Fractional Order Modeling
A Tutorial Introduction and An Application in Characterizing Complex Relaxation Processes

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Sept. 14, 2012. Friday 12:00-13:20
COB 267
Outline

• What is Fractional Calculus and Fractional Order Modeling/Controls

• A Worked Out Example on Fractional Order Modeling (FOM)

• Summary of Benefits of Using FOM

• An Application: Parameter-Distributed and DO FOM of Complex Relaxation Processes
Fractional (Noninteger)(order) operator

- First order differentiator: $s$
- First order integrator: $1/s$

What is $s^\alpha$ when $\alpha$ is a non-integer?
Fractional Order Integrator
(half order)

\[ G(s) = \frac{1}{\sqrt{s}} \]

Legal in MATLAB and everywhere?

```matlab
a=-0.5;w=logspace(-2,2,1000);fi=(j.*w).^a;
figure;subplot(2,1,1);semilogx(w,20*log10(abs(fi)));xlabel('frequency (rad./sec.)');ylabel('dB');grid on
subplot(2,1,2);semilogx(w,180*angle(fi)/pi);xlabel('frequency (rad./sec.)');ylabel('degree');grid on
```
Possible? Possible! Legal!!

Analog $\frac{1}{\sqrt{s}}$ using op-amps.

Fractor: Analogue device

Fractional Calculus Day at USU, April 19, 2005
Photo credit: Igor Podlubny

\[ G(s) = \frac{K}{R(s\tau)^\lambda} \]
Oustaloup’s Recursive Approximation for fractional order differentiators/integrator

\[
G(s) = \frac{1}{s^r} \approx \frac{B(s)}{A(s)}
\]

\[w_L=0.1; w_H=1000; r=-0.5; \text{figure; } N=3; \text{sys1=\text{tf}(1,[1,0]);}\]
\[
\text{sys_N_tf=ora_foc}(r,N,w_L,w_H); \text{bode(sys_N_tf,'k:',sys1,'r-');grid on;}
\]
\[
\text{title(\{'Oustaloup-Recursive-Approximation for } \{\text{it } s\}^\wedge{\text{\textbackslash{}^r}\}',num2str(r))};
\]

Chen’s impulse response invariant discretization for fractional order differentiators/integrator

\[
G(s) = \frac{1}{s^\gamma} \approx \frac{B(z^{-1})}{A(z^{-1})}
\]

What is $s^\alpha$ when $\alpha$ is a non-integer?

**Operator $aD_t^\alpha$**

A generalization of differential and integral operators:

$$ aD_t^\alpha = \begin{cases} 
  \frac{d^\alpha}{dt^\alpha} & \Re(\alpha) > 0, \\
  1 & \Re(\alpha) = 0, \\
  \int_a^t (d\tau)^{-\alpha} & \Re(\alpha) < 0.
\end{cases} \quad (7) $$

There are two commonly used definitions for the general fractional order differentiation and integral, i.e., the Grünwald-Letnikov definition and the Riemann-Liouville definition.
Example: Heaviside’s unit step

Example: $\sin(t)$
Fractional derivatives of ramp function.
Anything surprising so far?

Quite intuitive in fact.

For example,
... from integer to non-integer ...

\[ x^n = \underbrace{x \cdot x \cdots x}_n \]

\[ x^n = e^{n \ln x} \]

\[ n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n, \]

\[ \Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt, \quad x > 0, \]

\[ \Gamma(n + 1) = 1 \cdot 2 \cdot 3 \cdots n = n! \]
... from integer to non-integer ...

\[ D = 1 \quad \quad D = 2 \quad \quad D = 3 \]

\[ D = 1.26 \quad \quad D = 1.89 \quad \quad D = 2.73 \]
Interpolation of operations

\[ f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \ldots \]

\[ f, \quad \int f(t) \, dt, \quad \int dt \int f(t) \, dt, \quad \int dt \int dt \int f(t) \, dt, \quad \ldots \]

\[ \ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots \]
“Fractional Order Thinking”
or, “In Between Thinking”

• For example
  – Between integers there are non-integers;
  – Between logic 0 and logic 1, there is the “fuzzy logic”;
  – Between integer order splines, there are “fractional order splines”
  – Between integer high order moments, there are noninteger order moments (e.g. FLOS)
  – Between “integer dimensions”, there are fractal dimensions
  – Fractional Fourier transform (FrFT) – in-between time-n-freq.
  – Non-Integer order calculus (fractional order calculus – abuse of terminology.) (FOC)
Fractional Calculus was born in 1695

What if the order will be $n = \frac{1}{2}$?

