On the regional controllability of the sub-diffusion process with Caputo fractional derivative

Article in Fractional Calculus and Applied Analysis · January 2016

DOI: 10.1515/fca-2016-0065

4 authors, including:

Fudong Ge
China University of Geosciences

YangQuan Chen
University of California, Merced

Chunhai Kou
Donghua University

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 19, NUMBER 5 (2016) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

ON THE REGIONAL CONTROLLABILITY OF THE SUB-DIFFUSION PROCESS WITH CAPUTO FRACTIONAL DERIVATIVE

Fudong Ge[1], **YangQuan Chen**[2], **Chunhai Kou**[3], **Igor Podlubny**[4]

On the Occasion of Professor Richard L. Magin's 70th Birthday

Abstract

This paper is devoted to the investigation of regional controllability of the fractional order sub-diffusion process. We first derive the equivalent integral equations of the abstract sub-diffusion systems with Caputo and Riemann-Liouville fractional derivatives by utilizing the Laplace transform. The new definitions of regional controllability of the system studied are introduced by extending the existence contributions. Then we analyze the regional controllability of the fractional order sub-diffusion system with minimum energy control in two different cases: $B \in \mathcal{L}\left(\mathbf{R}^m, L^2(\Omega)\right)$ and $B \notin \mathcal{L}\left(\mathbf{R}^m, L^2(\Omega)\right)$. T he adjoint system of fractional order sub-diffusion system is also presented at the same time. Two applications are worked out in the end to verify the effectiveness of our results.

MSC 2010: Primary 26A33; Secondary 93B05, 60J60

Key Words and Phrases: fractional calculus, Caputo derivative, sub-diffusion, control, regional controllability

1. Introduction

Let Ω be an open bounded subset of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $Q = \Omega \times [0, b]$ and $\Sigma = \partial\Omega \times [0, b]$. In this paper, we consider the following abstract fractional order sub-diffusion system of order $\alpha \in (0, 1]$:

pp. 1262–1281 , DOI: 10.1515/fca-2016-0065 DE GRUYTER

$$\begin{cases} {}_0^C D_t^\alpha z(t) = Az(t) + Bu(t), \ t \in [0, b], \\ z(0) = z_0 \in D(A), \end{cases} \tag{1.1}$$

where $z \in L^2(0, b; L^2(\Omega))$, $D(A)$ holds for the domain of A and A generates a C_0 semigroup $\{\Phi(t)\}_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$. Besides, $u \in L^2(0, b; \mathbf{R}^p)$, $B : \mathbf{R}^p \to L^2(\Omega)$ is a linear operator (possibly unbounded) depending on the number and structure of actuators and here ${}_0^C D_t^\alpha$ denotes by the left-sided Caputo fractional order derivative.

Recently fractional order sub-diffusion systems have attracted increasing attention after the verification that they can be used to well characterize those anomalous diffusion processes in various real-world complex systems [24, 35, 11, 15]. As we all know, the anomalous diffusion processes in real world are essentially distributed and the continuous time random walk (CTRW) is useful to describe this phenomenon by allowing the incorporation of waiting time probability density function (PDF) and general jump PDF [14, 3, 38, 37]. More precisely, when the particles are assumed to jump at fixed time intervals with incorporating waiting times, the particles will experience the sub-diffusion processes, such as the flow through porous media microscopic process [33], or the swarm of robots moving through dense forest [31] etc. In this case, the mean squared displacement (MSD) is a power law of fractional exponent smaller than a linear function of the Gaussian diffusion process [32, 23]. Another case is that when the particles are supposed to jump following from a general, non-Gaussian jump distribution function, the particles then undergo the super-diffusion process [15, 3]. Now the MSD is a power law function of fractional exponent bigger than that of a Gaussian diffusion process.

It is worth noting that the controllability problem of a fractional order sub-diffusion system can be reformulated as a problem of infinite-dimensional control system. Moreover, in the case of diffusion systems, it should be pointed out that, in general, not all the states can be reached [12, 7]. So in this paper, to extend the existence results in [8, 39, 10], we introduce some notations on the regional controllability of the fractional order sub-diffusion systems with Caputo fractional derivative, where we are only concerned with the knowledge of the states in a sub-region along the spatial domain. This situation happens in many real dynamic systems, for example the pest spreading [1], crowd-pedestrian egress or evacuation [2], and etc. It is now widely believed that many real-world complex dynamics can be well characterized by using fractional calculus and fractional order controls can offer better performance not achievable before using integer order controls systems [32, 23], this is the reason why the fractional order models are superior in comparison with the integer order models.

Moreover, it is well known that the adjoint system plays an important role in many fields of mathematics, including mathematical physics and control theory [27, 4, 28, 6]. However, as the Definition 2.4 below indicates, the unique mild solution of the system (1.1) can be expressed as follows:

$$z(t) = S_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s)Bu(s)ds \tag{1.2}$$

and $S_\alpha(t) \neq K_\alpha(t)$, which makes it difficult to define the adjoint system of (1.1) and many methods introduced to analyze the controllability of integer order partial differential equations fails here. Then in this paper, we try to present the notion of adjoint system for the case of fractional order sub-diffusion equations with help of the integration by parts of fractional derivatives, which is consistently extended by Podlubny and Chen in [27]. Based on the semigroup theory [25], here we discuss the regional controllability of the fractional order sub-diffusion systems with minimum energy control in two different cases: $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$ and $B \notin \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$. More precisely, when $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$, our main result is derived by utilizing the Balder's theorem [34] and when $B \notin \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$, the Hilbert Uniqueness Methods (HUMs), which were first introduced by Lions [21] to study the controllability of partial differential equations [19, 30], are used to obtain the regional controllability of the system studied with minimum energy control.

The remainder of this paper is organized as follows: some basic knowledge of fractional calculous and some preliminary results are given in the next Section **2**. In Section **3**, our main results on the regional controllability analysis of the system studied in two different cases are presented. Two applications are worked out in Section **4** to test our obtained results.

2. Preliminaries

This section is devoted to introducing some definitions and preliminary results to be used afterwards.

Definition **2.1**. ([26, 17]) The left-sided and right-sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a function z on $[0, b]$ are given by

$${}_0I_t^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} z(\tau)d\tau \tag{2.1}$$

and

$${}_tI_b^\alpha z(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} z(\tau)d\tau, \tag{2.2}$$

respectively, provided that the right sides are pointwise defined on $[0, b]$.

