_H$
$\overline{y}_1 = x_{1H} + 2r_H$
$\underline{y}_2 = x_{2H} - 2r_H$
$\overline{y}_2 = x_{2H} + 2r_H$
$\underline{y}_3 = -1/3$
$\overline{y}_3 = +2r_H$
$r_H = 4.25/2$ in $= 4.25/24$ ft
$r_b = 1.68/2$ in $= 1.68/24$ ft

Table 56.1. *Putting Example Constants*

Chapter 57

qlin: **Quadratic-Linear Control**

Control of linear systems with a quadratic criteria, serve as the basis for the important topic of linear feedback [29, Chap. 5]. Four different examples with linear dynamics and quadratic objective function are given here.

Example 57.1 qlin01: MINIMUM ENERGY–LAGRANGE FORMULATION.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$t = 1000$

Differential Variables: $(x_1, x_2, x_3, x_4, x_5, x_6)$ ....

$x_1 = 1000$	$x_1 = 0$
$x_2 = 1000$	$x_2 = 0$
$x_3 = 1000$	$x_3 = 0$
$x_4 = -10$	$x_4 = 0$
$x_5 = 10$	$x_5 = 0$
$x_6 = -10$	$x_6 = 0$

Algebraic Variables: (u_1, u_2, u_3) ....

$-1 \le u_1 \le 1$	$-1 \le u_1 \le 1$	$-1 \le u_1 \le 1$
$-1 \le u_2 \le 1$	$-1 \le u_2 \le 1$	$-1 \le u_2 \le 1$
$-1 \le u_3 \le 1$	$-1 \le u_3 \le 1$	$-1 \le u_3 \le 1$

Differential-Algebraic Equations ....

$$\dot{x}_1 = x_4 \tag{57.1}$$

$$\dot{x}_2 = x_5 \tag{57.2}$$

$$\dot{x}_3 = x_6 \tag{57.3}$$
$$\dot{x}_4 = u_1 \tag{57.4}$$
$$\dot{x}_5 = u_2 \tag{57.5}$$
$$\dot{x}_6 = u_3 \tag{57.6}$$

Objective ....

Minimize $$J = \frac{1}{2}\int_0^{1000} \left(u_1^2 + u_2^2 + u_3^2\right) dt$$

$$\boxed{J^* = 5.58000000 \times 10^{-1}}$$

Example 57.2 qlin02: Minimum Energy–Mayer Formulation.

Repeat example 57.1 with the additional differential variable x_7 with initial value $x_7 = 0$ and augment the differential-algebraic equations (57.1)-(57.6) to include

$$\dot{x}_7 = \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2\right) \tag{57.7}$$

and replace the objective function by

$$J = x_7(1000) \tag{57.8}$$

$$\boxed{J^* = 5.57999981 \times 10^{-1}}$$

Example 57.3 qlin03: Minimum Energy, Path Constraint.

Repeat example 57.1 and augment the differential-algebraic equations (57.1)-(57.6) to include the algebraic constraint

$$-10^4 \le .1x_1 + .2x_2 \le 10^4 \tag{57.9}$$

$$\boxed{J^* = 5.58000000 \times 10^{-1}}$$

Example 57.4 qlin04: Minimum Deviation Control.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$ $t = 1$

Algebraic Variables: (u_1) ....

$-2 \le u_1 \le 2$ $-2 \le u_1 \le 2$ $-2 \le u_1 \le 2$

Objective ....

Minimize $$J = \frac{1}{2}\int_0^1 (\sin 2\pi t - u_1)^2 dt$$

$$\boxed{J^* = 2.88323851 \times 10^{-39}}$$

Chapter 58

rayl: **Rayleigh Problem**

Maurer and Augustin [68] present a series of examples that are simple enough to permit analytic expressions for the adjoint equations. As such direct and indirect solutions are readily available for testing purposes. Five different examples are discussed in reference [13, Sect. 4.11] and repeated here.

Example 58.1 rayl01: Control Constraints-Direct Formulation.

References: [13, Sect. 4.11],

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 4.5$	$t = 4.5$

Differential Variables: (y_1, y_2) ....

$y_1 = -5$	$y_1 = 0$
$y_2 = -5$	$y_2 = 0$

Algebraic Variables: (u) ....

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{58.1}$$

$$\dot{y}_2 = -y_1 + y_2(1.4 - py_2^2) + 4u \tag{58.2}$$

$$0 \geq u - 1 \tag{58.3}$$

$$0 \geq -u - 1 \tag{58.4}$$

where $p = 0.14$.

Objective ....

Maximize $$J = \int_0^{4.5} (u^2 + y_1^2) dt$$

$\boxed{J^* = 44.7209362}$

Example 58.2 rayl02: Control Constraints-Indirect Formulation.

Phase 1 .... *Boundary Arc 1* .... Phase 1

Parameters: $(t_F^{(1)})$ ....

$.01 \le t_F^{(1)}$

Independent Variable: (t) ....

$t = 0$ $\qquad$ $0 < t < t_F^{(1)}$ $\qquad$ $t = t_F^{(1)}$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = -5$
$y_2 = -5$

$\lambda_2 = -1/2$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{58.5}$$
$$\dot{y}_2 = -y_1 + y_2(1.4 - py_2^2) + 4u \tag{58.6}$$
$$\dot{\lambda}_1 = \lambda_2 - 2y_1 \tag{58.7}$$
$$\dot{\lambda}_2 = 3p\lambda_2 y_2^2 - 1.4\lambda_2 - \lambda_1. \tag{58.8}$$

where $p = 0.14$ and $u = 1$.

Phase 2 .... *Unconstrained Arc 1* .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....
Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$ $\qquad$ $t_I^{(2)} < t < t_F^{(2)}$ $\qquad$ $t = t_F^{(2)}$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = y_{1F}^{(1)}$
$y_2 = y_{2F}^{(1)}$
$\lambda_1 = \lambda_{1F}^{(1)}$
$\lambda_2 = -1/2$ $\qquad$ $\lambda_2 = 1/2$

Boundary Conditions ....

$t_F^{(2)} - t_I^{(2)} \geq .01$

Differential-Algebraic Equations ....

Equations (58.5) - (58.8)

where $p = 0.14$ and $u = -2\lambda_2$.

Phase 3 .... *Boundary Arc 2* .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....
Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} < t < t_F^{(3)}$	$t = t_F^{(3)}$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = y_{1F}^{(2)}$		
$y_2 = y_{2F}^{(2)}$		
$\lambda_1 = \lambda_{1F}^{(2)}$		
$\lambda_2 = 1/2$		$\lambda_2 = 1/2$

Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} \geq .01$

Differential-Algebraic Equations ....

Equations (58.5) - (58.8)

where $p = 0.14$ and $u = -1$.

Phase 4 .... *Unconstrained Arc 2* .... Phase 4

Parameters: $(t_I^{(4)})$ ....

$t_I^{(4)} \leq 4.49$

Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} < t < 4.5$	$t = 4.5$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = y_{1F}^{(2)}$
$y_2 = y_{2F}^{(2)}$
$\lambda_1 = \lambda_{1F}^{(2)}$
$\lambda_2 = 1/2$ $\qquad\qquad$ $\lambda_2 = 1/2$

Differential-Algebraic Equations ....

Equations (58.5) - (58.8)

where $p = 0.14$ and $u = -2\lambda_2$.

Example 58.3 rayl03: CONTROL BOUNDS-DIRECT FORMULATION.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 4.5$	$t = 4.5$

Differential Variables: (y_1, y_2) ....

$y_1 = -5$	$y_1 = 0$
$y_2 = -5$	$y_2 = 0$

Algebraic Variables: (u) ....

$-1 \le u \le 1$	$-1 \le u \le 1$	$-1 \le u \le 1$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{58.9}$$

$$\dot{y}_2 = -y_1 + y_2(1.4 - p y_2^2) + 4u \tag{58.10}$$

where $p = 0.14$.

Objective ....

Maximize $\qquad J = \int_0^{4.5} (u^2 + y_1^2)dt$

$\boxed{J^* = 44.7209362}$

Example 58.4 rayl04: Mixed State-Control Constraints-Direct Formulation.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\quad$ $0 < t < 4.5$ $\quad$ $t = 4.5$

Differential Variables: (y_1, y_2) ....

$y_1 = -5$
$y_2 = -5$

Algebraic Variables: (u) ....
Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{58.11}$$

$$\dot{y}_2 = -y_1 + y_2(1.4 - py_2^2) + 4u \tag{58.12}$$

$$0 \geq u + \frac{y_1}{6} \tag{58.13}$$

where $p = 0.14$.
Objective ....

Maximize $\quad J = \int_0^{4.5} (u^2 + y_1^2) dt$

$\boxed{J^* = 44.8044433}$

Example 58.5 rayl05: Mixed State-Control Constraints-Indirect Formulation.

Phase 1 .... *Boundary Arc 1* .... Phase 1

Parameters: $(t_F^{(1)})$ ....

$.01 \leq t_F^{(1)}$

Independent Variable: (t) ....

$t = 0$ $\quad$ $0 < t < t_F^{(1)}$ $\quad$ $t = t_F^{(1)}$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = -5$
$y_2 = -5$

Differential-Algebraic Equations . . . .

$$\dot{y}_1 = y_2 \tag{58.14}$$
$$\dot{y}_2 = -y_1 + y_2(1.4 - py_2^2) + 4u \tag{58.15}$$
$$\dot{\lambda}_1 = \lambda_2 - 2y_1 - \frac{\mu}{6} \tag{58.16}$$
$$\dot{\lambda}_2 = 3p\lambda_2 y_2^2 - 1.4\lambda_2 - \lambda_1 \tag{58.17}$$

where $p = 0.14$, $u(t) = -y_1/6$ and $\mu(t) = -2u - 4\lambda_2 = y_1/3 - 4\lambda_2$.

Phase 2 . . . . *Unconstrained Arc 1* . . . . Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(1)} = t_I^{(2)}$ $\qquad t_I^{(2)} < t < t_F^{(2)}$ $\qquad t = t_F^{(2)}$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ . . . .

$y_1 = y_{1F}^{(1)}$
$y_2 = y_{2F}^{(1)}$
$\lambda_1 = \lambda_{1F}^{(1)}$
$\lambda_2 = \lambda_{2F}^{(1)}$

Boundary Conditions . . . .

$t_F^{(2)} - t_I^{(2)} \geq .01$
$\mu_I^{(2)} = y_{1I}^{(2)}/3 - 4\lambda_{2I}^{(2)} = 0$

Differential-Algebraic Equations . . . .

Equations (58.14) - (58.17)

where $p = 0.14$, $u = -2\lambda_2$ and $\mu(t) = 0$.

Phase 3 . . . . *Boundary Arc 2* . . . . Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(2)} = t_I^{(3)}$ $\qquad t_I^{(3)} < t < t_F^{(3)}$ $\qquad t = t_F^{(3)}$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = y_{1F}^{(2)}$
$y_2 = y_{2F}^{(2)}$
$\lambda_1 = \lambda_{1F}^{(2)}$
$\lambda_2 = \lambda_{2F}^{(2)}$

Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} \geq .01$
$\mu_I^{(3)} = y_{1I}^{(3)}/3 - 4\lambda_{2I}^{(3)} = 0$

Differential-Algebraic Equations ....

Equations (58.14) - (58.17)

where $p = 0.14$, $u(t) = -y_1/6$ and $\mu(t) = -2u - 4\lambda_2 = y_1/3 - 4\lambda_2$.

Phase 4 .... *Unconstrained Arc 2* .... Phase 4

Parameters: $(t_I^{(4)})$ ....

$t_I^{(4)} \leq 4.49$

Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} < t < 4.5$	$t = 4.5$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = y_{1F}^{(2)}$	
$y_2 = y_{2F}^{(2)}$	
$\lambda_1 = \lambda_{1F}^{(2)}$	$\lambda_1 = 0$
$\lambda_2 = \lambda_{2F}^{(2)}$	$\lambda_2 = 0$

Differential-Algebraic Equations ....

Equations (58.14) - (58.17)

where $p = 0.14$, $u = -2\lambda_2$ and $\mu(t) = 0$.

Chapter 59

rbrm: **Robot Arm Control**

This model, that describes the motion of a robot arm, first appeared in the thesis of Monika Mössner-Beigel (Heidelberg University). By using a simple discretization technique a nonlinear programming test problem was created and incorporated into the COPS test suite [40] by Dolan, Moré and Munson.

Example 59.1 rbrm01: MINIMUM TIME MANEUVER.

Phase 1 .... Phase 1

Parameters: (t_F) ....
Independent Variable: (t) ....

$t = 0$	$0 \le t \le t_F$	$t = t_F$

Differential Variables: $(y_1, y_2, y_3, y_4, y_5, y_6)$ ....

$y_1 = 9/2$		$y_1 = 9/2$
$y_2 = 0$		$y_2 = 0$
$y_3 = 0$		$y_3 = 2\pi/3$
$y_4 = 0$		$y_4 = 0$
$y_5 = \pi/4$		$y_5 = \pi/4$
$y_6 = 0$		$y_6 = 0$

Algebraic Variables: (u_1, u_2, u_3) ....

$-1 \le u_1 \le 1$	$-1 \le u_1 \le 1$	$-1 \le u_1 \le 1$
$-1 \le u_2 \le 1$	$-1 \le u_2 \le 1$	$-1 \le u_2 \le 1$
$-1 \le u_3 \le 1$	$-1 \le u_3 \le 1$	$-1 \le u_3 \le 1$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{59.1}$$

$$\dot{y}_2 = u_1/L \tag{59.2}$$
$$\dot{y}_3 = y_4 \tag{59.3}$$
$$\dot{y}_4 = u_2/I_\theta \tag{59.4}$$
$$\dot{y}_5 = y_6 \tag{59.5}$$
$$\dot{y}_6 = u_3/I_\phi \tag{59.6}$$

where $L = 5$ and

$$I_\phi = \frac{1}{3}\left[(L - y_1)^3 + y_1^3\right] \tag{59.7}$$
$$I_\theta = I_\phi \left[\sin y_5\right]^2 \tag{59.8}$$

Objective ....

Minimize $J = t_F$

$$\boxed{J^* = 9.14093620}$$

Chapter 60

rcsp: **IUS/RCS Transfer to Geosynchronous Orbit**

The Inertial Upper Stage (IUS), was a two-stage solid-fueled rocket upper stage developed and used successfully from 1982 to 2004, for raising payloads from low Earth orbit to higher orbits primarily from the payload bay of the Space Shuttle. Although solid rocket stages were the primary source of propulsion, a liquid propellant reaction control system (RCS) was required to provide guidance and control capability. The *Gamma guidance* algorithm [55] implements a real-time control technique to correct errors in both magnitude and direction that are introduced by the solid propellant stages. The mission is designed with a constraint that ensures a high probability that the flight performance reserve (FPR) propellant used by the RCS system, is adequate. A second constraint ensures the RCS correction burn is applied in a posigrade (forward) direction. A complete discussion of the problem is found in references [12] and [7]. Example (60.1) formulates the problem using ten phases, with dynamics expressed in Cartesian coordinates. In example (60.2) the probability calculations are formulated as boundary conditions, eliminating two phases from the problem statement. Examples (60.3) and (60.4) repeat the first two examples, using modified equinoctial coordinates for the dynamic equations.

Example 60.1 rcsp01: Ten-phase, FPR Probability Formulation, (ECI).

Phase 1 .... *Coast in Park Orbit* .... Phase 1

Parameters: $(t_F^{(1)})$ ....

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = r_0$	$-\overline{r} \le r_x \le \overline{r}$	$-\overline{r} \le r_x \le \overline{r}$	ft
$r_y = 0$	$-\overline{r} \le r_y \le \overline{r}$	$-\overline{r} \le r_y \le \overline{r}$	ft

$r_z = 0$	$-\overline{r} \le r_z \le \overline{r}$	$-\overline{r} \le r_z \le \overline{r}$	ft
$v_x = 0$	$-\overline{v} \le v_x \le \overline{v}$	$-\overline{v} \le v_x \le \overline{v}$	ft/sec
$v_y = -v_o \cos i_0$	$-\overline{v} \le v_y \le \overline{v}$	$-\overline{v} \le v_y \le \overline{v}$	ft/sec
$v_z = v_o \sin i_0$	$-\overline{v} \le v_z \le \overline{v}$	$-\overline{v} \le v_z \le \overline{v}$	ft/sec

where

$$\mathbf{r}^\mathsf{T} = (r_x, r_y, r_z) \tag{60.1}$$

$$\mathbf{v}^\mathsf{T} = (v_x, v_y, v_z) \tag{60.2}$$

$$r_0 = h_0 + R_e \tag{60.3}$$

$$v_o = \sqrt{\frac{\mu}{r_0}} \tag{60.4}$$

with $\overline{r} = 4 \times 10^7$, $\overline{v} = 4 \times 10^4$ and the remaining problem parameters given in Table 60.1.

Differential-Algebraic Equations ....

$$\dot{\mathbf{r}} = \mathbf{v} \tag{60.5}$$

$$\dot{\mathbf{v}} = \mathbf{g} \tag{60.6}$$

where

$$r = \|\mathbf{r}\| \tag{60.7}$$

$$\mathbf{g} = -\frac{\mu}{r^3}\mathbf{r} \tag{60.8}$$

Phase 2 .... *First SRM Burn* .... Phase 2

Parameters: $(\psi^{(2)}, \theta^{(2)}, t_I^{(2)}, t_F^{(2)})$ ....

$-10^o \le \psi^{(2)} \le 0^o$ $\qquad -2^o \le \theta^{(2)} \le 2^o$

Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(1)}$	$-\overline{r} \le r_x \le \overline{r}$	$-\overline{r} \le r_x \le \overline{r}$	ft
$r_y = r_{yF}^{(1)}$	$-\overline{r} \le r_y \le \overline{r}$	$-\overline{r} \le r_y \le \overline{r}$	ft
$r_z = r_{zF}^{(1)}$	$-\overline{r} \le r_z \le \overline{r}$	$-\overline{r} \le r_z \le \overline{r}$	ft
$v_x = v_{xF}^{(1)}$	$-\overline{v} \le v_x \le \overline{v}$	$-\overline{v} \le v_x \le \overline{v}$	ft/sec
$v_y = v_{yF}^{(1)}$	$-\overline{v} \le v_y \le \overline{v}$	$-\overline{v} \le v_y \le \overline{v}$	ft/sec
$v_z = v_{zF}^{(1)}$	$-\overline{v} \le v_z \le \overline{v}$	$-\overline{v} \le v_z \le \overline{v}$	ft/sec
$0 \le w \le 38000$			lb

with $\overline{r} = 4 \times 10^7$, $\overline{v} = 4 \times 10^4$ and the remaining problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(2)} - t_I^{(2)} \geq 1$

Differential-Algebraic Equations ....

$$\dot{\mathbf{r}} = \mathbf{v} \tag{60.9}$$
$$\dot{\mathbf{v}} = \mathbf{g} + \mathbf{T} \tag{60.10}$$
$$\dot{w} = -T_c/I_{sp} \tag{60.11}$$

using the definitions in (60.7)-(60.8) and

$$\mathbf{Q}_v = \begin{bmatrix} \frac{\mathbf{v}}{\|\mathbf{v}\|} & \frac{\mathbf{v}\times\mathbf{r}}{\|\mathbf{v}\times\mathbf{r}\|} & \frac{\mathbf{v}}{\|\mathbf{v}\|} \times \left(\frac{\mathbf{v}\times\mathbf{r}}{\|\mathbf{v}\times\mathbf{r}\|}\right) \end{bmatrix} \tag{60.12}$$
$$\mathbf{T} = \frac{T_c g_0}{w}\mathbf{Q}_v \begin{pmatrix} \cos\theta\cos\psi \\ \cos\theta\sin\psi \\ \sin\theta \end{pmatrix} \tag{60.13}$$

where

$$\psi = \psi^{(2)} \tag{60.14}$$
$$\theta = \theta^{(2)} \tag{60.15}$$
$$T_c = T_1 \tag{60.16}$$
$$I_{sp} = \bar{I}_1 \tag{60.17}$$

Phase 3 .... *Coast Between SRM1 and RCS1* .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....
Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} < t < t_F^{(3)}$	$t = t_F^{(3)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = r_{xF}^{(2)}$	$-\overline{r} \leq r_x \leq \overline{r}$	$-\overline{r} \leq r_x \leq \overline{r}$	ft
$r_y = r_{yF}^{(2)}$	$-\overline{r} \leq r_y \leq \overline{r}$	$-\overline{r} \leq r_y \leq \overline{r}$	ft
$r_z = r_{zF}^{(2)}$	$-\overline{r} \leq r_z \leq \overline{r}$	$-\overline{r} \leq r_z \leq \overline{r}$	ft
$v_x = v_{xF}^{(2)}$	$-\overline{v} \leq v_x \leq \overline{v}$	$-\overline{v} \leq v_x \leq \overline{v}$	ft/sec
$v_y = v_{yF}^{(2)}$	$-\overline{v} \leq v_y \leq \overline{v}$	$-\overline{v} \leq v_y \leq \overline{v}$	ft/sec
$v_z = v_{zF}^{(2)}$	$-\overline{v} \leq v_z \leq \overline{v}$	$-\overline{v} \leq v_z \leq \overline{v}$	ft/sec

with $\overline{r} = 4 \times 10^7$, $\overline{v} = 4 \times 10^4$ and the remaining problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} = 100$

Differential-Algebraic Equations ....

