Caputo-Hadamard fractional differential equation: analysis and computation

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Introduction

Hadamard derivatives arise in many fields in mechanics, e.g., the fracture analysis of both planar and three dimensional elasticities. (N.I. Ioakimidis, Acta Mech. 1982)

Introduction

Definition

Hadamard fractional integral of order $\alpha > 0$ for a given function $f(x)$ is defined as (J. Hadamard, J. Math. Pures Appl. 1892)

$$ {}_H\mathrm{D}_{a^+}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{\mathrm{d}t}{t}, $$

where $x > a > 0$.

Definition

Riemann-Liouville fractional integral of order $\alpha > 0$ for a given function $f(x)$ is defined as (I. Podlubny, 1999)

$$ {}_{RL}\mathrm{D}_{a,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \mathrm{d}t, $$

where $x > a$.

Introduction

Definition

The Hadamard fractional derivative of order α for a given function $f(x)$ is defined as $(n-1<\alpha<n\in\mathbb{Z}^+,\ x>a>0)$

$$
\begin{aligned}
{}_H\mathrm{D}_{a^+}^{\alpha} f(x) =& \delta^n \left({}_H\mathrm{D}_{a^+}^{-(n-\alpha)} f(x) \right) \\
=& \left(x\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\log\frac{x}{t} \right)^{n-\alpha-1} f(t)\frac{\mathrm{d}t}{t}.
\end{aligned}
$$

Definition

The Riemann-Liouville fractional derivative of order $\alpha>0$ for a given function $f(x)$ is defined as $(n-1<\alpha<n\in\mathbb{Z}^+,\ x>a)$

$$
{}_{RL}\mathrm{D}_{a,x}^{\alpha} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f(t)\mathrm{d}t.
$$

Introduction

Table: Hadamard fractional calculus vs. Riemann-Liouville (R-L) fractional calculus

Fractional calculus	Kernel $(n-1<\alpha<n)$	Generalization
Hadamard	$(\log x - \log t)^{n-\alpha-1}$	$(x\frac{\mathrm{d}}{\mathrm{d}x})^n$
R-L	$(x-t)^{n-\alpha-1}$	$\frac{\mathrm{d}^n}{\mathrm{d}x^n}$

Remark

Hadamard fractional calculus is suitable for functions $f(x) \in \mathrm{span}\{1, \log x, \log^2 x, \log^3 x, \cdots\}_{x>0}$ *while R-L fractional calculus for functions* $f(x) \in \mathrm{span}\{1, x, x^2, x^3, \cdots\}_{x\in R}$

Function spaces

For ${}_H\mathrm{D}_{a+}^{-\alpha}(\cdot)$, we define space $L^p(a,\, b)$ $(1 \leq p \leq \infty)$ of those real-valued Lebesgue measurable functions f on $[a,\, b]$ for which $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} = \left(\int_a^b |f(t)|^p \mathrm{d}t\right)^{1/p},\ 1 \leq p < \infty,$$

and

$$\|f\|_{L^\infty} = \text{ess sup}_{a \leq x \leq b} |f(x)|.$$

Function spaces

For ${}_H\mathrm{D}^{\alpha}_{a+}(\cdot)$, we define space $AC^n_\delta[a,\, b]$ of those functions satisfying

$$AC^n_\delta[a,\, b] = \left\{ f : [a,\, b] \mapsto R \middle|\, \delta^{n-1} f(x) \in AC[a,\, b],\, \delta = x\frac{\mathrm{d}}{\mathrm{d}x} \right\},$$

where $n \in \mathbb{Z}^+$, and $AC[a,\, b]$ is the set of absolutely continuous functions on $[a,\, b]$.