REMARK **2.1**. By Definition 2.1, we see that only both $z(t)$ and ${}_tI_b^\alpha z(t)$ are continuous at the point $t=b$, we have

$$\lim_{t\to b^-} {}_tI_b^\alpha z(t) = 0.$$

If not so, for example, let $z(t)=(b-t)^{-\alpha}$, $\alpha\in(0,1)$, although ${}_tI_b^\alpha z(t)$ is continuous at the point $t=b$, we have

$$\lim_{t\to b^-} {}_tI_b^\alpha z(t) = \lim_{t\to b^-}\int_t^b \frac{(\tau-t)^{\alpha-1}}{\Gamma(\alpha)}(b-\tau)^{-\alpha}d\tau = \Gamma(1-\alpha)\neq 0. \tag{2.3}$$

DEFINITION **2.2**. ([26, 17]) For $t\in[0,b]$ and any given α $(n-1<\alpha<n)$, $n\in\mathbf{N}$, the left-sided and right-sided Riemann-Liouville fractional derivatives of order α of a function z are defined as

$${}_0D_t^\alpha z(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}z(s)ds \tag{2.4}$$

and

$${}_tD_b^\alpha z(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n\int_t^b (s-t)^{n-\alpha-1}z(s)ds, \tag{2.5}$$

respectively, provided that the right sides are pointwise defined.

DEFINITION **2.3**. ([26, 17]) The left-sided Caputo fractional derivative of order $\alpha>0$ of a function z is

$${}_0^CD_t^\alpha z(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\frac{d^n}{ds^n}z(s)ds, \tag{2.6}$$

where $t\geq 0$, $n-1<\alpha<n$, $n\in\mathbf{N}$ and the right side is pointwise defined.

DEFINITION **2.4**. For $t\in[0,b]$, any given $u\in L^2\left(0,b;\mathbf{R}^p\right)$, a function $z\in L^2\left(0,b;L^2(\Omega)\right)$ is said to be a mild solution of the system (1.1), denotes by $z(\cdot,u)$, if it satisfies

$$z(t,u) = S_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)Bu(s)ds, \tag{2.7}$$

where

$$S_\alpha(t) = \int_0^\infty \phi_\alpha(\theta)\Phi(t^\alpha\theta)d\theta \tag{2.8}$$

and

$$K_\alpha(t) = \alpha\int_0^\infty \theta\phi_\alpha(\theta)\Phi(t^\alpha\theta)d\theta. \tag{2.9}$$

Here $\{\Phi(t)\}_{t\geq 0}$ is the strongly continuous semigroup generated by A, $\phi_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\psi_\alpha(\theta^{-\frac{1}{\alpha}})$ and ψ_α is a probability density function defined by

$$\psi_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\quad \theta\in(0,\infty),$$

satisfying the following properties ([22])

$$\int_0^\infty e^{-\lambda\theta}\psi_\alpha(\theta)d\theta=e^{-\lambda^\alpha},\ \int_0^\infty\psi_\alpha(\theta)d\theta=1,\alpha\in(0,1) \tag{2.10}$$

and

$$\int_0^\infty\theta^\nu\phi_\alpha(\theta)d\theta=\frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)},\ \nu\geq 0. \tag{2.11}$$

Definition **2.5**. For $t\in[0,b]$, any given $f\in L^2\left(0,b;L^2(\Omega)\right)$, $0<\alpha<1$, a function $v\in L^2\left(0,b;L^2(\Omega)\right)$ is said to be a mild solution of the following system

$$\begin{cases} {}_0D_t^\alpha v(t)=Av(t)+f(t),\ t\in[0,b],\\ \lim\limits_{t\to 0^+}{}_0I_t^{1-\alpha}v(t)=v_0\in L^2(\Omega)\end{cases} \tag{2.12}$$

if it satisfies

$$z(t)=t^{\alpha-1}K_\alpha(t)v_0+\int_0^t(t-s)^{\alpha-1}K_\alpha(t-s)f(s)ds, \tag{2.13}$$

where $K_\alpha(t)$ is defined in Eq.(2.9).

For the results of Definition 2.4 and 2.5, we refer to Zhou and Jiao [36], where the authors studied a class of neutral evolution equations of fractional order with nonlocal conditions and obtained several criteria on the existence of mild solutions.

Let $\omega\subseteq\Omega$ be a given region of positive Lebesgue measure and $z_b\in L^2(\omega)$ (the target function) be a given element of the state space. We first state the following definition.

Definition **2.6**. The system (1.1) is called to be regionally exactly (respectively, approximately) controllable in ω at time b if for any $z_b\in L^2(\omega)$, given $\varepsilon>0$, there exists a control $u\in L^2(0,b;\mathbf{R}^p)$ such that

$$p_\omega z(b,u)=z_b\ \left(\text{respectively, }\ \|p_\omega z(b,u)-z_b\|_{L^2(\omega)}\leq\varepsilon\right), \tag{2.14}$$

where $p_\omega: L^2(\Omega)\to L^2(\omega)$, defined by $p_\omega z=z|_\omega$, is a projection operator.

In order to show our main results, the following lemmas are needed.

LEMMA **2.1**. ([36, 13])

(i) The operators $S_\alpha(t)$ and $K_\alpha(t)$ are linear bounded and for any $x \in L^2(\Omega)$, we have

$$\|S_\alpha(t)x\| \leq M\|x\| \text{ and } \|K_\alpha(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\|. \tag{2.15}$$

(ii) Operators $\{S_\alpha(t)\}_{t\geq 0}$ and $\{K_\alpha(t)\}_{t\geq 0}$ are strongly continuous, this is, for $\forall x \in L^2(\Omega)$ and $0 \leq t_1 \leq t_2 \leq b$, we have

$$\|S_\alpha(t_1)x - S_\alpha(t_2)x\| \to 0 \text{ and } \|K_\alpha(t_1)x - K_\alpha(t_2)x\| \to 0 \text{ as } t_1 \to t_2. \tag{2.16}$$

(iii) For $t > 0$, $S_\alpha(t)$ and $K_\alpha(t)$ are all compact operators if $\Phi(t)$ is compact.

LEMMA **2.2**. ([27]) *For $t \in [a, b]$ and α $(n-1 < \alpha < n, n \in \mathbf{N})$, the following formula holds*

$$\begin{aligned}\int_a^b f(t)\,{}_a^C D_t^\alpha g(t)dt = &\sum_{r=0}^{k-1} (-1)^{k-1-r} \left[g^{(r)}(t)\,{}_t D_b^{\alpha-1-r} f(t)\right]_{t=a}^{t=b} \\ &+ (-1)^k \int_a^b g(t)\,{}_t D_b^\alpha f(t)dt.\end{aligned}$$

In particular, if $0 < \alpha < 1$, we have

$$\int_a^b f(t)\,{}_a^C D_t^\alpha g(t)dt = \left[g(t)\,{}_t I_b^{1-\alpha} f(t)\right]_{t=a}^{t=b} - \int_a^b g(t)\,{}_t D_b^\alpha f(t)dt.$$

LEMMA **2.3**. ([18]) *Let the reflection operator Q on interval $[0, b]$ be as follows:*

$$Qf(t) := f(b-t).$$

Then the following equations hold:

$$\begin{aligned}&Q\,{}_0 I_t^\alpha f(t) = {}_t I_b^\alpha Q f(t), \;\; Q\,{}_0 D_t^\alpha f(t) = {}_t D_b^\alpha Q f(t) \quad \text{and} \\ &{}_0 I_t^\alpha Q f(t) = Q\,{}_t I_b^\alpha f(t), \;\; {}_0 D_t^\alpha Q f(t) = Q\,{}_t D_b^\alpha f(t).\end{aligned}$$

3. Main results

In this section, we will explore the possibility of finding a minimum energy control which steers the problem (1.1) from the initial state z_0 to a target function z_b on the sub-region ω.