It will lead to a paradox, from which one day useful consequences will be drawn.

G.F.A. de L'Hôpital (1661–1704)

G.W. Leibniz (1646–1716)
FOMs and Fractional Order Controls

- **IO Controller + IO Plant**
- **FO Controller + IO Plant**
- **FO Controller + FO Plant**
- **IO Controller + FO Plant**


"Fractional Order Modeling" @ EECS Seminar @ UC Merced
Outline

• What is Fractional Calculus and Fractional Order Modeling/Controls

• **A Worked Out Example on Fractional Order Modeling (FOM)**

• Summary of Benefits of Using FOM

• An Application: Parameter-Distributed and DO FOM of Complex Relaxation Processes
Modeling: heat transfer

\[
\frac{\partial^2 y(x, t)}{\partial x^2} = k^2 \frac{\partial y(x, t)}{\partial t},
\]
\[(t > 0, \ 0 < x < \infty)\]

Boundary condition:
\[
y(0, t) = m(t)
\]
\[
y(x, 0) = 0
\]
\[
\lim_{x \to \infty} y(x, t) < \infty
\]

Initial condition

Physical limit

Transfer function:
\[
\frac{d^2 Y(x, s)}{dx^2} = k^2 s Y(x, s)
\]
\[
Q(0, s) = M(s)
\]
\[
\lim_{x \to \infty} Y(x, s) < \infty
\]
\[ Y(x, s) = A(s)e^{-kx\sqrt{s}} + B(s)e^{kx\sqrt{s}} \]

\[ A(s) = Y(0, s) = M(s) \]

\[ B(s) = 0 \]

\[ Y(x, s) = M(s)e^{-kx\sqrt{s}} \]

\[ G(s) = \frac{Y(x, s)}{M(s)} = e^{-kx\sqrt{s}} \]

Irrational Transfer Function.
Taylor series expansion: polynomial of half order integrators!!
Ideal physical plant model:

First Order Plus Time Delay (FOPTD) Model:

\[ G_{p}(s) = e^{-\sqrt{s}} \]

Time Delay with Single Fractional Pole Model:

\[ G_{IO}(s) = \frac{K_1}{T_1 s + 1} e^{-L_1 s} \]

\[ G_{FO}(s) = \frac{K_2}{T_2 s^{0.5} + 1} e^{-L_2 s} \]

All models are wrong but some are useful.  
George E. P. Box

All models are wrong but some are dangerous …  
Leonard A. Smith
Step response of the “Ideal Plant”

\[ y(0, t) = m(t) = 1u(t), \quad M(s) = \frac{1}{s} \]

\[ Y(x, s) \bigg|_{x=1} = G(x, s) \bigg|_{x=1} \quad M(s) = G_p(s)M(s) = \frac{1}{s} e^{-\sqrt{s}} \]

So, “Reaction-Curve” or Step response of the “Ideal Plant”

\[ y(t) = L^{-1}\left[\frac{1}{s} e^{-\sqrt{s}}\right] \]
Magic code to do $y(t) = L^{-1}\left[\frac{1}{s} e^{-\sqrt{s}}\right]$}

% step response of normalized 1D heat equation when x=1
clear all; close all; alpha=.5; Ts=0.1;
F= @(s) exp(-s.^alpha)./s;
%-----------------------------------------------------------------
alfa=0; M=1024; P=20; Er=1e-10; tm=M*Ts; wmax0=2*pi/Ts/2; L = M;
Taxis=[0:L-1]*Ts; n=1:L-1; n=n*Ts ;
N=2*M; qd=2*P+1; t=linspace(0,tm,M); NT=2*tm*N/(N-2); omega=2*pi/NT;
c=alfa-log(Er)/NT; s=c-i*omega*(0:N+qd-1);
Fsc=feval(F,s); ft=fft(Fsc(1:N)); ft=ft(1:M);
q=Fsc(N+2:N+qd)./Fsc(N+1:N+qd-1); d=zeros(1,qd); e=d;
d(1)=Fsc(N+1); d(2)=-q(1); z=exp(-i*omega*t);
for r=2:2:qd-1; w=qd-r; e(1:w)=q(2:w+1)-q(1:w)+e(2:w+1); d(r+1)=e(1);
    if r>2; q(1:w-1)=q(2:w).*e(2:w)./e(1:w-1); d(r)=-q(1);
end
A2=zeros(1,M); B2=ones(1,M); A1=d(1)*B2; B1=B2;
for n=2:qd
    A=A1+d(n)*z.*A2; B=B1+d(n)*z.*B2;A2=A1; B2=B1; A1=A; B1=B;
end
ht=exp(c*t)/NT.*(2*real(ft+A./B)-Fsc(1));
%------------------------------------------------------------------
figure;tt=0:(length(ht)-1);tt=tt*Ts;plot(tt,ht);
xlabel('time (sec.)');ylabel('temperature (C)');grid on

