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Continuous fractional sliding mode-like control for exact rejection of non-differentiable Hölder disturbances

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Exploiting algebraic and topological properties of differintegral operators as well as a proposed principle of dynamic memory resetting, a uniform continuous sliding mode controller for a general class of integer order affine non-linear systems is proposed. The controller rejects a wide class of disturbances, enforcing in finite-time a sliding regime without chattering. Such disturbance is of Hölder type that is not necessarily differentiable in the usual (integer order) sense. The control signal is uniformly continuous in contrast to the classical (integer order) discontinuous scheme that has been proposed for both fractional and integer order systems. The proposed *principle of dynamic memory resetting* allows demonstrating robustness as well as: (i) finite-time convergence of the sliding manifold, (ii) asymptotic convergence of tracking errors, and (iii) exact disturbance observation. The validity of the proposed scheme is discussed in a representative numerical study.

Keywords: fractional-order control; sliding mode control; non-differentiable Hölder disturbances.

1. Introduction

Fractional Calculus dates back to the early years of the birth of Calculus, however, there have been reported the first few studies in modeling and control of dynamical systems only about two decades ago (Oustaloup *et al.*, 1995; Podlubny, 1999b), with hundreds of control application (Muñoz-Vázquez *et al.*, 2014a,b). Fractional Calculus provides novel and accurate methodologies to model some complex phenomena that conveys an understanding to reality considering properties such as memory and heritage (Podlubny, 1999a); these properties allow to handle in the control design some non-local and non-smooth physical phenomena, which in practice are more widespread than the smooth ones (Clarke, 1998). However, the integer order differentiability notion cannot be directly used to characterize non-smooth dynamics nor to design robust strategies to reject non-differentiable disturbances. Moreover, some non-differentiable functions have well-posed fractional derivatives, and this property can be considered to design chatter-less control strategies in virtue of such functions can be studied by using the notion of fractional differentiation (Ross *et al.*, 1994).

During the last two decades, continuous high-order sliding mode controllers have been proposed, (Emelyanov *et al.*, 1996; Dávila *et al.*, 2009; Levant & Michael, 2009; Moreno & Osorio, 2012), to get rid of chattering, however requiring differentiability at almost every point. Notice that non-differentiable disturbances are difficult to handle with high-order sliding mode control techniques (Levant & Michael, 2009; Moreno & Osorio, 2012), due to these require differentiability (at least in a weak sense) of the disturbance effects. Moreover, when conventional (Utkin, 1992), or integral, sliding mode control is considered (Utkin & Shi, 1996), difficulties arise to compensate those disturbances since the chattering phenomenon produces harmful effects. However, phenomena such as fractional noise, backlash, and turbulence can be represented by fractional differentiable functions (as solutions of fractional-order models) (Mandelbrot & Van Ness, 1968; Humphrey *et al.*, 1992; Barbosa & Tenreiro-Machado, 2002; Tenreiro-Machado, 2013). These effects can be modeled by functions that are not necessarily differentiable in the common (integer order) sense but comply, by construction, to the Hölder condition to conform a more general class of functions than Lipschitz ones. Notice that the fractional derivative considers an interval of non-zero measure that stores memory and provides heritage properties, instead of considering a single point such as in the case of an integer order derivative. Along this direction, we propose a low-frequency fractional order sliding mode controller to guarantee robustness against a wider variety of disturbances and uncertainties, which can be characterized by the Hölder continuity condition.

Fractional Calculus and sliding mode schemes have been combined to produce fractional order reaching laws for control of second-order dynamical systems in the pioneer works Vinagre & Calderón (2006) and Önder-Efe (2011), nevertheless the basic properties such as stability, robustness and finite-time convergence have not been fully demonstrated in general. Control of fractional order systems by means of fractional order sliding surfaces with an integer order reaching phase have been proposed, (Dadras & Momeni, 2013; Gao & Liao, 2013; Kamal *et al.*, 2013), with discontinuous control signals to assure stability with finite-time convergence of the sliding manifold, involving undesirable chattering effects. Also, in our previous works Muñoz-Vázquez *et al.* (2014a,b), the control of integer order plants is proposed with integer order reaching laws that induce fractional order sliding dynamics, with extended and improved dynamical features.