Let $U_b = \{u \in L^2(0, b; \mathbf{R}^p) : p_\omega z(b, u) = z_b\}$ and consider the following minimum energy problem

$$\inf_u J(u) = \inf_u \left\{ \int_0^b \|u(t)\|_{\mathbf{R}^p}^2 dt : u \in U_b \right\}. \tag{3.1}$$

Let $H : L^2(0, b; \mathbf{R}^p) \to L^2(\Omega)$ be

$$Hu = \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s)Bu(s)ds, \ \forall u \in L^2(0, b; \mathbf{R}^p). \tag{3.2}$$

Now we are ready to state our results in two different cases.

3.1. The case of $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$

Suppose that the semigroup $\{\Phi(t)\}_{t\geq 0}$ generated by operator A is is uniformly bounded, it follows from $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$ that there exist two constants $M_B, \ M > 0$ such that

$$\|B\| \leq M_B \ \text{ and } \ \sup_{t\geq 0} \|\Phi(t)\| \leq M.$$

Then we see the following results.

THEOREM **3.1**. *Suppose that $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$ and for any $t > 0$, $\Phi(t)$ is a compact operator, then the minimum energy problem* (3.1) *admits at least one optimal solution provided that the system* (1.1) *is regionally approximately controllable in ω at time b.*

P r o o f. Obviously, U_b is a closed and convex set. We first prove that the operator H in (3.2) is strongly continuous, which admits the existence of optimal control to the minimum energy problem (3.1) (see pp.597, [16]). For any $t \in [0, b], z_0 \in L^2(\Omega)$, by Lemma 2.1, we see that the term $S_\alpha(t)z_0$ in Eq. (2.7) is strongly continuous. Moreover, since the operator H is linear and continuous, according to the argument in [5], we only need to show that it is precompact.

Let $N : L^2(0, b; \mathbf{R}^p) \to L^2(\Omega)$ be

$$Nu(t) := \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s)Bu(s)ds, \ t \in [0, b]. \tag{3.3}$$

We next show that N is a relatively compact operator.

Let $\rho_r = \{u \in L^2(0, b; \mathbf{R}^p) : \|u\|_{L^2(0,b;\mathbf{R}^p)} \leq r\}$. For any fixed $t \in (0, b]$, $\varepsilon, \delta \in (0, t)$, $u \in \varrho_r$, let

$$\widetilde{N}_{(\varepsilon,\delta)}u(t) = \alpha \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{\alpha-1}\theta\phi(\theta)\Phi((t-s)^\alpha\theta)Bu(s)d\theta ds.$$

Since $\Phi(\varepsilon^q\delta)$ is compact and

$$\widetilde{N}_{(\varepsilon,\delta)}u(t) = \Phi(\varepsilon^\alpha\delta)\alpha\int_0^{t-\varepsilon}\int_\delta^\infty (t-s)^{\alpha-1}\theta\phi(\theta)\Phi((t-s)^\alpha\theta - \varepsilon^\alpha\delta)Bu(s)d\theta ds,$$

we get that $\widetilde{N}_{(\varepsilon,\delta)}$ is relatively compact. Together with

$$\|Bu(\cdot)\| \leq M_B r < \infty,$$

by (i) in Lemma 2.1, for any $t \in [0,b]$, we have

$$\begin{aligned}
\|\widetilde{N}u(t) - \widetilde{N}_{(\varepsilon,\delta)}u(t)\| = \alpha\| & \int_0^t \int_0^\delta (t-s)^{\alpha-1}\theta\phi(\theta)\Phi((t-s)^\alpha\theta)Bu(s)d\theta ds \\
& + \int_0^t \int_\delta^\infty (t-s)^{\alpha-1}\theta\phi(\theta)\Phi((t-s)^\alpha\theta)Bu(s)d\theta ds \\
& - \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{\alpha-1}\theta\phi(\theta)\Phi((t-s)^\alpha\theta)Bu(s)d\theta ds\| \\
& \leq \alpha \left\| \int_0^t \int_0^\delta (t-s)^{\alpha-1}\theta\phi(\theta)\Phi((t-s)^\alpha\theta)Bu(s)d\theta ds \right\| \\
& + \alpha \left\| \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{\alpha-1}\theta\phi(\theta)\Phi((t-s)^\alpha\theta)Bu(s)d\theta ds \right\| \\
& \leq MM_B r b^\alpha \int_0^\delta \theta\phi(\theta)d\theta + \frac{MM_B r\varepsilon^\alpha}{\Gamma(1+\alpha)} \to 0
\end{aligned}$$

as $\varepsilon, \delta \to 0$. Then $N\varrho_r$ is a relatively compact set in $L^2(\Omega)$.

Next, we shall prove that Nu is equicontinuous on $[0,b]$. For any $u \in \varrho_r$, $0 \leq \sigma_1 < \sigma_2 \leq b$,

$$\begin{aligned}
& \|Nu(\sigma_2) - Nu(\sigma_1)\| \leq \\
& \quad \leq \left\| \int_0^{\sigma_1} [(\sigma_2 - s)^{\alpha-1} - (\sigma_1 - s)^{\alpha-1}] K_\alpha(\sigma_2 - s)Bu(s)ds \right\| \\
& \quad + \left\| \int_0^{\sigma_1} (\sigma_1 - s)^{\alpha-1}[K_\alpha(\sigma_2 - s) - K_\alpha(\sigma_1 - s)]Bu(s)ds \right\| \\
& \quad + \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2 - s)^{\alpha-1} K_\alpha(\sigma_2 - s)Bu(s)ds \right\| \\
& \quad \leq \frac{MM_B r}{\Gamma(1+\alpha)}(\sigma_2^\alpha - \sigma_2^\alpha + (\sigma_2 - \sigma_1)^\alpha) + A + \frac{MM_B r}{\Gamma(1+\alpha)}(\sigma_2 - \sigma_1)^\alpha,
\end{aligned}$$

where

$$A = \left\| \int_0^{\sigma_1} (\sigma_1 - s)^{\alpha-1}[K_\alpha(\sigma_2 - s) - K_\alpha(\sigma_1 - s)]Bu(s)ds \right\|.$$