Function spaces

Generalized weighted space $C_{\gamma,\log}[a,\,b]$ defined as

$$C_{\gamma,\log}[a,\,b] =\left\{ f(x) \Big| \left(\log\frac{x}{a}\right)^{\gamma} f(x) \in C[a,\,b], \right.$$
$$\left. \|f\|_{C_{\gamma,\log}} = \left\| \left(\log\frac{x}{a}\right)^{\gamma} f(x) \right\|_{C} \right\},$$

where $0 \leq \gamma < 1$. Obviously, $C_{0,\log}[a,\,b] \equiv C[a,\,b]$.

Properties and assertions

Lemma

Consistency of Hadamard integral. For Hadamard fractional integral, if $0 < \alpha \to n \in \mathbb{Z}^+$, then

$$\lim_{\alpha \to n} {}_H\mathrm{D}_{a^+}^{-\alpha} f(x) = {}_H\mathrm{D}_{a^+}^{-n} f(x) = \frac{1}{(n-1)!} \int_a^x \left(\log \frac{x}{t}\right)^{n-1} f(t) \frac{\mathrm{d}t}{t}$$
$$= \int_a^x \frac{\mathrm{d}t_1}{t_1} \int_a^{t_1} \frac{\mathrm{d}t_2}{t_2} \cdots \int_a^{t_{n-1}} f(t_n) \frac{\mathrm{d}t_n}{t_n},$$

Moreover, as $\alpha \to 0^+$, one gets

$$\lim_{\alpha \to 0^+} {}_H\mathrm{D}_{a^+}^{-\alpha} f(x) = f(x).$$

Properties and assertions

Property

Semigroup property. Let $\alpha > 0$, $\beta > 0$, $1 \leq p \leq \infty$ and $0 < a < b < \infty$. For $f \in L^p(a, b)$, the semigroup property holds,

$$ {}_H\mathrm{D}_{a^+}^{-\alpha}\, {}_H\mathrm{D}_{a^+}^{-\beta} f(x) = {}_H\mathrm{D}_{a^+}^{-(\alpha+\beta)} f(x). $$

Properties and assertions

Lemma

Consistency of Hadamard derivative. For Hadamard fractional derivative, $n-1<\alpha<n\in\mathbb{Z}^+$, *if* $\alpha\to(n-1)^+$, *then*

$$\lim_{\alpha\to(n-1)^+} {}_H\mathrm{D}^{\alpha}_{a^+}f(x)=\delta^{n-1}f(x).$$

If $\alpha\to n^-$, *then*

$$\lim_{\alpha\to n^-} {}_H\mathrm{D}^{\alpha}_{a^+}f(x)=\delta^{n}f(x).$$

Properties and assertions

Lemma

If $0 < \alpha < 1$ *and* $\beta > 0$*, then the following relations hold.*

$$ {}_H\mathrm{D}_{a^+}^{-\alpha} \left(\log \frac{x}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\log \frac{x}{a}\right)^{\beta+\alpha-1}, $$

$$ {}_H\mathrm{D}_{a^+}^{\alpha} \left(\log \frac{x}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{x}{a}\right)^{\beta-\alpha-1}. $$

Properties and assertions

Property

"Inverse" operator. If $0 < \alpha < 1$, $f(x) \in L(a, b)$, and ${}_H\mathrm{D}_{a^+}^{-(1-\alpha)} f(x) \in AC_\delta^1[a, b]$, then

$$ {}_H\mathrm{D}_{a^+}^{-\alpha}\, {}_H\mathrm{D}_{a^+}^{\alpha} f(x) = f(x) + c \left(\log \frac{x}{a}\right)^{\alpha-1}, $$

where c is a constant associated with some initial value.

Properties and assertions

Property

Null space. If $0 < \alpha < 1$, $\gamma \geq 1-\alpha$ and $f(x) \in C_{\gamma,\log}[a, b]$. Then the relation ${}_H\mathrm{D}^{\alpha}_{a+} f(x) = 0$ is valid, if and only if,

$$f(x) = c\left(\log\frac{x}{a}\right)^{\alpha-1},$$

where c is a constant, i.e., $\mathcal{N}({}_H\mathrm{D}^{\alpha}_{a+}) = \left\{\left(\log\frac{x}{a}\right)^{\alpha-1}\right\}$.
More generally, if $n-1 < \alpha < n \in \mathbb{Z}^{+}$, then
$\mathcal{N}({}_H\mathrm{D}^{\alpha}_{a+}) = \left\{\left(\log\frac{x}{a}\right)^{\alpha-1}, \cdots, \left(\log\frac{x}{a}\right)^{\alpha-n}\right\}$.

