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Random-order fractional differential equation models ☆

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ABSTRACT

This paper proposes a new concept of random-order fractional differential equation model, in which a noise term is included in the fractional order. We investigate both a random-order anomalous relaxation model and a random-order time fractional anomalous diffusion model to demonstrate the advantages and the distinguishing features of the proposed models. From numerical simulation results, it is observed that the scale parameter and the frequency of the noise play a crucial role in the evolution behaviors of these systems. In addition, some potential applications of the new models are presented.

1. Introduction

When characterizing a complex system, most researchers are accustomed to applying nonlinear differential equation models. But it is difficult to solve a nonlinear problem, either numerically or theoretically, and even more difficult to establish a real nonlinear model for the complex system. Many assumptions have to be made artificially or unnecessarily to make practical engineering problems solvable, leading to loss of some important information. Since integer order differential equations cannot precisely describe the experimental and field measurement data, as an alternative approach, fractional order differential equation models are now being widely applied [1,2]. In the last few decades, fractional calculus has been widely used in tackling physical, biological, chemical, financial and physiological problems [3,4]. In particularly, different kinds of anomalous relaxation and diffusion processes have attracted much attention in fractional order dynamic modeling.

In the fractional order modeling of anomalous relaxation and diffusion phenomena, a number of scientists have proposed a variety of fractional order models, which involve different kinds of fractional derivative definitions [5,6]. Among these models, one of the successful models is the distributed-order fractional equation model, which can be used to describe diffusion processes which exhibit decelerating/accelerating behaviors in long-time range [7,8]. In addition, variable-order differential operator is also proposed to describe the diffusion process in which diffusion rate changes in the course of space or time [9–12]. In this study, we will consider the systems which suffer from some noises (e.g. fluctuations of the external pressure field in anomalous diffusion system, voltage fluctuations in the circuits, or unstable temperature field in the energy dissipation). These noises will inevitably cause the fluctuations of the considered systems. The evolution behaviors of considered systems will also show deviations from those of the constant-order fractional differential models. How to quantify the

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influence of the noise is important in system characterization. Therefore, we propose a fractional differential equation model in which the differential order includes a noise term to deal with the above noise effect quantification problem. Random-order model is a more generalized model compared with the simple case of a fractional order differential equation model with a single constant-order. Changing the constant-order fractional derivative into the random-order fractional derivative is an interesting and useful effort both in theoretical research and engineering application, as shown in this paper.

For simplicity, we will restrict our attention in the random-order time fractional differential equation models, this type of model looks like there are always some noises in system memory. The investigation in this study will focus on the system behavior when the fractional order contains a random noise term.

The structure of this paper is as follows. The definitions about random-order operators are presented in Section 2. In Section 3, the anomalous relaxation and diffusion equation models with time fractional differential order including random noise term are studied. We further discuss our findings about random-order equation models in Section 4. Some remarks are given in Section 5.

2. Definitions of random-order fractional derivative and integral

First of all, we briefly introduce the definitions of random-order derivative and integral. Here we only consider the random-order fractional derivative definition in Caputo basis, which is obtained via following the form of variable-order derivative definition [17,18]

$$ {}_tD_0^{\alpha_0+\varepsilon_t} f(t) = \frac{1}{\Gamma(1-(\alpha_0+\varepsilon_t))}\int_{0+}^{t}\frac{f'(\tau)d\tau}{(t-\tau)^{(\alpha_0+\varepsilon_t)}} + \frac{(f(0+)-f(0-))t^{-(\alpha_0+\varepsilon_t)}}{\Gamma(1-(\alpha_0+\varepsilon_t))}, \quad p(\varepsilon_t|0<\alpha_0+\varepsilon_t<1)=1, \tag{1} $$

where $p(\cdot)$ is probability density function, $\alpha_0\in(0,1)$ is a constant, $\varepsilon_t\in\mathbb{R}^+$ is a random noise term which changes with time and $\Gamma(\cdot)$ is Gamma function [9]. Obviously, this definition represents a derivative order that has no memory of its history. In addition, we assume the noise is random with the time evolution. It means that the memory rate at every time instant is independent, the system memory rate in this approach therefore jumps from $\alpha_0+\varepsilon_{t_1}$ to $\alpha_0+\varepsilon_{t_2}$ to $\alpha_0+\varepsilon_{t_3}$, and so on.