Since $\varepsilon > 0$ is small enough, we have

$$
\begin{aligned}
A \leq & \int_0^{\sigma_1-\varepsilon}(\sigma_1-s)^{\alpha-1}\|K_\alpha(\sigma_2-s)-K_\alpha(\sigma_1-s)\|\|Bu(s)\|ds \\
& +\int_{\sigma_1-\varepsilon}^{\sigma_1}(\sigma_1-s)^{\alpha-1}\|K_\alpha(\sigma_2-s)-K_\alpha(\sigma_1-s)\|\|Bu(s)\|ds \\
\leq & \left[\frac{M_B r}{\alpha}\left(\sigma_1^q-\varepsilon^q\right)\right]\sup_{s\in[0,\sigma_1-\varepsilon]}\|K_\alpha(\sigma_2-s)-K_\alpha(\sigma_1-s)\|+\frac{2MM_B r}{\Gamma(1+\alpha)}\varepsilon^q,
\end{aligned}
$$

and this expression tends to zero as $\sigma_2 \to \sigma_1$ due to the continuity of $K_\alpha(t)$ $(t>0)$ in the uniform operator topology. It follows from the Arzela-Ascoli theorem [29] that the operator N is precompact. Thus, H is strongly continuous, which guarantees the existence of optimal control to the minimum problem (3.1) under the fact that U_b is a closed and convex set.

Further, if the system (1.1) is approximately controllable in ω at time b, for any $z_b \in \omega$, suppose that $J(u^*)=\inf_u J(u)=\varepsilon<\infty$, by the definition of infimum, we can deduce that there exists a sequence $\{u_i\}_{i=1,2,\cdots}$ such that $p_\omega z(b,u_i)=z_b,\ u_i\in U_b\subseteq L^2\left(0,b;\mathbf{R}^p\right)(i=1,2,3,\cdots)$ and $J(u_i)\to J(u^*)$. Then we have $u_i \to_\omega u^*$ in $L^2(0,b,\mathbf{R}^p)$.

For any $t\in[0,b]$, by Definition 2.2 and Lemma 2.1,

$$
\begin{aligned}
& \|p_\omega z(t,u^*)-p_\omega z(t,u_i)\| \\
= & \left\|p_\omega\int_0^t(t-s)^{\alpha-1}K_\alpha(t-s)B(u^*(s)-u_i(s))ds\right\| \\
\leq & \left\|\int_0^t(t-s)^{\alpha-1}K_\alpha(t-s)B(u^*(s)-u_i(s))ds\right\| \\
\leq & \frac{\alpha MM_B}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}\|u^*-u_i\|_{L^2(0,b;\mathbf{R}^p)}ds,
\end{aligned}
$$

which yields that

$$p_\omega z(t,u_i)\to p_\omega z(t,u^*) \text{ in } C(0,b,\omega) \text{ as } i\to\infty.$$

And since U_b is closed and convex, from Marzur Lemma [29] we see that $u^*\in U_b$. Thus it follows from the Balder's theorem in [34] that

$$\varepsilon=J(u^*)=\lim_{i\to\infty}J(u_i)\geq J(u^*)\geq\varepsilon,$$

which means that u^* is the optimal solution of the minimization problem (3.1) and the proof is complete. □

3.2. The case of $B\notin\mathcal{L}\left(\mathbf{R}^p,L^2(\Omega)\right)$

If $B\notin\mathcal{L}\left(\mathbf{R}^p,L^2(\Omega)\right)$, we see that the operator N defined in Eq. (23) is unbounded and N is not relatively compact. Then Theorem **3.1** fails and new methods should be introduced. Moreover, we note that this case is also rich in physical systems, for example, when the actuator is pointwise or boundary actuator.

Here we will use the HUMs, which was first introduced by Lions in [21] to study the controllability problems of a linear distributed parameter systems. In order to study the regional controllability of (1.1), the following two assumptions are needed:

(A_1) B is a densely defined operator from $\mathbf{R}^p$ to $L^2(\Omega)$ and B^* exists.

(A_2) $(BK_\alpha(t))^*$ exists and $(BK_\alpha(t))^* = K_\alpha^*(t)B^*$, $t \in [0, b]$.

Take into account that the system (1.1) is line, by Definition 2.6, the system (1.1) is regionally approximately (exactly) controllable in ω at time b if and only if

$$\overline{Im\,(p_\omega H)} = L^2(\omega) \left(\text{ respectively, } Im\,(p_\omega H) = L^2(\omega) \right). \tag{3.4}$$

Suppose that $\{\Phi^*(t)\}_{t\geq 0}$, generated by the adjoint operator of A, is also a C_0 semigroup on $L^2(\Omega)$. For any $v \in L^2(\Omega)$, it follows from $\langle Hu, v\rangle = \langle u, H^*v\rangle$ and the assumptions $(A_1) - (A_2)$ that

$$H^*v = B^*(b-t)^{\alpha-1}K_\alpha^*(b-t)v, \tag{3.5}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing of space $L^2(\Omega)$, B^* is the adjoint operator of B and $K_\alpha^*(t) = \alpha\int_0^\infty \theta\phi_\alpha(\theta)\Phi^*(t^\alpha\theta)d\theta$. Then we have $\overline{Im\,(p_\omega H)} = L^2(\omega)$ is equivalent to

$$Ker\,(H^*p_\omega^*) = \{0\}, \tag{3.6}$$

where $p_\omega^* : L^2(\omega) \to L^2(\Omega)$, the adjoint operator of p_ω, is given by

$$p_\omega^* z(x) := \begin{cases} z(x), \ x \in \omega, \\ 0, \ x \in \Omega\backslash\omega. \end{cases} \tag{3.7}$$

Next, we shall explore the adjoint system of the system (1.1) and then use it to analyze the regional controllability of problem (1.1).

Denote A^* by the adjoint operator of A and for any $\varphi^0 \in L^2(\Omega)$, consider the following system

$$\begin{cases} {}_tD_b^\alpha Q\varphi(t) = -A^*Q\varphi(t), \\ \lim\limits_{t\to b^-} {}_tI_b^{1-\alpha}Q\varphi(t) = \varphi^0 \in D(A^*) \subseteq L^2(\Omega). \end{cases} \tag{3.8}$$

It follows from Lemma 2.2 that (3.8) can be rewritten as

$$\begin{cases} {}_0D_t^\alpha \varphi(t) = -A^*\varphi(t), \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha}\varphi(t) = \varphi^0 \in D(A^*) \subseteq L^2(\Omega) \end{cases} \tag{3.9}$$

and with the solution given by $\varphi(t) = -t^{\alpha-1}K_\alpha^*(t)\varphi^0$. Moreover, we have the following lemma.