Properties and assertions

If $\alpha > \beta > 0$, then:

Theorem

Differentiation after integration. Let $\alpha > \beta > 0$, $1 \leq p \leq \infty$ and $0 < a < b < \infty$. For $f \in L^p(a, b)$, there holds

$$_H\mathrm{D}^{\beta}_{a+} \, _H\mathrm{D}^{-\alpha}_{a+} f(x) = \, _H\mathrm{D}^{-(\alpha-\beta)}_{a+} f(x).$$

In particular, if $\beta = m \in \mathbb{Z}^+$, then

$$_H\mathrm{D}^{m}_{a+} \, _H\mathrm{D}^{-\alpha}_{a+} f(x) = \, _H\mathrm{D}^{-(\alpha-m)}_{a+} f(x).$$

Properties and assertions

If $\beta > \alpha > 0$ and $\beta - \alpha < 1$, then:

Theorem

Differentiation after integration. Let $n-1 < \alpha < \beta \le n \in \mathbb{Z}^+$, $1 \le p \le \infty$ *and* $0 < a < b < \infty$. *Then for* $f \in L^p(a, b)$ *and* ${}_H\mathrm{D}_{a+}^{-\alpha} f \in AC_\delta^n[a, b]$ *there holds*

$$ {}_H\mathrm{D}_{a+}^{\beta}\, {}_H\mathrm{D}_{a+}^{-\alpha} f(x) = {}_H\mathrm{D}_{a+}^{(\beta-\alpha)} f(x). $$

In particular, if $\beta = n \in \mathbb{Z}^+$, *then*

$$ {}_H\mathrm{D}_{a+}^{n}\, {}_H\mathrm{D}_{a+}^{-\alpha} f(x) = {}_H\mathrm{D}_{a+}^{(n-\alpha)} f(x). $$

Properties and assertions

If $\beta > \alpha > 0$ and $\beta - \alpha \geq 1$, then:

Theorem

Differentiation after integration. Let $n-1 < \alpha \leq n \leq m-1 < \beta \leq m\,(n, m \in \mathbb{Z}^+)$ *and* $1 \leq p \leq \infty$, $0 < a < b < \infty$. *Then for* $f \in L^p(a, b)$ *and* ${}_H\mathrm{D}_{a^+}^{-\alpha} f \in AC_\delta^m[a, b]$, *there holds*

$$ {}_H\mathrm{D}_{a^+}^{\beta}\, {}_H\mathrm{D}_{a^+}^{-\alpha} f(x) = {}_H\mathrm{D}_{a^+}^{(\beta-\alpha)} f(x). $$

In particular, if $\beta = m$, *then*

$$ {}_H\mathrm{D}_{a^+}^{m}\, {}_H\mathrm{D}_{a^+}^{-\alpha} f(x) = {}_H\mathrm{D}_{a^+}^{(m-\alpha)} f(x). $$

Properties and assertions

If $\beta \geq \alpha > 0$, then:

Theorem

Integration after differentiation. Let $\beta \geq \alpha > 0$, $n-1 < \alpha \leq n \in \mathbb{Z}^+$, $m-1 < \beta \leq m \in \mathbb{Z}^+$ *and* $0 < a < b < \infty$, $1 \leq p \leq \infty$. *Then for* $f \in AC_\delta^n[a, b]$ *and* ${}_H\mathrm{D}_{a+}^{\alpha} f \in L^p(a, b)$, *there holds*

$$ {}_H\mathrm{D}_{a+}^{-\beta}\, {}_H\mathrm{D}_{a+}^{\alpha} f(x) = {}_H\mathrm{D}_{a+}^{-(\beta-\alpha)} f(x). $$

In particular, if $\beta = \alpha$, *then*

$$ {}_H\mathrm{D}_{a+}^{-\alpha}\, {}_H\mathrm{D}_{a+}^{\alpha} f(x) = f(x). $$

⋆ The above introduction can be referred to the following references.