The definition of random-order fractional integral can be stated as follows:

$$ I_{0+}^{\alpha_0+\varepsilon_t} f(t) = \frac{1}{\Gamma(\alpha_0+\varepsilon_t)}\int_0^t (t-\tau)^{\alpha_0+\varepsilon_t-1} f(\tau)d\tau, \quad \alpha_0+\varepsilon_t>0, \ \forall t>0. \tag{2} $$

If $\varepsilon_t=0$, the above definitions (1) and (2) will become the definitions of constant-order derivative and integral. It should be pointed that, the properties of the random-order definitions can be obtained via the similar way in the constant-order definitions, we do not discuss the details in this paper [9,10].

3. Random-order fractional relaxation model and random-order time fractional diffusion model

Recently, fractional kinetic equations have attracted extensive attention as powerful tools for the description of anomalous relaxation and diffusion phenomena, see, e.g. [14–16]. Hereby, in this section, we focus on the anomalous relaxation and diffusion processes to introduce the random-order fractional differential equation models.

3.1. Random-order fractional relaxation model

Since a large number of experimental data, including data from mechanical, dielectric, volumetric, and magnetic relaxation, reaction kinetics, dynamic light and quasielastic neuron scattering exhibits the "anomalous" relaxation behavior, the fractional generalization of classical relaxation equation is proposed to depict this particular behavior [4]. The single constant-order fractional relaxation equation is stated as below

$$ {}_tD_*^{\alpha_0} u(t) = -\lambda u(t), \quad 0<\alpha_0<1, \tag{3} $$

where $u(t)$ is a field variable, α_0 is fractional differential order, λ is a positive coefficient denoting the inverse of some characteristic time.

Based on (3), we propose the fractional relaxation equation with differential order including random noise term ε_t:

$$ \begin{cases} D_*^{\alpha_0+\varepsilon_t} u(t) = -\lambda u(t), \\ p(\varepsilon_t|0<\alpha_0+\varepsilon_t<1)=1, \\ u(0)=1. \end{cases} \tag{4} $$

In the above equation, we assume the random noise term ε_t has no memory of past values of itself. Meanwhile, it can be known that the differential order also has no memory of its history from the expression (1) [9,18]. Therefore, we can perform the numerical simulation of (4) with the similar way of variable-order differential equation. In this study, we employ the Predictor-Corrector scheme to perform the numerical simulation of random-order fractional relaxation equation. The detailed description and convergence proof of this scheme can refer to [19,20]. The code for this method can be found in the Matlab Center [21].

Next, we consider the selection of random noise term, here we simply employ two types of random noises: the first one is uniform type noise which can be generalized by *Matlab* function, the second one is Gaussian type noise with mean value 0. For convenience, we apply the Box–Muller method to generate Gaussian type noise [22,23]

$$ \xi = 2\gamma\sqrt{-\ln u_1}\sin\phi, \tag{5} $$

where $\phi=\pi(u_2-1/2), u_1, u_2\in(0,1)$ are uniform random numbers, γ is the scale parameter.

The uniform type noise and the Gaussian type noise are shown in Fig. 1. Assuming $\lambda=0.5$ and $\alpha_0=0.5$ in (4), the relaxation behavior of random-order fractional relaxation model can be observed in Figs. 2–4.

From Figs. 2–4, we find that the system behavior exhibits more and more fluctuations with the time evolution. It demonstrates that the random noise plays