Equations (60.5) - (60.8)

Phase 4 .... *First RCS Burn* .... Phase 4

Parameters: $(\psi^{(4)}, \theta^{(4)}, t_I^{(4)}, t_F^{(4)})$ ....

$-10^o \leq \psi^{(4)} \leq 0^o$ $\qquad$ $-2^o \leq \theta^{(4)} \leq 2^o$

Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} < t < t_F^{(4)}$	$t = t_F^{(4)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(3)}$	$-\overline{r} \leq r_x \leq \overline{r}$	$-\overline{r} \leq r_x \leq \overline{r}$	ft
$r_y = r_{yF}^{(3)}$	$-\overline{r} \leq r_y \leq \overline{r}$	$-\overline{r} \leq r_y \leq \overline{r}$	ft
$r_z = r_{zF}^{(3)}$	$-\overline{r} \leq r_z \leq \overline{r}$	$-\overline{r} \leq r_z \leq \overline{r}$	ft
$v_x = v_{xF}^{(3)}$	$-\overline{v} \leq v_x \leq \overline{v}$	$-\overline{v} \leq v_x \leq \overline{v}$	ft/sec
$v_y = v_{yF}^{(3)}$	$-\overline{v} \leq v_y \leq \overline{v}$	$-\overline{v} \leq v_y \leq \overline{v}$	ft/sec
$v_z = v_{zF}^{(3)}$	$-\overline{v} \leq v_z \leq \overline{v}$	$-\overline{v} \leq v_z \leq \overline{v}$	ft/sec

with $\overline{r} = 4 \times 10^7$, $\overline{v} = 4 \times 10^4$ and the remaining problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(4)} - t_I^{(4)} \geq 1$

Differential-Algebraic Equations ....

Equations (60.9) - (60.13)

with

$$\psi = \psi^{(4)} \tag{60.18}$$

$$\theta = \theta^{(4)} \tag{60.19}$$

$$T_c = T_{r1} \tag{60.20}$$

$$I_{sp} = I_{r1} \tag{60.21}$$

Phase 5 .... *Coast Between RCS1 and SRM2* .... Phase 5

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....
Independent Variable: (t) ....

$t = t_F^{(4)} = t_I^{(5)}$	$t_I^{(5)} < t < t_F^{(5)}$	$t = t_F^{(5)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = r_{xF}^{(4)}$	$-\overline{r} \leq r_x \leq \overline{r}$	$-\overline{r} \leq r_x \leq \overline{r}$	ft
$r_y = r_{yF}^{(4)}$	$-\overline{r} \leq r_y \leq \overline{r}$	$-\overline{r} \leq r_y \leq \overline{r}$	ft
$r_z = r_{zF}^{(4)}$	$-\overline{r} \leq r_z \leq \overline{r}$	$-\overline{r} \leq r_z \leq \overline{r}$	ft
$v_x = v_{xF}^{(4)}$	$-\overline{v} \leq v_x \leq \overline{v}$	$-\overline{v} \leq v_x \leq \overline{v}$	ft/sec
$v_y = v_{yF}^{(4)}$	$-\overline{v} \leq v_y \leq \overline{v}$	$-\overline{v} \leq v_y \leq \overline{v}$	ft/sec
$v_z = v_{zF}^{(4)}$	$-\overline{v} \leq v_z \leq \overline{v}$	$-\overline{v} \leq v_z \leq \overline{v}$	ft/sec

with $\overline{r} = 2 \times 10^9$, $\overline{v} = 4 \times 10^5$ and the remaining problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(5)} - t_I^{(5)} \geq 1$

Differential-Algebraic Equations ....

Equations (60.5) - (60.8)

Phase 6 .... *Second SRM Burn* .... Phase 6

Parameters: $(\psi^{(6)}, \theta^{(6)}, t_I^{(6)}, t_F^{(6)})$ ....

$0^o \leq \psi^{(6)} \leq 40^o$ $\quad\quad -2^o \leq \theta^{(6)} \leq 2^o$

Independent Variable: (t) ....

$t = t_F^{(5)} = t_I^{(6)}$	$t_I^{(6)} < t < t_F^{(6)}$	$t = t_F^{(6)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(5)}$	$-\overline{r}_x \leq r_x \leq \overline{r}_x$	$-\overline{r}_x \leq r_x \leq \overline{r}_x$	ft
$r_y = r_{yF}^{(5)}$	$-\overline{r}_y \leq r_y \leq \overline{r}_y$	$-\overline{r}_y \leq r_y \leq \overline{r}_y$	ft
$r_z = r_{zF}^{(5)}$	$-\overline{r}_z \leq r_z \leq \overline{r}_z$	$-\overline{r}_z \leq r_z \leq \overline{r}_z$	ft

$v_x = v_{xF}^{(5)}$	$-\overline{v}_x \le v_x \le \overline{v}_x$	$-\overline{v}_x \le v_x \le \overline{v}_x$	ft/sec
$v_y = v_{yF}^{(5)}$	$-\overline{v}_y \le v_y \le \overline{v}_y$	$-\overline{v}_y \le v_y \le \overline{v}_y$	ft/sec
$v_z = v_{zF}^{(5)}$	$-\overline{v}_z \le v_z \le \overline{v}_z$	$-\overline{v}_z \le v_z \le \overline{v}_z$	ft/sec

with $\overline{r}_x = 2 \times 10^9$, $\overline{r}_y = 1 \times 10^8$, $\overline{r}_z = 1 \times 10^7$, $\overline{v}_x = 2 \times 10^5$, $\overline{v}_y = 2 \times 10^5$, $\overline{v}_z = 4 \times 10^4$ and the remaining problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(6)} - t_I^{(6)} \ge 1$

Differential-Algebraic Equations ....

Equations (60.9) - (60.13)

with

$$\psi = \psi^{(6)} \tag{60.22}$$

$$\theta = \theta^{(6)} \tag{60.23}$$

$$T_c = T_2 \tag{60.24}$$

$$I_{sp} = \bar{I}_2 \tag{60.25}$$

Phase 7 .... *Coast Between SRM2 and RCS2* .... Phase 7

Parameters: $(t_I^{(7)}, t_F^{(7)})$ ....
Independent Variable: (t) ....

$t = t_F^{(6)} = t_I^{(7)}$	$t_I^{(7)} < t < t_F^{(7)}$	$t = t_F^{(7)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z)$ ....

$r_x = r_{xF}^{(6)}$	$-\overline{r}_x \le r_x \le \overline{r}_x$	$-\overline{r}_x \le r_x \le \overline{r}_x$	ft
$r_y = r_{yF}^{(6)}$	$-\overline{r}_y \le r_y \le \overline{r}_y$	$-\overline{r}_y \le r_y \le \overline{r}_y$	ft
$r_z = r_{zF}^{(6)}$	$-\overline{r}_z \le r_z \le \overline{r}_z$	$-\overline{r}_z \le r_z \le \overline{r}_z$	ft
$v_x = v_{xF}^{(6)}$	$-\overline{v}_x \le v_x \le \overline{v}_x$	$-\overline{v}_x \le v_x \le \overline{v}_x$	ft/sec
$v_y = v_{yF}^{(6)}$	$-\overline{v}_y \le v_y \le \overline{v}_y$	$-\overline{v}_y \le v_y \le \overline{v}_y$	ft/sec
$v_z = v_{zF}^{(6)}$	$-\overline{v}_z \le v_z \le \overline{v}_z$	$-\overline{v}_z \le v_z \le \overline{v}_z$	ft/sec

with $\overline{r}_x = 2 \times 10^9$, $\overline{r}_y = 1 \times 10^8$, $\overline{r}_z = 1 \times 10^4$, $\overline{v}_x = 2 \times 10^5$, $\overline{v}_y = 2 \times 10^5$, $\overline{v}_z = 4 \times 10^3$ and the remaining problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(7)} - t_I^{(7)} = 100$

Differential-Algebraic Equations ....

Equations (60.5) - (60.8)

Phase 8 .... *Second RCS Burn* .... Phase 8

Parameters: $(\psi^{(8)}, \theta^{(8)}, w_{p1}, w_{p2}, w_{p3}, w_{p4}, w_{PL}, t_I^{(8)}, t_F^{(8)})$ ....

$0^o \le \psi^{(8)} \le 40^o$	$-1^o \le \theta^{(8)} \le 1^o$	$\overline{w}_{p1}/2 \le w_{p1} \le \overline{w}_{p1}$
$0 \le w_{p2} \le \overline{u}$	$\overline{w}_{p3}/2 \le w_{p3} \le \overline{w}_{p3}$	$0 \le w_{p4} \le \overline{u}$
$w_5/2 \le w_{PL}$		

Independent Variable: (t) ....

$t = t_F^{(7)} = t_I^{(8)}$	$t_I^{(8)} < t < t_F^{(8)}$	$t = t_F^{(8)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ ....

$r_x = r_{xF}^{(7)}$	$-\overline{r}_x \le r_x \le \overline{r}_x$	$-\overline{r}_x \le r_x \le \overline{r}_x$	ft
$r_y = r_{yF}^{(7)}$	$-\overline{r}_y \le r_y \le \overline{r}_y$	$-\overline{r}_y \le r_y \le \overline{r}_y$	ft
$r_z = r_{zF}^{(7)}$	$-\overline{r}_z \le r_z \le \overline{r}_z$	$r_z = 0$	ft
$v_x = v_{xF}^{(7)}$	$-\overline{v}_x \le v_x \le \overline{v}_x$	$-\overline{v}_x \le v_x \le \overline{v}_x$	ft/sec
$v_y = v_{yF}^{(7)}$	$-\overline{v}_y \le v_y \le \overline{v}_y$	$-\overline{v}_y \le v_y \le \overline{v}_y$	ft/sec
$v_z = v_{zF}^{(7)}$	$-\overline{v}_z \le v_z \le \overline{v}_z$	$v_z = 0$	ft/sec

with $\overline{r}_x = 2 \times 10^9$, $\overline{r}_y = 1 \times 10^8$, $\overline{r}_z = 1 \times 10^4$, $\overline{v}_x = 2 \times 10^5$, $\overline{v}_y = 2 \times 10^5$, $\overline{v}_z = 4 \times 10^3$ and the remaining problem parameters given in Table 60.1.

Boundary Conditions ....

$$
\begin{aligned}
w_I^{(2)} - w_{p1} - w_{p3} - w_{PL} &= w_{s1} + w_{s3} + \overline{u} \\
w_F^{(2)} - w_I^{(2)} + w_{p1} &= 0 \\
w_I^{(4)} - w_F^{(2)} &= 0 \\
w_F^{(4)} - w_I^{(4)} + w_{p2} &= 0 \\
w_F^{(4)} - w_I^{(6)} &= w_{s1} \\
w_F^{(6)} - w_I^{(6)} + w_{p3} &= 0 \\
w_I^{(8)} - w_F^{(6)} &= 0 \\
w_F^{(8)} - w_I^{(8)} + w_{p4} &= 0
\end{aligned}
$$

$$t_F^{(8)} - t_I^{(8)} \ge 1$$

$$
\begin{aligned}
\|\mathbf{r}\| &= r_F \\
\|\mathbf{v}\| &= v_F \\
\mathbf{r}^\mathsf{T}\mathbf{v}/(r_F v_F) &= 0
\end{aligned}
$$

Differential-Algebraic Equations ....

Equations (60.9) - (60.13)

with

$$\psi = \psi^{(8)} \tag{60.26}$$

$$\theta = \theta^{(8)} \tag{60.27}$$

$$T_c = T_{r2} \tag{60.28}$$

$$I_{sp} = I_{r2} \tag{60.29}$$

Phase 9 .... *FPR Probability Evaluation, Quadrants 1 and 4* .... Phase 9

Parameters: $(w_{p1}, w_{p2}, w_{p3}, w_{p4}, w_{PL}, I_U, I_M)$ ....

$1 \le w_{p1} \le \overline{w}_{p1}$ $\quad$ $0 \le w_{p2} \le \overline{u}$ $\quad$ $0 \le w_{p3} \le \overline{w}_{p3}$
$0 \le w_{p4} \le \overline{u}$ $\quad$ $0 \le w_{PL}$

Independent Variable: (I_1) ....
Differential Variables: (q) ....

$q = 0$

Boundary Conditions ....

$w_{p1} - w_{p1}^{(8)} = 0$
$w_{p2} - w_{p2}^{(8)} = 0$
$w_{p3} - w_{p3}^{(8)} = 0$
$w_{p4} - w_{p4}^{(8)} = 0$
$w_{PL} - w_{PL}^{(8)} = 0$
$I_1 - I_U = 0$

$I_1 - I_M = 0$

where the computational sequence (60.34)-(60.48) is executed prior to computing

$$v_{1U} = -a_3 \ln\left(1 - \frac{\overline{u}}{a_5}\right) \tag{60.30}$$

$$I_U = \frac{t_1 - v_{1U}}{a_1} \tag{60.31}$$

$$I_M = \frac{t_1}{a_1} \tag{60.32}$$

Differential-Algebraic Equations ....

$$\dot{q} = P(I_1) \tag{60.33}$$

where $P(I_1)$ is defined from the parameters $(w_{p1}, w_{p2}, w_{p3}, w_{p4}, w_{PL})$ and the values in Table 60.1 by the following sequence of calculations:

$$w_{i1} = w_{p1} + w_{s1} + w_{p3} + w_{s3} + w_{PL} + \overline{u} \tag{60.34}$$

$$w_{b1} = w_{i1} - w_{p1} \tag{60.35}$$

$$w_{i2} = w_{b1} \tag{60.36}$$

$$w_{b2} = w_{i2} - w_{p2} \tag{60.37}$$

$$w_{i3} = w_{b2} - w_{s1} \tag{60.38}$$

$$w_{b3} = w_{i3} - w_{p3} \tag{60.39}$$

$$w_{i4} = w_{b3} \tag{60.40}$$

$$w_{b4} = w_{i4} - w_{p4} \tag{60.41}$$

$$t_1 = g_0 \bar{I}_1 \ln\left[\frac{w_{i1}}{w_{b1}}\right] + g_0 I_{r1} \ln\left[\frac{w_{i2}}{w_{b2}}\right] \tag{60.42}$$

$$t_2 = g_0 \bar{I}_2 \ln\left[\frac{w_{i3}}{w_{b3}}\right] + g_0 I_{r2} \ln\left[\frac{w_{i4}}{w_{b4}}\right] \tag{60.43}$$

$$a_4 = w_{s3} + w_{PL} + \overline{u} \tag{60.44}$$

$$a_2 = w_{p3} + a_4 \tag{60.45}$$

$$a_5 = w_{s1} + a_2 \tag{60.46}$$

$$a_1 = g_0 \ln\left[\frac{w_{p1} + a_5}{a_5}\right] \tag{60.47}$$

$$a_3 = g_0 I_{r1} \tag{60.48}$$

$$a_6 = g_0 I_{r2} \tag{60.49}$$

$$v_1 = t_1 - a_1 I_1 \tag{60.50}$$

$$w_1 = a_5 \left[1 - \exp\left(\frac{-|v_1|}{a_3}\right)\right] \tag{60.51}$$

$$h = -a_6 \ln\left[1 - \frac{(\overline{u} - w_1)}{(a_4 - w_1)}\right] \tag{60.52}$$

$$D = g_0 \ln\left(\frac{a_2 - w_1}{a_4 - w_1}\right) \tag{60.53}$$

$$b_L = (t_2 - h)/D \tag{60.54}$$

$$b_U = (t_2 + h)/D \tag{60.55}$$

$$P(I_1) = \frac{1}{2\sqrt{2\pi}\sigma_1} \exp\left[-\frac{1}{2}\left(\frac{I_1 - \bar{I}_1}{\sigma_1}\right)^2\right]\left[\operatorname{erf}\left(\frac{b_U - \bar{I}_2}{\sqrt{2}\sigma_2}\right) - \operatorname{erf}\left(\frac{b_L - \bar{I}_2}{\sqrt{2}\sigma_2}\right)\right] \tag{60.56}$$

Phase 10 .... *FPR Probability Evaluation, Quadrants 2 and 3* .... Phase 10

Parameters: $(w_{p1}, w_{p2}, w_{p3}, w_{p4}, w_{PL}, s_P, s_N, I_U, I_M)$ ....

$1 \le w_{p1} \le \overline{w}_{p1}$	$0 \le w_{p2} \le \overline{u}$	$0 \le w_{p3} \le \overline{w}_{p3}$
$0 \le w_{p4} \le \overline{u}$	$0 \le w_{PL}$	$0 \le s_P \le .9$

$0 \leq s_N \leq .9$

Independent Variable: (I_1) ....
Differential Variables: (q) ....

$q = q_F^{(9)}$

Boundary Conditions ....

$w_{p1} - w_{p1}^{(9)} = 0$
$w_{p2} - w_{p2}^{(9)} = 0$
$w_{p3} - w_{p3}^{(9)} = 0$
$w_{p4} - w_{p4}^{(9)} = 0$
$w_{PL} - w_{PL}^{(9)} = 0$
$I_1 - I_M = 0$

$$I_1 - I_L = 0$$
$$q - \hat{q} + s_P - s_N = 0$$

where $\hat{q} = .9973$ and the computational sequence (60.34)-(60.48) is executed prior to computing

$$v_{1L} = -a_3 \ln\left(1 - \frac{\overline{u}}{a_5}\right) \tag{60.57}$$

$$I_L = \frac{t_1 + v_{1L}}{a_1} \tag{60.58}$$

$$I_M = \frac{t_1}{a_1} \tag{60.59}$$

Differential-Algebraic Equations ....

Equations (60.33) - (60.56)

Objective ....