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Definite conditions of Hadamard FDE

Question Can we impose the integer-order initial value conditions for the Hadamard fractional differential equations?

Answer: No!

Let us see a simple example.

Example

$$ {}_{H}\mathrm{D}^{\alpha}_{a^+}u(t) = f(t, u),\ 0 < a < t \le b. \tag{2.1} $$

If the initial value condition $u(a) = u_a$ is posed for equation (2.1), then

- (i) equation (2.1) has no solution provided that $f(t,\ u) \equiv 0$ and $u_a \neq 0$.
- (ii) equation (2.1) has two solutions $u(t) = 0$ and $u(t) = c\log\frac{t}{a}$, provided that $f(t,\ u) = u^{1-\alpha}$ and $u_a = 0$, where $c = (\Gamma(2-\alpha))^{\frac{1}{\alpha}}$.

So such a Cauchy problem is ill-posed.

Definite conditions of Hadamard FDE

Well-posed conditions for Hadamard FDE as $0 < \alpha < 1$

$$\begin{cases} {}_H\mathrm{D}_{a+}^{\alpha} u(t) = f(t, u),\ a < t \le b, \\ {}_H\mathrm{D}_{a+}^{\alpha-1} u(t)|_{t=a} = u_a, \end{cases}$$

or,

$$\begin{cases} {}_H\mathrm{D}_{a+}^{\alpha} u(t) = f(t, u),\ a < t \le b, \\ (\log \frac{t}{a})^{1-\alpha} u(t)|_{t=a} = \Gamma(2-\alpha) u_a, \end{cases}$$

or,

$$\begin{cases} {}_H\mathrm{D}_{a+}^{\alpha} u(t) = f(t, u),\ a < t \le b, \\ u(t_0) = u_0,\ a < t_0 \le b. \end{cases}$$

Introduction to Caputo-Hadamard derivative

In order to impose the integer-order initial value conditions for the Hadamard fractional differential equations, we use the modified form of Hadamard derivative, i.e., Caputo-Hadamard derivative.

Definition

The Caputo-Hadamard derivative is defined as (Y. Adjabi et al., J. Comput. Anal. Appl. 2016)

$$ {}_{CH}\mathrm{D}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t)\frac{\mathrm{d}t}{t}, $$

where $x > a > 0$, $n - 1 < \alpha < n \in \mathbb{Z}^+$.

★ **Characteristic of Caputo-Hadamard derivative:**

- Caputo-Hadamard derivative is obtained by changing the order of differentiation and integration of the Hadamard derivative.

Caputo-Hadamard FDE

In this case, we consider the following Caputo-Hadamard FDE

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a^+}u(t) = f(t,\,u),\, 0 < a < t,\, 0 < \alpha < 1, \\ u(a^+) = u_a. \end{cases} \tag{2.2}$$

Existence, uniqueness, and continuation theorems

Existence

Consider the following initial value problem (IVP)

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a+}u(t) = f(t,\, u),\, 0 < a < t,\, 0 < \alpha < 1, \\ u(a^+) = u_a. \end{cases} \tag{3.1}$$

- Hypothesis $[H_1]$

Let $f(t,\, u) : [a,\, +\infty)\times\Omega \to \mathbb{R}$ in IVP (3.1) be a continuous function, where $\Omega \subset \mathbb{R}$. Then $f(t, u)$ is a continuous bounded map defined on $[a,\, T] \times \overline{\Omega}_0$, where Ω_0 is a bounded subset of Ω.

