A Data-Driven Dynamic System Similarity Metric And Its Applications

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Abstract

This study investigates data-driven metrics for quantifying similarity between linear time-invariant (LTI) systems within the framework of behavioral system theory. A new index is proposed, which is based on projection matrices combined with the Maximum Mean Discrepancy (MMD). In addition to measuring system similarity, the proposed approach also supports system change detection, thereby facilitating improved controller design. The effectiveness of the proposed metrics is demonstrated through numerical simulations.

Backgroud

A dynamical system can be seen as a triple $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the time axis, $\mathbb{W}$ is the signal space, and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the behavior of the system [1]. The complexity of a LTI system is defined as $c = (m(\mathfrak{B}), \ell(\mathfrak{B}) n(\mathfrak{B}))$. The class of all the q-variate LTI systems with complexity c is denoted by $\mathcal{L}_c^q$. For the input/output representation, a trajectory w is given by $w = \mathrm{col}(u, y) \in \mathbb{R}^{m+p}, m+p = q$, where m and p denote the number of inputs and outputs, respectively. A finite length trajectory over the interval $[1, L]$ is defined as:

$$w|_L = \left[w(1)^T \; w(2)^T \cdots \; w(L)^T\right]^T.$$

The restricted behavior is then denoted as:

$$\mathfrak{B}|_L = \{w|_L \in \mathbb{R}^{L(m+p)} | w \in \mathfrak{B}\}.$$

For a restricted trajectory $w|_T$, an L-blocks Hankel matrix can be constructed as:

$$\mathfrak{H}_L(w|_T) = \begin{bmatrix} w(1) & w(2) & \cdots & w(T-L+1) \\ w(2) & w(3) & \cdots & w(T-L+2) \\ \vdots & \vdots & & \vdots \\ w(L) & w(L+1) & \cdots & w(T) \end{bmatrix}. \quad (1)$$

For LTI systems, when $L > \ell(\mathfrak{B})$, $\mathrm{image}(\mathfrak{H}_L(w|_T)) = \mathfrak{B}|_L$ if and only if $\mathrm{rank}(\mathfrak{H}_L) = \mathbf{m}(\mathfrak{B})L + \mathbf{n}(\mathfrak{B})$ [2].

The column space of the Hankel matrix spans the restricted behavior. Any finite-length trajectory $w|_L$ of $\mathfrak{B}$ can be expressed as a linear combination of the columns of $\mathfrak{H}_L(w|_T)$.

Methodology

The restricted behavior space can be spanned by the column space of $\mathfrak{H}_L$. Two restricted behavior spaces are equal if and only if the orthogonal projection matrices onto the column spaces of the corresponding Hankel matrices coincide.

Projection-based MMD

Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_c^q$, and $\mathfrak{B}_1|_L$ $\mathfrak{B}_2|_L$ denote restricted behavior spaces with orthogonal projection matrices onto the corresponding column spaces P_1, P_2. For $\sigma > 0$, the distance is defined as:

$$d^2(\mathfrak{B}_1|_L, \mathfrak{B}_2|_L) = \det\left(I + \frac{2}{\sigma^2}P_1\right)^{-1/2} + \det\left(I + \frac{2}{\sigma^2}P_2\right)^{-1/2} - 2\det\left(I + \frac{1}{\sigma^2}(P_1 + P_2)\right)^{-1/2}. \quad (2)$$

Probabilistic Explanation

From the probabilistic perspective, this index coincides with the Maximum Mean Discrepancy (MMD) between two Gaussian distributions under the RBF kernel. Let $x, x' \sim \mathcal{N}(0, P_1)$ and $y, y' \sim \mathcal{N}(0, P_2)$ be independent samples. Define $z_1 = x - x' \sim \mathcal{N}(0, 2P_1)$, $z_2 = x - y \sim \mathcal{N}(0, P_1 + P_2)$, and $z_3 = y - y' \sim \mathcal{N}(0, 2P_2)$. For the Gaussian kernel $k(x, x') = \exp(-\frac{\|x - x'\|^2}{2\sigma^2})$, MMD becomes

$$\mathrm{MMD}^2 = \mathbb{E}_{z_1}\left[\exp(-\frac{z_1^T z_1}{2\sigma^2})\right] - 2\mathbb{E}_{z_2}\left[\exp(-\frac{z_2^T z_2}{2\sigma^2})\right] + \mathbb{E}_{z_3}\left[\exp(-\frac{z_3^T z_3}{2\sigma^2})\right]. \quad (3)$$

