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ON THE REGIONAL CONTROLLABILITY OF THE SUB-DIFFUSION PROCESS WITH CAPUTO FRACTIONAL DERIVATIVE

Fudong Ge, YangQuan Chen, Chunhai Kou, Igor Podlubny

On the Occasion of Professor Richard L. Magin’s 70th Birthday

Abstract

This paper is devoted to the investigation of regional controllability of the fractional order sub-diffusion process. We first derive the equivalent integral equations of the abstract sub-diffusion systems with Caputo and Riemann-Liouville fractional derivatives by utilizing the Laplace transform. The new definitions of regional controllability of the system studied are introduced by extending the existence contributions. Then we analyze the regional controllability of the fractional order sub-diffusion system with minimum energy control in two different cases: $B \in L(R^m, L^2(\Omega))$ and $B \notin L(R^m, L^2(\Omega))$. The adjoint system of fractional order sub-diffusion system is also presented at the same time. Two applications are worked out in the end to verify the effectiveness of our results.

MSC 2010: Primary 26A33; Secondary 93B05, 60J60

Key Words and Phrases: fractional calculus, Caputo derivative, sub-diffusion, control, regional controllability

1. Introduction

Let $\Omega$ be an open bounded subset of $R^n$ with smooth boundary $\partial \Omega$, $Q = \Omega \times [0, b]$ and $\Sigma = \partial \Omega \times [0, b]$. In this paper, we consider the following abstract fractional order sub-diffusion system of order $\alpha \in (0, 1]$:
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\[
\begin{cases}
C_0 D_t^\alpha z(t) = Az(t) + Bu(t), \quad t \in [0, b], \\
z(0) = z_0 \in D(A),
\end{cases}
\]

where \( z \in L^2(0, b; L^2(\Omega)) \), \( D(A) \) holds for the domain of \( A \) and \( A \) generates a \( C_0 \) semigroup \( \{\Phi(t)\}_{t \geq 0} \) on the Hilbert space \( L^2(\Omega) \). Besides, \( u \in L^2(0, b; \mathbb{R}^p) \), \( B : \mathbb{R}^p \to L^2(\Omega) \) is a linear operator (possibly unbounded) depending on the number and structure of actuators and here \( C_0 D_t^\alpha \) denotes by the left-sided Caputo fractional order derivative.

Recently fractional order sub-diffusion systems have attracted increasing attention after the verification that they can be used to well characterize those anomalous diffusion processes in various real-world complex systems \[24, 35, 11, 15\]. As we all know, the anomalous diffusion processes in real world are essentially distributed and the continuous time random walk (CTRW) is useful to describe this phenomenon by allowing the incorporation of waiting time probability density function (PDF) and general jump PDF \[14, 3, 38, 37\]. More precisely, when the particles are assumed to jump at fixed time intervals with incorporating waiting times, the particles will experience the sub-diffusion processes, such as the flow through porous media microscopic process \[33\], or the swarm of robots moving through dense forest \[31\] etc. In this case, the mean squared displacement (MSD) is a power law of fractional exponent smaller than a linear function of the Gaussian diffusion process \[32, 23\]. Another case is that when the particles are supposed to jump following from a general, non-Gaussian jump distribution function, the particles then undergo the super-diffusion process \[15, 8\]. Now the MSD is a power law function of fractional exponent bigger than that of a Gaussian diffusion process.

It is worth noting that the controllability problem of a fractional order sub-diffusion system can be reformulated as a problem of infinite-dimensional control system. Moreover, in the case of diffusion systems, it should be pointed out that, in general, not all the states can be reached \[12, 7\]. So in this paper, to extend the existence results in \[8, 39, 10\], we introduce some notations on the regional controllability of the fractional order sub-diffusion systems with Caputo fractional derivative, where we are only concerned with the knowledge of the states in a sub-region along the spatial domain. This situation happens in many real dynamic systems, for example the pest spreading \[1\], crowd-pedestrian egress or evacuation \[2\], and etc. It is now widely believed that many real-world complex dynamics can be well characterized by using fractional calculus and fractional order controls can offer better performance not achievable before using integer order controls systems \[32, 23\], this is the reason why the fractional order models are superior in comparison with the integer order models.
Moreover, it is well known that the adjoint system plays an important role in many fields of mathematics, including mathematical physics and control theory [27, 4, 28, 6]. However, as the Definition 2.4 below indicates, the unique mild solution of the system (1.1) can be expressed as follows:

\[ z(t) = S_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)Bu(s)ds \]  
(1.2)

and \( S_\alpha(t) \neq K_\alpha(t) \), which makes it difficult to define the adjoint system of (1.1) and many methods introduced to analyze the controllability of integer order partial differential equations fails here. Then in this paper, we try to present the notion of adjoint system for the case of fractional order sub-diffusion equations with help of the integration by parts of fractional derivatives, which is consistently extended by Podlubny and Chen in [27]. Based on the semigroup theory [25], here we discuss the regional controllability of the fractional order sub-diffusion systems with minimum energy control in two different cases: \( B \in L(\mathbb{R}^p, L^2(\Omega)) \) and \( B \notin L(\mathbb{R}^p, L^2(\Omega)) \). More precisely, when \( B \in L(\mathbb{R}^p, L^2(\Omega)) \), our main result is derived by utilizing the Balder’s theorem [34] and when \( B \notin L(\mathbb{R}^p, L^2(\Omega)) \), the Hilbert Uniqueness Methods (HUMs), which were first introduced by Lions [21] to study the controllability of partial differential equations [19, 30], are used to obtain the regional controllability of the system studied with minimum energy control.

The remainder of this paper is organized as follows: some basic knowledge of fractional calculus and some preliminary results are given in the next Section 2. In Section 3, our main results on the regional controllability analysis of the system studied in two different cases are presented. Two applications are worked out in Section 4 to test our obtained results.