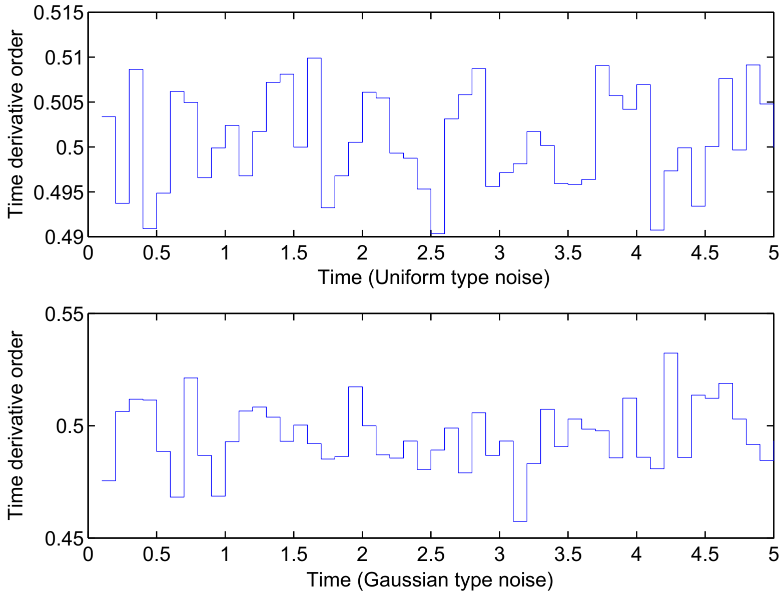

Fig. 1. Uniform type noise $\varepsilon_t \in (-0.01, 0.01)$ and Gaussian type noise $\gamma = 0.01$ in (5), time step $\Delta t = 0.1$.

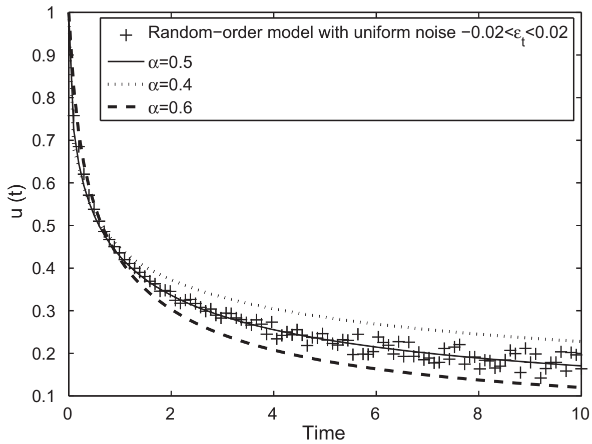

Fig. 2. The system behavior of the fractional relaxation model with uniform type noise included in differential order. Time step in numerical simulation $\Delta t = 0.1$, scale parameter $\gamma = 0.02$, random noise term $\varepsilon_t \in (-0.02, 0.02)$. The solid line represents the relaxation model with differential order is $\alpha = 0.5$, the dotted line $\alpha = 0.4$, and the dashed line $\alpha = 0.6$.

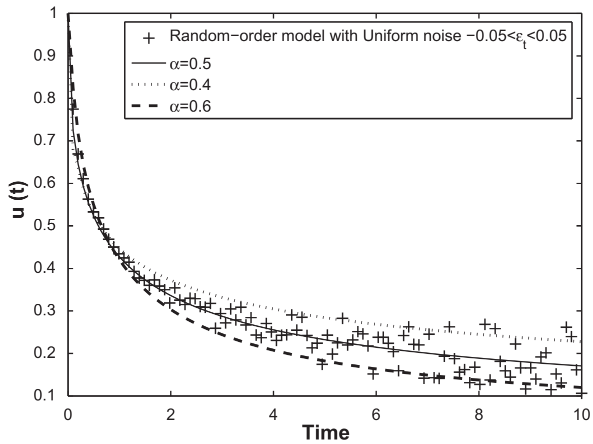

Fig. 3. The system behavior of the fractional relaxation model with uniform type noise included in differential order. Time step in numerical simulation $\Delta t = 0.1$, scale parameter $\gamma = 0.05$, random noise term $\varepsilon_t \in (-0.05, 0.05)$. The solid line represents the relaxation model with differential order is $\alpha = 0.5$, the dotted line $\alpha = 0.4$, and the dashed line $\alpha = 0.6$.

more and more important role in relaxation process with time evolution. Furthermore, the system behavior of random-order model with $\alpha_0 + \varepsilon_t$ and scale parameter γ is much more complex than that with differential orders $\alpha_0 \pm \gamma$. The relaxation curve of random-order model is not limited between the relaxation curves of constant-order models with differential orders: $\alpha_0 + \gamma$ and $\alpha_0 - \gamma$, no matter under uniform type noise or Gaussian type noise. Moreover, the system behavior will exhibit more fluctuations with larger scale parameter under the uniform type noise or Gaussian type noise, it can be partly confirmed by comparing Figs. 2 and 3. In addition, from the comparison of Figs. 2 and 4, It can be observed that the system behavior under the Gaussian type noise is more complex than that under uniform type noise with same scale parameter $\gamma = 0.02$. It is also pointed out that the decay curve is also influenced by the frequency of the noise, when the frequency of the noise ($f = 1/\Delta t$, Δt is time step) changes from small to large, the behavior of the system shows more and more fluctuations. Therefore, the frequency of the random noise is also an important factor to determine the system behavior.