Theorem

If Hypothesis $[H_1]$ is satisfied, then IVP (3.1) has at least one solution $u(t) \in C[a,\, h]$ for some $h \in [a,\, T]$.

Uniqueness

- Hypothesis $[H_2]$

Let $f(t, u) : [a, +\infty) \times \Omega \to \mathbb{R}$ in IVP (3.1) satisfy the Lipschitz condition with respect to the second variable, i.e.,

$$|f(t, u) - f(t, \tilde{u})| \leq L|u - \tilde{u}|, \; L > 0.$$

Theorem

If Hypotheses $[H_1]$ and $[H_2]$ are satisfied, then IVP (3.1) has a unique solution $u(t) \in C[a, h]$, where $h \in [a, T]$.

Continuation

Theorem

Assume that Hypotheses $[H_1]$ and $[H_2]$ hold. Then $u = u(t)$, $t \in [a, \beta)$ is noncontinuable if and only if for some $\xi \in (a, \frac{a+\beta}{2})$ and for any bounded closed subset $V \subset [\xi, +\infty) \times \mathbb{R}$ there exists a $t^* \in [\xi, \beta)$ such that $(t^*, u(t^*)) \notin V$.

This continuation theorem guarantees that we can consider the stability of solution to IVP (3.1).

★ The above discussions come from the following references.

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Dynamics of Hadamard FDE

Gronwall inequality

Theorem

Let $\alpha > 0$. Suppose that $u(t)$ and $b(t)$ are non-negative functions and locally integrable on $a \leq t < T \leq +\infty$, and $g(t)$ is a non-negative, non-decreasing, bounded continuous function on $[a, T]$, satisfying

$$u(t) \leq b(t) + g(t) \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s)\mathrm{d}s, \; t \in [a, T]. \tag{4.1}$$

Then, for $t \in [a, T]$, it derives

$$u(t) \leq b(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log \frac{t}{s}\right)^{n\alpha-1} b(s)\right] \frac{\mathrm{d}s}{s}. \tag{4.2}$$

Gronwall inequality

Define an operator A as

$$A\psi(t) = g(t)\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\psi(s)\frac{\mathrm{d}s}{s}.$$

One immediately gets that

$$u(t) \le b(t) + Au(t).$$

After iterating the above inequality, it's easy to derive that

$$u(t) \le b(t) + \sum_{k=1}^{n-1} A^k b(t) + A^n u(t).$$

Gronwall inequality

By induction method, it claims that

$$A^n u(t) \leq \int_a^t \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log \frac{t}{s}\right)^{n\alpha-1} u(s) \frac{\mathrm{d}s}{s}.$$

In addition, $g(t)$ is bounded, there exists a constant C such that

$$A^n u(t) \leq \int_a^t \frac{(C\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log \frac{t}{s}\right)^{n\alpha-1} u(s) \frac{\mathrm{d}s}{s} \to 0,$$

as $n \to +\infty$, $t \in [a, T]$.
Therefore the proof is completed.

Continuous dependence of solution on initial condition and derivative parameter

Consider the following two IVPs

$$\begin{cases} {}_{CH}\mathrm{D}_{a^+}^{\alpha}x(t) = f(t, x), \; t > a > 0, \\ x(a^+) = x_a, \end{cases} \tag{4.3}$$

and

$$\begin{cases} {}_{CH}\mathrm{D}_{a^+}^{\alpha-\epsilon}y(t) = f(t, y), \; t > a > 0, \\ y(a^+) = x_a + \delta. \end{cases} \tag{4.4}$$

Here f is a continuous function and satisfies the Lipschitz condition with regard to the second variable. i.e.,

$$|f(t, x) - f(t, y)| \leq L|x - y|, \; L > 0,$$

where the constant L is independent of t, x, y.