Lemma **3.1**. *When u spans space $L^2(0, b; \mathbf{R}^p)$, then the solution $z(b, u)$ is dense in $L^2(\Omega)$.*

P r o o f. On the contrary, if $z(b,u)$ is not dense in $L^2(\Omega)$, then there exists an element $\varphi^0 \in L^2(\Omega)$, $\varphi^0 \neq 0$ such that

$$(z(b,u),\varphi^0) = 0 \quad \text{for all} \quad u \in L^2(0,b;\mathbf{R}^p). \tag{3.10}$$

Multiplying both sides of (3.8) by $z(t)$ and integrating in Q gives

$$\int_\Omega \int_0^b z(t)\,{}_tD_b^\alpha Q\varphi(t)dtdx = \int_0^b (z(t), -A^*Q\varphi(t))dt = -\int_0^b (Az(t), Q\varphi(t))dt.$$

From Lemma 2.3 we see that

$$\int_\Omega \int_0^b z(t)_t D_b^\alpha Q\varphi(t)dtdx = (z(b,u), \lim_{t\to b} {}_tI_b^{1-\alpha}Q\varphi(t)) - (z_0, \lim_{t\to 0} {}_tI_b^{1-\alpha}Q\varphi(t))$$
$$- \int_0^b (Q\varphi(t), {}_0^C D_t^\alpha z(t))dt.$$

It follows from $z_0 = 0$ that

$$(z(b,u),\varphi^0) = \int_0^b (Q\varphi(t), Bu(t))dt. \tag{3.11}$$

By (3.10), since $u \in L^2(0,b;\mathbf{R}^b)$ is arbitrary, we have $Q\varphi(t) = \varphi(b-t) \equiv 0$ in $L^2(\Omega)$ for all $t \in [0,b]$, then $\varphi^0 = 0$, a contradiction. The proof is complete. □

By Lemma 3.1, we then conclude that the system (3.8) is the adjoint system of the problem (1.1). Next, we shall try to explore the regional controllability of the system (1.1) based this duality lemma and the HUMs.

Let $Z = Im\,(p_\omega H) \subseteq L^2(\omega)$, by duality $Z \subseteq L^2(\omega) \subseteq Z^*$ and for any $f \in Z^*$, define

$$\|f\|_{Z^*} := \int_0^b \|B^*(b-s)^{\alpha-1}K_\alpha^*(b-s)p_\omega^* f\|^2 ds, \tag{3.12}$$

where p_ω^* is defined in Eq. (3.7).

Lemma **3.2**. *$\|\cdot\|_{Z^*}$ is a norm of space Z^* provided that the system* (1.1) *is approximately controllable in ω at time b.*

P r o o f. If the system (1.1) is approximately controllable in ω at time b, we get that $Ker\,(H^*p_\omega^*) = \{0\}$, i.e.,

$$B^*(b-s)^{\alpha-1}K_\alpha^*(b-s)p_\omega^* f = 0 \Rightarrow f = 0. \tag{3.13}$$

Hence, for any $f \in Z^*$, it follows from

$$\|f\|_{Z^*} = \int_0^b \|B^*(b-s)^{\alpha-1}K_\alpha^*(b-s)p_\omega^* f\|^2 ds = 0 \Leftrightarrow$$
$$\Leftrightarrow \; B^*(b-s)^{\alpha-1}K_\alpha^*(b-s)p_\omega^* f = 0$$

that $\|\cdot\|_{Z^*}$ is a norm of space Z^* and the proof is complete. □

Denote the completion of the set Z^* with the norm $\|\cdot\|_{Z^*}$ again by Z^*. For each $f \in Z^*$, since f is a linear bounded functional on Z, by the Riesz representation theorem, there exists a unique element in $L^2(\Omega)$, denoted by Pf, such that

$$f(y) = (Pf, y) \quad \text{for all } \; y \in Z. \tag{3.14}$$

Then $P : Z^* \to Z$ is a linear operator and the following lemma holds.

LEMMA **3.3**. *The operator $P : Z^* \to Z$ is isometry.*

P r o o f. For any $f \in Z^*$, it follows from (3.14) that

$$\|Pf\|_Z = \sup_{\|y\|_Z=1} (Pf, y) = \sup_{\|y\|_Z=1} \|f(y)\| = \|f\|_{Z^*}.$$

Then $Im\,(P) \subseteq Z$ is a closed subspace. To complete the proof, we should only show that $Im\,(P) = Z$. If not so, then there exists a $y_0 \in Z$, $y_0 \neq 0$ such that $(Px, y_0) = 0$. By (3.14), we have

$$f(y_0) = 0 \quad \text{for all } f \in Z^*,$$

which implies that $y_0 = 0$, a contradiction. Then the proof is complete. □

Further, let $\Lambda : Z^* \to Z$ be

$$\Lambda f = p_\omega \varphi_1(b), \tag{3.15}$$

where $\varphi_1(t)$ is defined by

$$\begin{cases} {}_0^C D_t^\alpha \varphi_1(t) = A\varphi_1(t) + BB^*(b-t)^{\alpha-1}K_\alpha^*(b-t)f, \\ \varphi_1(0) = 0. \end{cases} \tag{3.16}$$

Since for any $f \in Z^*$, $y \in Z$, by Hölder's inequality, we have

$$(\Lambda f, y) =$$
$$= \int_\Omega p_\omega \int_0^b (b-s)^{\alpha-1}K_\alpha(b-s)BB^*(b-s)^{\alpha-1}K_\alpha^*(b-s)p_\omega^* f(x) ds y(x) dx$$
$$\leq \int_0^b \|B^*(b-s)^{\alpha-1}K_\alpha^*(b-s)p_\omega^* f\|^2 ds \|y\| \leq \|f\|_{Z^*}\|y\|$$

and $\|\Lambda f\| \leq \|f\|_{Z^*}$. Further, for any $f \in Z^*$, we have

$$
\begin{aligned}
&(\Lambda f, f) = \\
&= \int_\Omega p_\omega \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s) BB^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega^* f(x) ds f(x) dx \\
&= \int_0^b \int_\Omega \left[B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega^* f(x)\right]^2 dx ds.
\end{aligned}
$$

Then if the system (1.1) is regionally approximately controllable in ω at b, we get that $f = 0$. Thus it follows from the uniqueness of P that Λ is an isomorphism from Z^* to Z.

Next, suppose that $\varphi_0(t)$ satisfies

$$
\begin{cases} {}_0^C D_t^\alpha \varphi_0(t) = A\varphi_0(t), \\ \varphi_0(0) = z_0 \in D(A), \end{cases} \tag{3.17}
$$

for all $z_b \in L^2(\omega)$, we have $z_b = p_\omega [\varphi_1(b) + \varphi_0(b)]$. Further, let f be the solution of the following equation

$$
\Lambda f := z_b - p_\omega \varphi_0(b). \tag{3.18}
$$

Then we are ready to state the following theorem.

