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Bifurcation dynamics of the tempered fractional Langevin equation

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Tempered fractional processes offer a useful extension for turbulence to include low frequencies. In this paper, we investigate the stochastic phenomenological bifurcation, or stochastic P-bifurcation, of the Langevin equation perturbed by tempered fractional Brownian motion. However, most standard tools from the well-studied framework of random dynamical systems cannot be applied to systems driven by non-Markovian noise, so it is desirable to construct possible approaches in a non-Markovian framework. We first derive the spectral density function of the considered system based on the generalized Parseval's formula and the Wiener-Khinchin theorem. Then we show that it enjoys interesting and diverse bifurcation phenomena exchanging between or among explosive-like, unimodal, and bimodal kurtosis. Therefore, our procedures in this paper are not merely comparable in scope to the existing theory of Markovian systems but also provide a possible approach to discern P-bifurcation dynamics in the non-Markovian settings. *Published by AIP Publishing.*
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The nonlinear dynamical systems with random uncertainty occur ubiquitously in nature. The noise perturbation can drastically change the deterministic dynamics when it acts as a driving term in the equations of considered objects. The well-known framework of random dynamical systems (RDS) bridges the gap between dynamical systems and stochastic analysis, and offers useful tools to solve problems such as stochastic stability, random attractor, invariant manifolds, and stochastic bifurcation. There is a rich literature on the complex bifurcation dynamics of stochastic systems driven by Markovian noise (including Gaussian white noise and Lévy noise) within the framework of RDS. However, there is no *a priori* reason to assume that the random forces are independent over disjoint time intervals, which motivates us to re-evaluate and improve the crucial results from Markovian settings to non-Markovian counterparts. It is very difficult to follow and hard to analyze because of the long-range correlations of the non-Markovian noise. Recently, fractional noise has been found to destroy or induce dynamical bifurcation (D-bifurcation) under the framework of non-Markovian systems. Unfortunately, it seems that there is no result available in the literature on phenomenological bifurcation (P-bifurcation) for non-Markovian systems. The main difficulties below are twofold. First, such systems cannot reduce to a Markovian case without endowing infinitely many degrees of freedom and the driving noise cannot be white in the limit of large time rescalings. Second, one does not have the exact analytical solution for the steady Fokker-Planck equation of general non-Markovian systems. Nevertheless, the well-known Wiener-Khinchin theorem states that the spectral density and the autocorrelation function form a Fourier pair for a stationary stochastic process. This actually offers us an alternative idea to explore P-bifurcation based on the spectral density. In this paper, we intend to discern such complex dynamics through Ornstein-Uhlenbeck process driven by a tempered fractional noise. We show that such a system shares diverse and interesting phenomena, that is, the shape of its spectral density changes between or among explosive-like, unimodal, and bimodal curves. Consequently, this paper supplies an effective approach to analyze P-bifurcation dynamics in the non-Markovian framework.

I. INTRODUCTION

Bifurcation theory provides a strategy for investigating the qualitative or topological changes in the orbit structure of parameterized dynamical systems. The basic and key question that arises is what one may expect to occur in the dynamics with a given number of parameters allowed to vary. The general term "bifurcation" was first introduced by Henri Poincaré in mathematics.[1] Since then, it has been developed progressively up to now, and several good books with an excellent level of background and techniques have appeared on this subject.[2–9] For the deterministic systems, they are fully determined by the parameter values and initial conditions, but not with the ensemble of external constraints acting on the system. Contrary to internal fluctuations, the fluctuations generated by the environment or the system parameters cannot be ignored. Obviously, the natural world is buffeted by stochasticity and actually subjected to noise perturbations. It is worth noting that the present noise is used to describe the interaction between the (small) system and its

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(large) environment. The driving noise could be white or colored, Gaussian or non-Gaussian, semimartingale or non-semimartingale, and Markovian or non-Markovian. In complete analogy to the deterministic case, Arnold established the so-called stochastic bifurcation theory based on the framework of random dynamical systems (RDS), which is tailor-made to cover all important systems under randomness which are presently of interest.[10]

In general, there are now two well-established mechanisms, "trajectory based approach" and "distribution based approach," to formalize the issue of stochastic bifurcation. More precisely, the former is called Dynamical bifurcation (D-bifurcation), which is connected to the stability of an RDS under an invariant measure and reduces to the deterministic definition of bifurcation in the absence of noise. And the latter is called Phenomenological bifurcation (P-bifurcation), where one can search for changes of the shape of the stationary probability density function of the stationary measure. However, these two kinds of definitions may lead to different results. For instance, Baxendale presented an example of the one-parameter family of stochastic flow on the two-dimensional torus in which the largest Lyapunov exponent changes its sign while the shape of stationary probability density does not depend on the bifurcation parameter.[11] In contrary, Crauel and Flandoli analyzed the effect of additive noise on a pitchfork bifurcation and found that the largest Lyapunov exponent does not change its sign while the stationary probability density function does change its shape from a mono-peak one into double-peak one at a critical parameter value.[12] Therefore, the stochastic bifurcation theory is still in its infancy. As pointed out in Ref. 13, these two approaches are not sufficient in the following questions: What has really happened for stochastic bifurcation? What is the topological property of a stochastic system? What kind of invariance is suitable for predicting stochastic bifurcation?