**Application of numerical inverse Laplace transform algorithms in fractional calculus**

*Journal of the Franklin Institute, Volume 348, Issue 2, March 2011, Pages 315-330*

Hu Sheng, Yan Li, YangQuan Chen http://dx.doi.org/10.1016/j.jfranklin.2010.11.009 (Check ref [8])

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So, let us do fitting!

Ideal physical plant model:

\[ G_p(s) = e^{-\sqrt{s}} \]

First Order Plus Time Delay (FOPTD) Model:

\[ G_{IO}(s) = \frac{K_1}{T_1s + 1} e^{-L_1s} \]

Time Delay with Single Fractional Pole Model:

\[ G_{FO}(s) = \frac{K_2}{T_2s^{0.5} + 1} e^{-L_2s} \]

All models are wrong but some are useful. *George E. P. Box*
FOPDT optimal fitting result $J=0.13541$

TDWFP optimal fitting result $J=0.020817$

$J=0.13541$

$J=0.020817$

$K_1 = 0.9120, \quad T_1 = 2.2393, \quad L_1 = 0$

$K_2 = 1.0197, \quad T_2 = 1.2312, \quad L_2 = 0.0001$
Fitting code for

\[ G_{IO}(s) = \frac{K_1}{T_1 s + 1} e^{-L_1 s} \]

% fitting using FOPTD model - integral of error square (ISE)
function \[ J = \text{foptdfit}(x, y0, Ts) \]
\% Ts: sampling period; ht: step response (from NILT numerical inverse
\% Laplace transform)
\% previously we got Ts and ht array (reaction curve)
options = optimset('TolX', 1e-10, 'TolFun', 1e-10);
Tic; [x, FVAL, EXITFLAG] = fminsearch(@fopdtfit(x, ht, Ts), [1, 1, 0], options);
Toc
% May need to wait half minute
K1 = x(1); T1 = x(2); L1 = x(3); T = (0:length(ht)-1)*Ts; if L1<0; L1 = 0; end
sysfoptd = tf([K1], [T1, 1], 'iodelay', L1);
y = step(sysfoptd, T); plot(T, ht, 'r', t, y, 'k:'); grid on;
title(['FOPDT optimal fitting result J=', num2str(FVAL)]);
xlabel('time (sec.)'); ylabel('step response'); legend('ideal', 'FOPDT')

% fitting using FOPTD model - integral of error square (ISE)
function \[ J = \text{foptdfit}(x, y0, Ts) \]
K1 = x(1); T1 = x(2); L1 = x(3); T = (0:length(y0)-1)*Ts; if L1<0; L1 = 0; end
sysfoptd = tf([K1], [T1, 1], 'iodelay', L1);
y = step(sysfoptd, T);
J = (y' - y0) * (y - y0') * Ts;
Fitting code for

\[ G_{FO}(s) = \frac{K_2}{T_2 s^{0.5} + 1} e^{-L_2 s} \]

```MATLAB
options=optimset('TolX',1e-10,'TolFun',1e-10);
Tic;[x,FVAL,EXITFLAG] =fminsearch(@(x) tdwfpfit(x,ht,Ts),[1,2,0],options);toc
% May need to wait 1000+ seconds!
K1=x(1);T1=x(2);L1=x(3);Np=length(ht);T=(0:Np-1)*Ts;if L1<0; L1=0; end
y=mlf(0.5,1.5,-T.^0.5/T1);y=(K1/T1)*(T.^0.5).*y;
Nstep=floor(L1/Ts);
y1=zeros(size(y));y1(Nstep+1:Np)=y(1:Np-Nstep);
y=y1;plot(T,ht,'r',t,y,'k:');grid on;
title(['TDWFP optimal fitting result J=',num2str(FVAL)]);
xlabel('time (sec.)');ylabel('step response'); legend('ideal', 'TDWFP model')

% fitting using TDWFP model - integral of error square (ISE)
function [J]=tdwfpfit(x,y0,Ts);
K1=x(1);T1=x(2);L1=x(3);Np=length(y0);T=(0:Np-1)*Ts;if L1<0; L1=0; end
y=mlf(0.5,1.5,-T.^0.5/T1);y=(K1/T1)*(T.^0.5).*y;
Nstep=floor(L1/Ts);y1=zeros(size(y));y1(Nstep+1:Np)=y(1:Np-Nstep);
J=(y1-y0)*(y1-y0)'*Ts;
% get MLF.m from
% www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-function
```