THEOREM **3.2**. *If the system* (1.1) *is regionally approximately controllable in* ω *at time* b*, then for any* $z_b \in L^2(\omega)$*,* (3.18) *has a unique solution* $f \in Z^*$ *and the control*

$$
u^* = B^*(b-\cdot)^{\alpha-1} K_\alpha^*(b-\cdot) p_\omega^* f
$$

steers the system to z_b *at time* b *in* ω*. Moreover,* u^* *solves the minimization problem* (3.1).

P r o o f. By Lemma 3.1, we get that if the system (1.1) is regionally approximately controllable in ω at time b, then $\|\cdot\|_{Z^*}$ is a norm of space Z^*. Let the completion of Z^* with respect to the norm $\|\cdot\|_{Z^*}$ again by Z^*. Then next we show that the equation (3.18) has a unique solution in Z^*.

For any $f \in Z^*$, by the definition of operator Λ in (3.15), we get that

$$
\begin{aligned}
\langle f, \Lambda f \rangle &= \langle f, p_\omega \varphi_1(b) \rangle \\
&= \int_0^b \left\langle f, p_\omega (b-s)^{\alpha-1} K_\alpha(b-s) Bu^*(s) \right\rangle ds \\
&= \int_0^b \|B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega^* f\|^2 ds = \|f\|_{Z^*}^2.
\end{aligned}
$$

Hence, it follows from Theorem 2.1 in [20] that the equation (3.18) admits a unique solution in Z^*. Further, let $u = u^*$ in problem (1.1), we see that $p_\omega z(b, u^*) = z_b$.

For any $u_1 \in L^2(0, b, \mathbf{R}^p)$ with $p_\omega z(b, u_1) = z_b$, we obtain that

$$p_\omega \left[z(b, u^*) - z(b, u_1)\right] = 0,$$

and for any $f \in Z^*$ we have

$$\langle f, p_\omega \left[z(b, u^*) - z(b, u_1)\right]\rangle = 0.$$

It follows that

$$\int_0^b \big\langle u^*(s) - u_1(s), B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega^* f \big\rangle ds = 0.$$

Moreover, since

$$\begin{aligned} J'(u^*)(u^* - u_1) &= 2\int_0^b \langle u^*(s), u^*(s) - u_1(s)\rangle ds \\ &= 2\int_0^b \big\langle B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega^* f, u^*(s) - u_1(s)\big\rangle ds \\ &= 0, \end{aligned}$$

by Theorem 1.3 in [20], we conclude that u^* solves the minimum energy problem (3.1) and the proof is complete. □

4. Examples

In this section, two examples which are reachable on a sub-region but not on the whole domain are introduced with $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$ and $B \notin \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$, respectively.

Example **4.1**. Let us consider the following one dimension fractional order sub-diffusion system with a zone actuator $Bu = p_{[a_1,a_2]}u$, $0 \le a_1 \le a_2 \le 1$ and $B \in \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$:

$$\begin{cases} {}_0^C D_t^{0.7} z(x,t) = \frac{\partial^2}{\partial x^2} z(x,t) + p_{[a_1,a_2]} u(t) \text{ in } [0,1] \times [0,b], \\ z(x,0) = z_0 \text{ in } [0,1], \\ z(0,t) = z(1,t) = 0 \text{ in } [0,b]. \end{cases} \tag{4.1}$$

We see that B is a bounded continuous operator with $M_B = 1$, $A = \frac{\partial^2}{\partial x^2}$ and $\Phi(t)z(x) = \sum_{i=1}^{\infty} \exp(\lambda_i t)(z, \xi_i)\xi_i(x),\ x \in [0,1]$, where

$$\lambda_i = -i^2\pi^2 \quad \text{and} \quad \xi_i(x) = \sqrt{2}\sin(i\pi x),\ i = 1, 2, \cdots,\ x \in [0,1].$$

Then $\{\Phi(t)\}_{t\ge0}$ generated by A is uniformly bounded with $M = 1$. Further, we have

$$\begin{aligned}
&K_{0.7}(t)z(x) = 0.7\int_0^\infty \theta\phi_{0.7}(\theta)\Phi(t^{0.7}\theta)zd\theta \\
&= 0.7\sum_{i=1}^{\infty}\int_0^\infty \theta\phi_{0.7}(\theta)\exp(\lambda_i t^{0.7}\theta)d\theta(z,\xi_i)\xi_i(x) \\
&= 0.7\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\int_0^\infty \frac{(\lambda_i t^{0.7})^j}{j!}\theta^{j+1}\phi_{0.7}(\theta)d\theta(z,\xi_i)\xi_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{0.7(j+1)(\lambda_i t^{0.7})^j}{\Gamma(1+0.7j+0.7)}(z,\xi_i)\xi_i(x) \\
&= \sum_{i=1}^{\infty}E_{0.7,0.7}(\lambda_i t^{0.7})(z,\xi_i)\xi_i(x),
\end{aligned} \tag{4.2}$$

where $E_{\alpha,\beta}(z) := \sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i+\beta)}$, $\mathbf{Re}\,\alpha > 0,\ \beta, z \in \mathbf{C}$ is known as the generalized Mittag-Leffler function in two parameters. Similarly, one has

$$S_{0.7}(t)z(x) = \int_0^\infty \phi_{0.7}(\theta)\Phi(t^{0.7}\theta)d\theta = \sum_{i=1}^{\infty}(z,\xi_i)E_{0.7,1}(\lambda_i t^{0.7})\xi_i(x). \tag{4.3}$$

What is more, since $A = \frac{\partial^2}{\partial x^2}$ is a self-adjoint operator, we have

$$\begin{aligned}
(H^*z)(t) &= B^*(b-t)^{-0.3}K_{0.7}^*(b-t)z(t) \\
&= (b-t)^{-0.3}\sum_{i=1}^{\infty}E_{0.7,0.7}(\lambda_i(b-t)^{0.7})(z,\xi_i)\int_{a_1}^{a_2}\xi_i(x)dx.
\end{aligned}$$

Then from

$$\int_{a_1}^{a_2}\xi_i(x)dx = \frac{\sqrt{2}}{i\pi}\sin\frac{i\pi(a_1+a_2)}{2}\sin\frac{i\pi(a_1-a_1)}{2}$$

it follows that

$$Ker\,(H^*) \neq \{0\} \quad (\overline{Im\,(H)} \neq L^2(\omega))$$

when $a_2 - a_1 \in Q$, i.e., the system (4.1) is not weakly controllable when $a_2 - a_1 \in Q$.