2. Preliminaries

This section is devoted to introducing some definitions and preliminary results to be used afterwards.

**Definition 2.1.** ([26, 17]) The left-sided and right-sided Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( z \) on \([0, b]\) are given by

\[ 0I_0^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} z(\tau)d\tau \]  
(2.1)

and

\[ tI_b^\alpha z(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} z(\tau)d\tau, \]  
(2.2)

respectively, provided that the right sides are pointwise defined on \([0, b]\).
Remark 2.1. By Definition 2.1, we see that only both \( z(t) \) and \( \mathcal{I}_b^\alpha z(t) \) are continuous at the point \( t = b \), we have
\[
\lim_{t \to b^-} \mathcal{I}_b^\alpha z(t) = 0.
\]

If not so, for example, let \( z(t) = (b - t)^{-\alpha} \), \( \alpha \in (0, 1) \), although \( \mathcal{I}_b^\alpha z(t) \) is continuous at the point \( t = b \), we have
\[
\lim_{t \to b^-} \mathcal{I}_b^\alpha z(t) = \lim_{t \to b^-} \int_t^b \frac{(\tau - t)^{\alpha - 1}}{\Gamma(\alpha)} (b - \tau)^{-\alpha} d\tau = \Gamma(1 - \alpha) \neq 0. \tag{2.3}
\]

Definition 2.2. (\cite{26,17}) For \( t \in [0, b] \) and any given \( \alpha \) \( (n - 1 < \alpha < n) \), \( n \in \mathbb{N} \), the left-sided and right-sided Riemann-Liouville fractional derivatives of order \( \alpha \) of a function \( z \) are defined as
\[
\mathcal{D}_t^\alpha z(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t - s)^{n-\alpha-1} z(s) ds, \tag{2.4}
\]
and
\[
\mathcal{D}_t^\alpha z(t) = \frac{1}{\Gamma(n - \alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b (s - t)^{n-\alpha-1} z(s) ds, \tag{2.5}
\]
érespectively, provided that the right sides are pointwise defined.

Definition 2.3. (\cite{26,17}) The left-sided Caputo fractional derivative of order \( \alpha > 0 \) of a function \( z \) is
\[
\mathcal{C}_0D_t^\alpha z(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} \frac{d^n}{ds^n} z(s) ds, \tag{2.6}
\]
where \( t \geq 0 \), \( n - 1 < \alpha < n \), \( n \in \mathbb{N} \) and the right side is pointwise defined.

Definition 2.4. For \( t \in [0, b] \), any given \( u \in L^2 \left( 0, b; \mathbb{R}^p \right) \), a function \( z \in L^2 \left( 0, b; L^2(\Omega) \right) \) is said to be a mild solution of the system (1.1), denoted by \( z(\cdot, u) \), if it satisfies
\[
z(t, u) = S_\alpha(t)z_0 + \int_0^t (t - s)^{\alpha - 1} K_\alpha(t - s)Bu(s) ds, \tag{2.7}
\]
where
\[
S_\alpha(t) = \int_0^\infty \phi_\alpha(\theta) \Phi(t^\alpha \theta) d\theta \tag{2.8}
\]
and
\[
K_\alpha(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi(t^\alpha \theta) d\theta. \tag{2.9}
\]
Here \( \{\Phi(t)\}_{t \geq 0} \) is the strongly continuous semigroup generated by \( A \), 
\[ \phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-\frac{1}{\alpha}} \psi_\alpha(\theta^{-\frac{1}{\alpha}}) \]
and \( \psi_\alpha \) is a probability density function defined by 
\[ \psi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n-1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty), \]
satisfying the following properties (2.22)
\[ \int_0^\infty e^{-\lambda \theta} \psi_\alpha(\theta) d\theta = e^{-\lambda \alpha}, \quad \int_0^\infty \psi_\alpha(\theta) d\theta = 1, \alpha \in (0, 1) \quad (2.10) \]
and
\[ \int_0^\infty \theta^\nu \phi_\alpha(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \quad \nu \geq 0. \quad (2.11) \]

**Definition 2.5.** For \( t \in [0, b] \), any given \( f \in L^2(0, b; L^2(\Omega)) \), \( 0 < \alpha < 1 \), a function \( v \in L^2(0, b; L^2(\Omega)) \) is said to be a mild solution of the following system 
\[ 0D_t^\alpha v(t) = Av(t) + f(t), \quad t \in [0, b], \]
\[ \lim_{t \to 0^+} 0I_t^{1-\alpha} v(t) = v_0 \in L^2(\Omega) \quad (2.12) \]
if it satisfies 
\[ z(t) = t^{\alpha-1} K_\alpha(t)v_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s)f(s)ds, \quad (2.13) \]
where \( K_\alpha(t) \) is defined in Eq.(2.9).

For the results of Definition 2.4 and 2.5, we refer to Zhou and Jiao [36], where the authors studied a class of neutral evolution equations of fractional order with nonlocal conditions and obtained several criteria on the existence of mild solutions.

Let \( \omega \subseteq \Omega \) be a given region of positive Lebesgue measure and \( z_b \in L^2(\omega) \) (the target function) be a given element of the state space. We first state the following definition.

**Definition 2.6.** The system (1.1) is called to be regionally exactly (respectively, approximately) controllable in \( \omega \) at time \( b \) if for any \( z_b \in L^2(\omega) \), given \( \varepsilon > 0 \), there exists a control \( u \in L^2(0, b; \mathbb{R}^p) \) such that 
\[ p_\omega z(b, u) = z_b \quad ( \text{respectively, } \|p_\omega z(b, u) - z_b\|_{L^2(\omega)} \leq \varepsilon ), \quad (2.14) \]
where \( p_\omega : L^2(\Omega) \to L^2(\omega), \) defined by \( p_\omega z = z|_{\omega} \), is a projection operator.