Maximize $\qquad J = 10^{-3} w_{PL} - 100 s_P - 100 s_N$

$$\boxed{J^* = 4.90751915; \quad s_P^* = s_N^* = 0}$$

Example 60.2 rcsp02: Point Function, FPR Probability Formulation, (ECI).

Repeat the first seven phases of example 60.1.

Phase 8 .... *Second RCS Burn* .... Phase 8

Parameters: $(\psi^{(8)}, \theta^{(8)}, w_{p1}, w_{p2}, w_{p3}, w_{p4}, w_{PL}, t_I^{(8)}, t_F^{(8)})$ ....

$$0^o \le \psi^{(8)} \le 40^o \qquad -1^o \le \theta^{(8)} \le 1^o \qquad \overline{w}_{p1}/2 \le w_{p1} \le \overline{w}_{p1}$$
$$0 \le w_{p2} \le \overline{u} \qquad \overline{w}_{p3}/2 \le w_{p3} \le \overline{w}_{p3} \qquad 0 \le w_{p4} \le \overline{u}$$
$$w_5/2 \le w_{PL}$$

Independent Variable: (t) . . . .

$t = t_F^{(7)} = t_I^{(8)}$	$t_I^{(8)} < t < t_F^{(8)}$	$t = t_F^{(8)}$	sec

Differential Variables: $(r_x, r_y, r_z, v_x, v_y, v_z, w)$ . . . .

$r_x = r_{xF}^{(7)}$	$-\overline{r}_x \le r_x \le \overline{r}_x$	$-\overline{r}_x \le r_x \le \overline{r}_x$	ft
$r_y = r_{yF}^{(7)}$	$-\overline{r}_y \le r_y \le \overline{r}_y$	$-\overline{r}_y \le r_y \le \overline{r}_y$	ft
$r_z = r_{zF}^{(7)}$	$-\overline{r}_z \le r_z \le \overline{r}_z$	$r_z = 0$	ft
$v_x = v_{xF}^{(7)}$	$-\overline{v}_x \le v_x \le \overline{v}_x$	$-\overline{v}_x \le v_x \le \overline{v}_x$	ft/sec
$v_y = v_{yF}^{(7)}$	$-\overline{v}_y \le v_y \le \overline{v}_y$	$-\overline{v}_y \le v_y \le \overline{v}_y$	ft/sec
$v_z = v_{zF}^{(7)}$	$-\overline{v}_z \le v_z \le \overline{v}_z$	$v_z = 0$	ft/sec

with $\overline{r}_x = 2 \times 10^9$, $\overline{r}_y = 1 \times 10^8$, $\overline{r}_z = 1 \times 10^4$, $\overline{v}_x = 2 \times 10^5$, $\overline{v}_y = 2 \times 10^5$, $\overline{v}_z = 4 \times 10^3$ and the remaining problem parameters given in Table 60.1.

Boundary Conditions . . . .

$$w_I^{(2)} - w_{p1} - w_{p3} - w_{PL} = w_{s1} + w_{s3} + \overline{u}$$
$$w_F^{(2)} - w_I^{(2)} + w_{p1} = 0$$
$$w_I^{(4)} - w_F^{(2)} = 0$$
$$w_F^{(4)} - w_I^{(4)} + w_{p2} = 0$$
$$w_F^{(4)} - w_I^{(6)} = w_{s1}$$
$$w_F^{(6)} - w_I^{(6)} + w_{p3} = 0$$
$$w_I^{(8)} - w_F^{(6)} = 0$$
$$w_F^{(8)} - w_I^{(8)} + w_{p4} = 0$$
$$t_F^{(8)} - t_I^{(8)} \ge 1$$
$$\|\mathbf{r}\| = r_F$$
$$\|\mathbf{v}\| = v_F$$
$$\mathbf{r}^\mathsf{T}\mathbf{v}/(r_F v_F) = 0$$
$$\ln(q_1) = \ln(.9973)$$
$$\ln(q_2) \ge \ln(.97)$$

The values of q_1 and q_2 are computed from $(w_{p1}, w_{p2}, w_{p3}, w_{p4}, w_{PL})$ as follows:
(a) Execute the sequence (60.34)-(60.48)
(b) Define the bounds I_U, I_M, and I_L from (60.31), (60.32), and (60.58) respectively. Define $r_L = \bar{I}_1 - (5.6)\sigma_1$.
(c) Define $N = 2^{(n_b-1)} = 64$ for $n_b = 7$ and set

$$\alpha = (I_M - I_U)/N \tag{60.60}$$
$$\beta = (I_L - I_M)/N \tag{60.61}$$
$$\gamma = (I_M - r_L)/N \tag{60.62}$$

(d) For $k = 0, 1, \ldots, N$ evaluate

$$\hat{q}_k = P(I_U + k\alpha) \tag{60.63}$$
$$\tilde{q}_k = P(I_M + k\beta) \tag{60.64}$$
$$r_k = R(r_L + k\gamma) \tag{60.65}$$

where $P(I_1)$ is computed by the sequence (60.34)-(60.56) and

$$\Gamma(I_1) = \frac{t_2}{g_0}\left[\ln\left(\frac{a_2 - w_1}{a_4 - w_1}\right)\right]^{-1} \tag{60.66}$$
$$R(I_1) = \frac{1}{2\sqrt{2\pi}\sigma_1}\exp\left[-\frac{1}{2}\left(\frac{I_1 - \bar{I}_1}{\sigma_1}\right)^2\right]\left[1 + \text{erf}\left(\frac{\Gamma(I_1) - \bar{I}_2}{\sqrt{2}\sigma_2}\right)\right] \tag{60.67}$$

(e) Using Romberg quadrature with the values $\hat{q}_k$ and $\tilde{q}_k$ evaluate

$$q_1 = \int_{I_U}^{I_M} P(I_1)dI_1 + \int_{I_M}^{I_L} P(I_1)dI_1 \tag{60.68}$$

and the values r_k evaluate

$$q_2 = \int_{r_L}^{I_M} R(I_1)dI_1 \tag{60.69}$$

Differential-Algebraic Equations . . . .

Equations (60.9) - (60.13)

with

$$\psi = \psi^{(8)} \tag{60.70}$$
$$\theta = \theta^{(8)} \tag{60.71}$$
$$T_c = T_{r2} \tag{60.72}$$
$$I_{sp} = I_{r2} \tag{60.73}$$

Objective . . . .

Maximize $J = w_{PL}$

$\boxed{J^* = 4907.51941}$

Example 60.3 rcsp03: Ten-phase, FPR Probability Formulation, (MEE).

Phase 1 . . . . *Coast in Park Orbit* . . . . Phase 1

Parameters: $(t_F^{(1)})$ . . . .
Independent Variable: (t) . . . .

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$	sec

Differential Variables: (p, f, g, h, k, L) ....

$p = p_1$	$\underline{p}_1 \le p \le \overline{p}_1$	$\underline{p}_1 \le p \le \overline{p}_1$	ft
$f = 0$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = 0$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_1$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = 0$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = 180^o$	$\underline{L}_1 \le L \le \overline{L}_1$	$\underline{L}_1 \le L \le \overline{L}_1$	rad

Differential-Algebraic Equations ....

$$\dot{\mathbf{y}} = \mathbf{A}(\mathbf{y})\boldsymbol{\Delta} + \mathbf{b} \tag{60.74}$$

where $\mathbf{y}^\mathsf{T} = (p, f, g, h, k, L)$ and the right hand side given by (53.5)-(53.15) with $\boldsymbol{\Delta} = 0$ using the problem constants in Table 60.1.

Phase 2 .... *First SRM Burn* .... Phase 2

Parameters: $(\psi^{(2)}, \theta^{(2)}, t_I^{(2)}, t_F^{(2)})$ ....

$-10^o \le \psi^{(2)} \le 0^o$ $\quad$ $-2^o \le \theta^{(2)} \le 2^o$

Independent Variable: (t) ....

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$	sec

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(1)}$	$\underline{p}_2 \le p \le \overline{p}_2$	$\underline{p}_2 \le p \le \overline{p}_2$	ft
$f = f_F^{(1)}$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = g_F^{(1)}$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_F^{(1)}$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_F^{(1)}$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_F^{(1)}$	$\underline{L}_2 \le L \le \overline{L}_2$	$\underline{L}_2 \le L \le \overline{L}_2$	rad
$0 \le w \le 38000$			lb

with problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(2)} - t_I^{(2)} \ge 1$

Differential-Algebraic Equations ....

$$\dot{\mathbf{y}} = \mathbf{A}(\mathbf{y})\boldsymbol{\Delta} + \mathbf{b} \tag{60.75}$$

$$\dot{w} = -T_c/I_{sp} \tag{60.76}$$

using the definitions in (53.5)-(53.15) and

$$\boldsymbol{\Delta} = \frac{T_c g_0}{w}\mathbf{Q}_v \begin{pmatrix} \cos\theta\cos\psi \\ \cos\theta\sin\psi \\ \sin\theta \end{pmatrix} \tag{60.77}$$

where

$$\psi = \psi^{(2)} \tag{60.78}$$
$$\theta = \theta^{(2)} \tag{60.79}$$
$$T_c = T_1 \tag{60.80}$$
$$I_{sp} = \bar{I}_1 \tag{60.81}$$

Phase 3 .... *Coast Between SRM1 and RCS1* .... Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ ....
Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} < t < t_F^{(3)}$	$t = t_F^{(3)}$	sec

Differential Variables: (p, f, g, h, k, L) ....

$p = p_F^{(2)}$	$\underline{p}_3 \le p \le \overline{p}_3$	$\underline{p}_3 \le p \le \overline{p}_3$	ft
$f = f_F^{(2)}$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = g_F^{(2)}$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_F^{(2)}$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_F^{(2)}$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_F^{(2)}$	$\underline{L}_3 \le L \le \overline{L}_3$	$\underline{L}_3 \le L \le \overline{L}_3$	rad

using the problem constants in Table 60.1.

Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} = 100$

Differential-Algebraic Equations ....

Equation (60.74)

Phase 4 .... *First RCS Burn* .... Phase 4

Parameters: $(\psi^{(4)}, \theta^{(4)}, t_I^{(4)}, t_F^{(4)})$ ....

$-10^o \leq \psi^{(4)} \leq 0^o$ $\qquad$ $-2^o \leq \theta^{(4)} \leq 2^o$

Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} < t < t_F^{(4)}$	$t = t_F^{(4)}$	sec

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(3)}$	$\underline{p}_4 \leq p \leq \overline{p}_4$	$\underline{p}_4 \leq p \leq \overline{p}_4$	ft
$f = f_F^{(3)}$	$-1 \leq f \leq 1$	$-1 \leq f \leq 1$	
$g = g_F^{(3)}$	$-1 \leq g \leq 1$	$-1 \leq g \leq 1$	
$h = h_F^{(3)}$	$-1 \leq h \leq 1$	$-1 \leq h \leq 1$	
$k = k_F^{(3)}$	$-1 \leq k \leq 1$	$-1 \leq k \leq 1$	
$L = L_F^{(3)}$	$\underline{L}_4 \leq L \leq \overline{L}_4$	$\underline{L}_4 \leq L \leq \overline{L}_4$	rad

with problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(4)} - t_I^{(4)} \geq 1$

Differential-Algebraic Equations ....

Equations (60.75) - (60.77)

with

$$\psi = \psi^{(4)} \tag{60.82}$$

$$\theta = \theta^{(4)} \tag{60.83}$$

$$T_c = T_{r1} \tag{60.84}$$

$$I_{sp} = I_{r1} \tag{60.85}$$

Phase 5 .... *Coast Between RCS1 and SRM2* .... **Phase 5**

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....

Independent Variable: (t) ....

$t = t_F^{(4)} = t_I^{(5)}$	$t_I^{(5)} < t < t_F^{(5)}$	$t = t_F^{(5)}$	sec

Differential Variables: (p, f, g, h, k, L) ....

$p = p_F^{(4)}$	$\underline{p}_5 \leq p \leq \overline{p}_5$	$\underline{p}_5 \leq p \leq \overline{p}_5$	ft
$f = f_F^{(4)}$	$-1 \leq f \leq 1$	$-1 \leq f \leq 1$	
$g = g_F^{(4)}$	$-1 \leq g \leq 1$	$-1 \leq g \leq 1$	

$h = h_F^{(4)}$	$-1 \leq h \leq 1$	$-1 \leq h \leq 1$	
$k = k_F^{(4)}$	$-1 \leq k \leq 1$	$-1 \leq k \leq 1$	
$L = L_F^{(4)}$	$\underline{L}_5 \leq L \leq \overline{L}_5$	$\underline{L}_5 \leq L \leq \overline{L}_5$	rad

using the problem constants in Table 60.1.

Boundary Conditions ....

$t_F^{(5)} - t_I^{(5)} \geq 1$

Differential-Algebraic Equations ....

Equation (60.74)

Phase 6..........................*Second SRM Burn*..........................Phase 6

Parameters: $(\psi^{(6)}, \theta^{(6)}, t_I^{(6)}, t_F^{(6)})$ ....

$0^o \leq \psi^{(6)} \leq 40^o$ $\quad -2^o \leq \theta^{(6)} \leq 2^o$

Independent Variable: (t) ....

$t = t_F^{(5)} = t_I^{(6)}$	$t_I^{(6)} < t < t_F^{(6)}$	$t = t_F^{(6)}$	sec

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(5)}$	$\underline{p}_6 \leq p \leq \overline{p}_6$	$\underline{p}_6 \leq p \leq \overline{p}_6$	ft
$f = f_F^{(5)}$	$-1 \leq f \leq 1$	$-1 \leq f \leq 1$	
$g = g_F^{(5)}$	$-1 \leq g \leq 1$	$-1 \leq g \leq 1$	
$h = h_F^{(5)}$	$-1 \leq h \leq 1$	$-1 \leq h \leq 1$	
$k = k_F^{(5)}$	$-1 \leq k \leq 1$	$-1 \leq k \leq 1$	
$L = L_F^{(5)}$	$\underline{L}_6 \leq L \leq \overline{L}_6$	$\underline{L}_6 \leq L \leq \overline{L}_6$	rad

with problem parameters given in Table 60.1.

Boundary Conditions ....

$t_F^{(6)} - t_I^{(6)} \geq 1$

Differential-Algebraic Equations ....

Equations (60.75) - (60.77)

with

$$\psi = \psi^{(6)} \tag{60.86}$$

$$\theta = \theta^{(6)} \tag{60.87}$$

$$T_c = T_2 \tag{60.88}$$
$$I_{sp} = \bar{I}_2 \tag{60.89}$$

Phase 7 .... *Coast Between SRM2 and RCS2* .... Phase 7

Parameters: $(t_I^{(7)}, t_F^{(7)})$ ....

Independent Variable: (t) ....

$t = t_F^{(6)} = t_I^{(7)}$	$t_I^{(7)} < t < t_F^{(7)}$	$t = t_F^{(7)}$	sec

Differential Variables: (p, f, g, h, k, L) ....

$p = p_F^{(6)}$	$\underline{p}_7 \le p \le \overline{p}_7$	$\underline{p}_7 \le p \le \overline{p}_7$	ft
$f = f_F^{(6)}$	$-1 \le f \le 1$	$-1 \le f \le 1$	
$g = g_F^{(6)}$	$-1 \le g \le 1$	$-1 \le g \le 1$	
$h = h_F^{(6)}$	$-1 \le h \le 1$	$-1 \le h \le 1$	
$k = k_F^{(6)}$	$-1 \le k \le 1$	$-1 \le k \le 1$	
$L = L_F^{(6)}$	$\underline{L}_7 \le L \le \overline{L}_7$	$\underline{L}_7 \le L \le \overline{L}_7$	rad

using the problem constants in Table 60.1.

Boundary Conditions ....

$t_F^{(7)} - t_I^{(7)} = 100$

Differential-Algebraic Equations ....

Equation (60.74)

Phase 8 .... *Second RCS Burn* .... Phase 8

Parameters: $(\psi^{(8)}, \theta^{(8)}, w_{p1}, w_{p2}, w_{p3}, w_{p4}, w_{PL}, t_I^{(8)}, t_F^{(8)})$ ....

$0^o \le \psi^{(8)} \le 40^o$	$-1^o \le \theta^{(8)} \le 1^o$	$\overline{w}_{p1}/2 \le w_{p1} \le \overline{w}_{p1}$
$0 \le w_{p2} \le \overline{u}$	$\overline{w}_{p3}/2 \le w_{p3} \le \overline{w}_{p3}$	$0 \le w_{p4} \le \overline{u}$
$w_5/2 \le w_{PL}$		

Independent Variable: (t) ....

$t = t_F^{(7)} = t_I^{(8)}$	$t_I^{(8)} < t < t_F^{(8)}$	$t = t_F^{(8)}$	sec

Differential Variables: (p, f, g, h, k, L, w) ....

$p = p_F^{(7)}$	$\underline{p}_8 \leq p \leq \overline{p}_8$	$p = p_8$	ft
$f = f_F^{(7)}$	$-1 \leq f \leq 1$	$f = 0$	
$g = g_F^{(7)}$	$-1 \leq g \leq 1$	$g = 0$	
$h = h_F^{(7)}$	$-1 \leq h \leq 1$	$h = 0$	
$k = k_F^{(7)}$	$-1 \leq k \leq 1$	$k = 0$	
$L = L_F^{(7)}$	$\underline{L}_8 \leq L \leq \overline{L}_8$	$\underline{L}_8 \leq L \leq \overline{L}_8$	rad

with problem parameters given in Table 60.1.

Boundary Conditions . . . .

$$\begin{aligned} w_I^{(2)} - w_{p1} - w_{p3} - w_{PL} &= w_{s1} + w_{s3} + \overline{u} \\ w_F^{(2)} - w_I^{(2)} + w_{p1} &= 0 \\ w_I^{(4)} - w_F^{(2)} &= 0 \\ w_F^{(4)} - w_I^{(4)} + w_{p2} &= 0 \\ w_F^{(4)} - w_I^{(6)} &= w_{s1} \\ w_F^{(6)} - w_I^{(6)} + w_{p3} &= 0 \\ w_I^{(8)} - w_F^{(6)} &= 0 \\ w_F^{(8)} - w_I^{(8)} + w_{p4} &= 0 \end{aligned}$$

$t_F^{(8)} - t_I^{(8)} \geq 1$

Differential-Algebraic Equations . . . .

Equations (60.75) - (60.77)

with

$$\psi = \psi^{(8)} \tag{60.90}$$

$$\theta = \theta^{(8)} \tag{60.91}$$

$$T_c = T_{r2} \tag{60.92}$$

$$I_{sp} = I_{r2} \tag{60.93}$$

Repeat phases 9 and 10 of example 60.1.

$$\boxed{J^* = 4909.23796; \quad s_P^* = s_N^* = 0}$$

Example 60.4 rcsp04: Point Function, FPR Probability Formulation, (MEE).

Repeat the first seven phases of example 60.3.

Modify phase eight of example 60.2 as follows:

(a) Define
Differential Variables: (p, f, g, h, k, L, w) . . . .

$p = p_F^{(7)}$	$\underline{p}_8 \leq p \leq \overline{p}_8$	$p = p_8$	ft
$f = f_F^{(7)}$	$-1 \leq f \leq 1$	$f = 0$	
$g = g_F^{(7)}$	$-1 \leq g \leq 1$	$g = 0$	
$h = h_F^{(7)}$	$-1 \leq h \leq 1$	$h = 0$	
$k = k_F^{(7)}$	$-1 \leq k \leq 1$	$k = 0$	
$L = L_F^{(7)}$	$\underline{L}_8 \leq L \leq \overline{L}_8$	$\underline{L}_8 \leq L \leq \overline{L}_8$	rad

with problem parameters given in Table 60.1.