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Benefits of FOM

- Captures (more) physics

\[ G_p(s) = e^{-\sqrt{s}} \]

\[ G_{FO}(s) = \frac{K_2}{T_2 s^{0.5} + 1} e^{-L_2 s} \]

- Reaction curve fitting: Better than the best FOPDT model

\[ G_{io}(s) = \frac{K_1}{T_1 s + 1} e^{-L_1 s} \]

- Could be a nice starting point for better controller design?
FOMs and Fractional Order Controls

• IO Controller + IO Plant
• FO Controller + IO Plant
• FO Controller + FO Plant
• IO Controller + FO Plant


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Fractional order PID control

- 90% are PI/PID type in industry. (Ubiquitous)

\[ u(t) = K_p(e(t) + T_i D_t^{-\lambda} e(t) + \frac{1}{T_d} D_t^\mu e(t)) \]  \hspace{1cm} (D_t^*(*) \equiv_0 D_t^(*)).


YangQuan Chen, Dingyu Xue, and Huifang Dou. "Fractional Calculus and Biomimetic Control”. IEEE Int. Conf. on Robotics and Biomimetics (RoBio04), August 22-25, 2004, Shengyang, China.
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Complex relaxation in NMR


http://en.wikipedia.org/wiki/Metastasis
T2 relaxation in NMR

http://hs.doversherborn.org/hs/bridgerj/DSHS/appphysics/NMR/T2.htm

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Carr–Purcell–Meiboom–Gill (CPMG) pulse sequence, as shown in Fig. 1, is widely used to measure spin–spin relaxation time $T_2$.
Complex relaxation: How to characterize or model it?

- Debye relaxation: \( \exp\left(-\frac{t}{\tau}\right) \)
- Distributed-parameter (infinite # of time constants): 
  \[
  \int_0^T \frac{f(\tau)}{\tau s + 1} d\tau
  \]
  (H. Fröhlich, 1949)
Complex relaxation: How to better characterize or model it?

- Cole-Cole (1941)
  \[
  \frac{1}{1 + \tau \mathcal{S}^\alpha}
  \]

- Distributed-parameter (infinite \# time constants)
  \[
  \int_0^\tau \frac{f(\tau)}{\tau \mathcal{S}^\alpha + 1} d\tau
  \]

(Hu, Li and Chen, IEEE CDC 2010)
More complex relaxation models

- Cole-Davidson

\[
H_{C-D}(s) = \int_0^T \frac{f(\tau)}{(1 + \tau s)^\beta} \, d\tau
\]

- Havriliak-Negami

\[
H_{H-N}(s) = \int_0^T \frac{f(\tau)}{(1 + \tau s^\alpha)^\beta} \, d\tau
\]

- Distributed-order case? Sure!

\[
H(s) = \int_0^1 \frac{f(\gamma)}{\tau s^\gamma + 1} \, d\gamma
\]
More complex relaxation models:

Why?

- More complexities captured
- More informing (useful) parameters extracted
- Closer to the nature of the real (fractional) dynamics
- Not much added computational overhead

increase model complexities! When?
An illustration

- Distributed-parameter (infinite time constants)

\[ \int_0^T \frac{f(\tau)}{\tau s^\alpha + 1} d\tau \]

- \( \alpha = 0.75, \ T=1 \ \text{sec.} \)

\[ f(\tau) = 1 + 2\tau \]
The order information might be important in diagnosis of cancer, trauma etc. for treatment responses ...
A take home message

- Go “Fractional Order Models” if (note: not \textit{iff})

  - The signals are from a complex “system”
    - biological,
    - physiological,
    - behavioral,
    - social (crowds, community emergence, alcohol diffusion …)
    - man-made, (transportation/comm/power networks, etc.)
    - natural (geological, ecological, environmental etc.)
    - cosmological
    - etc.
Go “Fractional Order Models”, too?

\[ H(s) = \int_0^1 \frac{f(\gamma)}{\tau s^\gamma + 1} d\gamma \]
Thank you for attending my talk!

For more information, check
http://mechatronics.ece.usu.edu/foc/afc
http://www.youtube.com/user/FractionalCalculus
coming soon
http://mechatronics.ucmerced.edu/