When the driving noise is considered as Gaussian white noise, the theory of stochastic bifurcation has attracting increasing interests within the framework of RDS. For the basic concepts and established results on this issue before 1999, we refer the reader to the monograph[10] and the references therein and also the review paper.[14] Afterwards, the theory of stochastic bifurcation has been advanced to a new level,[15–23] but much more systematic work needs to be done to gain more insights. When the considered systems are driven by α-stable Lévy noise, stochastic P-bifurcation was further discussed in some typical examples. Particularly, P-bifurcation of a simple dynamical system was numerically analyzed by solving a nonlocal Fokker-Planck equation;[24] the stochastic bifurcation of a stochastic Lorenz-Stenflo system with Lévy noise was studied using the theory of random attractors.[25] Noting that both Gaussian white noise and Lévy noise are Markovian, so the well-studied general framework of RDS and the corresponding stationary density function could be applied to understand the stochastic bifurcation dynamics. When the driving noise is non-Markovian, however, it seems far more arduous since the RDS framework is no longer applicable. This situation motivates us to explore such complex dynamics. In this respect, a recent paper was devoted to stochastic D-bifurcation of the Black-Scholes model with mixed fractional Brownian motion (FBM).[26] The main results showed that fractional noise not only destroys but also induces a stochastic bifurcation under some suitable conditions. Indeed, such dynamics belongs in the category of D-bifurcation since authors adopted the stability change in the considered system. As far as we know, however, it seems that there is no result available in the literature on P-bifurcation for non-Markovian systems. So it is interesting and natural to investigate the impact on stochastic P-bifurcation induced by non-Markovian noise. We will proceed this topic mainly by way of an instructive example, namely, tempered fractional Ornstein-Uhlenbeck (TFOU) process, whose precise definition can be found in Section II below.

Recently, Meerschaert and Sabzikar have proposed a new stochastic process, tempered fractional Brownian motion (TFBM), which multiplies the power law kernel in the moving average representation of a fractional Brownian motion (FBM) by an exponentially tempering factor.[27] They further presented some pleasant properties of TFBM, including moving average and spectral representations, covariance structure and scaling feature, and indicated an important application to modeling wind speed near the earth's surface.

Now let us start by reviewing the definition and some basic properties of TFBM. Denote $\{B(t)\}_{t\in\mathbb{R}}$ by a two-sided Brownian motion with mean zero and variance $|t|$ for all $t\in\mathbb{R}$. Define an independently scattered Gaussian random measure $B(dx)$ with control measure $m(dx)=dx$ by setting $B[a,b]=B(b)-B(a)$ for any real numbers $a<b$, and then extending to all Borel sets. For any $\alpha<1/2$ and $\lambda\geq 0$, a TFBM is defined by the following integral:

$$B_{\alpha,\lambda}(t)=\int_{-\infty}^{+\infty}[e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha}-e^{-\lambda(-x)_+}(-x)_+^{-\alpha}]B(dx),\tag{1}$$

where $(x)_+=x1_{\{x>0\}}$, $0^0=0$ and λ is called tempered parameter. It follows from the Proposition 2.3 in Ref. 27 that TFBM has the covariance function

$$Cov\left[B_{\alpha,\lambda}(t),B_{\alpha,\lambda}(s)\right]=\frac{1}{2}\left[C_t^2|t|^{2H}+C_s^2|s|^{2H}-C_{t-s}^2|t-s|^{2H}\right],\tag{2}$$

where $H=1/2-\alpha$, and

$$C_t^2=\frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}}-\frac{2\Gamma(H+1/2)}{\sqrt{\pi}}\frac{1}{(2\lambda|t|)^{H}}K_H(\lambda|t|),\quad t\neq 0,\tag{3}$$

in which $K_H(\cdot)$ is the modified Bessel function of the second kind, and $C_0^2=0$. Generally speaking, when $\alpha<-1/2$ and $\lambda=0$, TFBM (1) does not exist because the integrand in the right hand of (1) is not square integrable. When $-1/2<\alpha<1/2$ and $\lambda=0$, TFBM (1) reduces to FBM with Hurst scaling index $H=1/2-\alpha$. There have been numerous attempts to define a (stochastic) integral with respect to FBM (see the survey paper[28] for more details). When $\alpha<-1/2$ and $\lambda>0$, or when $\alpha=0$ and $\lambda>0$, TFBM (1) is a continuous semimartingale. So the classical Itô stochastic calculus is applicable to TFBM in these cases. When $\alpha\in(-1/2,0)\cup$

$(0/1/2)$ and $\lambda > 0$, TFBM is neither a semimartingale nor a Markov process. Inspiring by the seminal work by Pipiras and Taqqu,[29] the theory of stochastic integration for TFBM has recently been developed based on the tempered fractional calculus.[30] It is also worth pointing out that the main advantage of TFBM is that the tempering parameter can be chosen arbitrarily light for modifying the power law kernel to any desired degree of accuracy over a finite interval. This significant property makes TFBM and its stochastic integrals natural candidates to modeling realistic data. For example, the spectral density of FGN blows up at the origin like a power law, while the spectral density of TFGN follows the same power law at moderate frequencies, but remains bounded at very low frequencies, a behavior typically seen in wind speed data.[27]

Fractional Langevin equation was shown to describe the anomalous diffusion in the non-Markovian framework.[31] Also, the fractional Langevin equation was applied to model the financial time series to characterize the inverse power-law correlation.[32] Based on the fractional calculus, the fractional Langevin equation involving both Caputo and Riemann-Liouville derivatives was introduced by using the generalized Newtonian law.[33] However, these above mentioned results focused on the time-fractional Langevin equation or Langevin equation with a fractional noise.