(b) Omit the boundary conditions

$$\|\mathbf{r}\| = r_F$$
$$\|\mathbf{v}\| = v_F$$
$$\mathbf{r}^\mathsf{T}\mathbf{v}/(r_F v_F) = 0$$

$$\boxed{J^* = 4909.23795}$$

$\overline{w}_{p1} = 21586.7$	$\overline{w}_{p3} = 6059$
$w_{s1} = 2500.3$	$w_{s3} = 2304.887099$
$\sigma_1 = .5365$	$\sigma_2 = .6088$
$\bar{I}_1 = 291.9306600$	$\bar{I}_2 = 300.7969263$
$I_{r1} = 216.2099000$	$I_{r2} = 223.0743000$
$T_1 = 41655.2$	$T_2 = 17676.4$
$T_{r1} = 130.$	$T_{r2} = 100.$
$\overline{u} = 189.7$	$w_5 = 5288.107204$
$r_F = 138586325.00510725$	$v_F = 10078.281956575302$
$h_0 = 150$ nm $= 911417.32283464505$	$R_e = 20925662.73$
$\mu = .1407645794 \times 10^{17}$	$i_0 = 28.5^o$
$p_1 = 21837080.05283464$	$p_8 = 138334442.2575590$
$\mu = .1407645794 \times 10^{17}$	$h_1 = -0.2539676464749437$
$\underline{p}_1 = \underline{p}_2 = 2183708.005283465$	$\overline{p}_1 = 109185400.2641732$
$\underline{p}_3 = \underline{p}_4 = \underline{p}_5 = 3776664.197643460$	$\overline{p}_2 = \overline{p}_3 = \overline{p}_4 = 188833209.8821730$
$\underline{p}_6 = 7535181.112615490$	$\overline{p}_5 = 376759055.6307745$
$\underline{p}_7 = \underline{p}_8 = 13833444.22575590$	$\overline{p}_6 = \overline{p}_7 = \overline{p}_8 = 691672211.2877948$
$\underline{L}_2 = \underline{L}_3 = \underline{L}_4 = \underline{L}_5 = 270^o$	$\overline{L}_1 = \overline{L}_2 = \overline{L}_3 = \overline{L}_4 = \underline{L}_6 = \underline{L}_7 = \underline{L}_8 = 450^o$
$\underline{L}_1 = 90^o$	$\overline{L}_5 = \overline{L}_6 = \overline{L}_7 = \overline{L}_8 = 630^o$

Table 60.1. *IUS/RCS example constants.*

Chapter 61

rivr: **River Crossing**

Ernst Zermelo was a German mathematician who first presented the problem that now bears his name. "Zermelo navigation" has been used to describe the motion of many things including aircraft, ships, birds, robots, and even light waves leading to an analog of "Snell's Law." As such, it is considered a "classical" example of optimal control. Bryson and Ho [29, Sect. 2.7] describe the situation as follows:

> A ship must travel through a region of strong currents. . . . The problem is to steer the ship in such a way as to minimize the time necessary to go from a point A to a point B.

Two examples, using an analytic function to model the river current are posed here as described in reference [16]. A simple version of this problem is given as example (74.1).

Example 61.1 rivr01: Minimum Time–Downstream Crossing.

Phase 1 . . . . Phase 1

Parameters: (t_F) . . . .

$0 \leq t_F$

Independent Variable: (t) . . . .

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: (x, y) . . . .

$x = 0$	$x = 2\pi$
$y = -1$	$y = 1$

Algebraic Variables: (V, s_θ, c_θ) ....

$0 \le V \le \hat{V}$ $\qquad$ $0 \le V \le \hat{V}$ $\qquad$ $0 \le V \le \hat{V}$

Boundary Conditions ....

$\dot{x} = 0$
$\dot{y} = 0$

$\dot{x} = 0$
$\dot{y} = 0$

Differential-Algebraic Equations ....

$$\dot{x} = V c_\theta + u(x, y) \tag{61.1}$$
$$\dot{y} = V s_\theta + v(x, y) \tag{61.2}$$
$$1 = s_\theta^2 + c_\theta^2 \tag{61.3}$$
$$0 \le c_n(x) - y \tag{61.4}$$
$$0 \le y - c_s(x) \tag{61.5}$$
$$-\dot{V}_{max} \le \dot{V} \le \dot{V}_{max} \tag{61.6}$$

where

$$u(x, y) = \frac{\bar{R}}{\sqrt{1 + \cos^2(x)}} \exp\left[-\left(\frac{y - \sin(x)}{\hat{w}} \right)^2 \right] \tag{61.7}$$
$$v(x, y) = \frac{\bar{R}\cos(x)}{\sqrt{1 + \cos^2(x)}} \exp\left[-\left(\frac{y - \sin(x)}{\hat{w}} \right)^2 \right] \tag{61.8}$$
$$c_n(x) = \sum_{k=1}^{N} a_k B_k(x) \tag{61.9}$$
$$c_s(x) = \sum_{k=1}^{N} b_k B_k(x) \tag{61.10}$$

and the coefficients a_k b_k of the monotonic cubic splines are computed such that

$$c_n(x_k) = \hat{c}_n(x_k) \tag{61.11}$$
$$c_s(x_k) = \hat{c}_s(x_k) \tag{61.12}$$

where $x_k = 2\pi(k-1)/(N-1)$ for $k = 1, \ldots, N$ and $N = 21$. The data points are

$$\hat{c}_n(x_k) = \begin{cases} \sin x_k + \hat{w} - \Delta & \text{for } k = 1, \ldots, (N-1) \\ \sin x_k + \hat{w} + \epsilon & \text{for } k = N \end{cases} \tag{61.13}$$

and

$$\hat{c}_s(x_k) = \begin{cases} \sin x_k - \hat{w} + \Delta & \text{for } k = 2, \ldots, N \\ \sin x_k - \hat{w} - \epsilon & \text{for } k = 1. \end{cases} \tag{61.14}$$

where $\dot{V}_{max} = 100$, $\hat{w} = 1$, $\Delta = .1$, $\epsilon = 10^{-5}$, $\hat{V} = 4$ and $\bar{R} = 3$.
Objective . . . .

Minimize $$J = t_F$$

$$\boxed{J^* = 1.29620614}$$

Example 61.2 rivr02: Minimum Time–Upstream Crossing.
Repeat example 61.1 with $\bar{R} = -3$.

$$\boxed{J^* = 2.82601443}$$

Chapter 62

robo: **Industrial Robot**

In his doctoral thesis, Oskar von Stryk [85] presents an interesting example that describes the motion of an industrial robot called the Manutec r3. The multibody dynamics are defined by over 4000 lines of machine derived code [75], and a detailed description of the example problems given here is found in reference [13, Sect. 6.9]. In addition to the fact that the control appears linearly which suggests a solution that is either bang-bang or has singular arcs, state constraints on the angular velocity can lead to an index two DAE system. Four different versions of the problem are posed, including the final example (62.4) that incorporates the switching structure using a nine phase formulation.

Example 62.1 robo01: Mayer Formulation.

Phase 1 . . . . Phase 1

Independent Variable: (t) . . . .

$t = 0$	$0 \le t \le .53$	$t = t_F = .53$

Differential Variables: $(q_1, q_2, q_3, v_1, v_2, v_3, E)$ . . . .

$q_1 = 0$	$q_{1L} \le q_1 \le q_{1U}$	$q_1 = 1$	rad
$q_2 = -1.5$	$q_{2L} \le q_2 \le q_{2U}$	$q_2 = -1.95$	rad
$q_3 = 0$	$q_{3L} \le q_3 \le q_{3U}$	$q_3 = 1$	rad
$v_1 = 0$	$v_{1L} \le v_1 \le v_{1U}$	$v_1 = 0$	rad/sec
$v_2 = 0$	$v_{2L} \le v_2 \le v_{2U}$	$v_2 = 0$	rad/sec
$v_3 = 0$	$v_{3L} \le v_3 \le v_{3U}$	$v_3 = 0$	rad/sec
$E = 0$	$0 \le E$	$0 \le E$	

Algebraic Variables: (u_1, u_2, u_3) . . . .

$u_{1L} \le u_1 \le u_{1U}$	$u_{1L} \le u_1 \le u_{1U}$	$u_{1L} \le u_1 \le u_{1U}$

$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$
$u_{3L} \le u_3 \le u_{3U}$	$u_{3L} \le u_3 \le u_{3U}$	$u_{3L} \le u_3 \le u_{3U}$

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{q}} = \mathbf{v} \tag{62.1}$$
$$\dot{\mathbf{v}} = \mathbf{F}(\mathbf{v}, \mathbf{q}, \mathbf{u}) \tag{62.2}$$
$$\dot{E} = \mathbf{u}^{\mathsf{T}}\mathbf{u} \tag{62.3}$$

where Table 62.1 defines the constants with $\mathbf{q}^{\mathsf{T}} = (q_1, q_2, q_3)$, $\mathbf{v}^{\mathsf{T}} = (v_1, v_2, v_3)$, and $\mathbf{u}^{\mathsf{T}} = (u_1, u_2, u_3)$. Simulation software described in [85, 75] is used to implement complicated expressions for the matrix $\mathbf{M}$ and function $\mathbf{f}(\mathbf{v}, \mathbf{q}, \mathbf{u})$ that define the function

$$\mathbf{F}(\mathbf{v}, \mathbf{q}, \mathbf{u}) = \mathbf{M}^{-1}(\mathbf{q})\mathbf{f}(\mathbf{v}, \mathbf{q}, \mathbf{u}) \tag{62.4}$$

Objective . . . .

Minimize $\quad J = E(t_F)$

$$\boxed{J^* = 20.4042462}$$

Example 62.2 robo02: Lagrange Formulation.

Phase 1 . . . . Phase 1

Independent Variable: (t) . . . .

$t = 0$	$0 \le t \le .53$	$t = t_F = .53$

Differential Variables: $(q_1, q_2, q_3, v_1, v_2, v_3)$ . . . .

$q_1 = 0$	$q_{1L} \le q_1 \le q_{1U}$	$q_1 = 1$	rad
$q_2 = -1.5$	$q_{2L} \le q_2 \le q_{2U}$	$q_2 = -1.95$	rad
$q_3 = 0$	$q_{3L} \le q_3 \le q_{3U}$	$q_3 = 1$	rad
$v_1 = 0$	$v_{1L} \le v_1 \le v_{1U}$	$v_1 = 0$	rad/sec
$v_2 = 0$	$v_{2L} \le v_2 \le v_{2U}$	$v_2 = 0$	rad/sec
$v_3 = 0$	$v_{3L} \le v_3 \le v_{3U}$	$v_3 = 0$	rad/sec

Algebraic Variables: (u_1, u_2, u_3) . . . .

$u_{1L} \le u_1 \le u_{1U}$	$u_{1L} \le u_1 \le u_{1U}$	$u_{1L} \le u_1 \le u_{1U}$
$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$
$u_{3L} \le u_3 \le u_{3U}$	$u_{3L} \le u_3 \le u_{3U}$	$u_{3L} \le u_3 \le u_{3U}$

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{q}} = \mathbf{v} \tag{62.5}$$

$$\dot{\mathbf{v}} = \mathbf{F}(\mathbf{v}, \mathbf{q}, \mathbf{u}) \tag{62.6}$$

where Table 62.1 defines the constants with $\mathbf{q}^\mathsf{T} = (q_1, q_2, q_3)$, $\mathbf{v}^\mathsf{T} = (v_1, v_2, v_3)$, and $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_3)$.

Objective . . . .

Minimize $$J = \int_0^{t_F} \mathbf{u}^\mathsf{T}\mathbf{u}\, dt$$

$$\boxed{J^* = 20.4042452}$$

Example 62.3 robo03: Minimum Time With Regularization.

Repeat example 62.2 with $\rho = 10^{-5}$ in the following modified definition:

Objective . . . .

Minimize $$J = t_F + \rho \int_0^{t_F} \mathbf{u}^\mathsf{T}\mathbf{u}\, dt$$

$$\boxed{J^* = .494994960}$$

Example 62.4 robo04: Minimum Time With Switching Structure.

Phase 1 . . . . (u_{1L}, u_{2L}, u_{3U}) . . . . Phase 1

Parameters: $(t_F^{(1)})$ . . . .
Independent Variable: (t) . . . .

$t = 0$	$0 \le t \le t_F^{(1)}$	$0 \le t \le t_F^{(1)}$	sec

Differential Variables: $(q_1, q_2, q_3, v_1, v_2, v_3)$ . . . .

$q_1 = 0$	$q_{1L} \le q_1 \le q_{1U}$	$q_{1L} \le q_1 \le q_{1U}$	rad
$q_2 = -1.5$	$q_{2L} \le q_2 \le q_{2U}$	$q_{2L} \le q_2 \le q_{2U}$	rad
$q_3 = 0$	$q_{3L} \le q_3 \le q_{3U}$	$q_{3L} \le q_3 \le q_{3U}$	rad
$v_1 = 0$	$v_{1L} \le v_1 \le v_{1U}$	$v_{1L} \le v_1 \le v_{1U}$	rad/sec
$v_2 = 0$	$v_{2L} \le v_2 \le v_{2U}$	$v_2 = -1.5$	rad/sec
$v_3 = 0$	$v_{3L} \le v_3 \le v_{3U}$	$v_{3L} \le v_3 \le v_{3U}$	rad/sec

Differential-Algebraic Equations . . . .

Equations (62.5) - (62.6)

where $\mathbf{v}^\mathsf{T} = (v_1, v_2, v_3)$ and $\mathbf{u}^\mathsf{T} = (u_{1L}, u_{2L}, u_{3U})$.

Phase 2 . . . . (v_{2L}, u_{1L}, u_{3U}) . . . . Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} \le t \le t_F^{(2)}$	$t_I^{(2)} \le t \le t_F^{(2)}$	sec

Differential Variables: $(q_1, q_2, q_3, v_1, v_3)$ . . . .

$q_1 = q_{1F}^{(1)}$	$q_{1L} \le q_1 \le q_{1U}$	$q_{1L} \le q_1 \le q_{1U}$	rad
$q_2 = q_{2F}^{(1)}$	$q_{2L} \le q_2 \le q_{2U}$	$q_{2L} \le q_2 \le q_{2U}$	rad
$q_3 = q_{3F}^{(1)}$	$q_{3L} \le q_3 \le q_{3U}$	$q_{3L} \le q_3 \le q_{3U}$	rad
$v_1 = v_{1F}^{(1)}$	$v_{1L} \le v_1 \le v_{1U}$	$v_{1L} \le v_1 \le v_{1U}$	rad/sec
$v_3 = v_{3F}^{(1)}$	$v_{3L} \le v_3 \le v_{3U}$	$v_3 = 5.2$	rad/sec

Algebraic Variables: (u_2) . . . .

$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$

Boundary Conditions . . . .

$t_F^{(2)} - t_I^{(2)} \ge .001$

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{q}} = \mathbf{v} \tag{62.7}$$

$$\dot{v}_1 = \mathbf{F}_1 \tag{62.8}$$

$$\dot{v}_3 = \mathbf{F}_3 \tag{62.9}$$

$$0 = \mathbf{F}_2 \tag{62.10}$$

where $\mathbf{v}^\mathsf{T} = (v_1, v_{2L}, v_3)$ and $\mathbf{u}^\mathsf{T} = (u_{1L}, u_2, u_{3U})$.

Phase 3 . . . . (v_{2L}, v_{3U}, u_{1L}) . . . . Phase 3

Parameters: $(t_I^{(3)}, t_F^{(3)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} \le t \le t_F^{(3)}$	$t_I^{(3)} \le t \le t_F^{(3)}$	sec

Differential Variables: (q_1, q_2, q_3, v_1) ....

$q_1 = q_{1F}^{(2)}$	$q_{1L} \le q_1 \le q_{1U}$	$q_{1L} \le q_1 \le q_{1U}$	rad
$q_2 = q_{2F}^{(2)}$	$q_{2L} \le q_2 \le q_{2U}$	$q_{2L} \le q_2 \le q_{2U}$	rad
$q_3 = q_{3F}^{(2)}$	$q_{3L} \le q_3 \le q_{3U}$	$q_{3L} \le q_3 \le q_{3U}$	rad
$v_1 = v_{1F}^{(2)}$	$v_{1L} \le v_1 \le v_{1U}$	$v_1 = 3$	rad/sec

Algebraic Variables: (u_2, u_3) ....

$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$
$u_{3L} \le u_3 \le u_{3U}$	$u_{3L} \le u_3 \le u_{3U}$	$u_{3L} \le u_3 \le u_{3U}$

Boundary Conditions ....

$t_F^{(3)} - t_I^{(3)} \ge .001$

Differential-Algebraic Equations ....

$$\dot{\mathbf{q}} = \mathbf{v} \tag{62.11}$$

$$\dot{v}_1 = \mathbf{F}_1 \tag{62.12}$$

$$0 = \mathbf{F}_2 \tag{62.13}$$

$$0 = \mathbf{F}_3 \tag{62.14}$$

where $\mathbf{v}^\mathsf{T} = (v_1, v_{2L}, v_{3U})$ and $\mathbf{u}^\mathsf{T} = (u_{1L}, u_2, u_3)$.

Phase 4 .... (v_{1U}, v_{2L}, v_{3U}) .... Phase 4

Parameters: $(t_I^{(4)}, t_F^{(4)})$ ....
Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} \le t \le t_F^{(4)}$	$t_I^{(4)} \le t \le t_F^{(4)}$	sec

Differential Variables: (q_1, q_2, q_3) ....

$q_1 = q_{1F}^{(3)}$	$q_{1L} \le q_1 \le q_{1U}$	$q_{1L} \le q_1 \le q_{1U}$	rad
$q_2 = q_{2F}^{(3)}$	$q_{2L} \le q_2 \le q_{2U}$	$q_{2L} \le q_2 \le q_{2U}$	rad
$q_3 = q_{3F}^{(3)}$	$q_{3L} \le q_3 \le q_{3U}$	$q_{3L} \le q_3 \le q_{3U}$	rad

Algebraic Variables: (u_1, u_2, u_3) ....

$u_{1L} \le u_1 \le u_{1U}$	$u_{1L} \le u_1 \le u_{1U}$	$u_{1L} \le u_1 \le u_{1U}$
$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$	$u_{2L} \le u_2 \le u_{2U}$
$u_{3L} \le u_3 \le u_{3U}$	$u_{3L} \le u_3 \le u_{3U}$	$u_{3L} \le u_3 \le u_{3U}$

Boundary Conditions ....

$t_F^{(4)} - t_I^{(4)} \geq .001$

Differential-Algebraic Equations ....

$$\dot{\mathbf{q}} = \mathbf{v} \tag{62.15}$$
$$0 = \mathbf{F}_1 \tag{62.16}$$
$$0 = \mathbf{F}_2 \tag{62.17}$$
$$0 = \mathbf{F}_3 \tag{62.18}$$

where $\mathbf{v}^\mathsf{T} = (v_{1U}, v_{2L}, v_{3U})$ and $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_3)$.

Phase 5 .... (v_{1U}, v_{2L}, u_{3L}) .... Phase 5

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....

Independent Variable: (t) ....

$t = t_F^{(4)} = t_I^{(5)}$	$t_I^{(5)} \leq t \leq t_F^{(5)}$	$t_I^{(5)} \leq t \leq t_F^{(5)}$	sec

Differential Variables: (q_1, q_2, q_3, v_3) ....