In order to show our main results, the following lemmas are needed.
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**Lemma 2.1.** ([35, 13])

(i) The operators \( S_\alpha(t) \) and \( K_\alpha(t) \) are linear bounded and for any \( x \in L^2(\Omega) \), we have
\[
\| S_\alpha(t) x \| \leq M \| x \| \quad \text{and} \quad \| K_\alpha(t) x \| \leq \frac{\alpha M}{(1 + \alpha)} \| x \|. \tag{2.15}
\]

(ii) Operators \( \{ S_\alpha(t) \}_{t \geq 0} \) and \( \{ K_\alpha(t) \}_{t \geq 0} \) are strongly continuous, this is, for \( \forall x \in L^2(\Omega) \) and \( 0 \leq t_1 \leq t_2 \leq b \), we have
\[
\| S_\alpha(t_1) x - S_\alpha(t_2) x \| \to 0 \quad \text{and} \quad \| K_\alpha(t_1) x - K_\alpha(t_2) x \| \to 0 \quad \text{as} \ t_1 \to t_2. \tag{2.16}
\]

(iii) For \( t > 0 \), \( S_\alpha(t) \) and \( K_\alpha(t) \) are all compact operators if \( \Phi(t) \) is compact.

**Lemma 2.2.** ([27]) For \( t \in [a, b] \) and \( \alpha \) (\( n - 1 < \alpha < n, n \in \mathbb{N} \)), the following formula holds
\[
\int_a^b f(t) C_t^\alpha D_t^\alpha g(t) dt = \sum_{r=0}^{k-1} (-1)^{k-1-r} \left[ g^{(r)}(t) C_t^{\alpha-1-r} f(t) \right]_{t=a}^{t=b} + (-1)^k \int_a^b g(t) C_t^{\alpha} f(t) dt.
\]
In particular, if \( 0 < \alpha < 1 \), we have
\[
\int_a^b f(t) C_t^\alpha D_t^\alpha g(t) dt = \left[ g(t) C_t^{1-\alpha} f(t) \right]_{t=a}^{t=b} - \int_a^b g(t) C_t^{\alpha} f(t) dt.
\]

**Lemma 2.3.** ([18]) Let the reflection operator \( Q \) on interval \([0, b]\) be as follows:
\[
Qf(t) := f(b - t).
\]
Then the following equations hold:
\[
Q_0 C_t^\alpha f(t) = C_0^\alpha Qf(t), \quad Q_0 D_t^\alpha f(t) = D_0^\alpha Qf(t) \quad \text{and} \quad 0 C_t^\alpha f(t) = C_t^\alpha Qf(t), \quad 0 D_t^\alpha f(t) = D_t^\alpha Qf(t).
\]

## 3. Main results

In this section, we will explore the possibility of finding a minimum energy control which steers the problem (1.1) from the initial state \( z_0 \) to a target function \( z_b \) on the sub-region \( \omega \).

Let \( U_b = \{ u \in L^2(0, b; \mathbb{R}^p) : p_\omega z(b, u) = z_b \} \) and consider the following minimum energy problem
\[
\inf_{u} J(u) = \inf_{u} \left\{ \int_0^b \| u(t) \|^2_{\mathbb{R}^p} dt : u \in U_b \right\}. \tag{3.1}
\]
Let $H : L^2(0, b; \mathbb{R}^p) \to L^2(\Omega)$ be

$$Hu = \int_0^b (b - s)^{\alpha - 1} K_\alpha (b - s) B u(s) ds, \ \forall u \in L^2(0, b; \mathbb{R}^p).$$

(3.2)

Now we are ready to state our results in two different cases.

### 3.1. The case of $B \in \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$

Suppose that the semigroup $\{\Phi(t)\}_{t \geq 0}$ generated by operator $A$ is uniformly bounded, it follows from $B \in \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$ that there exist two constants $M_B, M > 0$ such that $\|B\| \leq M_B$ and $\sup_{t \geq 0} \|\Phi(t)\| \leq M$.

Then we see the following results.

**Theorem 3.1.** Suppose that $B \in \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$ and for any $t > 0$, $\Phi(t)$ is a compact operator, then the minimum energy problem (3.1) admits at least one optimal solution provided that the system (1.1) is regionally approximately controllable in $\omega$ at time $b$.

**Proof.** Obviously, $U_b$ is a closed and convex set. We first prove that the operator $H$ in (3.2) is strongly continuous, which admits the existence of optimal control to the minimum energy problem (3.1) (see pp.597, [16]).

For any $t \in [0, b]$, $z_0 \in L^2(\Omega)$, by Lemma 2.1, we see that the term $S_\alpha (t) z_0$ in Eq. (2.7) is strongly continuous. Moreover, since the operator $H$ is linear and continuous, according to the argument in [5], we only need to show that it is precompact.

Let $N : L^2(0, b; \mathbb{R}^p) \to L^2(\Omega)$ be

$$Nu(t) := \int_0^t (t - s)^{\alpha - 1} K_\alpha (t - s) B u(s) ds, \ t \in [0, b].$$

(3.3)

We next show that $N$ is a relatively compact operator.