This paper concentrates on stochastic P-bifurcation induced by tempered fractional noise. To this end, we will introduce a concept of a class of stochastic bifurcation based on the spectral density function. Furthermore, we will prove and examine the conditions for an emerging explosive-like unimodal or bimodal curve from the P-bifurcation point of view. And finally, we will manifest that the tempered fractional processes possess very diverse and interesting bifurcation phenomena. In addition to the usual unimodal → bimodal transition, we shall detect several new transitions including explosive-like → unimodal, explosive-like → bimodal, and explosive-like → unimodal → bimodal.

The remainder of this paper is organized as follows. In Sec. II, the tempered fractional Ornstein-Uhlenbeck process is formulated by the idea that simple system with complex dynamics. In Sec. III, the main results are stated and proved in details through the spectral density function as the assigned bifurcation parameter varies. Some relevant remarks are also provided to discuss the implications and extensions of the obtained results. Finally, the conclusions are drawn in Sec. IV.

II. MODEL DERIVATION

In statistical physics, the classical Ornstein-Uhlenbeck (OU) process is understood to describe the velocity of a massive Brownian particle under the influence of friction. This stochastic process with parameters $\rho > 0$ and $\sigma > 0$ solves the following equation:[34]

$$dX(t) = -\rho X(t)dt + \sigma dB(t), \tag{4}$$

where $1/\rho$ and σ^2 represent, respectively, the relaxation time and the diffusion constant, and $B(t)$ is a standard one-dimensional Brownian motion. By the Langevin's hypothesis and Newton's second law, the OU process (4) can be rewritten as the Langevin equation,[35]

$$m\frac{dV(t)}{dt} = -\gamma V(t) + F(t),$$

with relaxation time and diffusion constant

$$\frac{1}{\rho} = \frac{Dm}{k_B T}, \quad \sigma = \sqrt{\frac{2}{D}}\frac{k_B T}{m},$$

in which $V(t)$ denotes the velocity of the Brownian particle of mass m and diffusion coefficient D at absolute temperature T, γ denotes the damping constant, k_B is the Boltzmann's constant, and $F(t)$ is a noise term representing the effect of the collisions with the molecules of the fluid.

Noting that the OU process (4) becomes a driftless Brownian motion with diffusion constant σ^2 under the limit $\rho \to 0$. Setting the initial condition $\xi = \sigma\int_{-\infty}^{0}\exp(\rho\tau)dB(\tau)$, the stationary solution of (4) is given by

$$X(t) = \sigma\int_{-\infty}^{t} e^{-\rho(t-\tau)}dB(\tau),$$

which is an almost surely continuous, centered Gaussian Markov process.

Recently, there has been a growing interest in studying stochastic systems driven by fractional Brownian motion (FBM) in view of their applications to long memory property. Cheridito *et al.* proposed the so-called fractional Ornstein-Uhlenbeck (FOU) process, which is the solution of the following fractional Langevin equation,[36]

$$dY(t) = -\rho Y(t)dt + \sigma dB^H(t), \tag{5}$$

where $B^H(t)$ is a standard FBM with Hurst parameter $0 < H < 1$. Similarly, the almost surely continuous process,

$$Y(t) = \sigma\int_{-\infty}^{t} e^{-\rho(t-\tau)}dB^H(\tau),$$

solves (5) with initial condition $\xi = \sigma\int_{-\infty}^{0}\exp(\rho\tau)dB^H(\tau)$. Indeed, $Y(t)$ is a stationary, centered Gaussian, but non-Markov process when $H \neq 1/2$.

Now we are ready to introduce the Langevin equation driven by a TFBM of the form

$$dZ(t) = -\rho Z(t)dt + \sigma dB_{\alpha,\lambda}(t), \tag{6}$$

where $B_{\alpha,\lambda}(t)$ is a TFBM defined in (1). In particular, if $\alpha = 0$ and $\lambda = 0$, (6) becomes OU process (4); if $-/12 < \alpha < 1/2$ and $\lambda = 0$, (6) reduces to FOU process (5), where $\alpha = 1/2 - H$. By applying the Conjugate method,[36] one can obtain a unique almost surely continuous solution of (6) as

$$Z(t) = \sigma\int_{-\infty}^{t} e^{-\rho(t-\tau)}dB_{\alpha,\lambda}(\tau),$$

under the initial condition

$$\xi = \sigma\int_{-\infty}^{0} e^{\rho\tau}dB_{\alpha,\lambda}(\tau).$$

Herein, we call $\{Z(t)\}_{t\in\mathbb{R}}$ a *tempered fractional Ornstein-Uhlenbeck (TFOU) process*. Noting that when $\rho \to 0$, TFOU reduces to TFBM, therefore we assume that $\sigma > 0$, $\rho \geq 0$, $\lambda \geq 0$, and $-1/2 < \alpha < 1/2$ through the rest of the paper.

Although the framework of the stochastic integral with respect to the TFBM was established in Ref. 30, the corresponding Itô formula and Fokker-Planck-Kolmogorov equation of TFBM-driving stochastic systems are still open. The key technical difficulties are due to the non-local kernel t^{2H} and the modified Bessel function of the second kind $k_v(z)$ in the covariance function of the TFBM. Consequently, the classical stochastic tools cannot be directly applied to study P-bifurcation, which is concerned with change in the shape of the stationary probability densities when the parameters vary. Since the TFOU is stationary stochastic process, its spectral density and the autocorrelation form a Fourier pair by the well-known Wiener-Khinchin theorem.[37] This fact motivates us to open a possible way to discuss the P-bifurcation based on the spectral density function.