Since the moment generating function of a quadratic form $Z^\top AZ$ with $Z \sim \mathcal{N}(0, C)$ and $C, A \succeq 0$ is

$$\mathbb{E}\left[e^{tZ^\top AZ}\right] = \det(I - 2tAC)^{-1/2}. \quad (4)$$

Applying this result with $t = -\frac{1}{2\sigma^2}$, $A = I$, and $C \in \{2P_1, 2P_2, P_1 + P_2\}$ yields a closed-form expression for the MMD, which coincides with the distance defined in (2).

Since the Gaussian RBF kernel is characteristic, MMD defines a metric on probability measures. Consequently, the proposed index is a metric on restricted behavior spaces.

Geometric Explanation

The proposed index can also be related to the classical notion of principal angles. Let $\mathbb{U}_i = \mathrm{range}(P_i)$ with $r_i = \dim(\mathbb{U}_i)$, $k = \dim(\mathbb{U}_1 \cap \mathbb{U}_2)$ be the number of zero principal angles, $m = \min(r_1 - k,\ r_2 - k)$ the number of nonzero principal angles, $\alpha = 1/\sigma^2$ and $\ell = r_1 + r_2 - 2k - 2m$. Since $r = r_1 = r_2$, the expression (2) reduces to

$$d^2(\mathfrak{B}_1|_L, \mathfrak{B}_2|_L) = 2\,(1 + 2\alpha)^{-r/2} \times \left[1 - \left(\prod_{i=1}^{m} \frac{1 + 2\alpha}{1 + 2\alpha + \alpha^2 \sin^2 \theta_{k+i}}\right)^{1/2}\right]. \quad (5)$$

To avoid the collapse of the index in high dimensions, the normalized index is defined as

$$\tilde{d}^2 = d^2/(2(1 + 2\alpha)^{-r/2}).$$

When the principal angles are small, the normalized index can be approximated as

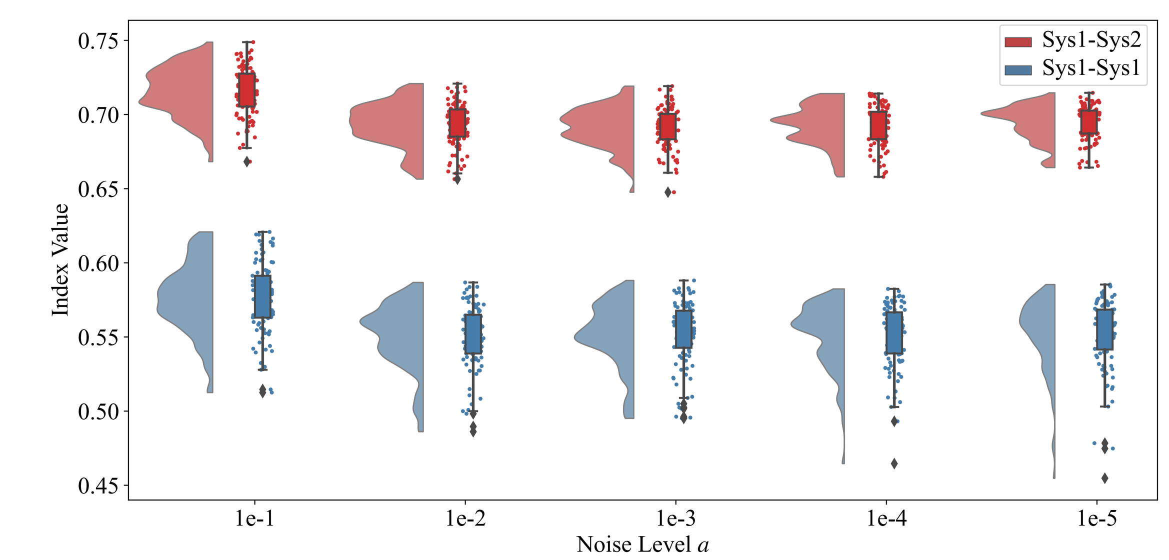
$$\tilde{d}^2 \approx \frac{\alpha^2}{2(1 + 2\alpha)} \sum_{i=1}^{m} \sin^2(\theta_{k+i}). \quad (6)$$