Let $\rho_r = \{u \in L^2(0, b; \mathbb{R}^p) : \|u\|_{L^2(0, b; \mathbb{R}^p)} \leq r\}$. For any fixed $t \in (0, b]$, $\varepsilon, \delta \in (0, t)$, $u \in \rho_r$, let

$$\tilde{N}(\varepsilon, \delta) u(t) = \alpha \int_0^{t - \varepsilon} \int_\delta^\infty (t - s)^{\alpha - 1} \theta \phi(\theta) \Phi((t - s)^{\alpha} \theta) B u(s) d\theta ds.$$

Since $\Phi(\varepsilon \delta)$ is compact and

$$\tilde{N}(\varepsilon, \delta) u(t) = \Phi(\varepsilon \delta) \alpha \int_0^{t - \varepsilon} \int_\delta^\infty (t - s)^{\alpha - 1} \theta \phi(\theta) \Phi((t - s)^{\alpha} \theta - \varepsilon \delta) B u(s) d\theta ds,$$
we get that \( \tilde{N}(\epsilon, \delta) \) is relatively compact. Together with
\[
\| B u(\cdot) \| \leq M_{B r} < \infty,
\]
by (i) in Lemma 2.1, for any \( t \in [0, b] \), we have
\[
\| \tilde{N} u(t) - \tilde{N}(\epsilon, \delta) u(t) \| = \alpha \| \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \Phi((t-s)^\alpha \theta) B u(s) d\theta ds \\
+ \int_0^t \int_\delta^\infty (t-s)^{\alpha-1} \theta \Phi((t-s)^\alpha \theta) B u(s) d\theta ds \\
- \int_0^{t-\epsilon} \int_\delta^\infty (t-s)^{\alpha-1} \theta \Phi((t-s)^\alpha \theta) B u(s) d\theta ds \|
\leq \alpha \| \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \Phi((t-s)^\alpha \theta) B u(s) d\theta ds \|
+ \alpha \| \int_1^{t-\epsilon} \int_\delta^\infty (t-s)^{\alpha-1} \theta \Phi((t-s)^\alpha \theta) B u(s) d\theta ds \|
\leq M_{B r} b^\alpha \int_0^\delta \theta \Phi(\theta) d\theta + \frac{M M_{B r} \epsilon^\alpha}{\Gamma(1+\alpha)} \to 0
\]
as \( \epsilon, \delta \to 0 \). Then \( N_{\theta r} \) is a relatively compact set in \( L^2(\Omega) \).

Next, we shall prove that \( N u \) is equicontinuous on \([0, b] \). For any \( u \in \varrho_r \), \( 0 \leq \sigma_1 < \sigma_2 \leq b \),
\[
\| N u(\sigma_2) - N u(\sigma_1) \| \leq \\
\leq \left\| \int_0^{\sigma_1} [(\sigma_2-s)^{\alpha-1} - (\sigma_1-s)^{\alpha-1}] K_\alpha(\sigma_2-s) B u(s) ds \right\| \\
+ \left\| \int_0^{\sigma_1} (\sigma_1-s)^{\alpha-1} [K_\alpha(\sigma_2-s) - K_\alpha(\sigma_1-s)] B u(s) ds \right\| \\
+ \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2-s)^{\alpha-1} K_\alpha(\sigma_2-s) B u(s) ds \right\| \\
\leq \frac{M M_{B r}}{\Gamma(1+\alpha)} (\sigma_2^\alpha - \sigma_1^\alpha + (\sigma_2 - \sigma_1)^\alpha) + A + \frac{M M_{B r}}{\Gamma(1+\alpha)} (\sigma_2 - \sigma_1)^\alpha,
\]
where
\[
A = \left\| \int_0^{\sigma_1} (\sigma_1-s)^{\alpha-1} [K_\alpha(\sigma_2-s) - K_\alpha(\sigma_1-s)] B u(s) ds \right\| .
\]
Since \( \epsilon > 0 \) is small enough, we have
Then we have \( \sigma \) and this expression tends to zero as \( \sigma \to \sigma_1 \), which yields that

\[
\text{If } B/\Omega \text{, which means that}
\]

and this expression tends to zero as \( \sigma_2 \to \sigma_1 \) due to the continuity of \( K_\alpha(t) \) \( (t > 0) \) in the uniform operator topology. It follows from the Arzela-Ascoli theorem \[29\] that the operator \( N \) is precompact. Thus, \( H \) is strongly continuous, which guarantees the existence of optimal control to the minimum problem \((3.1)\) under the fact that \( U_b \) is a closed and convex set.

Further, if the system \((1.1)\) is approximately controllable in \( \omega \) at time \( b \), for any \( z_b \in \omega \), suppose that \( J(u^*) = \inf_u J(u) = \varepsilon < \infty \), by the definition of infimum, we can deduce that there exists a sequence \( \{u_i\}_{i=1,2,...} \) such that \( p_\omega z(b, u_i) = z_b, u_i \in U_b \subseteq L^2(0,b; R^p)(i = 1,2,3,...) \) and \( J(u_i) \to J(u^*) \). Then we have \( u_i \to u^* \) in \( L^2(0,b; R^p) \).

For any \( t \in [0,b] \), by Definition 2.2 and Lemma 2.1,

\[
\|p_\omega z(t, u^*) - p_\omega z(t, u_i)\| = \left\| p_\omega \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) B(u^*(s) - u_i(s)) \text{ds} \right\|
\leq \left\| \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s) B(u^*(s) - u_i(s)) \text{ds} \right\|
\leq \frac{\alpha M M_B}{\Gamma(1 + \alpha)} \int_0^t (t-s)^{\alpha-1} \|u^* - u_i\|_{L^2(0,b; R^p)} \text{ds},
\]

which yields that

\[ p_\omega z(t, u_i) \to p_\omega z(t, u^*) \text{ in } C(0,b,\omega) \text{ as } i \to \infty. \]

And since \( U_b \) is closed and convex, from Marzur Lemma \[29\] we see that \( u^* \in U_b \). Thus it follows from the Balder’s theorem in \[3.1\] that

\[ \varepsilon = J(u^*) = \lim_{i \to \infty} J(u_i) \geq J(u^*) \geq \varepsilon, \]

which means that \( u^* \) is the optimal solution of the minimization problem \((3.1)\) and the proof is complete. \( \square \)

### 3.2. The case of \( B \notin \mathcal{L}(R^p, L^2(\Omega)) \)

If \( B \notin \mathcal{L}(R^p, L^2(\Omega)) \), we see that the operator \( N \) defined in Eq. (23) is unbounded and \( N \) is not relatively compact. Then Theorem 3.1 fails and new methods should be introduced. Moreover, we note that this case is
also rich in physical systems, for example, when the actuator is pointwise or boundary actuator.