Continuous dependence of solution on initial condition and derivative parameter

Theorem

Let $0 < \epsilon < \alpha < 1$. Suppose that $x(t)$, $y(t) \in C[a, T]$ are the solutions of IVPs (4.3) *and* (4.4)*, respectively. Then, the following inequality holds on $[a, h]$, $h < T$.*

$$
\begin{aligned}
&|x(t) - y(t)| \\
\leq &G(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{[(L\Gamma(\alpha-\epsilon))/\Gamma(\alpha)]^n}{\Gamma(n(\alpha-\epsilon))} \left(\log \frac{t}{s} \right)^{n(\alpha-\epsilon)-1} G(s) \right] \frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Continuous dependence of solution on initial condition and derivative parameter

Here

$$
\begin{aligned}
G(t) =& |\delta| + \left| \frac{\left(\log \frac{t}{a}\right)^{\alpha-\epsilon}}{\alpha-\epsilon} \left(\frac{1}{\Gamma(\alpha-\epsilon)} - \frac{1}{\Gamma(\alpha)} \right) \right| \cdot \|f\| \\
&+ \left| \frac{\left(\log \frac{t}{a}\right)^{\alpha-\epsilon}}{(\alpha-\epsilon)\Gamma(\alpha)} - \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)} \right| \cdot \|f\|.
\end{aligned}
$$

Continuous dependence of solution on initial condition and derivative parameter

The solutions of IVPs (4.3) and (4.4) are given by

$$x(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s,\, x(s)) \frac{\mathrm{d}s}{s},$$

and

$$y(t) = x_a + \delta + \frac{1}{\Gamma(\alpha-\epsilon)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-\epsilon-1} f(s,\, y(s)) \frac{\mathrm{d}s}{s},$$

respectively.

Continuous dependence of solution on initial condition and derivative parameter

Then it follows

$$|x(t) - y(t)| \leq G(t) + \frac{L}{\Gamma(\alpha - \epsilon)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-\epsilon-1} |x(s) - y(s)| \frac{\mathrm{d}s}{s},$$

where $G(t)$ is a function of t given by

$$\begin{aligned} G(t) = & |\delta| + \left| \frac{\left(\log \frac{t}{a}\right)^{\alpha-\epsilon}}{\alpha - \epsilon} \left(\frac{1}{\Gamma(\alpha - \epsilon)} - \frac{1}{\Gamma(\alpha)} \right) \right| \cdot \|f\| \\ & + \left| \frac{\left(\log \frac{t}{a}\right)^{\alpha-\epsilon}}{(\alpha - \epsilon)\Gamma(\alpha)} - \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha + 1)} \right| \cdot \|f\|. \end{aligned}$$

Continuous dependence of solution on initial condition and derivative parameter

With the help of Gronwall inequality, it yields

$$
\begin{aligned}
&|x(t)-y(t)| \\
&\leq G(t)+\int_a^t\left[\sum_{n=1}^{\infty}\frac{[(L\Gamma(\alpha-\epsilon))/\Gamma(\alpha)]^n}{\Gamma(n(\alpha-\epsilon))}\left(\log\frac{t}{s}\right)^{n(\alpha-\epsilon)-1}G(s)\right]\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

All this ends the proof.

Estimates of the upper bound of the Lyapunov exponents

Consider the IVP involving Caputo-Hadamard derivative as follows

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a^+}\mathtt{x}(t) = \mathtt{f}(t,\, \mathtt{x}),\ t > a > 0, \\ \mathtt{x}(a^+) = \mathtt{x}_a, \end{cases} \tag{4.5}$$

where $(t,\, \mathtt{x}) \in (a,\, +\infty) \times \Omega \subset (a,\, +\infty) \times \mathbb{R}^n$.

Estimates of the upper bound of the Lyapunov exponents

Theorem

The first variation equation of (4.5) *is*

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a^+}\Phi(t) = \mathbf{f}_{\mathbf{x}}(t, \mathbf{x})\Phi(t), \ t > a > 0, \\ \Phi(a^+) = I, \end{cases} \tag{4.6}$$

where I is the identity matrix.