3.2. Random-order time fractional diffusion model

Fractional diffusion equations account for typical anomalous features which are observed in many systems, e.g. the processes of dispersive transport in amorphous

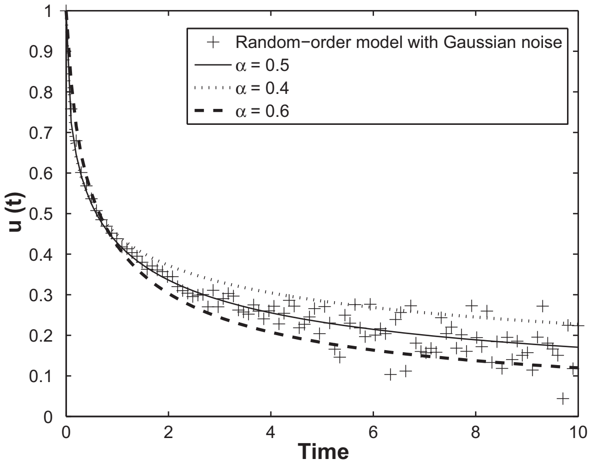

Fig. 4. The system behavior of the fractional relaxation model with Gaussian type noise included in differential order. Time step in numerical simulation $\Delta t = 0.1$, scale parameter $\gamma = 0.02$. The solid line represents the relaxation model with differential order is $\alpha = 0.5$, the dotted line $\alpha = 0.4$, and the dashed line $\alpha = 0.6$.

semiconductors, disordered media, liquid crystals, turbulence, biopolymers, proteins, biosystems and some other systems [24].

To exploit the characteristics of random-order time fractional diffusion model, we consider the following time fractional diffusion equation which depicts the sub-diffusion process

$$\begin{cases} \dfrac{\partial^{\alpha_0+\varepsilon_{t,x}} u(x,t)}{\partial t^{\alpha_0+\varepsilon_{t,x}}} = K\dfrac{\partial^2 u(x,t)}{\partial x^2}, & p(\varepsilon_{t,x}|0<\alpha_0+\varepsilon_{t,x}<1)=1, \\ u(0,t)=u(L,t)=0, & t>0, \\ u(x,0)=\sin(x\pi/L), & 0<x<L, \end{cases} \tag{6}$$

where K is generalized diffusion coefficient, $u(x,t)$ represents mass or other quantities of interest, $\alpha_0+\varepsilon_{t,x}$ is the time derivative order and $\varepsilon_{t,x}$ is random noise term, $p(\cdot)$ denotes probability density function. We should point out that, Kobelev et al. have investigated variable-order diffusion model with weak-memory derivative order $\alpha_0+\varepsilon(t,x)$ in [25], where weak memory term $\varepsilon(t,x)$ is a depending function of time and coordinate, and $\varepsilon(t,x)$ should satisfy $\varepsilon(t,x) \ll 1$ which is needed for theoretical analysis. The new model (6) has extended $\varepsilon_{t,x}$ to be a random-noise term which is random with time or location and $\varepsilon_{t,x}$ is not limited by $\varepsilon_{t,x} \ll 1$. Next, we will accomplish the numerical simulation by finite difference method to illustrate the performance of the random-order time fractional diffusion model. Let K=1.0 and L=10.0 in (6), we adopt Gaussian type noise in the time fractional order term and employ the Crank–Nicholson scheme for the discretization of above diffusion Eq. (6). The Crank–Nicholson scheme can be verified to be stable and efficient by the methods presented in [26,27]. Then, we can draw the sub-diffusion curve which is shown in Fig. 5. In this numerical simulation, for different time instants, if $m > n \geq 0$, then the following inequality should be satisfied

$$\int_0^L u(x,t_m)\,dx \leq \int_0^L u(x,t_n)\,dx. \tag{7}$$

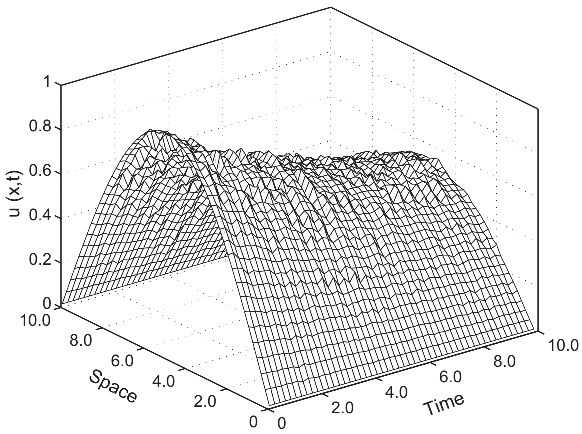

Fig. 5. The diffusion evolution behavior of the random-order time fractional diffusion model. The scale parameter of Gaussian type noise (5) included in the time derivative order is $\gamma = 0.05$. In the numerical simulation: time step $\Delta t = 0.2$, space step $\Delta x = 0.2$ and $\alpha_0 = 0.8$.