Here we will use the HUMs, which was first introduced by Lions in [21] to study the controllability problems of a linear distributed parameter systems. In order to study the regional controllability of (1), the following two assumptions are needed:

(A1) $B$ is a densely defined operator from $\mathbb{R}^p$ to $L^2(\Omega)$ and $B^*$ exists.

(A2) $(BK_\alpha(t))^*$ exists and $(BK_\alpha(t))^* = K_{\alpha}^*(t)B^*$, $t \in [0, b]$.

Take into account that the system (1) is line, by Definition 2.6, the system (1) is regionally approximately (exactly) controllable in $\omega$ at time $b$ if and only if

\[ Im(p_\omega H) = L^2(\omega) \]  \hspace{0.5cm} \text{(3.4)}

Suppose that $\{\Phi^*(t)\}_{t \geq 0}$, generated by the adjoint operator of $A$, is also a $C_0$ semigroup on $L^2(\Omega)$. For any $v \in L^2(\Omega)$, it follows from $\langle Hu, v \rangle = \langle u, H^*v \rangle$ and the assumptions (A1) - (A2) that

\[ H^*v = B^*(b - t)^{\alpha - 1}K_{\alpha}^*(b - t)v, \]  \hspace{0.5cm} \text{(3.5)}

where $\langle \cdot, \cdot \rangle$ is the duality pairing of space $L^2(\Omega)$, $B^*$ is the adjoint operator of $B$ and $K_{\alpha}^*(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta)\Phi^*(t^\alpha \theta)d\theta$. Then we have $Im(p_\omega H) = L^2(\omega)$ is equivalent to

\[ Ker(H^*p_\omega) = \{0\}, \]  \hspace{0.5cm} \text{(3.6)}

where $p_\omega : L^2(\omega) \to L^2(\Omega)$, the adjoint operator of $p_\omega$, is given by

\[ p_\omega^*z(x) = \begin{cases} z(x), & x \in \omega, \\ 0, & x \in \Omega \setminus \omega. \end{cases} \]  \hspace{0.5cm} \text{(3.7)}

Next, we shall explore the adjoint system of the system (1) and then use it to analyze the regional controllability of problem (1).

Denote $A^*$ by the adjoint operator of $A$ and for any $\varphi^0 \in L^2(\Omega)$, consider the following system

\[ \begin{cases} tD^\alpha_0Q\varphi(t) = -A^*Q\varphi(t), \\ \lim_{t \to b^-} tI^1_0 - \alpha Q\varphi(t) = \varphi^0 \in D(A^*) \subseteq L^2(\Omega). \end{cases} \]  \hspace{0.5cm} \text{(3.8)}

It follows from Lemma 2.2 that (3.8) can be rewritten as

\[ \begin{cases} \varphi^0 \in D(A^*) \subseteq L^2(\Omega) \\ \lim_{t \to 0^+} tI^1_0 - \alpha \varphi(t) = \varphi^0 \end{cases} \]  \hspace{0.5cm} \text{(3.9)}

and with the solution given by $\varphi(t) = -t^{\alpha - 1}K_{\alpha}^*(t)\varphi^0$. Moreover, we have the following lemma.

**Lemma 3.1.** When $u$ spans space $L^2(0, b; \mathbb{R}^p)$, then the solution $z(b, u)$ is dense in $L^2(\Omega)$.
Proof. On the contrary, if \( z(b, u) \) is not dense in \( L^2(\Omega) \), then there exists an element \( \varphi^0 \in L^2(\Omega), \varphi^0 \neq 0 \) such that

\[
(z(b, u), \varphi^0) = 0 \quad \text{for all} \quad u \in L^2(0, b; \mathbb{R}^p).
\]  

(3.10)

Multiplying both sides of (3.8) by \( z(t) \) and integrating in \( Q \) gives

\[
\int_{\Omega} \int_0^b z(t) D_b^\alpha Q \varphi(t) dt dx = \int_0^b (z(t), -A^* Q \varphi(t)) dt = -\int_0^b (A z(t), Q \varphi(t)) dt.
\]

From Lemma 2.3 we see that

\[
\int_{\Omega} \int_0^b z(t) D_b^\alpha Q \varphi(t) dt dx = (z(b, u), \lim_{t \to b} I_b^{1-\alpha} Q \varphi(t)) - (z_0, \lim_{t \to 0} I_b^{1-\alpha} Q \varphi(t))
\]

\[
- \int_0^b (Q \varphi(t), C_t D_t^\alpha z(t)) dt.
\]

It follows from \( z_0 = 0 \) that

\[
(z(b, u), \varphi^0) = \int_0^b (Q \varphi(t), B u(t)) dt.
\]  

(3.11)

By (3.10), since \( u \in L^2(0, b; \mathbb{R}^p) \) is arbitrary, we have \( Q \varphi(t) = \varphi(b - t) \equiv 0 \) in \( L^2(\Omega) \) for all \( t \in [0, b] \), then \( \varphi^0 = 0 \), a contradiction. The proof is complete.

By Lemma 3.1, we then conclude that the system (3.8) is the adjoint system of the problem (1.1). Next, we shall try to explore the regional controllability of the system (1.1) based this duality lemma and the HUMs.

Let \( Z = Im(p_\omega H) \subseteq L^2(\omega) \), by duality \( Z \subseteq L^2(\omega) \subseteq Z^* \) and for any \( f \in Z^* \), define

\[
\|f\|_{Z^*} := \int_0^b \|B^*(b - s)^{\alpha - 1} K_\alpha^*(b - s) p_*^\omega f\|^2 ds,
\]  

(3.12)

where \( p_*^\omega \) is defined in Eq. (3.7).

**Lemma 3.2.** \( \| \cdot \|_{Z^*} \) is a norm of space \( Z^* \) provided that the system (1.1) is approximately controllable in \( \omega \) at time \( b \).