Next, we show that there exists a sub-region $\omega \subseteq \Omega$ such that the system (4.1) is regionally controllable in ω at time b. For example, let $a_1 = 0,\ a_2 = 1/2,\ z_* = \xi_k, (k = 4j, j = 1, 2, 3, \cdots)$. Obviously, z_* is not reachable on $\Omega = [0, 1]$. However, since

$$E_{\alpha,\alpha}(t) > 0\ (t \geq 0) \text{ and } \int_0^{1/2}\xi_i(x)dx = \frac{\sqrt{2}}{i\pi}\left(1-\cos(i\pi/2)\right),\ i = 1, 2, \cdots,$$

let $\omega = [1/4, 3/4]$, we see that

$$\begin{aligned}
(H^*p_\omega^*p_\omega z_*)(t) &= \sum_{i=1}^{\infty}\frac{E_{0.7,0.7}(\lambda_i(b-t)^{0.7})}{(b-t)^{0.3}}(\xi_i,\xi_k)_{L^2(\frac{1}{4},\frac{3}{4})}\int_0^{1/2}\xi_i(x)dx \\
&= \sum_{i\neq 4j}\frac{\sqrt{2}E_{0.7,0.7}(\lambda_i(b-t)^{0.7})}{i\pi(b-t)^{0.3}}\int_{1/4}^{3/4}\xi_i(x)\xi_{4j}(x)dx\,[1-\cos(i\pi/2)] \neq 0.
\end{aligned}$$

Then z_* is regionally controllable on $\omega = [1/4, 3/4]$ at time b.

Moreover, according to Theorem **3.1**, if the system (4.1) is approximately controllable in $\omega = [1/4, 3/4]$ at time b, then the minimum energy problem (3.1) admits at least one optimal solution.

Example **4.2**. Consider the following fractional order sub-diffusion system with a pointwise actuator $Bu = u(t)\delta(x-\sigma)$, $0 < \sigma < 1$. Obviously, $B \notin \mathcal{L}\left(\mathbf{R}^p, L^2(\Omega)\right)$ is unbounded.

$$\begin{cases} {}_0^C D_t^{0.7} z(x,t) = \frac{\partial^2}{\partial x^2} z(x,t) + u(t)\delta(x-\sigma) \text{ in } [0,1]\times[0,b], \\ z(x,0) = z_0 \text{ in } [0,1], \\ z(0,t) = z(1,t) = z_0 \text{ in } [0,b]. \end{cases} \tag{4.4}$$

Similar to the argument above, we see that

$$\begin{aligned} (H^*z)(t) &= B^*(b-t)^{-0.3} K_{0.7}^*(b-t) z(t) \\ &= \sum_{i=1}^{\infty} \frac{E_{0.7,0.7}(\lambda_i (b-t)^{0.7})}{(b-t)^{0.3}} (z, \xi_i)\xi_i(\sigma). \end{aligned}$$

Then the system (4.4) is not weakly controllable in Ω if $\sigma \in Q$. Moreover, let $\sigma = 1/2$, by the argument above, there exists a sub-region $\omega = [1/4, 3/4] \subseteq \Omega$ such that the system (4.4) is regionally controllable in ω at time b.

Further, since $A = \frac{\partial^2}{\partial x^2}$ is a self-adjoint operator, suppose that the system (4.4) is approximately controllable in ω at time b, by Lemma 3.1, we get that

$$\begin{aligned} f \to \|f\|_{Z^*} &= \int_0^b \left\|(b-s)^{-0.3} K_\alpha^*(b-s) p_\omega^* f(\sigma)\right\|^2 ds \\ &= \int_0^b \left\|(b-t)^{-0.3} \sum_{i=1}^{\infty} E_{0.7,0.7}(\lambda_i (b-t)^{0.7})(z,\xi_i) p_\omega^* f(\sigma)\right\|^2 ds \end{aligned}$$

defines a norm on Z^*. It follows from Lemma 3.2 that

$$\Lambda f = p_\omega \varphi_1(\cdot, \sigma), \tag{4.5}$$

is a isometry form Z^* to Z, where $\varphi_1(x,t)$ is the solution of the following system

$$\begin{cases} {}_0^C D_t^{0.7} \varphi_1(x,t) = \frac{\partial^2}{\partial x^2} \varphi_1(x,) + (b-t)^{-0.3} K_\alpha^*(b-t) f(\sigma), \\ \varphi_1(x,0) = 0, \\ \varphi_1(0,t) = \varphi_1(1,t) = 0. \end{cases} \tag{4.6}$$

Then by Theorem **3.2**, we see that the control

$$u^*(t) = (b-t)^{-0.3} \sum_{i=1}^{\infty} E_{0.7,0.7}(\lambda_i (b-t)^{0.7})(z,\xi_i) p_\omega^* f(\sigma)$$

steers the system to z_b at time b in ω, where f is the solution of equations

$$\Lambda f = z_b - p_\omega \varphi_0(\cdot, \sigma), \tag{4.7}$$

and $\varphi_0(x,t)$ solves

$$\begin{cases} {}_0^C D_t^{0.7} \varphi_0(x,t) = \frac{\partial^2}{\partial x^2} \varphi_0(x,t), \\ \varphi_0(x,0) = z_0(x) \in D(A), \\ \varphi_0(0,t) = \varphi_0(1,t) = 0. \end{cases} \tag{4.8}$$

Moreover, u^* is the solution of the minimum energy problem (3.1).

5. Conclusions

This paper deals with the regional controllability problems of the fractional sub-diffusion equations with Caputo fractional derivative in two different cases: $B \in \mathcal{L}\left(\mathbf{R}^m, L^2(\Omega)\right)$ and $B \notin \mathcal{L}\left(\mathbf{R}^m, L^2(\Omega)\right)$. The duality result of the fractional sub-diffusion is also derived at the same time. The results here can be regarded as the extensions of the results in [7, 8, 39] and can also be extended to complex fractional order distributed parameter systems. For instance, the problem of regional gradient controllability/observability of fractional order distributed parameter systems as well as the case of fractional order super-diffusion systems with more complicated dynamics are of great interest. For more information on the potential topics related to fractional DPSs, we refer the readers to [9] and the references therein.

Acknowledgements

This work was supported in parts by Chinese Universities Scientific Fund No. CUSF-DH-D-2014061, Natural Science Foundation of Shanghai No. 15ZR1400800, APVV-14-0892, ARO W911NF-15-1-0228, VEGA 1/0908/15 and 1/0552/14.

References

[1] J. Cao, Y.Q. Chen, C. Li, Multi-UAV-based optimal crop-dusting of anomalously diffusing infestation of crops. In: *2015 American Control Conference*, Palmer House Hilton, July 1-3, 2015, Chicago, IL-USA; See also: arXiv:1411.2880.

[2] K. Cao, Y.Q. Chen, D. Stuart, D. Yue, Cyber-physical modeling and control of crowd of pedestrians: a review and new framework. *IEEE/CAA J. of Automatica Sinica* **2**, No 3 (2015), 334–344.

[3] A. Cartea, D. del Castillo-Negrete, Fluid limit of the continuous-time random walk with general Lévy jump distribution functions. *Phys. Review E* **76**, No 4 (2007), 041105.