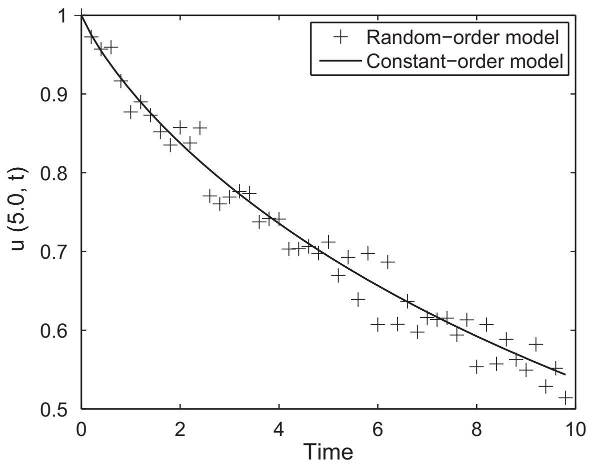

Fig. 6. The decay behavior of the time fractional diffusion model with order including Gaussian type noise at x=5.0 with $\gamma = 0.05$. In the numerical simulation: time step $\Delta t = 0.2$, space step $\Delta x = 0.2$ and $\alpha_0 = 0.8$.

From the diffusion image shown in Fig. 5, one can observe that though the diffusion trend is similar with the constant-order fractional diffusion model, the new model exhibits more fluctuations than constant-order model. We can also observe obvious fluctuations characteristics of the diffusion curve from Fig. 6. In addition, the frequencies of the noise ($f_t = 1/\Delta t$ and $f_x = 1/\Delta x$, Δt and Δx are time and space steps) are also important factors to determine the diffusion behavior. This feature is same with the random-order fractional relaxation model.

4. Discussions

4.1. Physical implications

In usual anomalous physical cases, the considered system may be influenced by small noise or the noise term plays a negligible role, so we can neglect the effect of noise term when analyzing the system behavior. In these cases, the extensively applied constant-order fractional dynamic model is sufficient. On the other hand, if the system endures un-negligible noise, in order to fully estimate the system and analyze the long time system behavior under noise field, the random-order fractional derivative model is a very suitable approach.

From the statement in Section 3, we have found that the random-order fractional differential equation can be employed to describe the anomalous relaxation and diffusion processes enduring random noise. In addition, different types of random noise will produce diverse system behaviors. In this study, we have shown that if the random noise obey uniform distribution, the influence of the noise term is easy to estimate. Whereas, if the random noise obeys Gaussian distribution, the evolution behavior of the system is rather complex.

4.2. Potential applications

In many cases of the relaxation or diffusion processes, the experimental or field measurement data do not match the constant-order fractional equation models [28–31]. The reason for this issue may lie in the measurement technology or system error, but the underlying reason may lie in that the constant-order fractional equation model has its own limitations. It cannot characterize the fluctuation feature of system. However, the random-order fractional differential equation model can well capture this characteristic. So employing random-order fractional differential equation model is helpful in fully understanding such kinds of stochastic systems.

Based on the above discussions, we believe that this new model will be helpful in estimating the stability of the system and useful in system control. The potential applications will relate to the pollution control, system stability analysis, finance and stock market modeling etc. [13].

5. Conclusive remarks

In this study, we offer a concise introduction to the random-order fractional differential equation model. The analysis and numerical simulation in this study have shown that the new model can characterize some anomalous relaxation or diffusion processes enduring random noise. In summary, we believe that the random-order fractional differential equation model can serve as a useful tool for the characterization of complex stochastic processes in the real world.

However, further investigations are needed to establish fast numerical methods for the random-order fractional differential equation model. Furthermore, in order to better reflect the real world system, the random noise term in which the scale parameter is dependent on the time and space should also be considered in the further investigation.

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