Proof. If the system (1.1) is approximately controllable in \( \omega \) at time \( b \), we get that \( Ker(H^* p_*^\omega) = \{0\} \), i.e.,

\[
B^*(b - s)^{\alpha - 1} K_\alpha^*(b - s) p_*^\omega f = 0 \Rightarrow f = 0.
\]  

(3.13)

Hence, for any \( f \in Z^* \), it follows from
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\[ \| f \|_{Z^*} = \int_0^b \| B^*(b-s)\alpha^{-1}K_\alpha^*(b-s)p_\omega^*f \|^2 ds = 0 \Leftrightarrow \]
\[ \Leftrightarrow B^*(b-s)\alpha^{-1}K_\alpha^*(b-s)p_\omega^*f = 0 \]
that \( \| \cdot \|_{Z^*} \) is a norm of space \( Z^* \) and the proof is complete. \( \square \)

Denote the completion of the set \( Z^* \) with the norm \( \| \cdot \|_{Z^*} \) again by \( Z^* \).

For each \( f \in Z^* \), since \( f \) is a linear bounded functional on \( Z \), by the Riesz representation theorem, there exists a unique element in \( L^2(\Omega) \), denoted by \( Pf \), such that
\[ f(y) = (Pf, y) \text{ for all } y \in Z. \tag{3.14} \]

Then \( P : Z^* \to Z \) is a linear operator and the following lemma holds.

**Lemma 3.3.** The operator \( P : Z^* \to Z \) is isometry.

**Proof.** For any \( f \in Z^* \), it follows from (3.14) that
\[ \| Pf \|_Z = \sup_{\| y \|_Z = 1} (Pf, y) = \sup_{\| y \|_Z = 1} \| f(y) \| = \| f \|_{Z^*}. \]

Then \( \text{Im}(P) \subseteq Z \) is a closed subspace. To complete the proof, we should only show that \( \text{Im}(P) = Z \). If not so, there exists a \( y_0 \in Z \), \( y_0 \neq 0 \) such that \( (Px, y_0) = 0 \). By (3.14), we have
\[ f(y_0) = 0 \text{ for all } f \in Z^*, \]
which implies that \( y_0 = 0 \), a contradiction. Then the proof is complete. \( \square \)

Further, let \( \Lambda : Z^* \to Z \) be
\[ \Lambda f = p_\omega \varphi_1(b), \tag{3.15} \]
where \( \varphi_1(t) \) is defined by
\[ \begin{cases}
C_0D_t^\alpha \varphi_1(t) = A\varphi_1(t) + BB^*(b-t)\alpha^{-1}K_\alpha^*(b-t)f, \\
\varphi_1(0) = 0.
\end{cases} \tag{3.16} \]

Since for any \( f \in Z^*, y \in Z \), by Hölder’s inequality, we have
\[ (\Lambda f, y) = \]
\[ = \int_\Omega p_\omega \int_0^b (b-s)\alpha^{-1}K_\alpha(b-s)BB^*(b-s)\alpha^{-1}K_\alpha^*(b-s)p_\omega^*f(x)dy(x)dx \]
\[ \leq \int_0^b \| B^*(b-s)\alpha^{-1}K_\alpha^*(b-s)p_\omega^*f \|^2 ds \| y \| \leq \| f \|_{Z^*} \| y \| \]
and \( \| \Lambda f \| \leq \| f \|_{Z^*} \). Further, for any \( f \in Z^* \), we have

\[
\langle \Lambda f, f \rangle = \int_0^b \int_{\Omega} p_\omega (b-s)^{\alpha-1} K_\alpha(b-s)B B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega f(x) ds f(x) dx
\]

\[
= \int_0^b \int_{\Omega} \left[ B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega f(x) \right]^2 dx ds.
\]

Then if the system (1.1) is regionally approximately controllable in \( \omega \) at \( b \), we get that \( f = 0 \). Thus it follows from the uniqueness of \( P \) that \( \Lambda \) is an isomorphism from \( Z^* \) to \( Z \).

Next, suppose that \( \varphi_0(t) \) satisfies

\[
\begin{cases}
  C_0 D_t^\alpha \varphi_0(t) = A \varphi_0(t), \\
  \varphi_0(0) = z_0 \in D(A),
\end{cases}
\] (3.17)

for all \( z_0 \in L^2(\omega) \), we have \( z_0 = p_\omega [\varphi_1(b) + \varphi_0(b)] \). Further, let \( f \) be the solution of the following equation

\[
\Lambda f := z_0 - p_\omega \varphi_0(b).
\] (3.18)

Then we are ready to state the following theorem.

**Theorem 3.2.** If the system (1.1) is regionally approximately controllable in \( \omega \) at time \( b \), then for any \( z_0 \in L^2(\omega) \), (3.18) has a unique solution \( f \in Z^* \) and the control

\[
u^* = B^*(b-\cdot)^{\alpha-1} K_\alpha^*(b-\cdot) p_\omega f
\]

steers the system to \( z_0 \) at time \( b \) in \( \omega \). Moreover, \( u^* \) solves the minimization problem (3.1).

**Proof.** By Lemma 3.1, we get that if the system (1.1) is regionally approximately controllable in \( \omega \) at time \( b \), then \( \| \cdot \|_{Z^*} \) is a norm of space \( Z^* \). Let the completion of \( Z^* \) with respect to the norm \( \| \cdot \|_{Z^*} \) again by \( Z^* \). Then next we show that the equation (3.18) has a unique solution in \( Z^* \).