[4] Y. Chitour, E. Trélat, Controllability of partial differential equations. In: *Advanced Topics in Control Systems Theory*, Springer (2006), 171–198.

[5] A. Debbouche, J.J. Nieto, Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controls. *Appl. Math. and Computation* **245** (2014), 74–85.

[6] S. Dolecki, D.L. Russell, A general theory of observation and control. *SIAM J. on Control and Optimization* **15**, No 2 (1977), 185–220.

[7] A. El Jai, A.J. Pritchard, *Sensors and Controls in the Analysis of Distributed Systems*. Halsted Press (1988).

[8] A. El Jai, M. Simon, E. Zerrik, A. Pritchard, Regional controllability of distributed parameter systems. *Internat. J. of Control* **62**, No 6 (1995), 1351–1365.

[9] F. Ge, Y.Q. Chen, C. Kou, Cyber-physical systems as general distributed parameter systems: three types of fractional order models and emerging research opportunities. *IEEE/CAA J. of Automatica Sinica* **2**, No 4 (2015), 353–357.

[10] F. Ge, Y.Q. Chen, C. Kou, Regional controllability of anomalous diffusion generated by the time fractional diffusion equations, *ASME IDETC/CIE 2015, Boston, Aug. 2-5, 2015*, DETC2015-46697.

[11] F. Ge, Y.Q. Chen, C. Kou, Regional gradient controllability of subdiffusion processes. *J. Math. Anal. Appl.* **440**, No 2 (2016), 865–884.

[12] F. Ge, Y.Q. Chen, C. Kou, Regional boundary controllability of time fractional diffusion processes. *IMA J. of Math. Control and Information* (2016), 1–18.

[13] F. Ge, H. Zhou, C. Kou, Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions via an approximating technique. *Appl. Math. and Computation* **275** (2016), 107–120.

[14] R. Gorenflo, F. Mainardi, Fractional diffusion processes: Probability distributions and continuous time random walk. *Springer Lecture Notes in Physics* **621** (2003), 148–166.

[15] G. Gradenigo, A. Sarracino, D. Villamaina, A. Vulpiani, Einstein relation in systems with anomalous diffusion. *Acta Phys. Polonica B* **44**, No 5 (2013), 899–912.

[16] S. Hu, N. S. Papageorgiou, *Handbook of Multivalued Analysis: Volume II: Applications*. Springer (2013).

[17] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier (2006).

[18] M. Klimek, *On Solutions of Linear Fractional Differential Equations of a Variational Type*. Publ. Office of Czestochowa Univ. of Techn. (2009).

[19] J. E. Lagnese, The Hilbert uniqueness method: a retrospective. In: *Optimal Control of Partial Differential Equations*, Springer (1991), 158–181.

[20] J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*. Springer (1971).

[21] J. Lions, Exact controllability, stabilization and perturbations for distributed systems. *SIAM Review* **30**, No 1 (1988), 1–68.

[22] F. Mainardi, P. Paradisi, R. Gorenflo, Probability distributions generated by fractional diffusion equations (2007). *arXiv*:0704.0320.

[23] B. Mandelbrot, *The Fractal Geometry of Nature*. Freeman & Co. (1982).

[24] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Physics Reports* **339**, No 1 (2000), 1–77.

[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer (2012).

[26] I. Podlubny, *Fractional Differential Equations*. Academic Press (1999).

[27] I. Podlubny, Y.Q. Chen, Adjoint fractional differential expressions and operators. In: *ASME 2007 Intern. Design Engineering Techn. Conferences and Computers and Information in Engineering Conference*, American Society of Mechanical Engineers (2007), 1385–1390.

[28] A. Pritchard, A. Wirth, Unbounded control and observation systems and their duality. *SIAM J. on Control and Optimization* **16**, No 4 (1978), 535–545.

[29] M. Renardy, R.C. Rogers, *An Introduction to Partial Differential Equations*. Springer (2006).

[30] K. Schlacher, M. Schöberl, Modelling, analysis and control of distributed parameter systems. *Mathematical and Computer Modelling of Dynamical Systems* **17**, No 1 (2011), 1–2.

[31] W.M. Spears, D.F. Spears, *Physicomimetics: Physics-Based Swarm Intelligence*. Springer (2012).

[32] P.J. Torvik, R.L. Bagley, On the appearance of the fractional derivative in the behavior of real materials. *J. of Appl. Mechanics* **51**, No 2 (1984), 294–298.

[33] V. Uchaikin, R. Sibatov, *Fractional Kinetics in Solids*. World Sci. (2013).

[34] M. Valadier, Young measures, weak and strong convergence and the visintin-balder theorem. *Set-Valued Anal.* **2**, No 1-2 (1994), 357–367.

[35] M. Weiss, M. Elsner, F. Kartberg, T. Nilsson, Anomalous sub-diffusion is a measure for cytoplasmic crowding in living cells. *Biophysical J.* **87**, No 5 (2004), 3518–3524.

[36] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations. *Computers and Math. with Applications* **59**, No 3 (2010), 1063–1077.

[37] C. Zeng, Y.Q. Chen, Q. Yang, Almost sure and moment stability properties of fractional order Black-Scholes model. *Fract. Calc. Appl. Anal.* **16**, No 2 (2013), 317–331; DOI: 10.2478/s13540-013-0020-0; http://www.degruyter.com/view/j/fca.2013.16.issue-2/issue-files/fca.2013.16.issue-2.xml.

[38] C. Zeng, Y.Q. Chen, Optimal random search, fractional dynamics and fractional calculus. *Fract. Calc. Appl. Anal.* **17**, No 2 (2014), 321–332; DOI: 10.2478/s13540-014-0171-7; http://www.degruyter.com/view/j/fca.2014.17.issue-2/issue-files/fca.2014.17.issue-2.xml.

[39] E. Zerrik, A. Boutoulout, A. E. Jai, Actuators and regional boundary controllability of parabolic systems. *International J. of Systems Science* **31**, No 1 (2000), 73–82.

[1] *School of Computer Science, China University of Geosciences*
Wuhan 430074, PR CHINA
e-mail: gefd2011@gmail.com

[2] *Mechatronics, Embedded Systems and Automation Lab – MESA Lab,*
University of California, Merced, CA 95343, USA
e-mail: ychen53@ucmerced.edu

[3] *Department of Applied Mathematics, Donghua University*
Shanghai 201620, PR CHINA
e-mail: kouchunhai@dhu.edu.cn

[4] *Institute of Control and Informatization of Production Processes*
BERG Faculty, Technical University of Kosice
B. Nemcovej 3, 04200 Kosice, SLOVAKIA
e-mail: igor.podlubny@tuke.sk

Received: January 2, 2016

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **19**, No 5 (2016), pp. 1261–1281,
DOI: 10.1515/fca-2016-0065