$q_1 = q_{1F}^{(4)}$	$q_{1L} \leq q_1 \leq q_{1U}$	$q_{1L} \leq q_1 \leq q_{1U}$	rad
$q_2 = q_{2F}^{(4)}$	$q_{2L} \leq q_2 \leq q_{2U}$	$q_{2L} \leq q_2 \leq q_{2U}$	rad
$q_3 = q_{3F}^{(4)}$	$q_{3L} \leq q_3 \leq q_{3U}$	$q_{3L} \leq q_3 \leq q_{3U}$	rad
$v_3 = 5.2$	$v_{3L} \leq v_3 \leq v_{3U}$	$v_{3L} \leq v_3 \leq v_{3U}$	rad/sec

Algebraic Variables: (u_1, u_2) ....

$u_{1L} \leq u_1 \leq u_{1U}$	$u_{1L} \leq u_1 \leq u_{1U}$	$u_{1L} \leq u_1 \leq u_{1U}$
$u_{2L} \leq u_2 \leq u_{2U}$	$u_{2L} \leq u_2 \leq u_{2U}$	$u_{2L} \leq u_2 \leq u_{2U}$

Boundary Conditions ....

$t_F^{(5)} - t_I^{(5)} \geq .001$

Differential-Algebraic Equations ....

$$\dot{\mathbf{q}} = \mathbf{v} \tag{62.19}$$
$$\dot{v}_3 = \mathbf{F}_3 \tag{62.20}$$
$$0 = \mathbf{F}_1 \tag{62.21}$$
$$0 = \mathbf{F}_2 \tag{62.22}$$

where $\mathbf{v}^\mathsf{T} = (v_{1U}, v_{2L}, v_3)$ and $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_{3L})$.

Phase 6 . . . . (v_{1U}, u_{2U}, u_{3L}) . . . . Phase 6

Parameters: $(t_I^{(6)}, t_F^{(6)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(5)} = t_I^{(6)}$	$t_I^{(6)} \leq t \leq t_F^{(6)}$	$t_I^{(6)} \leq t \leq t_F^{(6)}$	sec

Differential Variables: $(q_1, q_2, q_3, v_2, v_3)$ . . . .

$q_1 = q_{1F}^{(5)}$	$q_{1L} \leq q_1 \leq q_{1U}$	$q_{1L} \leq q_1 \leq q_{1U}$	rad
$q_2 = q_{2F}^{(5)}$	$q_{2L} \leq q_2 \leq q_{2U}$	$q_{2L} \leq q_2 \leq q_{2U}$	rad
$q_3 = q_{3F}^{(5)}$	$q_{3L} \leq q_3 \leq q_{3U}$	$q_{3L} \leq q_3 \leq q_{3U}$	rad
$v_2 = -1.5$	$v_{2L} \leq v_2 \leq v_{2U}$	$v_{2L} \leq v_2 \leq v_{2U}$	rad/sec
$v_3 = v_{3F}^{(5)}$	$v_{3L} \leq v_3 \leq v_{3U}$	$v_{3L} \leq v_3 \leq v_{3U}$	rad/sec

Algebraic Variables: (u_1) . . . .

$u_{1L} \leq u_1 \leq u_{1U}$	$u_{1L} \leq u_1 \leq u_{1U}$	$u_{1L} \leq u_1 \leq u_{1U}$

Boundary Conditions . . . .

$t_F^{(6)} - t_I^{(6)} \geq .001$

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{q}} = \mathbf{v} \tag{62.23}$$

$$\dot{v}_2 = \mathbf{F}_2 \tag{62.24}$$

$$\dot{v}_3 = \mathbf{F}_3 \tag{62.25}$$

$$0 = \mathbf{F}_1 \tag{62.26}$$

where $\mathbf{v}^\mathsf{T} = (v_{1U}, v_2, v_3)$ and $\mathbf{u}^\mathsf{T} = (u_1, u_{2U}, u_{3L})$.

Phase 7 . . . . (u_{1U}, u_{2U}, u_{3L}) . . . . Phase 7

Parameters: $(t_I^{(7)}, t_F^{(7)})$ . . . .
Independent Variable: (t) . . . .

$t = t_F^{(6)} = t_I^{(7)}$	$t_I^{(7)} \leq t \leq t_F^{(7)}$	$t_I^{(7)} \leq t \leq t_F^{(7)}$	sec

Differential Variables: $(q_1, q_2, q_3, v_1, v_2, v_3)$ . . . .

$q_1 = q_{1F}^{(6)}$	$q_{1L} \le q_1 \le q_{1U}$	$q_{1L} \le q_1 \le q_{1U}$	rad
$q_2 = q_{2F}^{(6)}$	$q_{2L} \le q_2 \le q_{2U}$	$q_{2L} \le q_2 \le q_{2U}$	rad
$q_3 = q_{3F}^{(6)}$	$q_{3L} \le q_3 \le q_{3U}$	$q_{3L} \le q_3 \le q_{3U}$	rad
$v_1 = 3$	$v_{1L} \le v_1 \le v_{1U}$	$v_{1L} \le v_1 \le v_{1U}$	rad/sec
$v_2 = v_{2F}^{(6)}$	$v_{2L} \le v_2 \le v_{2U}$	$v_{2L} \le v_2 \le v_{2U}$	rad/sec
$v_3 = v_{3F}^{(6)}$	$v_{3L} \le v_3 \le v_{3U}$	$v_{3L} \le v_3 \le v_{3U}$	rad/sec

Boundary Conditions ....

$t_F^{(7)} - t_I^{(7)} \ge .001$

Differential-Algebraic Equations ....

Equations (62.5) - (62.6)

where $\mathbf{v}^\mathsf{T} = (v_1, v_2, v_3)$ and $\mathbf{u}^\mathsf{T} = (u_{1U}, u_{2U}, u_{3L})$.

Phase 8 .... (u_{1U}, u_{2U}, u_{3U}) .... Phase 8

Parameters: $(t_I^{(8)}, t_F^{(8)})$ ....
Independent Variable: (t) ....

$t = t_F^{(7)} = t_I^{(8)}$	$t_I^{(8)} \le t \le t_F^{(8)}$	$t_I^{(8)} \le t \le t_F^{(8)}$	sec

Differential Variables: $(q_1, q_2, q_3, v_1, v_2, v_3)$ ....

$q_1 = q_{1F}^{(7)}$	$q_{1L} \le q_1 \le q_{1U}$	$q_{1L} \le q_1 \le q_{1U}$	rad
$q_2 = q_{2F}^{(7)}$	$q_{2L} \le q_2 \le q_{2U}$	$q_{2L} \le q_2 \le q_{2U}$	rad
$q_3 = q_{3F}^{(7)}$	$q_{3L} \le q_3 \le q_{3U}$	$q_{3L} \le q_3 \le q_{3U}$	rad
$v_1 = v_{1F}^{(7)}$	$v_{1L} \le v_1 \le v_{1U}$	$v_{1L} \le v_1 \le v_{1U}$	rad/sec
$v_2 = v_{2F}^{(7)}$	$v_{2L} \le v_2 \le v_{2U}$	$v_{2L} \le v_2 \le v_{2U}$	rad/sec
$v_3 = v_{3F}^{(7)}$	$v_{3L} \le v_3 \le v_{3U}$	$v_{3L} \le v_3 \le v_{3U}$	rad/sec

Boundary Conditions ....

$t_F^{(8)} - t_I^{(8)} \ge .001$

Differential-Algebraic Equations ....

Equations (62.5) - (62.6)

where $\mathbf{v}^\mathsf{T} = (v_1, v_2, v_3)$ and $\mathbf{u}^\mathsf{T} = (u_{1U}, u_{2U}, u_{3U})$.

Phase 9 .... (u_{1U}, u_{2L}, u_{3U}) .... Phase 9

Parameters: $(t_I^{(9)}, t_F^{(9)})$ ....
Independent Variable: (t) ....

$t = t_F^{(8)} = t_I^{(9)}$	$t_I^{(9)} \le t \le t_F^{(9)}$	$t_I^{(9)} \le t \le t_F^{(9)}$	sec

Differential Variables: $(q_1, q_2, q_3, v_1, v_2, v_3)$ ....

$q_1 = q_{1F}^{(8)}$	$q_{1L} \le q_1 \le q_{1U}$	$q_1 = 1$	rad
$q_2 = q_{2F}^{(8)}$	$q_{2L} \le q_2 \le q_{2U}$	$q_2 = -1.95$	rad
$q_3 = q_{3F}^{(8)}$	$q_{3L} \le q_3 \le q_{3U}$	$q_3 = 1$	rad
$v_1 = v_{1F}^{(8)}$	$v_{1L} \le v_1 \le v_{1U}$	$v_1 = 0$	rad/sec
$v_2 = v_{2F}^{(8)}$	$v_{2L} \le v_2 \le v_{2U}$	$v_2 = 0$	rad/sec
$v_3 = v_{3F}^{(8)}$	$v_{3L} \le v_3 \le v_{3U}$	$v_3 = 0$	rad/sec

Boundary Conditions ....

$t_F^{(9)} - t_I^{(9)} \ge .001$

Differential-Algebraic Equations ....

Equations (62.5) - (62.6)

where $\mathbf{v}^\mathsf{T} = (v_1, v_2, v_3)$ and $\mathbf{u}^\mathsf{T} = (u_{1U}, u_{2L}, u_{3U})$.

Objective ....

Minimize $J = t_F^{(9)}$

$\boxed{J^* = .49518904}$

$q_{1L} = -2.97$	$q_{1U} = 2.97$
$q_{2L} = -2.01$	$q_{2U} = 2.01$
$q_{3L} = -2.86$	$q_{3U} = 2.86$
$v_{1L} = -3$	$v_{1U} = 3$
$v_{2L} = -1.5$	$v_{2U} = 1.5$
$v_{3L} = -5.2$	$v_{3U} = 5.2$
$u_{1L} = -7.5$	$u_{1U} = 7.5$
$u_{2L} = -7.5$	$u_{2U} = 7.5$
$u_{3L} = -7.5$	$u_{3U} = 7.5$

Table 62.1. *Industrial Robot example constants.*

Chapter 63

skwz: **Andrew's Squeezer Mechanism**

Hairer and Wanner [53, pp. 530–542] describe an example of a *multibody system* called "Andrew's squeezer mechanism" and have supplied a software implementation of the relevant equations. The problem is used as a benchmark for testing a number of different multibody simulation codes as described in [82]. When the torque appearing in the equations is a constant, the problem is simply and IVP. However, an optimal control problem can be posed by treating the torque as a variable to be minimized, as discussed in [13, Sect. 6.10].

Example 63.1 skwz01: Initial Value Problem.

Phase 1 . . . . Phase 1

Independent Variable: (t) . . . .

$t = 0$ $\quad\quad$ $0 \leq t \leq .03$ $\quad\quad$ $t = .03$

Differential Variables: $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ . . . .

$p_1 = \beta_0$
$p_2 = \Theta_0$
$p_3 = \gamma_0$
$p_4 = \Phi_0$
$p_5 = \delta_0$
$p_6 = \Omega_0$
$p_7 = \varepsilon_0$
$\mathbf{v} = \mathbf{0}$

Algebraic Variables: $(q_1, q_2, q_3, q_4, q_5, q_6, q_7, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$ . . . .

Differential-Algebraic Equations . . . .

$$\dot{\mathbf{p}} = \mathbf{v} \tag{63.1}$$

$$\dot{\mathbf{v}} = \mathbf{q} \tag{63.2}$$
$$\mathbf{0} = \mathbf{M}(\mathbf{p})\mathbf{q} - \mathbf{f}(\mathbf{v},\mathbf{p},u) + \mathbf{G}^{\mathsf{T}}(\mathbf{p})\boldsymbol{\lambda} \tag{63.3}$$
$$\mathbf{0} = \mathbf{g}_{pp}(\mathbf{p})(\mathbf{v},\mathbf{v}) + \mathbf{G}(\mathbf{p})\mathbf{q} \tag{63.4}$$

where

$$\mathbf{p}^{\mathsf{T}} = (p_1, p_2, p_3, p_4, p_5, p_6, p_7) \tag{63.5}$$
$$\mathbf{v}^{\mathsf{T}} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7) \tag{63.6}$$
$$\mathbf{q}^{\mathsf{T}} = (q_1, q_2, q_3, q_4, q_5, q_6, q_7) \tag{63.7}$$
$$\boldsymbol{\lambda}^{\mathsf{T}} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \tag{63.8}$$

For this example $u = u_0 = 0.033$ and the derivation and implementation of software to calculate the DAE functions $\mathbf{M}$, $\mathbf{f}$, $\mathbf{G}$, $\mathbf{g}$, and $\mathbf{g}_{pp}$ is given in [53, pp. 530–542].

Example 63.2 skwz02: Minimum Energy.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 \le t \le .03$	$t = .03$

Differential Variables: $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ ....

$p_1 = \beta_0$		$p_1 = 15.8106$
$p_2 = \Theta_0$		
$p_3 = \gamma_0$		
$p_4 = \Phi_0$		
$p_5 = \delta_0$		
$p_6 = \Omega_0$		
$p_7 = \varepsilon_0$		
$\mathbf{v} = \mathbf{0}$		

Algebraic Variables: $(q_1, q_2, q_3, q_4, q_5, q_6, q_7, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, u)$ ....

$0 \le u \le .066$	$0 \le u \le .066$	$0 \le u \le .066$

Differential-Algebraic Equations ....

Equations (63.1) - (63.8)

Objective ....

Minimize $$J = \frac{1}{t_F u_0^2} \int_0^{t_F} u^2(t) dt$$

$\boxed{J^* = .667075654}$

Example 63.3 skwz03: Minimum Time.

Repeat example 63.2 with the following changes:

Parameters: (t_F) ....

$10^{-4} \leq t_F \leq .045$

Independent Variable: (t) ....

$t = 0$ $\quad 0 \leq t \leq t_F$ $\quad t = t_F$

Objective ....

Minimize $$J = t_F$$

$\boxed{J^* = .0250513707}$

Example 63.4 skwz04: Multiphase Minimum Energy.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\quad 0 \leq t \leq .01$ $\quad t = .01$

Differential Variables: $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ ....

$p_1 = \beta_0$
$p_2 = \Theta_0$
$p_3 = \gamma_0$
$p_4 = \Phi_0$
$p_5 = \delta_0$
$p_6 = \Omega_0$
$p_7 = \varepsilon_0$
$\mathbf{v} = \mathbf{0}$

Algebraic Variables: $(q_1, q_2, q_3, q_4, q_5, q_6, q_7, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, u)$ ....

$0 \le u \le .066$ | $0 \le u \le .066$ | $0 \le u \le .066$

Differential-Algebraic Equations ....

Equations (63.1) - (63.8)

Phase 2 .... Phase 2

Independent Variable: (t) ....

$t = .01$ | $.01 \le t \le .02$ | $t = .02$

Differential Variables: $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ ....

$p_1 = p_{1F}^{(1)}$
$v_1 = v_{1F}^{(1)}$

Algebraic Variables: $(q_1, q_2, q_3, q_4, q_5, q_6, q_7, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, u)$ ....

$0 \le u \le .066$ | $0 \le u \le .066$ | $0 \le u \le .066$

Boundary Conditions ....

$\mathbf{g}(\mathbf{p}) = \mathbf{0}$
$\mathbf{G}(\mathbf{p})\mathbf{v} = \mathbf{0}$

Differential-Algebraic Equations ....

Equations (63.1) - (63.8)

Phase 3 .... Phase 3

Independent Variable: (t) ....

$t = .02$ | $.02 \le t \le .03$ | $t = .03$

Differential Variables: $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ ....

$p_1 = p_{1F}^{(2)}$ | | $p_1 = 15.8106$
$v_1 = v_{1F}^{(2)}$

Algebraic Variables: $(q_1, q_2, q_3, q_4, q_5, q_6, q_7, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, u)$ ....

$0 \leq u \leq .066$ $\qquad$ $0 \leq u \leq .066$ $\qquad$ $0 \leq u \leq .066$

Boundary Conditions ....

$\mathbf{g}(\mathbf{p}) = \mathbf{0}$
$\mathbf{G}(\mathbf{p})\mathbf{v} = \mathbf{0}$

Differential-Algebraic Equations ....

Equations (63.1) - (63.8)

Objective ....

Minimize
$$J = \frac{1}{t_F u_0^2} \int_0^{t_F} u^2(t) dt$$

$\boxed{J^* = .666960939}$

β_0	=	$-0.61713890014276449635894845800 1 \times 10^{-1}$
Θ_0	=	0
γ_0	=	$0.455279819163070380255912382449$
Φ_0	=	$0.222668390165885884674473185609$
δ_0	=	$0.487364979543842550225598953530$
Ω_0	=	$-0.222668390165885884674473185609$
ε_0	=	$0.123054744454982119249735015568 \times 10^{1}$

Table 63.1. *Dynamic Model Parameters*

Chapter 64

soar: **Dynamic Soaring**

Yiyuan Zhao [87] describes a collection of optimal control problems that define optimal patterns of glider dynamic soaring utilizing wind gradients. The example given here computes the minimum wind gradient slope that can sustain an energy-neutral dynamic soaring flight.

Example 64.1 soar01: Minimum Wind Factor.

Phase 1 . . . . Phase 1

Parameters: (β, t_F) . . . .

$0 \le \beta \le 0.15$	$10 \le t_F \le 30$	

Independent Variable: (t) . . . .

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: $(x, y, h, v, \gamma, \psi)$ . . . .

$x = 0$	$-1500 \le x \le 1500$	$x = 0$
$y = 0$	$-1000 \le y \le 1000$	$y = 0$
$h = 0$	$0 \le h \le 1000$	$h = 0$
$10 \le v \le 350$	$10 \le v \le 350$	$10 \le v \le 350$
$-75^0 \le \gamma \le 75^0$	$-75^0 \le \gamma \le 75^0$	$-75^0 \le \gamma \le 75^0$
$-450^0 \le \psi \le 0^0$	$-450^0 \le \psi \le 0^0$	$-450^0 \le \psi \le 0^0$

Algebraic Variables: (C_L, σ) . . . .

$0 \le C_L \le \overline{C}_L$	$0 \le C_L \le \overline{C}_L$	$0 \le C_L \le \overline{C}_L$
$-75^0 \le \sigma \le 0^0$	$-75^0 \le \sigma \le 0^0$	$-75^0 \le \sigma \le 0^0$

Boundary Conditions . . . .

$$0 \le \int_0^{t_F} C_L^2 dt \ \le 10^5$$

$$\psi_I - \psi_F = 360^0$$

$$v_F = v_I$$
$$\gamma_F = \gamma_I$$

Differential-Algebraic Equations . . . .

$$\dot{x} = v \cos\gamma \sin\psi + W_x \tag{64.1}$$
$$\dot{y} = v \cos\gamma \cos\psi \tag{64.2}$$
$$\dot{h} = v \sin\gamma \tag{64.3}$$
$$\dot{v} = -D/m - g_0 \sin\gamma - \dot{W}_x \cos\gamma \sin\psi \tag{64.4}$$
$$\dot{\gamma} = (L \cos\sigma - w \cos\gamma + m\dot{W}_x \sin\gamma \sin\psi)/(mv) \tag{64.5}$$
$$\dot{\psi} = (L \sin\sigma - m\dot{W}_x \cos\psi)/(mv \cos\gamma) \tag{64.6}$$
$$-2 \le \frac{L}{w} \le 5 \tag{64.7}$$

where Table 64.1 defines the problem constants and

$$w = mg_0 \tag{64.8}$$
$$q = \frac{1}{2}\rho_0 v^2 \tag{64.9}$$
$$C_D = C_{D0} + KC_L^2 \tag{64.10}$$
$$L = qSC_L \tag{64.11}$$
$$D = qSC_D \tag{64.12}$$
$$W_x = \beta h + W_0 \tag{64.13}$$
$$\dot{W}_x = \beta \dot{h} \tag{64.14}$$

Objective . . . .