For any \( f \in Z^* \), by the definition of operator \( \Lambda \) in (3.15), we get that

\[
\langle f, \Lambda f \rangle = \langle f, p_\omega \varphi_1(b) \rangle
\]

\[
= \int_0^b \int_{\Omega} \langle f, p_\omega (b-s)^{\alpha-1} K_\alpha(b-s)Bu^*(s) \rangle ds
\]

\[
= \int_0^b \| B^*(b-s)^{\alpha-1} K_\alpha^*(b-s) p_\omega f \|^2 ds = \| f \|_{Z^*}^2.
\]
Hence, it follows from Theorem 2.1 in [20] that the equation (3.18) admits a unique solution in $Z^*$. Further, let $u = u^*$ in problem (1.1), we see that $p_\omega z(b, u^*) = z_b$.

For any $u_1 \in L^2(0, b, \mathbb{R}^p)$ with $p_\omega z(b, u_1) = z_b$, we obtain that

$$p_\omega [z(b, u^*) - z(b, u_1)] = 0,$$

and for any $f \in Z^*$ we have

$$\langle f, p_\omega [z(b, u^*) - z(b, u_1)] \rangle = 0.$$

It follows that

$$\int_0^b \langle u^*(s) - u_1(s), B^*(b - s)^{\alpha - 1} K_\alpha^*(b - s)p_\omega f \rangle ds = 0.$$

Moreover, since

$$J'(u^*)(u^* - u_1) = 2 \int_0^b \langle u^*(s), u^*(s) - u_1(s) \rangle ds$$

$$= 2 \int_0^b \langle B^*(b - s)^{\alpha - 1} K_\alpha^*(b - s)p_\omega f, u^*(s) - u_1(s) \rangle ds$$

$$= 0,$$

by Theorem 1.3 in [20], we conclude that $u^*$ solves the minimum energy problem (3.1) and the proof is complete.

4. Examples

In this section, two examples which are reachable on a sub-region but not on the whole domain are introduced with $B \in \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$ and $B^* \in \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$, respectively.

**Example 4.1.** Let us consider the following one dimension fractional order sub-diffusion system with a zone actuator $Bu = p_{[a_1, a_2]}u$, $0 \leq a_1 \leq a_2 \leq 1$ and $B \in \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$:

$$\begin{cases}
\mathcal{C}D_0^{1/2}z(x, t) = \frac{\partial^2}{\partial x^2}z(x, t) + p_{[a_1, a_2]}u(t) \text{ in } [0, 1] \times [0, b], \\
z(x, 0) = z_0 \text{ in } [0, 1], \\
z(0, t) = z(1, t) = 0 \text{ in } [0, b].
\end{cases} \quad (4.1)$$

We see that $B$ is a bounded continuous operator with $M_B = 1$, $A = \frac{\partial^2}{\partial x^2}$ and $\Phi(t)z(x) = \sum_{i=1}^\infty \exp(\lambda_i t)(z, \xi_i(x))\xi_i(x)$, $x \in [0, 1]$, where

$$\lambda_i = -i^2 \pi^2 \quad \text{and} \quad \xi_i(x) = \sqrt{2}\sin(i\pi x), \ i = 1, 2, \cdots, \ x \in [0, 1].$$

Then $\{\Phi(t)\}_{t \geq 0}$ generated by $A$ is uniformly bounded with $M = 1$. Further, we have
\[ K_{0.7}(t)z(x) = 0.7 \int_0^\infty \theta \phi_{0.7}(\theta) \Phi(t^{0.7} \theta) z d\theta \]
\[ = 0.7 \sum_{i=1}^\infty \int_0^\infty \theta \phi_{0.7}(\theta) \exp(\lambda_i t^{0.7} \theta) d\theta (z, \xi_i) \xi_i(x) \]
\[ = 0.7 \sum_{i=1}^\infty \sum_{j=0}^\infty \int_0^\infty (\lambda_i t^{0.7})^j \theta^{j+1} \phi_{0.7}(\theta) d\theta (z, \xi_i) \xi_i(x) \]
\[ = \sum_{i=1}^\infty E_{0.7,0.7}(\lambda_i t^{0.7})(z, \xi_i) \xi_i(x), \tag{4.2} \]

where \( E_{\alpha,\beta}(z) := \sum_{i=0}^{\infty} \frac{z^i}{i!(\alpha+\beta)} \), \( \text{Re} \alpha > 0, \beta, z \in \mathbb{C} \) is known as the generalized Mittag-Leffler function in two parameters. Similarly, one has

\[ S_{0.7}(t)z(x) = \int_0^\infty \phi_{0.7}(\theta) \Phi(t^{0.7} \theta) d\theta = \sum_{i=1}^\infty (z, \xi_i) E_{0.7,1}(\lambda_i t^{0.7}) \xi_i(x). \tag{4.3} \]

What is more, since \( A = \frac{\partial^2}{\partial x^2} \) is a self-adjoint operator, we have

\[ (H^*z)(t) = B^*(b - t)^{-0.3} K_{0.7}^*(b - t)z(t) \]
\[ = (b - t)^{-0.3} \sum_{i=1}^\infty E_{0.7,0.7}(\lambda_i(b - t)^{0.7})(z, \xi_i) \int_{a_1}^{a_2} \xi_i(x) dx. \]

Then from
\[ \int_{a_1}^{a_2} \xi_i(x) dx = \frac{\sqrt{2}}{i\pi} \sin \frac{i\pi(a_1 + a_2)}{2} \sin \frac{i\pi(a_1 - a_2)}{2} \]

it follows that
\[ \text{Ker} (H^*) \neq \{0\} \quad (\text{Im} (H) \neq L^2(\omega)) \]

when \( a_2 - a_1 \in Q \), i.e., the system (4.1) is not weakly controllable when \( a_2 - a_1 \in Q \).