Our task in Section III is to determine the stochastic bifurcation phenomena of (6) with the help of its spectral density function.

III. MAIN RESULTS AND REMARKS

Obviously there are two more free parameters, Hurst exponent H and tempered factor λ, in TFBM-driving stochastic differential equations than the Itô (or Stratonovich) counterpart. For instance, one can choose the coefficient parameter ρ as the bifurcation parameter while fix the Hurst parameter H and tempered factor λ. And, of course, we could choose H or λ as the bifurcation parameter but fix the other two parameters.

Recall that in the Markov framework, the concept of P-bifurcation was used to describe the transition from a unimodal (mono-peak) to a bimodal (double-peaks) or crater-like form in the stationary density. This leads to the following definition of P-bifurcation, which is intentionally kept as close as possible to the definition of P-bifurcation in Ref. 10.

Definition III.1. *A parameterized stochastic system is called to undergo a bifurcation of spectral density if there exists "qualitative change" in the shape of its spectral density when a bifurcation parameter varies.*

Remark III.1. *The bifurcation of spectral density belongs in the category of P-bifurcation.*

Following the developed theory of stochastic integrals for TFBM,[30] one could construct a generalized isometry formula similar to Lemma 2.1 in Ref. 36 since the covariance function (2) of a TFBM is valid. However, this formula is much complicated due to the requirement of computing the first and second derivatives of the Bessel function term (3). Alternatively, we utilize the harmonizable representation of the integral with a TFBM, and then obtain the spectral density function by using the generalized Parseval's formula (i.e., Theorem 3.15 in Ref. 30) and the Wiener-Khinchin theorem.[37]

Lemma III.1. *TFOU process (6) admits the following spectral density function:*

$$h(\omega) = \frac{\sigma^2\Gamma(1-\alpha)^2}{\sqrt{2\pi}}\frac{1}{\rho^2+\omega^2}\frac{\omega^2}{(\lambda^2+\omega^2)^{1-\alpha}}. \tag{7}$$

Proof. The Fourier transform of a function $f(t)$ is defined as

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\omega t} f(t)dt,$$

and the inverse Fourier transform as

$$\breve{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega t} f(\omega)d\omega.$$

Then the autocorrelation function yields that

$$\begin{aligned}
r_Z(t) &= E[Z(0)Z(t)] \\
&= E\left[\left(\sigma\int_{-\infty}^{0} e^{\rho\tau_1}dB_{\alpha,\lambda}(\tau_1)\right)\left(\sigma\int_{-\infty}^{t} e^{-\rho(t-\tau_2)}dB_{\alpha,\lambda}(\tau_2)\right)\right] \\
&= \sigma^2 e^{-\rho t}E\left[\left(\int_{-\infty}^{+\infty} 1_{\{\tau_1\leq 0\}}e^{\rho\tau_1}dB_{\alpha,\lambda}(\tau_1)\right)\right. \\
&\quad\times\left.\left(\int_{-\infty}^{+\infty} 1_{\{\tau_2\leq t\}}e^{\rho\tau_2}dB_{\alpha,\lambda}(\tau_2)\right)\right] \\
&= \sigma^2\Gamma(1-\alpha)^2 e^{-\rho t}\int_{-\infty}^{+\infty} \widehat{1_{\{\omega\leq 0\}}e^{\rho\omega}}\,\overline{\widehat{1_{\{\omega\leq t\}}e^{\rho\omega}}} \\
&\quad\times\frac{\omega^2}{(\lambda^2+\omega^2)^{1-\alpha}}d\omega \\
&= \frac{\sigma^2\Gamma(1-\alpha)^2}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}\frac{1}{\rho^2+\omega^2}\frac{\omega^2}{(\lambda^2+\omega^2)^{1-\alpha}}d\omega.
\end{aligned}$$

This shows that (7) holds and therefore the proof of Lemma III.1 is complete. □

In order to discern the P-bifurcation phenomena of (6), we discuss first the singularity and extrema of spectral density $h(\omega)$ in a straightforward way.

Case 1: When $\lambda = \rho = 0$ and $\alpha \in (-1/2, 1/2)$, $h(\omega)$ has singularity at $\omega^* = 0$, due to

$$h(\omega) = \frac{\sigma^2\Gamma(1-\alpha)^2}{\sqrt{2\pi}}\frac{1}{\omega^{2-2\alpha}}, \quad 1 < 2-2\alpha < 3.$$

Case 2: When $\lambda = 0$, $\rho > 0$ and $\alpha \in (-1/2, 0)$, $h(\omega)$ also has singularity at $\omega^* = 0$, since

$$h(\omega) = \frac{\sigma^2\Gamma(1-\alpha)^2}{\sqrt{2\pi}}\frac{1}{\rho^2+\omega^2}\frac{1}{\omega^{-2\alpha}}, \quad 0 < -2\alpha < 1.$$

Case 3: For other nonsingular cases, it is easy to observe that

$$h'(\omega) = \frac{\sigma^2\Gamma(1-\alpha)^2}{\sqrt{2\pi}}\frac{((\alpha-1)\lambda^4+\alpha\rho^2\omega^2+\rho^2\lambda^2)\omega}{(\rho^2+\omega^2)^2(\lambda^2+\omega^2)^{2-\alpha}}.$$