Minimize $$J = \beta$$

$$\boxed{J^* = 6.35863657 \times 10^{-2}; \quad t_F^* = 25.366666}$$

$W_0 = 0$	$m = 5.6$
$g_0 = 32.2$	$S = 45.09703$
$C_{D0} = .00873$	$K = .045$
$\rho_0 = .002378$	$\overline{C}_L = 1.5$

Table 64.1. *Dyanmic Soaring example parameters.*

Chapter 65

ssmd: **Space Station Attitude Control**

In his Master's thesis, Pietz [76] presents results for an application that arises when trying to control the attitude of the International Space Station. A modified minimum energy objective, that is more well-behaved than the original formulation, is given in reference [13, Sect. 6.7]. The formulation of this problem utilizes Euler-Rodriguez parameters to define the vehicle attitude, in contrast to the more commonly used *quaternions* (cf (10.1), (24.1)).

Example 65.1 ssmd01: INTERNATIONAL SPACE STATION MOMENTUM DUMPING.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 1800$	$t = 1800$

Differential Variables: $(\boldsymbol{\omega}^\mathsf{T}, \mathbf{r}^\mathsf{T}, \mathbf{h}^\mathsf{T})$ ....

$\boldsymbol{\omega} = \overline{\boldsymbol{\omega}}_0$	$-.002 \le \omega \le .002$	$-.002 \le \omega \le .002$
$\mathbf{r} = \overline{\mathbf{r}}_0$	$-1 \le r \le 1$	$-1 \le r \le 1$
$\mathbf{h} = \overline{\mathbf{h}}_0$	$-15000 \le h \le 15000$	$-15000 \le h \le 15000$

Algebraic Variables: $(\mathbf{u}^\mathsf{T})$ ....

$-150 \le u \le 150$	$-150 \le u \le 150$	$-150 \le u \le 150$

Boundary Conditions ....

$$\mathbf{0} = \dot{\boldsymbol{\omega}}$$
$$\mathbf{0} = \dot{\mathbf{r}}$$

Differential-Algebraic Equations . . . .

$$\dot{\boldsymbol{\omega}} = \mathbf{J}^{-1}\left\{\boldsymbol{\tau}_{gg}(\mathbf{r}) - \boldsymbol{\omega}^{\otimes}\left[\mathbf{J}\boldsymbol{\omega} + \mathbf{h}\right] - \mathbf{u}\right\} \tag{65.1}$$

$$\dot{\mathbf{r}} = \frac{1}{2}\left[\mathbf{r}\mathbf{r}^{\mathsf{T}} + \mathbf{I} + \mathbf{r}^{\otimes}\right]\left[\boldsymbol{\omega} - \boldsymbol{\omega}_0(\mathbf{r})\right] \tag{65.2}$$

$$\dot{\mathbf{h}} = \mathbf{u} \tag{65.3}$$

$$0 \le h_{max} - \|\mathbf{h}\| \tag{65.4}$$

where

$$\mathbf{a}^{\otimes} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{65.5}$$

$$\boldsymbol{\tau}_{gg}(\mathbf{r}) = 3\omega_{orb}^2 \mathbf{C}_3^{\otimes}\mathbf{J}\mathbf{C}_3 \tag{65.6}$$

$$\boldsymbol{\omega}_0(\mathbf{r}) = -\omega_{orb}\mathbf{C}_2 \tag{65.7}$$

where $\mathbf{C}_2$ and $\mathbf{C}_3$ are the second and third columns respectively of

$$\mathbf{C} = \mathbf{I} + \frac{2}{1 + \mathbf{r}^{\mathsf{T}}\mathbf{r}}\left(\mathbf{r}^{\otimes}\mathbf{r}^{\otimes} - \mathbf{r}^{\otimes}\right). \tag{65.8}$$

In addition to the values given in Table 65.1, the problem constants are $h_{max} = 10000$, $\omega_{orb} = .06511(\pi/180)$ and

$$\mathbf{J} = \begin{pmatrix} 2.80701911616 \times 10^7 & 4.822509936 \times 10^5 & -1.71675094448 \times 10^7 \\ 4.822509936 \times 10^5 & 9.5144639344 \times 10^7 & 6.02604448 \times 10^4 \\ -1.71675094448 \times 10^7 & 6.02604448 \times 10^4 & 7.6594401336 \times 10^7 \end{pmatrix}. \tag{65.9}$$

Objective . . . .

Minimize $$J = 10^{-6}\int_0^{1800} \mathbf{u}^{\mathsf{T}}(t)\mathbf{u}(t)dt$$

$$\boxed{J^* = 3.58688358}$$

$\overline{\boldsymbol{\omega}}_0$	$\overline{\mathbf{r}}_0$	$\overline{\mathbf{h}}_0$
-9.5380685844896$\times 10^{-6}$	2.9963689649816$\times 10^{-3}$	5000.
-1.1363312657036$\times 10^{-3}$	1.5334477761054$\times 10^{-1}$	5000.
5.3472801108427$\times 10^{-6}$	3.8359805613992$\times 10^{-3}$	5000.

Table 65.1. *Space Station Model Parameters*

Chapter 66

stgl: Innate Immune Response

Stengel, Ghigliazza, Kulkarni, and Laplace [83] present an example that incorporates a delay-differential equation model for a biomedical application. When formulated using the method of steps, an optimal control problem with 40 states, 40 controls, and 72 boundary conditions is obtained.

Example 66.1 stgl01: INNATE IMMUNE RESPONSE.

Phase 1 .... *DDE: Method of Steps* .... Phase 1

Independent Variable: (t) ....

$t = 0$ $\qquad 0 < t < \delta \qquad t = \delta = 1$

Differential Variables: $(y_1, \ldots, y_{LN} \qquad LN = 40)$ ....

$y_1 = 3$
$y_2 = 2$
$y_3 = 4/3$
$y_4 = 0$

where for $t_F = 10$, $N = t_F/\delta = 10$, $L = 4$ and $M = 4$.

Algebraic Variables: $(v_1, \ldots, v_{MN} \qquad MN = 40)$ ....

Boundary Conditions ....

$y_{j+kL}(\delta) = y_{j+L+kL}(0)$
$v_{j+kM}(\delta) = v_{j+M+kM}(0)$

for $k = 0, 1, \ldots, N-2$ and $j = 1, 2, 3, 4$.

Differential-Algebraic Equations ....

$$\dot{y}_{1+kL} = (a_{11} - a_{12}y_{3+kL})y_{1+kL} + b_1 v_{1+kM} \tag{66.1}$$

$$\dot{y}_{2+kL} = a_{21}(y_{4+kL})a_{22}y_{1+kL-L}y_{3+kL-L} - a_{23}(y_{2+kL} - x_2^*) + b_2v_{2+kM} \tag{66.2}$$
$$\dot{y}_{3+kL} = a_{31}y_{2+kL} - (a_{32} + a_{33}y_{1+kL})y_{3+kL} + b_3v_{3+kM} \tag{66.3}$$
$$\dot{y}_{4+kL} = a_{41}y_{1+kL} - a_{42}y_{4+kL} + b_4v_{4+kM} \tag{66.4}$$

for $k = 0, 1, \ldots, N-1$, where $L = 4$ and $M = 4$. In addition for $0 \leq t \leq \delta$

$$y_{1-L}(t) = 0 \tag{66.5}$$
$$y_{3-L}(t) = 3 \tag{66.6}$$

The problem coefficients are defined as

$$a_{11} = 1 \quad a_{12} = 1 \quad a_{22} = 3 \quad a_{23} = 1 \tag{66.7}$$
$$a_{31} = 1 \quad a_{32} = 1.5 \quad a_{33} = .5 \quad a_{41} = 1 \tag{66.8}$$
$$a_{42} = 1 \quad b_1 = -1 \quad b_2 = 1 \quad b_3 = 1 \tag{66.9}$$
$$b_4 = -1 \quad x_2^* = 2 \tag{66.10}$$

Objective ....

Minimize

$$J = \frac{1}{2}\left[y_{1+(N-1)L}^2 + y_{4+(N-1)L}^2\right] + \frac{1}{2}\int_0^\delta \sum_{k=0}^{N-1}\left[y_{1+kL}^2 + y_{4+kL}^2 + \left(\sum_{j=1}^{4} v_{j+kM}^2\right)\right] dt \tag{66.11}$$

$$\boxed{J^* = 4.42844156}$$

Chapter 67

tb2s: Two-Strain Tuberculosis Model

In their paper Jung, Lenhart, and Feng [62], present an optimal control model for two-strain tuberculosis treatment. Reference [13, Sect. 6.16] describes the example given here.

Example 67.1 tb2s01: Minimum Infectious Strain and Cost.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 5$	$t = 5$

Differential Variables: $(S, T, L_1, I_1, L_2, I_2)$ ....

$S = 76N/120$
$T = N/120$
$L_1 = 36N/120$
$I_1 = 4N/120$
$L_2 = 2N/120$
$I_2 = N/120$

Algebraic Variables: (u_1, u_2) ....

$.05 \le u_1 \le .95$	$.05 \le u_1 \le .95$	$.05 \le u_1 \le .95$
$.05 \le u_2 \le .95$	$.05 \le u_2 \le .95$	$.05 \le u_2 \le .95$

Differential-Algebraic Equations ....

$$\dot{S} = \Lambda - \beta_1 S \frac{I_1}{N} - \beta^* S \frac{I_2}{N} - \mu S \tag{67.1}$$

$$\dot{T} = u_1 r_1 L_1 - \mu T + (1 - (1 - u_2)(p + q)) r_2 I_1 - \beta_2 T \frac{I_1}{N} - \beta^* T \frac{I_2}{N} \tag{67.2}$$

$$\dot{L}_1 = \beta_1 S \frac{I_1}{N} - (\mu + k_1) L_1 - u_1 r_1 L_1 + (1 - u_2) p r_2 I_1 + \beta_2 T \frac{I_1}{N} - \beta^* L_1 \frac{I_2}{N} \tag{67.3}$$

$$\dot{L}_2 = (1 - u_2) q r_2 I_1 - (\mu + k_2) L_2 + \beta^* (S + L_1 + T) \frac{I_2}{N} \tag{67.4}$$

$$\dot{I}_1 = k_1 L_1 - (\mu + d_1) I_1 - r_2 I_1 \tag{67.5}$$

$$\dot{I}_2 = k_2 L_2 - (\mu + d_2) I_2 \tag{67.6}$$

Objective . . . .

Minimize $$J = \int_0^5 \left[L_2 + I_2 + \frac{1}{2} B_1 u_1^2 + \frac{1}{2} B_2 u_2^2 \right] dt$$

$\boxed{J^* = 5152.07310}$

$\beta_1 = 13$	$\beta_2 = 13$	$\mu = .0143$	$d_1 = 0$
$d_2 = 0$	$k_1 = .5$	$k_2 = 1$	$r_1 = 2$
$r_2 = 1$	$p = .4$	$q = .1$	$N = 30000$
$\beta^* = .029$	$B_1 = 50$	$B_2 = 500$	$\Lambda = \mu N$

Table 67.1. *Tuberculosis Model Parameters*

Chapter 68

tmpr: **Temperature Control**

A model defined by a partial differential equation can be transformed to a system of ordinary differential equations using the method of lines. Optimal control techniques can then be applied to the resulting system of ODE's. Three different problems that demonstrate this technique are given the test collection. Example (34.1) first appeared in reference [22]. A more complex process given as example (34.2) was first discussed by Heinkenschloss in reference [56] and is also addressed in reference [13, Sect. 4.6.10]. The example given here describes the solution of a system described by a partial differential equation with two spatial dimensions in addition to time. The resulting large-scale optimal control problem was first presented in [49]

.

Example 68.1 tmpr01: Minimum Deviation Heating, Boundary Control.

Phase 1 .... *PDE using Method of Lines* .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 2.0$	$t = 2.0$

Differential Variables: $(T_{i,j};\ i = 0, \ldots, m;\ j = 0, \ldots, n)$ ....

$0 \le T_{i,j} \le .7$	$0 \le T_{i,j} \le .7$	$0 \le T_{i,j} \le .7$

Algebraic Variables: (u) ....

$0 \le u \le 1$	$0 \le u \le 1$	$0 \le u \le 1$

Differential-Algebraic Equations ....

$$\dot{T}_{i,j} = \frac{\alpha_{i,j}}{(\Delta x)^2}\left[T_{i-1,j} - 2T_{i,j} + T_{i+1,j}\right] + \frac{\alpha_{i,j}}{(\Delta y)^2}\left[T_{i,j-1} - 2T_{i,j} + T_{i,j+1}\right] + S_{i,j} \tag{68.1}$$

for $i = 0, \ldots, m$ and $j = 0, \ldots, n$. The spatial discretization of the domain $0 \le x \le x_{max}$ is given by

$$x_i = i\Delta x \tag{68.2}$$

where $\Delta x = x_{max}/m$, and similarly the domain $0 \le y \le y_{max}$ is discretized by

$$y_j = j\Delta y \tag{68.3}$$

where $\Delta y = y_{max}/n$. The source term is given by

$$S_{i,j} = S_{max} \exp\left[\frac{-\beta_1}{\beta_2 + T_{i,j}}\right] \tag{68.4}$$

The boundary controls are given by

$$u_1(x,t) = \begin{cases} u(t) & \text{for } 0 \le x \le .2 \\ \left(1 - \dfrac{x - .2}{1.2}\right) u(t) & \text{for } .2 \le x \le .8 \end{cases} \tag{68.5}$$

$$u_2(y,t) = \begin{cases} u(t) & \text{for } 0 \le y \le .4 \\ \left(1 - \dfrac{y - .4}{2.4}\right) u(t) & \text{for } .4 \le y \le 1.6 \end{cases} \tag{68.6}$$

Values outside of the domain $\Omega = \{(x,y) \mid 0 \le x \le x_{max}, 0 \le y \le y_{max}\}$ are eliminated using the boundary conditions leading to the following expressions for $i = 0, \ldots, m$

$$\sigma_y = (2\Delta y)/\lambda \tag{68.7}$$

$$T_{i,n+1} = T_{i,n-1} \tag{68.8}$$

$$T_{i,-1} = \sigma_y \left[u_1(x_i,t) - T_{i,0}\right] + T_{i,1}/\sigma_y \tag{68.9}$$

and for $j = 0, \ldots, n$

$$\sigma_x = (2\Delta x)\lambda \tag{68.10}$$

$$T_{m+1,j} = T_{m-1,j} \tag{68.11}$$

$$T_{-1,j} = \sigma_x \left[u_2(y_j,t) - T_{0,j}\right] + T_{1,j}/\sigma_x. \tag{68.12}$$

For example we set $m = 4$, $n = 8$, $\alpha_{i,j} = 1$, $\beta_1 = .2$, $\beta_2 = .05$, $\lambda = .5$, $S_{max} = .5$, $x_{max} = .8$, $x_c = .6$, $y_{max} = 1.6$, and $y_c = .6$.

Objective ....

Minimize

$$J = \sum_{i=1}^{M} \sum_{j=1}^{N} \int_0^2 c_{i,j} \left[T_{m-M+i,n-N+j}(t) - \tau(t)\right]^2 dt$$

where

$$c_{i,j} = w_i v_j \Delta x \Delta y \tag{68.13}$$

$$w_i = \begin{cases} .5 & \text{for } i = 1 \text{ or } i = M \\ 1 & \text{otherwise} \end{cases} \tag{68.14}$$

$$v_j = \begin{cases} .5 & \text{for } j = 1 \text{ or } j = N \\ 1 & \text{otherwise} \end{cases} \tag{68.15}$$

$$\tau(t) = \begin{cases} 1.25(t - .2) & \text{for } .2 < t \leq .6 \\ .5 & \text{for } .6 < t \leq 1 \\ .5 - .75(t - 1) & \text{for } 1 < t \leq 1.4 \\ .2 & \text{for } 1.4 < t \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{68.16}$$

$$M = \text{nint}\left[\frac{x_{max} - x_c}{\Delta x}\right] + 1 \tag{68.17}$$

$$N = \text{nint}\left[\frac{y_{max} - y_c}{\Delta y}\right] + 1 \tag{68.18}$$

and "nint" denotes the "nearest integer."

$$\boxed{J^* = 5.25049005 \times 10^{-4}}$$

Chapter 69

traj: **Trajectory Examples**

A collection of common trajectory optimization problems, all formulated using *intrinsic or flight path* coordinates are given here. Using these coordinates, example (69.1) defines a two-burn orbit transfer, that addresses the same physical application as a number of other examples, namely (14.1)-(14.4), and (35.1)-(35.2). Construction of the reentry trajectory for the space shuttle is a classic example of an optimal control problem. The problem is of considerable practical interest and is nearly intractable using a simple shooting method because of its nonlinear behavior. Early results were presented by Bulirsch [30] on one version of the problem, as well by Dickmanns [39]. Ascher, Mattheij, and Russell present a similar problem [2, p. 23] and Brenan, Campbell, and Petzold discuss a closely related path control problem [27, p. 157]. Four different versions of the optimal reentry trajectory for the space shuttle are given. Example (69.2) is a maximum downrange reentry [17, 5, 88]. Examples (69.3)-(69.5) all define maximum crossrange cases [13, Sect. 6.1], where examples (69.4) and (69.5) add constraints on the control and aerodynamic heating, respectively.

Example 69.1 traj03: Two-Burn Orbit Transfer.

Phase 1 .... *Park Orbit Coast* .... Phase 1

Parameters: $(t_F^{(1)})$ ....

$-1000 \le t_F^{(1)} \le 25000$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = 150$	100 nm $\le h \le$ 30000 nm	ft

$\phi = -5^o$	$-90^o \le \phi \le +270^o$	rad
$\theta = -3^o$	$-89^o \le \theta \le +89^o$	rad
$v = \sqrt{\mu/r} \approx 25402.539$	$100 \le v \le 35000$	ft/sec
$\gamma = 0$	$-89^o \le \gamma \le +89^o$	rad
$\psi = 61.5^o$	$0^o \le \psi \le 180^o$	rad

Differential-Algebraic Equations . . . .

$$\dot{h} = v \sin\gamma \tag{69.1}$$

$$\dot{\phi} = \frac{v}{r\cos\theta}\cos\gamma\sin\psi \tag{69.2}$$

$$\dot{\theta} = \frac{v}{r}\cos\gamma\cos\psi \tag{69.3}$$

$$\dot{v} = -g\sin\gamma \tag{69.4}$$

$$\dot{\gamma} = \cos\gamma\left(\frac{v}{r} - \frac{g}{v}\right) \tag{69.5}$$

$$\dot{\psi} = \frac{v}{r\cos\theta}\cos\gamma\sin\psi\sin\theta \tag{69.6}$$

where $r = R_e + h$, $R_e = 20902900$ ft, $g = \mu/r^2$, and $\mu = 0.14076539 \times 10^{17}$ ft^3/sec^2.

Phase 2 . . . . *First Burn* . . . . Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ . . . .

$-1000 \le t_I^{(2)} \le 25000$ $\quad$ $-1000 \le t_F^{(2)} \le 25000$

Independent Variable: (t) . . . .

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ . . . .

$h = h_F^{(1)}$	$100 \text{ nm} \le h \le 30000 \text{ nm}$		ft
$\phi = \phi_F^{(1)}$	$-90^o \le \phi \le +270^o$		rad
$\theta = \theta_F^{(1)}$	$-89^o \le \theta \le +89^o$		rad
$v = v_F^{(1)}$	$100 \le v \le 35000$		ft/sec
$\gamma = \gamma_F^{(1)}$	$-89^o \le \gamma \le +89^o$		rad
$\psi = \psi_F^{(1)}$	$0^o \le \psi \le 180^o$		rad
$w = 33500$	$w \le 50000$	$11000 \le w \le 50000$	lb

Algebraic Variables: (α, β) . . . .

$0^o \le \alpha \le +88^o$	rad
$0^o \le \beta \le 175^o$	rad

Differential-Algebraic Equations ....

$$\dot{h} = v \sin\gamma \tag{69.7}$$

$$\dot{\phi} = \frac{v}{r\cos\theta}\cos\gamma\sin\psi \tag{69.8}$$

$$\dot{\theta} = \frac{v}{r}\cos\gamma\cos\psi \tag{69.9}$$

$$\dot{v} = \frac{T_c\cos\alpha}{m} - g\sin\gamma \tag{69.10}$$

$$\dot{\gamma} = \frac{T_c\sin\alpha\cos\beta}{mv} + \cos\gamma\left(\frac{v}{r} - \frac{g}{v}\right) \tag{69.11}$$

$$\dot{\psi} = \frac{T_c\sin\alpha\sin\beta}{mv\cos\gamma} + \frac{v}{r\cos\theta}\cos\gamma\sin\psi\sin\theta \tag{69.12}$$

$$\dot{w} = -\dot{w}_c \tag{69.13}$$

where $T_c = 43500$, $\dot{w}_c = 150$, $m = w/g_0$, and $g_0 = 32.174$.

Phase 3 .... *Transfer Orbit First Leg* .... Phase 3

Parameters: $(t_I^{(3)})$ ....

$-1000 \le t_I^{(3)} \le 25000$

Independent Variable: (t) ....

$t = t_F^{(2)} = t_I^{(3)}$	$t_I^{(3)} < t < 6000$	$t = 6000$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = h_F^{(2)}$	$100 \text{ nm} \le h \le 30000 \text{ nm}$	ft
$\phi = \phi_F^{(2)}$	$-90^o \le \phi \le +270^o$	rad
$\theta = \theta_F^{(2)}$	$-89^o \le \theta \le +89^o$	rad
$v = v_F^{(2)}$	$100 \le v \le 35000$	ft/sec
$\gamma = \gamma_F^{(2)}$	$-89^o \le \gamma \le +89^o$	rad
$\psi = \psi_F^{(2)}$	$0^o \le \psi \le 180^o$	rad

Differential-Algebraic Equations ....

Equations (69.1) - (69.6)

Phase 4 .... *Transfer Orbit Second Leg* .... Phase 4

Parameters: $(t_F^{(4)})$ ....

$-1000 \leq t_F^{(4)} \leq 25000$

Independent Variable: (t) ....

$t = 6000$	$6000 < t < t_F^{(4)}$	$t = t_F^{(4)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi)$ ....

$h = h_F^{(3)}$	$100 \text{ nm} \leq h \leq 30000 \text{ nm}$	ft
$\phi = \phi_F^{(3)}$	$-90^o \leq \phi \leq +270^o$	rad
$\theta = \theta_F^{(3)}$	$-89^o \leq \theta \leq +89^o$	rad
$v = v_F^{(3)}$	$100 \leq v \leq 35000$	ft/sec
$\gamma = \gamma_F^{(3)}$	$-89^o \leq \gamma \leq +89^o$	rad
$\psi = \psi_F^{(3)}$	$0^o \leq \psi \leq 180^o$	rad

Differential-Algebraic Equations ....

Equations (69.1) - (69.6)

Phase 5 .... *Second Burn* .... Phase 5

Parameters: $(t_I^{(5)}, t_F^{(5)})$ ....

$-1000 \leq t_I^{(5)} \leq 25000 \qquad -1000 \leq t_F^{(5)} \leq 25000$

Independent Variable: (t) ....

$t = t_F^{(4)} = t_I^{(5)}$	$t_I^{(5)} < t < t_F^{(5)}$	$t = t_F^{(5)}$	sec

Differential Variables: $(h, \phi, \theta, v, \gamma, \psi, w)$ ....

$h = h_F^{(4)}$	$100 \text{ nm} \leq h \leq 30000 \text{ nm}$	$h = 19323 \text{ nm}$	ft
$\phi = \phi_F^{(4)}$	$-90^o \leq \phi \leq +270^o$		rad
$\theta = \theta_F^{(4)}$	$-89^o \leq \theta \leq +89^o$	$\theta = 0^o$	rad
$v = v_F^{(4)}$	$100 \leq v \leq 35000$	$v = \sqrt{\mu/r} \approx 10088.312$	ft/sec
$\gamma = \gamma_F^{(4)}$	$-89^o \leq \gamma \leq +89^o$	$\gamma = 0^o$	rad
$\psi = \psi_F^{(4)}$	$0^o \leq \psi \leq 180^o$	$\psi = 90^o$	rad
$w = 11000$	$1000 \leq w$	$1000 \leq w$	lb

Algebraic Variables: (α, β) ....

$0^o \leq \alpha \leq +88^o$	rad
$0^o \leq \beta \leq 175^o$	rad

Differential-Algebraic Equations ....

Equations (69.7) - (69.13)

where $T_c = 18300$, $\dot{w}_c = 60$, $m = w/g_0$, and $g_0 = 32.174$.

Objective ....

Maximize $$J = w(t_F^{(5)})$$

$$\boxed{J^* = 6469.4662}$$

Example 69.2 traj09: SHUTTLE MAXIMUM DOWNRANGE.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$t_F \leq 4000$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$	sec

Differential Variables: (h, ϕ, v, γ) ....

$h = 260000$	$0 \leq h$	$h = 80000$	ft
$\phi = 0$			rad
$v = 25600$	$1 \leq v$	$v = 2500$	ft/sec
$\gamma = -1^o$	$-89^o \leq \gamma \leq +89^o$	$\gamma = -5^o$	rad

Algebraic Variables: (α) ....

	$-90^o \leq \alpha \leq +90^o$	rad

Differential-Algebraic Equations ....

$$\dot{h} = v \sin \gamma \tag{69.14}$$

$$\dot{\phi} = \frac{v}{r} \cos \gamma \tag{69.15}$$

$$\dot{v} = -\frac{D}{m} - g \sin \gamma \tag{69.16}$$

$$\dot{\gamma} = \frac{L}{mv} + \cos \gamma \left(\frac{v}{r} - \frac{g}{v} \right) \tag{69.17}$$

using the parameter definitions given in Table 69.1.

Objective ....

Maximize $J = \phi(t_F)$

$$\boxed{J^* = 3.2726493; \quad t_F^* = 3.6337108 \times 10^3}$$

Example 69.3 traj21: Shuttle Maximum Crossrange.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$t_F \leq 2500$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$	sec

Differential Variables: $(h, \theta, v, \gamma, \psi)$ ....

$h = 260000$	$0 \leq h$	$h = 80000$	ft
$\theta = 0$	$-89^o \leq \theta \leq +89^o$		rad
$v = 25600$	$1 \leq v$	$v = 2500$	ft/sec
$\gamma = -1^o$	$-89^o \leq \gamma \leq +89^o$	$\gamma = -5^o$	rad
$\psi = 90^o$			rad

Algebraic Variables: (α, β) ....

	$-90^o \leq \alpha \leq +90^o$		rad
	$-90^o \leq \beta \leq 1^o$		rad

Differential-Algebraic Equations ....

$$\dot{h} = v \sin\gamma \tag{69.18}$$

$$\dot{\theta} = \frac{v}{r} \cos\gamma \cos\psi \tag{69.19}$$

$$\dot{v} = -\frac{D}{m} - g \sin\gamma \tag{69.20}$$

$$\dot{\gamma} = \frac{L}{mv} \cos\beta + \cos\gamma \left(\frac{v}{r} - \frac{g}{v} \right) \tag{69.21}$$

$$\dot{\psi} = \frac{1}{mv\cos\gamma} L \sin\beta + \frac{v}{r\cos\theta} \cos\gamma \sin\psi \sin\theta \tag{69.22}$$

$$q \leq q_U \tag{69.23}$$

for $q_U = \infty$ and parameter definitions given in Table 69.1.

Objective ....

Maximize $$J = \theta(t_F)$$

$$\boxed{J^* = 5.9587608 \times 10^{-1}; \quad t_F^* = 2.0085881 \times 10^3}$$

$q = q_a q_r$	$m = w/g_0$
$D = \frac{1}{2} c_D S \rho v^2$	$a_0 = -0.20704$
$L = \frac{1}{2} c_L S \rho v^2$	$a_1 = 0.029244$
$g = \mu/r^2$	$\mu = 0.14076539 \times 10^{17}$
$r = R_e + h$	$b_0 = 0.07854$
$\rho = \rho_0 \exp[-h/h_r]$	$b_1 = -0.61592 \times 10^{-2}$
$\rho_0 = 0.002378$	$b_2 = 0.621408 \times 10^{-3}$
$h_r = 23800$	$q_r = 17700\sqrt{\rho}(0.0001v)^{3.07}$
$c_L = a_0 + a_1 \hat{\alpha}$	$q_a = c_0 + c_1\hat{\alpha} + c_2\hat{\alpha}^2 + c_3\hat{\alpha}^3$
$c_D = b_0 + b_1\hat{\alpha} + b_2\hat{\alpha}^2$	$c_0 = 1.0672181$
$\hat{\alpha} = (180/\pi)\alpha$	$c_1 = -0.19213774 \times 10^{-1}$
$R_e = 20902900$	$c_2 = 0.21286289 \times 10^{-3}$
$S = 2690$	$c_3 = -0.10117249 \times 10^{-5}$
$w = 203000$	$g_0 = 32.174$

Table 69.1. *Shuttle reentry example parameters.*

Example 69.4 traj22: Shuttle Maximum Crossrange with Control Bound. Repeat example 69.3 with the algebraic bound

$$-70^o \leq \beta \leq 1^o. \tag{69.24}$$

$$\boxed{J^* = 5.9574673 \times 10^{-1}; \quad t_F^* = 2.0346546 \times 10^3}$$

Example 69.5 traj36: Shuttle Maximum Crossrange with Heat Limit. Repeat example 69.3 with $q_U = 70$.

$$\boxed{J^* = 5.3451536 \times 10^{-1}; \quad t_F^* = 2.1986660 \times 10^3}$$

Chapter 70

tran: **Train Problem**

Vanderbei [84] poses a simple formulation that describes the motion of a train on a track defined by a terrain function. Although the terrain function used here is rather simple, the approach can be extended to important real world applications by incorporating high fidelity models of real terrain geometry and train dynamics.

Example 70.1 tran01: Minimum Fuel Cost.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 4.8$	$t = 4.8$

Differential Variables: (x, v) ....

$x = 0$		$x = 6$
$v = 0$		$v = 0$

Algebraic Variables: (u_a, u_b) ....

$0 \le u_a \le 10$	$0 \le u_a \le 10$	$0 \le u_a \le 10$
$0 \le u_b \le 2$	$0 \le u_b \le 2$	$0 \le u_b \le 2$

Differential-Algebraic Equations ....

$$\dot{x} = v \tag{70.1}$$

$$\dot{v} = h(x) - (a + bv + cv^2) + u_a - u_b \tag{70.2}$$

where

$$h(x) = \sum_{j=1}^{2} \left[\frac{s_{j+1} - s_j}{\pi} \right] \tan^{-1} \left[\frac{x - z_j}{\epsilon} \right] \tag{70.3}$$

Objective . . . .

Minimize
$$J = \int_0^{4.8} \left[u_a v + \rho(u_a^2 + u_b^2) \right] \; dt$$

where $\rho = 10^{-3}$.

$\boxed{J^* = 4.95569943}$

$a = .3$	$b = .14$	$c = .16$
$z_1 = 2$	$z_2 = 4$	$\epsilon = .05$
$s_1 = 2$	$s_2 = 0$	$s_3 = -2$

Table 70.1. *Train model constants*

Chapter 71

tumr: **Tumor Anti-angiogenesis**

Ledzewicz and Schättler [65] present a model that describes the growth of a tumor. In this process, called *angiogenesis*, there is a bi-directional signaling between tumor cells and endothelial cells: tumour cells produce vascular endothelial growth factor to stimulate endothelial cell growth; endothelial cells in turn provide the lining for the newly forming blood vessels that supply nutrients to the tumour and thus sustain tumour growth. This model describes a treatment that inhibits the growth, thereby causing regression of the tumor. A complete discussion is given in reference [13, Sect. 6.17].

Example 71.1 tumr01: Minimum Tumor Size–One Phase Formulation.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$.01 \le t_F$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: (p, q, y) ....

$p = p_0$	$.01 \le p \le \bar{p}$	$.01 \le p \le \bar{p}$
$q = q_0$	$.01 \le q \le \bar{q}$	$.01 \le q \le \bar{q}$
$y = 0$	$0 \le y$	$0 \le y \le A$

where $p_0 = \bar{p}/2$, $q_0 = \bar{q}/4$, and $\bar{p} = \bar{q} = [(b-\mu)/d]^{3/2}$. The problem constants are given in Table (71.1).

Algebraic Variables: (u) ....

$0 \le u \le a$	$0 \le u \le a$	$0 \le u \le a$

Differential-Algebraic Equations ....

$$\dot{p} = -\xi p \ln\left(\frac{p}{q}\right) \tag{71.1}$$

$$\dot{q} = q\left[b - (\mu + dp^{\frac{2}{3}} + Gu)\right] \tag{71.2}$$

$$\dot{y} = u \tag{71.3}$$

Objective ....

Minimize $J = p(t_F)$

$\boxed{J^* = 7571.67075}$

Example 71.2 tumr02: Minimum Tumor Size–Two Phase Formulation.

Phase 1 .... Phase 1

Parameters: $(t_F^{(1)})$ ....

$.01 \le t_F$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F^{(1)}$

Differential Variables: (p, q, y) ....

$p = p_0$	$.01 \le p \le \bar{p}$	$.01 \le p \le \bar{p}$
$q = q_0$	$.01 \le q \le \bar{q}$	$.01 \le q \le \bar{q}$
$y = 0$	$0 \le y$	$0 \le y \le A$

where $p_0 = \bar{p}/2$, $q_0 = \bar{q}/4$, and $\bar{p} = \bar{q} = [(b-\mu)/d]^{3/2}$. The problem constants are given in Table (71.1).

Differential-Algebraic Equations ....

$$\dot{p} = -\xi p \ln\left(\frac{p}{q}\right) \tag{71.4}$$

$$\dot{q} = q\left[b - (\mu + dp^{\frac{2}{3}} + Ga)\right] \tag{71.5}$$

$$\dot{y} = a \tag{71.6}$$

Phase 2 . . . . Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ . . . .

Independent Variable: (t) . . . .

$t = t_I^{(2)} = t_F^{(1)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$

Differential Variables: (p, q, y) . . . .

$p = p_F^{(1)}$	$.01 \le p \le \bar{p}$	$.01 \le p \le \bar{p}$
$q = q_F^{(1)}$	$.01 \le q \le \bar{q}$	$.01 \le q \le \bar{q}$
$y = y_F^{(1)}$	$0 \le y$	$0 \le y \le A$

where $p_0 = \bar{p}/2$, $q_0 = \bar{q}/4$, and $\bar{p} = \bar{q} = [(b - \mu)/d]^{3/2}$.

Differential-Algebraic Equations . . . .

$$\dot{p} = -\xi p \ln\left(\frac{p}{q}\right) \tag{71.7}$$

$$\dot{q} = q\left[b - (\mu + dp^{\frac{2}{3}})\right] \tag{71.8}$$

$$\dot{y} = 0 \tag{71.9}$$

Objective . . . .

Minimize $\qquad J = p(t_F^{(2)})$

$$\boxed{J^* = 7571.67158}$$

Example 71.3 tumr03: Minimum Tumor Size–Indirect Formulation.

Phase 1 . . . . Phase 1

Parameters: $(t_F^{(1)})$ . . . .

$.01 \le t_F$

Independent Variable: (t) . . . .

$t = 0$	$0 < t < t_F$	$t = t_F^{(1)}$

Differential Variables: $(p, q, y, \lambda_p, \lambda_q, \lambda_y)$ ....

$p = p_0$	$.01 \le p \le \bar{p}$	$.01 \le p \le \bar{p}$
$q = q_0$	$.01 \le q \le \bar{q}$	$.01 \le q \le \bar{q}$
$y = 0$	$0 \le y$	$0 \le y \le A$

where $p_0 = \bar{p}/2$, $q_0 = \bar{q}/4$, and $\bar{p} = \bar{q} = [(b-\mu)/d]^{3/2}$. The problem constants are given in Table (71.1).

Differential-Algebraic Equations ....

$$\dot{p} = -\xi p \ln\left(\frac{p}{q}\right) \tag{71.10}$$

$$\dot{q} = q\left[b - (\mu + dp^{\frac{2}{3}} + Ga)\right] \tag{71.11}$$

$$\dot{y} = a \tag{71.12}$$

$$\dot{\lambda}_p = \xi\lambda_p\left[\ln\left(\frac{p}{q}\right) + 1\right] + \frac{2}{3}\lambda_q dqp^{-\frac{1}{3}} \tag{71.13}$$

$$\dot{\lambda}_q = -\xi\lambda_p\frac{p}{q} + \lambda_q\left[b - (\mu + dp^{\frac{2}{3}} + Ga)\right] \tag{71.14}$$

$$\dot{\lambda}_y = 0 \tag{71.15}$$

Phase 2 .... Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ ....

Independent Variable: (t) ....

$t = t_I^{(2)} = t_F^{(1)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$

Boundary Conditions ....

$\Phi(t_I^{(2)}) = 0$		
		$H(t_F^{(2)}) = 0$

where

$$\Phi = \lambda_y - \lambda_q G q \tag{71.16}$$

$$H = -\lambda_p \xi p \ln\left(\frac{p}{q}\right) + \lambda_q q\left[b - (\mu + dp^{\frac{2}{3}} + Gu)\right] + \lambda_y u \tag{71.17}$$

Differential Variables: $(p, q, y, \lambda_p, \lambda_q, \lambda_y)$ ....

$p = p_F^{(1)}$	$.01 \le p \le \bar{p}$	$.01 \le p \le \bar{p}$
$q = q_F^{(1)}$	$.01 \le q \le \bar{q}$	$.01 \le q \le \bar{q}$

$y = y_F^{(1)}$	$0 \le y$	$y = A$
$\lambda_p = \lambda_{pF}^{(1)}$		$\lambda_p = 1$
$\lambda_q = \lambda_{qF}^{(1)}$		$\lambda_q = 0$
$\lambda_y = \lambda_{yF}^{(1)}$		

where $p_0 = \bar{p}/2$, $q_0 = \bar{q}/4$, and $\bar{p} = \bar{q} = [(b-\mu)/d]^{3/2}$.

Differential-Algebraic Equations ....

$$\dot{p} = -\xi p \ln\left(\frac{p}{q}\right) \tag{71.18}$$

$$\dot{q} = q\left[b - (\mu + dp^{\frac{2}{3}})\right] \tag{71.19}$$

$$\dot{y} = 0 \tag{71.20}$$

$$\dot{\lambda}_p = \xi\lambda_p\left[\ln\left(\frac{p}{q}\right) + 1\right] + \frac{2}{3}\lambda_q dqp^{-\frac{1}{3}} \tag{71.21}$$

$$\dot{\lambda}_q = -\xi\lambda_p\frac{p}{q} + \lambda_q\left[b - (\mu + dp^{\frac{2}{3}})\right] \tag{71.22}$$

$$\dot{\lambda}_y = 0 \tag{71.23}$$

Objective ....