Next, we show that there exists a sub-region \( \omega \subseteq \Omega \) such that the system (4.1) is regionally controllable in \( \omega \) at time \( b \). For example, let \( a_1 = 0, a_2 = 1/2, z_\ast = \xi_k, (k = 4j, j = 1, 2, 3, \cdots) \). Obviously, \( z_\ast \) is not reachable on \( \Omega = [0, 1] \). However, since

\[ E_{\alpha,\alpha}(t) > 0 \quad (t \geq 0) \quad \text{and} \quad \int_0^{1/2} \xi_i(x) dx = \frac{\sqrt{2}}{i\pi} (1 - \cos(i\pi/2)) \quad , \quad i = 1, 2, \cdots, \]

let \( \omega = [1/4, 3/4] \), we see that

\[ (H^* p_{\omega} p_{\omega} z_\ast)(t) = \sum_{i=1}^\infty \frac{E_{0.7,0.7}(\lambda_i(b - t)^{0.7})}{(b - t)^{0.3}} (\xi_i, \xi_k)_{L^2(1/4, 3/4)} \int_0^{1/2} \xi_i(x) dx \]
\[ = \sum_{i \neq 4j} \frac{\sqrt{2} E_{0.7,0.7}(\lambda_i(b - t)^{0.7})}{i\pi(b - t)^{0.3}} \int_{1/4}^{3/4} \xi_i(x) \xi_{4j}(x) dx \left[ 1 - \cos(i\pi/2) \right] \neq 0. \]
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Then $z_*$ is regionally controllable on $\omega = [1/4, 3/4]$ at time $b$.

Moreover, according to Theorem 3.1, if the system (4.1) is approximately controllable in $\omega = [1/4, 3/4]$ at time $b$, then the minimum energy problem (3.1) admits at least one optimal solution.

Example 4.2. Consider the following fractional order sub-diffusion system with a pointwise actuator $Bu = u(t)\delta(x - \sigma)$, $0 < \sigma < 1$. Obviously, $B \notin \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$ is unbounded.

$$\begin{cases}
\frac{C}{6} D_t^{0.7} z(x, t) = \frac{\partial^2}{\partial x^2} z(x, t) + u(t)\delta(x - \sigma) \text{ in } [0, 1] \times [0, b], \\
z(x, 0) = z_0 \text{ in } [0, 1], \\
z(0, t) = z(1, t) = z_0 \text{ in } [0, b],
\end{cases}
$$

(4.4)

Similar to the argument above, we see that $(H^* z)(t) = B^*(b - t)^{-0.3} K_0^{*}(b - t)z(t)$

$$= \sum_{i=1}^{\infty} E_{0.7, 0.7}(\lambda_i(b - t)^{0.7})(z, \xi_i)\xi_i(\sigma).$$

Then the system (4.4) is not weakly controllable in $\Omega$ if $\sigma \in Q$. Moreover, let $\sigma = 1/2$, by the argument above, there exists a sub-region $\omega = [1/4, 3/4] \subseteq \Omega$ such that the system (4.4) is regionally controllable in $\omega$ at time $b$.

Further, since $A = \frac{\partial^2}{\partial x^2}$ is a self-adjoint operator, suppose that the system (4.4) is approximately controllable in $\omega$ at time $b$, by Lemma 3.1, we get that

$$f \to \|f\|_{Z^*} = \int_0^b \left\| (b - s)^{-0.3} K_0^{*}(b - s)p_i^* f(\sigma) \right\|^2 ds$$

$$= \int_0^b \left\| (b - t)^{-0.3} \sum_{i=1}^{\infty} E_{0.7, 0.7}(\lambda_i(b - t)^{0.7})(z, \xi_i)p_i^* f(\sigma) \right\|^2 ds$$

defines a norm on $Z^*$. It follows from Lemma 3.2 that

$$\Lambda f = p_\omega \varphi_1(\cdot, \sigma),$$

(4.5)

is a isometry form $Z^*$ to $Z$, where $\varphi_1(x, t)$ is the solution of the following system

$$\begin{cases}
\frac{C}{6} D_t^{0.7} \varphi_1(x, t) = \frac{\partial^2}{\partial x^2} \varphi_1(x, t) + (b - t)^{-0.3} K_0^{*}(b - t)f(\sigma), \\
\varphi_1(x, 0) = 0, \\
\varphi_1(0, t) = \varphi_1(1, t) = 0.
\end{cases}
$$

(4.6)

Then by Theorem 3.2, we see that the control
\[ u^*(t) = (b - t)^{-0.3} \sum_{i=1}^{\infty} E_{0,7,0.7}(\lambda_i(b - t)^{0.7})(z, \xi_i)p_{\omega}^*f(\sigma) \]

steers the system to \( z_b \) at time \( b \) in \( \omega \), where \( f \) is the solution of equations

\[ Af = z_b - p_{\omega} \varphi_0(\cdot, \sigma), \quad (4.7) \]

and \( \varphi_0(x, t) \) solves

\[
\begin{cases}
\mathcal{C}_0 D_t^{0.7} \varphi_0(x, t) = \frac{\partial^2}{\partial^2 x} \varphi_0(x, t), \\
\varphi_0(x, 0) = z_0(x) \in D(A), \\
\varphi_0(0, t) = \varphi_0(1, t) = 0.
\end{cases}
\quad (4.8)
\]

Moreover, \( u^* \) is the solution of the minimum energy problem \( (3.1) \).

5. Conclusions

This paper deals with the regional controllability problems of the fractional sub-diffusion equations with Caputo fractional derivative in two different cases: \( B \in \mathcal{L}(\mathbb{R}^m, L^2(\Omega)) \) and \( B / \in \mathcal{L}(\mathbb{R}^m, L^2(\Omega)) \). The duality result of the fractional sub-diffusion is also derived at the same time. The results here can be regarded as the extensions of the results in \([7, 8, 39]\) and can also be extended to complex fractional order distributed parameter systems. For instance, the problem of regional gradient controllability/observability of fractional order distributed parameter systems as well as the case of fractional order super-diffusion systems with more complicated dynamics are of great interest. For more information on the potential topics related to fractional DPSs, we refer the readers to \([9]\) and the references therein.

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