By the First Derivative Test, we further have

(3a) $\rho = 0$, $\lambda > 0$ and $-1/2 < \alpha < 1/2$: $h(\omega)$ has a unique global maxima at $\omega^* = 0$;

(3b) $\rho > 0$, $\lambda = 0$ and $\alpha = 0$: $h(\omega)$ has a unique global maxima at $\omega^* = 0$;

(3c) $\rho > 0$, $\lambda = 0$ and $0 < \alpha < 1/2$: $h(\omega)$ has a unique global minima at $\omega^* = 0$ and two global maxima at $\omega^* = \pm\sqrt{\alpha/(1-\alpha)}\rho$;

(3d) $\rho > 0$, $\lambda > 0$ and $-1/2 < \alpha < 1/2$: $h(\omega)$ has a unique global minima at $\omega^* = 0$ and two global maxima at

$$\omega^* = \pm\sqrt{(\alpha\rho^2 + \sqrt{\alpha^2\rho^2 + 4(1-\alpha)\lambda^2}\rho)/(2-2\alpha)}.$$

With these discussions at hand, we are now ready to present our main result of bifurcation of spectral density of non-Markovian system (6).

Theorem III.1. *Assume that parameters α and ρ are fixed, then*

(i) *if $-1/2 < \alpha < 1/2$ and $\rho = 0$, model (6) undergoes P-bifurcation at $\lambda^* = 0$;*

(ii) *if $-1/2 < \alpha \le 0$ and $\rho > 0$, model (6) undergoes P-bifurcation at $\lambda^* = 0$;*

(iii) *if $0 < \alpha < 1/2$ and $\rho > 0$, there is no P-bifurcation in model (6).*

Proof. Based on the discussion of the singularity and extrema of spectral density $h(\omega)$, we establish the corresponding proofs of these three cases, respectively.

(i) When $-1/2 < \alpha < 1/2$ and $\rho = 0$, the non-Markovian system (6) has a singular point at $\omega^* = 0$ if $\lambda = 0$, however, this singular point becomes the unique global maxima if $\lambda > 0$. More precisely, $h(\omega)$ explodes for $\lambda = 0$ in the neighborhood of $\omega^* = 0$, but it is unimodal for $\lambda > 0$. Therefore, $h(\omega)$ evolves from explosive-like to unimodal as λ increases from zero to positive. This implies that model (6) undergoes P-bifurcation at $\lambda^* = 0$.

(ii) This case is divided into two situations. First, when $-1/2 < \alpha < 0$ and $\rho > 0$, the non-Markovian system (6) has a singular point at $\omega^* = 0$ if $\lambda = 0$, however, this singular point becomes the unique global minima and there are two more global maxima at $\omega^* = \pm\sqrt{(\alpha\rho^2 + \sqrt{\alpha^2\rho^2 + 4(1-\alpha)\lambda^2}\rho)/(2-2\alpha)}$ if $\lambda > 0$. Second, when $\alpha = 0$ and $\rho > 0$, the non-Markovian system (6) has a unique global maxima at $\omega^* = 0$ if $\lambda = 0$, but this point becomes the unique global minima and there are two more global maxima at $\omega^* = \pm\lambda\rho$ if $\lambda > 0$. Then, in the former situation, $h(\omega)$ explodes for $\lambda = 0$ in the neighborhood of $\omega^* = 0$, but it is unimodal for $\lambda > 0$. That is, $h(\omega)$ evolves from explosive-like to unimodal as λ increases from zero to positive. And in the latter situation, $h(\omega)$ is unimodal for $\lambda = 0$, but it is bimodal for $\lambda > 0$. That is, $h(\omega)$ evolves from unimodal to bimodal as λ increases from zero to positive. Both situations mean that model (6) undergoes P-bifurcation at $\lambda^* = 0$.

(iii) When $0 < \alpha < 1/2$ and $\rho > 0$, there are a unique minima and two global maxima in (6) for all $\lambda \ge 0$. Then $h(\omega)$ is always bimodal, and thus there is no P-bifurcation. □

Theorem III.2. *Assume that parameters α and λ are fixed, then*

(i) *if $-1/2 < \alpha < 0$ and $\lambda = 0$, there is no P-bifurcation in model (6);*

(ii) *if $0 \le \alpha < 1/2$ and $\lambda = 0$, model (6) undergoes P-bifurcation at $\rho^* = 0$;*

(iii) *if $-1/2 < \alpha < 1/2$ and $\lambda > 0$, model (6) undergoes P-bifurcation at $\rho^* = 0$.*

Proof. Similar to the proof of Theorem III.1, we also construct the proof as follows.

(i) When $-1/2 < \alpha < 0$ and $\lambda = 0$, $h(\omega)$ always explodes in the neighborhood of $\omega^* = 0$ for all $\rho \ge 0$. Thus there is no P-bifurcation in model (6) in this case.

(ii) This case is divided into two situations. First, when $\alpha = 0$ and $\lambda = 0$, $h(\omega)$ evolves from explosive-like to unimodal as ρ increases from zero to positive. And in another situation, $h(\omega)$ evolves from explosive-like to bimodal as ρ increases from zero to positive. Both situations imply that model (6) undergoes P-bifurcation at $\rho^* = 0$.