Boundary Value Problem (BVP)

$\xi = 0.084$	$b = 5.85$	$d = 0.00873$
$G = 0.15$	$\mu = 0.02$	$a = 75$
	$A = 15$	

Table 71.1. *Tumor Model Parameters*

Chapter 72

vpol: **Van der Pol Oscillator**

Maurer and Augustin [68] discuss a version of the Van der Pol Oscillator problem with a constraint on one of the state variables. Three different versions of the problem are given here and described more fully in reference [13, pp 187-191]. The first two examples introduce the constraint as a simple bound and as a path constraint, respectively. The third example requires solution of the boundary value problem that results from an indirect formulation of the same example.

Example 72.1 vpol01: STATE BOUND FORMULATION.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 5$	$t = 5$

Differential Variables: (y_1, y_2) ....

$y_1 = 1$		
$y_2 = 0$	$-.4 \leq y_2$	$-.4 \leq y_2$

Algebraic Variables: (u) ....

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{72.1}$$

$$\dot{y}_2 = (1 - y_1^2)y_2 - y_1 + u \tag{72.2}$$

Objective ....

Minimize $$J = \int_0^5 (u^2 + y_1^2 + y_2^2)\ dt$$

$\boxed{J^* = 2.95369916}$

Example 72.2 vpol04: Path Constraint Formulation.

Phase 1 .... Phase 1

Independent Variable: (t) ....

$t = 0$	$0 < t < 5$	$t = 5$

Differential Variables: (y_1, y_2) ....

$y_1 = 1$		
$y_2 = 0$	$-.4 \le y_2$	$-.4 \le y_2$

Algebraic Variables: (u, v) ....

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{72.3}$$

$$\dot{y}_2 = v - y_1 + u \tag{72.4}$$

$$0 = v - (1 - y_1^2)y_2 \tag{72.5}$$

Objective ....

Minimize $$J = \int_0^5 (u^2 + y_1^2 + y_2^2)\ dt$$

$\boxed{J^* = 2.95369919}$

Example 72.3 vpol07: Indirect Formulation.

Phase 1 .... Phase 1

Parameters: $(t_F^{(1)})$ ....

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = 1$		
$y_2 = 0$		$y_2 = -.4$

Boundary Conditions . . . .

$$(y_1^2 - 1)y_2 + y_1 + \lambda_2/2 = 0$$

Differential-Algebraic Equations . . . .

$$\dot{y}_1 = y_2 \tag{72.6}$$
$$\dot{y}_2 = (1 - y_1^2)y_2 - y_1 + u \tag{72.7}$$
$$\dot{\lambda}_1 = -2y_1 + 2y_1 y_2 \lambda_2 + \lambda_2 \tag{72.8}$$
$$\dot{\lambda}_2 = -2y_2 - \lambda_1 + \lambda_2(y_1^2 - 1) \tag{72.9}$$

where

$$u = -\lambda_2/2 \tag{72.10}$$

Phase 2 . . . . Phase 2

Parameters: $(t_I^{(2)}, t_F^{(2)})$ . . . .

Independent Variable: (t) . . . .

$t = t_I^{(2)} = t_F^{(1)}$	$t_I^{(2)} < t < t_F^{(2)}$	$t = t_F^{(2)}$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ . . . .

$y_1 = y_{1F}^{(1)}$
$y_2 = -.4$
$\lambda_1 = \lambda_{1F}^{(1)}$

Boundary Conditions . . . .

$$(y_1^2 - 1)y_2 + y_1 + \lambda_2/2 = 0$$

Differential-Algebraic Equations . . . .

$$\dot{y}_1 = y_2 \tag{72.11}$$
$$\dot{y}_2 = (1 - y_1^2)y_2 - y_1 + u \tag{72.12}$$
$$\dot{\lambda}_1 = -2y_1 + 2y_1 y_2 \lambda_2 + \lambda_2 - \mu(2y_1 y_2 + 1) \tag{72.13}$$
$$\dot{\lambda}_2 = -2y_2 - \lambda_1 + \lambda_2(y_1^2 - 1) + \mu(1 - y_1^2) \tag{72.14}$$

where

$$u = (y_1^2 - 1)y_2 + y_1 \tag{72.15}$$

$$\mu = 2u + \lambda_2 \tag{72.16}$$

Phase 3 .... Phase 3

Parameters: $(t_I^{(3)})$ ....

Independent Variable: (t) ....

$t = t_I^{(3)} = t_F^{(2)}$	$t_I^{(3)} < t < 5$	$t = 5$

Differential Variables: $(y_1, y_2, \lambda_1, \lambda_2)$ ....

$y_1 = y_{1F}^{(2)}$	
$y_2 = -.4$	
$\lambda_1 = \lambda_{1F}^{(2)}$	$\lambda_1 = 0$
	$\lambda_2 = 0$

Boundary Conditions ....

$(y_1^2 - 1)y_2 + y_1 + \lambda_2/2 = 0$

Differential-Algebraic Equations ....

$$\dot{y}_1 = y_2 \tag{72.17}$$
$$\dot{y}_2 = (1 - y_1^2)y_2 - y_1 + u \tag{72.18}$$
$$\dot{\lambda}_1 = -2y_1 + 2y_1 y_2 \lambda_2 + \lambda_2 \tag{72.19}$$
$$\dot{\lambda}_2 = -2y_2 - \lambda_1 + \lambda_2(y_1^2 - 1) \tag{72.20}$$

where

$$u = -\lambda_2/2 \tag{72.21}$$

Objective ....

Boundary Value Problem (BVP)

Chapter 73

wind: **Abort Landing in the Presence of Windshear**

The dynamic behavior of an aircraft landing in the presence of a windshear was first formulated as an optimal control problem by Miele, Wang, and Melvin [70]. A number of other authors investigated the problem including Bulirsch, Montrone, and Pesch [31, 32] who introduce the problem as follows:

> One of the most dangerous situations for a passenger aircraft in take-off and landing is caused by the presence of low altitude windshears. This meteorological phenomenon, which is more common in subtropical regions, is usually associated with high ground temperatures leading to a so-called downburst. This downburst involves a column of descending air which spreads horizontally near the ground. Even for a highly skilled pilot, an inadvertent encounter with a windshear can be a fatal problem, since the aircraft might encounter a headwind followed by a tailwind, both coupled with a downdraft. The transition from headwind to tailwind yields an acceleration so that the resulting windshear inertia force can be as large as the drag of the aircraft, and sometimes as large as the thrust of the engines. This explains why the presence of low altitude windshears is a threat to safety in aviation. Some 30 aircraft accidents over the past 20 years have been attributed to windshear, and this attests to the perilousness of this occurrence. Among these accidents, the most disastrous ones happened in 1982 in New Orleans, where 153 people were killed, and in 1985 in Dallas, where 137 people were killed.

A complete discussion of the multi-phase formulation given here is found in reference [13, Sect. 6.6].

Example 73.1 wind01: Maximize Minimum Altitude.

Phase 1 .... Phase 1

Parameters: $(h_{min}^{(1)}, t_F^{(1)})$ ....

$.01 \le t_F^{(1)} \le 3.0875$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F^{(1)}$	$t = t_F^{(1)}$	sec

Differential Variables: (x, h, v, γ) ....

$x = x_0$	$0 \le x \le 10000$	$x = 500$	ft
$h = h_0$	$0 \le h \le 1500$	$0 \le h \le 1500$	ft
$v = v_0$	$10 \le v \le 500$	$10 \le v \le 500$	ft/sec
$\gamma = \gamma_0$	$-20^o \le \gamma \le 20^o$	$-20^o \le \gamma \le 20^o$	rad

Algebraic Variables: (α) ....

$\alpha = \alpha_0$	$0 \le \alpha \le \alpha_{max}$	$0 \le \alpha \le \alpha_{max}$	rad

Differential-Algebraic Equations ....

$$\dot{x} = v\cos\gamma + w_x \tag{73.1}$$

$$\dot{h} = v\sin\gamma + w_h \tag{73.2}$$

$$\dot{v} = \frac{1}{m}\left[T\cos(\alpha+\delta) - D\right] - g\sin\gamma - (\dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma) \tag{73.3}$$

$$\dot{\gamma} = \frac{1}{mv}\left[T\sin(\alpha+\delta) + L\right] - \frac{g}{v}\cos\gamma + (\dot{w}_x\sin\gamma - \dot{w}_h\cos\gamma) \tag{73.4}$$

$$0 \le \alpha_{max} - \alpha \tag{73.5}$$

$$-u_{max} \le \dot{\alpha} \le u_{max} \tag{73.6}$$

$$0 \le h - h_{min} \le 1500 \tag{73.7}$$

where

$$T = \beta T_* \tag{73.8}$$

$$T_* = a_0 + a_1 v + a_2 v^2 \tag{73.9}$$

$$D = \frac{1}{2}C_D \rho S v^2 \tag{73.10}$$

$$C_D(\alpha) = b_0 + b_1\alpha + b_2\alpha^2 \tag{73.11}$$

$$L = \frac{1}{2}C_L \rho S v^2 \tag{73.12}$$

$$C_L(\alpha) = \begin{cases} c_0 + c_1\alpha & \alpha \le \alpha_* \\ c_0 + c_1\alpha + c_2(\alpha - \alpha_*)^2 & \alpha_* \le \alpha \le \alpha_{max} \end{cases} \tag{73.13}$$

$$w_x = A(x) \tag{73.14}$$

$$w_h = \frac{h}{h_*}B(x) \tag{73.15}$$

with

$$A(x) = -50 + ax^3 + bx^4 \tag{73.16}$$
$$B(x) = dx^3 + ex^4 \tag{73.17}$$
$$\beta(t) = \beta_0 + \dot{\beta}_0 t \tag{73.18}$$

Phase 2 . . . . Phase 2

Parameters: $(h_{min}^{(2)}, t_I^{(2)})$ . . . .

$h_{min}^{(1)} = h_{min}^{(2)}$	$t_I^{(2)} \leq t_\beta$

Independent Variable: (t) . . . .

$t = t_F^{(1)} = t_I^{(2)}$	$t_I^{(2)} < t < t_\beta$	$t = t_\beta = (1 - \beta_0)/\dot{\beta}_0$	sec

Differential Variables: (x, h, v, γ) . . . .

$x = 500$	$0 \leq x \leq 10000$	$0 \leq x \leq 10000$	ft
$h = h_F^{(1)}$	$0 \leq h \leq 1500$	$0 \leq h \leq 1500$	ft
$v = v_F^{(1)}$	$10 \leq v \leq 500$	$10 \leq v \leq 500$	ft/sec
$\gamma = \gamma_F^{(1)}$	$-20^o \leq \gamma \leq 20^o$	$-20^o \leq \gamma \leq 20^o$	rad

Algebraic Variables: (α) . . . .

$\alpha = \alpha_F^{(1)}$	$0 \leq \alpha \leq \alpha_{max}$	$0 \leq \alpha \leq \alpha_{max}$	rad

Differential-Algebraic Equations . . . .

Equations (73.1) - (73.18)

Phase 3 . . . . Phase 3

Parameters: $(h_{min}^{(3)}, t_F^{(3)})$ . . . .

$h_{min}^{(2)} = h_{min}^{(3)}$	$t_\beta \leq t_F^{(3)}$

Independent Variable: (t) . . . .

$t = t_\beta$	$t_\beta < t < t_F^{(3)}$	$t = t_F^{(3)}$	sec

Differential Variables: (x, h, v, γ) ....

$x = x_F^{(2)}$	$0 \le x \le 10000$	$x = 4100$	ft
$h = h_F^{(2)}$	$0 \le h \le 1500$	$0 \le h \le 1500$	ft
$v = v_F^{(2)}$	$10 \le v \le 500$	$10 \le v \le 500$	ft/sec
$\gamma = \gamma_F^{(2)}$	$-20^o \le \gamma \le 20^o$	$-20^o \le \gamma \le 20^o$	rad

Algebraic Variables: (α) ....

$\alpha = \alpha_F^{(2)}$	$0 \le \alpha \le \alpha_{max}$	$0 \le \alpha \le \alpha_{max}$	rad

Differential-Algebraic Equations ....

Equations (73.1) - (73.15)

Replace (73.16)-(73.18) with

$$A(x) = \frac{1}{40}(x - 2300) \tag{73.19}$$

$$B(x) = -51 \exp\left[-c(x - 2300)^4\right] \tag{73.20}$$

$$\beta(t) = 1 \tag{73.21}$$

Phase 4 .... Phase 4

Parameters: $(h_{min}^{(4)}, t_I^{(4)}, t_F^{(4)})$ ....

$h_{min}^{(3)} = h_{min}^{(4)}$

Independent Variable: (t) ....

$t = t_F^{(3)} = t_I^{(4)}$	$t_I^{(4)} < t < t_F^{(4)}$	$t = t_F^{(4)}$	sec

Differential Variables: (x, h, v, γ) ....

$x = 4100$	$0 \le x \le 10000$	$x = 4600$	ft
$h = h_F^{(3)}$	$0 \le h \le 1500$	$0 \le h \le 1500$	ft
$v = v_F^{(3)}$	$10 \le v \le 500$	$10 \le v \le 500$	ft/sec
$\gamma = \gamma_F^{(3)}$	$-20^o \le \gamma \le 20^o$	$-20^o \le \gamma \le 20^o$	rad

Algebraic Variables: (α) ....

$\alpha = \alpha_F^{(3)}$	$0 \le \alpha \le \alpha_{max}$	$0 \le \alpha \le \alpha_{max}$	rad

Boundary Conditions ....

$t_F^{(4)} - t_I^{(4)} \ge .001$

Differential-Algebraic Equations ....

Equations (73.1) - (73.15)

Replace (73.16)-(73.18) with

$$A(x) = 50 - a(4600 - x)^3 - b(4600 - x)^4 \tag{73.22}$$

$$B(x) = d(4600 - x)^3 - e(4600 - x)^4 \tag{73.23}$$

$$\beta(t) = 1 \tag{73.24}$$

Phase 5 .... Phase 5

Parameters: ($h_{min}^{(5)}, t_I^{(5)}$) ....

$h_{min}^{(4)} = h_{min}^{(5)}$

Independent Variable: (t) ....

$t = t_F^{(4)} = t_I^{(5)}$	$t_I^{(5)} < t < t_F$	$t = t_F$	sec

Differential Variables: (x, h, v, γ) ....

$x = 4600$	$0 \le x \le 10000$	$0 \le x \le 10000$	ft
$h = h_F^{(4)}$	$0 \le h \le 1500$	$0 \le h \le 1500$	ft
$v = v_F^{(4)}$	$10 \le v \le 500$	$10 \le v \le 500$	ft/sec
$\gamma = \gamma_F^{(4)}$	$-20^o \le \gamma \le 20^o$	$\gamma = \gamma_F$	rad

Algebraic Variables: (α) ....

$\alpha = \alpha_F^{(4)}$	$0 \le \alpha \le \alpha_{max}$	$0 \le \alpha \le \alpha_{max}$	rad

Differential-Algebraic Equations ....

Equations (73.1) - (73.15)

Replace (73.16)-(73.18) with

$$A(x) = 50 \tag{73.25}$$

$$B(x) = 0 \tag{73.26}$$

$$\beta(t) = 1 \tag{73.27}$$

Objective ....

Maximize $J = h^{(5)}_{min}$

$\boxed{J^* = 491.852293}$

t_F	40 sec	u_{max}	3 deg/sec
α_{max}	17.2 deg	ρ	$.2203\times10^{-2}$ lb sec^2 ft^{-4}
S	$.1560\times10^4$ ft^2	g	3.2172×10^1 ft sec^{-2}
mg	150000 lb	δ	2 deg
a_0	$.4456\times10^5$ lb	a_1	$-.2398\times10^2$ lb sec/ft
a_2	$.1442\times10^{-1}$ lb sec^2 ft^{-2}	β_0	.3825
$\dot\beta_0$	.2 sec^{-1}	b_0	.1552
b_1	.12369 rad^{-1}	b_2	2.4203 rad^{-2}
c_0	.7125	c_1	6.0877 rad^{-1}
c_2	-9.0277 rad^{-2}	a_*	12 deg
h_*	1000 ft	a	6×10^{-8} sec^{-1} ft^{-2}
b	-4×10^{-11} sec^{-1} ft^{-3}	c	$-\ln(25/30.6)\times10^{-12}$ ft^{-4}
d	-8.02881×10^{-8} sec^{-1} ft^{-2}	e	6.28083×10^{-11} sec^{-1} ft^{-3}
x_0	0 ft	γ_0	-2.249 deg
h_0	600 ft	α_0	7.353 deg
v_0	239.7 ft/sec	γ_F	7.431 deg

Table 73.1. *Dynamic Model Parameters*

Chapter 74

zrml: **Zermelo's Problem**

Bryson and Ho [29, Sect. 2.7] describe the classical Zermelo's problem as follows:

> A ship must travel through a region of strong currents. ... The problem is to steer the ship in such a way as to minimize the time necessary to go from a point A to a point B.

A very simple model for the current function is used here and examples (61.1) and (61.2) illustrate the solution with more realistic current descriptions.

Example 74.1 zrml01: MINIMUM TIME.

Phase 1 .... Phase 1

Parameters: (t_F) ....

$0 \le t_F$

Independent Variable: (t) ....

$t = 0$	$0 < t < t_F$	$t = t_F$

Differential Variables: (x, y) ....

$x = 3.5$	$x = 0$
$y = -1.8$	$y = 0$

Algebraic Variables: (θ) ....
Differential-Algebraic Equations ....

$$\dot{x} = V\cos\theta + cy \tag{74.1}$$

$$\dot{y} = V\sin\theta \tag{74.2}$$

where $V = 1$, and $c = -1$.

Objective . . . .

Minimize $J = t_F$

$$\boxed{J^* = 5.26493205}$$

Appendix

Conversion Factors

$g_0 = 32.174$ ft/sec^2
1 hr = 3600. sec
1 nm = 6076.1154855643 ft
1 au = 149597870.691 km
1 knot = 6076.1154855643/3600 = 1.6878098571011944 ft/sec
1 rad = $(180/\pi)$ deg = 57.29577951308232^o

Table A.1. *Conversion Factors*

Appendix

Software

Numerical solutions have been obtained for all problems documented in this book. All of the software used to compute these results is publicly available as described in the following two sections.

A.1 Optimal Control Test Suite

The following items are available at no cost:

1. Sparse Optimization Suite $\mathbb{SOS}$ User's Guide .... **sosdoc.pdf**
2. FORTRAN 90 test suite main program using $\mathbb{SOS}$ software format ..... **cdsosx.f**
3. FORTRAN 90 source code implementations in $\mathbb{SOS}$ format for all test problems .... **prblms.tar**
4. Test Problem Data files .... **prblmsAdat.tar**
5. $\mathbb{SOS}$ Input Options for each problem .... **options.tar**
6. Test suite performance summary file .... **sumrey.ref**

They can be downloaded from

- The (AMA) Applied Mathematical Analysis L.L.C. web site at <http://www.appliedmathematicalanalysis.com/>

A.2 $\mathbb{SOS}$ Optimal Control Algorithm

The following items are available for license to the public:

1. Sparse Optimization Suite $\mathbb{SOS}$ library
2. GESOP Graphical User Interface

For license information contact:

Astos Solutions GmbH, E-mail: service@astos.de http://www.astos.de

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