In the small-angle regime, this index is the chordal distance scaled by a factor decided by α, making its sensitivity tunable and offering more flexibility than the classical chordal distance.

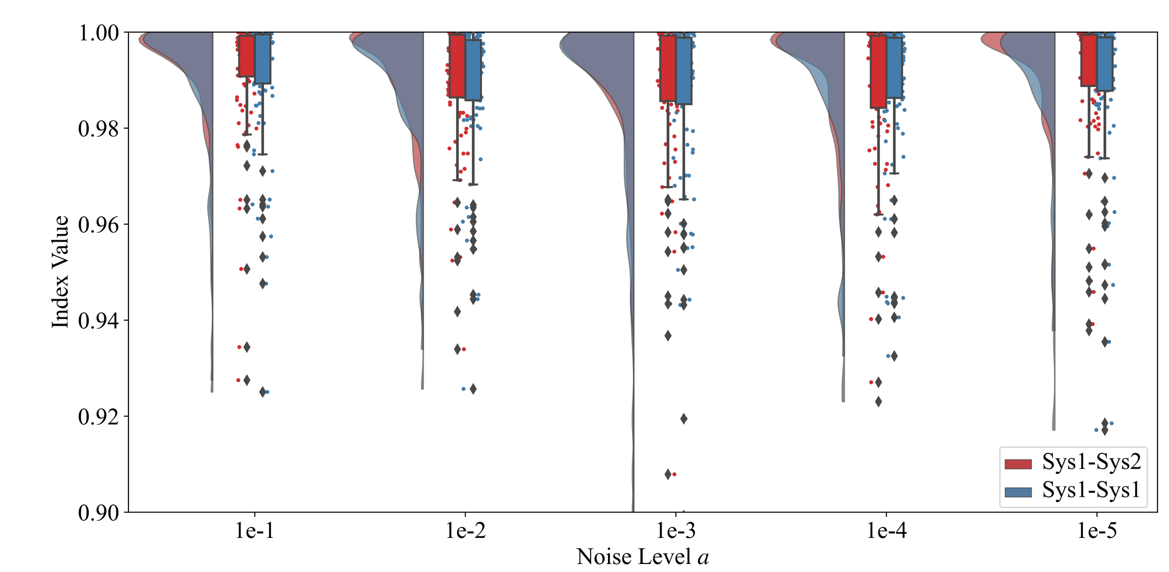
Simulation Results

Two systems are used in this study:

- System 1: $y_t = 0.2y_{t-1} + 0.24y_{t-2} + 2u_{t-1}$,
- System 2: $y_t = 0.7y_{t-1} - 0.12y_{t-2} + u_{t-1}$.



(a) Results for the proposed index under different noise levels.



(b) Results for L-gap under different noise levels.

Figure: Results of the two-sample test under different noise levels. The horizontal axis represents the noise level a (standard deviation of the Gaussian noise $\mathcal{N}(0, a^2)$), while the vertical axis shows the corresponding index values.

References

[1] Jan C Willems and Jan W Polderman. *Introduction to Mathematical Systems Theory: A Behavioral Approach*, volume 26. Springer Science & Business Media, 1997.

[2] Ivan Markovsky and Florian Dörfler. Identifiability in the behavioral setting. *IEEE Transactions on Automatic Control*, 68(3):1667–1677, 2022.