(iii) When $-1/2 < \alpha < 1/2$ and $\lambda > 0$, $h(\omega)$ evolves from unimodal to bimodal as ρ increases from zero to positive. Thus, model (6) undergoes P-bifurcation at $\rho^* = 0$. □

Theorem III.3. *Assume that parameters ρ and λ are fixed, then*

(i) *if $\rho = 0$ and $\lambda = 0$, there is no P-bifurcation in model (6);*

(ii) *if $\rho > 0$ and $\lambda = 0$, model (6) undergoes P-bifurcation at $\alpha^* = 0$;*

(iii) *if $\rho \ge 0$ and $\lambda > 0$, there is no P-bifurcation in model (6).*

Proof. Based on the proof procedure in Theorems III.1 and III.2, it is also straightforward to establish the proof as follows.

(i) When $\rho = 0$ and $\lambda = 0$, this claim holds since $h(\omega)$ is always explode-like.

(ii) When $\rho > 0$ and $\lambda = 0$, $h(\omega)$ evolves from explosive-like to unimodal to bimodal as α increases on the open interval $(-1/2, 1/2)$. This indicates the required result.

(iii) Since $h(\omega)$ is always unimodal for $\rho = 0$ and $\lambda > 0$, and it is bimodal for $\rho > 0$ and $\lambda > 0$, respectively. Therefore this claim is true in this case. □

Remark III.2. *Actually, the triple (α, ρ, λ) could be divided in twelve situations:*

$$\Omega_1 = \left\{(\alpha,\rho,\lambda) \middle| -\frac{1}{2} < \alpha < 0, \rho = 0, \lambda = 0\right\},$$

$$\Omega_2 = \left\{(\alpha,\rho,\lambda) \middle| -\frac{1}{2} < \alpha < 0, \rho = 0, \lambda > 0\right\},$$

$$\Omega_3 = \left\{(\alpha,\rho,\lambda)| -\frac{1}{2} < \alpha < 0, \rho > 0, \lambda = 0\right\},$$
$$\Omega_4 = \left\{(\alpha,\rho,\lambda)| -\frac{1}{2} < \alpha < 0, \rho > 0, \lambda > 0\right\},$$
$$\Omega_5 = \{(\alpha,\rho,\lambda)|\alpha = 0, \rho = 0, \lambda = 0\},$$
$$\Omega_6 = \{(\alpha,\rho,\lambda)|\alpha = 0, \rho = 0, \lambda > 0\},$$
$$\Omega_7 = \{(\alpha,\rho,\lambda)|\alpha = 0, \rho > 0, \lambda = 0\},$$
$$\Omega_8 = \{(\alpha,\rho,\lambda)|\alpha = 0, \rho > 0, \lambda > 0\},$$
$$\Omega_9 = \left\{(\alpha,\rho,\lambda)|0 < \alpha < \frac{1}{2}, \rho = 0, \lambda = 0\right\},$$
$$\Omega_{10} = \left\{(\alpha,\rho,\lambda)|0 < \alpha < \frac{1}{2}, \rho = 0, \lambda > 0\right\},$$
$$\Omega_{11} = \left\{(\alpha,\rho,\lambda)|0 < \alpha < \frac{1}{2}, \rho > 0, \lambda = 0\right\},$$
$$\Omega_{12} = \left\{(\alpha,\rho,\lambda)|0 < \alpha < \frac{1}{2}, \rho > 0, \lambda > 0\right\}.$$

Then the above results are illustrated in Table I.

We next conduct numerical simulations to illustrate our results. It is evident from (7) that the noise intensity σ is one multiple for $h(\omega)$. Thus when the positive noise intensity σ increases, the spectral density $h(\omega)$ just becomes flatter (or less spiky) for fixed parameters α, ρ, and λ, and its number and the shape do not change. So we take $\sigma = 1$ without generality in the following simulation. On the other hand, since $-1/2 < \alpha < 1/2$, $\rho \geq 0$, and $\lambda \geq 0$, we choose $\alpha = -1/4, 0, 1/4$, $\rho = 0, 1$, and $\lambda = 0, 1$, and thus it yields 12 different types (α,ρ,λ). By setting the iterated interval $\omega \in [-100, 100]$, iterated points as 2×10^8, the spectral density function (7) is shown in Fig. 1. In particular, Figs. 1(a)–1(l) correspond to situations Ω_i, $i = 1, 2, \ldots, 12$, respectively. Also, we magnify each situation to look at the details of a small region $[-4, 4]$. Here, we observe that $h(\omega)$ is explosive-like for $\Omega_1, \Omega_3, \Omega_5, \Omega_9$, unimodal for $\Omega_2, \Omega_6, \Omega_7, \Omega_{10}$, and bimodal for $\Omega_4, \Omega_8, \Omega_{11}, \Omega_{12}$. And this is more evident within the small interval $[-4, 4]$.