Estimates of the upper bound of the Lyapunov exponents

Lemma

Assume that $0 < \alpha < 2$, β *is an arbitrary complex number,* μ *is a real number satisfying*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}.$$

Then, for any integer $p \geq 1$, *the following expansion formula holds*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right),$$

where $|z| \to \infty$, $|\arg(z)| \leq \mu$.

Estimates of the upper bound of the Lyapunov exponents

Definition

Let $\lambda_k(t)(k = 1, 2, \cdots, n)$ be the eigenvalues of $\Phi(t)$ in system (4.6), which satisfy $|\lambda_1(t)| \leq |\lambda_2(t)| \leq \cdots \leq |\lambda_n(t)|$. Then the Lyapunov exponents l_k are defined by

$$l_k = \lim_{t\to\infty} \sup \frac{1}{t} \log|\lambda_k(t)|, \ k = 1, 2, \cdots, n.$$

Theorem

The Lyapunov exponents of system (4.5) *satisfy*

$$-\infty \leq l_1 \leq \cdots \leq l_n \leq 0.$$

Estimates of the upper bound of the Lyapunov exponents

Performing integrating transformation on both sides of (4.6), one gets

$$\Phi(t) = I + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \mathbf{f}_{\mathbf{x}}(s, \mathbf{x}(s))\Phi(s)\frac{\mathrm{d}s}{s}.$$

Taking the matrix norm on the above equation, and combining that the function $\mathtt{f}$ satisfies $\|\mathtt{f}_{\mathtt{x}}(t, \mathtt{x})\| \leq M$, it follows that

$$\|\Phi(t)\| \leq 1 + \frac{M}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|\Phi(s)\|\frac{\mathrm{d}s}{s}.$$

It follows from the Gronwall inequality that

$$\|\Phi(t)\| \leq E_{\alpha,\alpha}\left(M\left(\log t - \log a\right)^{\alpha}\right).$$

Estimates of the upper bound of the Lyapunov exponents

Using the fact that the spectral radius of a given matrix is no bigger than its norm, which gives

$$|\lambda_n(t)| \leq \|\Phi(t)\| \leq E_{\alpha,\alpha}\left(M\left(\log t - \log a\right)^{\alpha}\right).$$

By the expansion formula of the Mittag-Leffler function, one has

$$E_{\alpha,\alpha}\left(M\left(\log\frac{t}{a}\right)^{\alpha}\right) \approx \frac{1}{\alpha}M^{(1-\alpha)/\alpha}\left(\log\frac{t}{a}\right)^{1-\alpha}\left(\frac{t}{a}\right)^{M^{1/\alpha}},$$

as $t \to +\infty$.

Estimates of the upper bound of the Lyapunov exponents

From the definition of the Lyapunov exponents, one has

$$
\begin{aligned}
l_n &= \lim_{t\to\infty} \sup \frac{1}{t} \log |\lambda_n(t)| \\
&\leq \lim_{t\to\infty} \sup \frac{1}{t} \log \|\Phi(t)\| \\
&\leq \lim_{t\to\infty} \sup \frac{1}{t} \log \left(E_{\alpha,\alpha} \left(M \left(\log \frac{t}{a} \right)^{\alpha} \right) \right) \\
&= \lim_{t\to\infty} \sup \frac{1}{t} \log \left(\frac{1}{\alpha} M^{(1-\alpha)/\alpha} \left(\log \frac{t}{a} \right)^{1-\alpha} \left(\frac{t}{a} \right)^{M^{1/\alpha}} \right) \\
&= 0.
\end{aligned}
$$

Hence, the proof is completed.

Estimates of the upper bound of the Lyapunov exponents

- The leading Lyapunov exponent is negative in the sense of the above definition. In this case, such a system seems not to be chaotic in the sense of the positivity of the leading exponent.
- The lower bound is not available.

Thank you for your attention!