Alternatively, we can also summarize the bifurcation transitions in a more precise way. We use a directed node-link diagram layout with Graphviz,[38] a popular open-source graph drawing package developed at AT&T Labs, to show the relationship among these bifurcation transitions. Let us now set off the attributes of necessary nodes and edges. More specifically, the nodes $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_5, \boldsymbol{\Omega}_9$ are assigned shape *triangle* filled with a shade of *red*, the nodes $\boldsymbol{\Omega}_2, \boldsymbol{\Omega}_6, \boldsymbol{\Omega}_7, \boldsymbol{\Omega}_{10}$ are specified shape *circle* filled with a shade of *lightskyblue*, and the nodes $\boldsymbol{\Omega}_4, \boldsymbol{\Omega}_8, \boldsymbol{\Omega}_{11}, \boldsymbol{\Omega}_{12}$ are appointed shape *doublecircle* filled with a shade of *lightgreen*. Different shapes and colors of the shade are used to distinguish the three kurtosis: explosive-like, unimodal, bimodal. Besides, the edge labeled λ is drawn as a dashed line to represent undergoing bifurcation by varying λ, the edge labeled ρ is set as a dotted line to represent undergoing bifurcation by varying ρ, and the edge labeled α is established as a solid line to represent undergoing bifurcation by varying α. Moreover, edge points are listed left-to-right regardless of the orientation of the edge. Based on the above settings of the attributes of nodes and edges, we make the edges from node $\boldsymbol{\Omega}_\mathbf{i}$ to node $\boldsymbol{\Omega}_\mathbf{j}$ to depict the transition between these kurtosis shape according to Theorems III.1, III.2, and III.3, where $i, j = 1, 2, \ldots, 12$. Then the required nodes and edges are created with the tool *dot*, and the corresponding layout (bifurcation diagram) is illustrated in Fig. 2.

TABLE I. The relationship between (α,ρ,λ) and $h(\omega)$.

(α,ρ,λ)	$h(\omega)$
$\Omega_1 \cup \Omega_3 \cup \Omega_5 \cup \Omega_9$	Explosive-like
$\Omega_2 \cup \Omega_6 \cup \Omega_7 \cup \Omega_{10}$	Unimodal
$\Omega_4 \cup \Omega_8 \cup \Omega_{11} \cup \Omega_{12}$	Bimodal

Remark III.3. *From the above theoretical and numerical results, the non-Markovian system (6) possesses very diverse and interesting bifurcation phenomena. In addition to the usual unimodal $\rightarrow$ bimodal transition, we also detect several new transitions including explosive-like $\rightarrow$ unimodal, explosive-like $\rightarrow$ bimodal and explosive-like $\rightarrow$ unimodal $\rightarrow$ bimodal.*

Remark III.4. P-bifurcation under Brownian motion *(Theorem III.2 (ii) with $\alpha = 0$): in this situation TFOU process (6) reduces to the classical OU process (4), and there is only one bifurcation parameter ρ. Our obtained results imply that the P-bifurcation occurs with the transition from explosive-like to unimodal.*

Remark III.5. P-bifurcation under fractional Brownian motion *(Theorem III.2 (ii) with $0 < \alpha < 1/2$ and Theorem III.3 (ii)): in these situations, TFOU process (6) becomes FOU process (5), and the parameters ρ and α have been chosen as bifurcation parameter. It has been shown that the former undergoes a P-bifurcation with the transition from explosive-like to unimodal, while the latter evolves from explosive-like to unimodal to bimodal.*

Remark III.6. *The adopted approach in Theorem III.1 needs to fix the parameters α and ρ, so it is impossible to study the stochastic bifurcation phenomena of models (4) and (5), since there is no bifurcation parameter in this situation. However, stochastic P-bifurcation does occur in model (6) under some appropriate conditions by choosing the tempered parameter λ as bifurcation parameter.*

Remark III.7. *As stated in Theorem III.2 (ii) above, P-bifurcation does occur in OU process (4) and FOU process (5), discussed in Remarks III.4 and III.5, respectively. Since TFOU process (6) includes models (4) and (5) as special cases, (6) shares the same bifurcation phenomena in Theorem III.2 (ii). In addition, there is one more free positive parameter λ, and thus TFOU process (6) exhibits more complex bifurcation dynamics in comparison to models (4) and (5), given in Theorem III.2 (iii) to supply more details.*

Remark III.8. *It is necessary to point out that we limit ourselves to detecting P-bifurcation as a single active parameter varies. As a matter of fact, one can extend our results to codimension one (two) P-bifurcations when two (three) parameters are designated to be active. As an example, there is codimension one P-bifurcation as active parameter pair (ρ,λ) changes from (0, 0) to (1, 1) for fixed $-1/2 < \alpha < 1/2$, and codimension two P-bifurcation occurs*

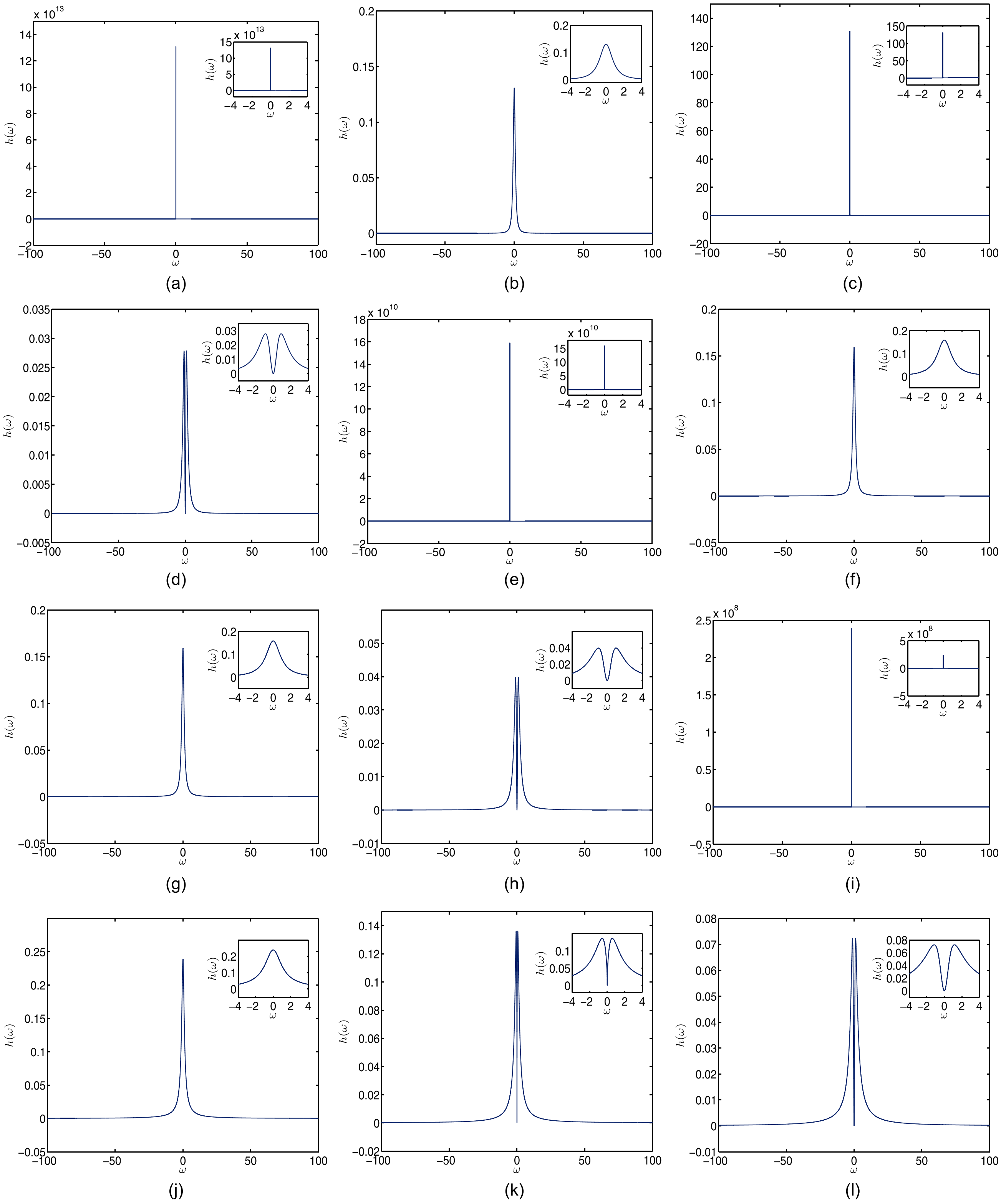

FIG. 1. Spectral density (7) with different triple (α, ρ, λ).

when active parameter triple (α, ρ, λ) *shifts from (−1/4, 0, 0) to (1/4, 1, 1). This extension is straightforward and can be obtained by simply rereading the paper carefully.*

IV. CONCLUSION

To effectively characterize the P-bifurcation of non-Markovian systems, we have proposed a new approach based on the spectral density function, which opens a possible way to understand the theory of stochastic P-bifurcation from a Markovian framework to a non-Markovian framework. We have demonstrated that the TFOU process exhibits very diverse and interesting bifurcation phenomena. Our approach in this paper not only results in unambiguously recovering the usual unimodal → bimodal transition within Markovian

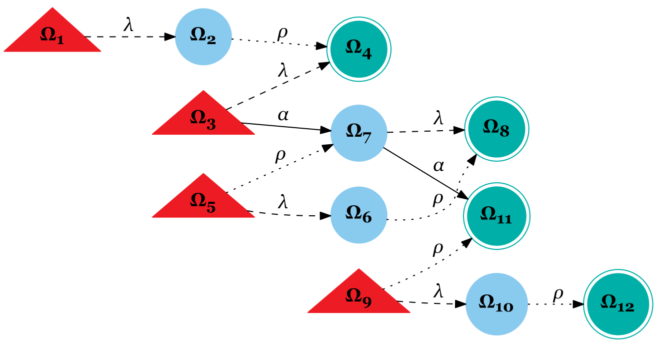

FIG. 2. The stochastic P-bifurcation diagram of TFOU (6).

framework but also leads to the benefits of detecting several new transitions including explosive-like → unimodal, explosive-like → bimodal, and explosive-like → unimodal → bimodal under the framework of non-Markovian systems.

Note that the classical Kolmogorov 5/3 spectral model was proposed for turbulence in the inertial range,[39] and then FBM with $H = 1/3$ was shown to exhibit the Kolmogorov spectrum, which indicates that FBM provides a stochastic process model for turbulence in the inertial range.[40] Recently, TFBM was demonstrated to fit turbulent velocity data over the entire frequency range.[41] Based on the spectral density of the current model, we can recover the Kolmogorov spectrum with special triple (α, ρ, λ). Hence, the current model can provide another alternative model for turbulence. Additionally, TFGN can be considered as the Yaglom noise, which is the tempered fractional integral of a white noise, but different from the time-fractional or space-fractional turbulence model. The former considers the memory effect from the external random environment to exhibit the Kolmogorov spectrum, while the latter involves the memory effect from the internal dynamical evolution. Both of them can recover the Kolmogorov spectrum for turbulence.

The proposed method could apply to a wider class of the nonlinear stochastic differential equations within the non-Markovian framework provided that the corresponding spectral density is available. We also provide some remarks on possible extension of our results to situations (codimension one P-bifurcation and codimension two P-bifurcation) that are not covered here, so much more work needs to be done to gain more insights into this issue. Last but not least, we do believe that it may fertilize the well-studied stochastic bifurcation theory from one side, and it can open a possible way to discern the theory of stochastic bifurcation within the non-Markovian framework from the other.

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