In contrast to these schemes, we address in this paper the design of a chatter-less and fractional order sliding mode controller for non-linear systems subject to Hölder disturbances. It is shown that a fractional order reaching phase is induced such that the control signal is uniformly continuous and preserves the order of the system in the sliding motion. In contrast to these schemes, and by taking advantage of the algebraic and topological properties of the differintegral operators, such as memory and heritage, in this paper we address the design of a novel fractional order sliding mode controller for non-linear systems subject to Hölder disturbances, where the following salient features can be enlisted:

- A fractional order controller for integer order systems
- A continuous sliding mode based controller which guarantees a finite-time sliding regime
- Robustness against Hölder continuous disturbances (not necessarily differentiable)
- Exact invariance of both the sliding manifold and its derivative
- Exact estimation of disturbances and uncertainties

The rest of this paper is organized as follows: Section 2 introduces some preliminaries of differintegral operators as well as on Hölder spaces. Section 3 presents the methodology to design fractional order sliding mode controllers, with remarks provided in Section 4. Section 5 shows a numerical study for a representative second-order system, and finally, conclusions are discussed in Section 6.

2. Preliminaries

2.1 *On differintegral operators*

Consider the following differintegral operators (Podlubny, 1999a):

- *Riemann–Liouville fractional integral*

$$ {}_{t_0}I_t^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\nu}} \, \mathrm{d}\tau \tag{2.1} $$

- *Riemann–Liouville fractional derivative*

$$ {}_{t_0}D_t^{\nu} f(t) = \frac{\mathrm{d}^{\lceil \nu \rceil}}{\mathrm{d}t^{\lceil \nu \rceil}} \, {}_{t_0}I_t^{\lceil \nu \rceil - \nu} f(t) \tag{2.2} $$

- *Caputo fractional derivative*

$$ {}_{t_0}^{C}D_t^{\nu} f(t) = {}_{t_0}I_t^{\lceil \nu \rceil - \nu} f^{(\lceil \nu \rceil)}(t) \tag{2.3} $$

- *Grünwald–Letnikov differintegral*

$$ {}_{t_0}D_t^{\nu} f(t) = \lim_{h \to 0} h^{-\nu} \sum_{k=0}^{\lfloor (t-t_0)/h \rfloor} w_k^{\nu} f(t-kh) \tag{2.4} $$

  where $\lceil \nu \rceil = \min\{x \in \mathbb{Z} : x \geqslant \nu\}$ and $\lfloor \nu \rfloor = \max\{x \in \mathbb{Z} : x \leqslant \nu\}$ are the ceil and floor functions, respectively,

$$ \Gamma(x) = \int_0^{\infty} z^{x-1} e^{-z} \, \mathrm{d}z $$

  is the Euler Gamma function, and w_k^{ν} are coefficients which for numerical purposes can be computed recursively as $w_0^{\nu} = 1$ and $w_k^{\nu} = (1 - (\nu+1)/k) w_{k-1}^{\nu}$.

2.2 *On Hölder Spaces*

A Hölder space is a set of functions that comply to the following Hölder condition (Samko *et al.*, 1993),

$$ |\varphi(t_2) - \varphi(t_1)| \leqslant H |t_2 - t_1|^{\nu}. \tag{2.5} $$

Thus, the function $\varphi : \Omega \to \mathbb{R}$ is called Hölder continuous on a bounded interval $\Omega \subset \mathbb{R}$ for the Hölder fractional exponent $\nu \in (0,1)$ if $\exists H > 0$ such that $\forall t_1, t_2 \in \Omega$, the inequality (2.5) holds. Also, the critical exponent of φ is defined as the maximum of such ν such that φ complies to (2.5). Now, the following definition is useful.

DEFINITION 2.1 The Hölder space $\mathscr{H}^{\nu}(\Omega)$ is the set of those functions that comply to the continuity condition (2.5) over Ω.

It is clear that for $\nu = 1$, (2.5) implies the well-known Lipschitz condition, which is related with the differentiability of a function in the distributional sense. In addition, for every $\nu > 0$, the Hölder condition implies uniform continuity since for $\epsilon > 0$, $|t_2 - t_1| < \delta \Rightarrow |\varphi(t_2) - \varphi(t_1)| < \epsilon$ in virtue of $\delta \leqslant (\epsilon/H)^{1/\nu}$.

The space $\mathscr{H}^{\nu}(\Omega)$ is equipped with the norm

$$\|\varphi(t)\|_{\mathscr{H}^{\nu}(\Omega)} = \sup_{t\in\Omega}|\varphi(t)| + \sup_{\substack{t_1,t_2\in\Omega\\ t_1\neq t_2}} \frac{|\varphi(t_2)-\varphi(t_1)|}{|t_2-t_1|^{\nu}}, \tag{2.6}$$

for $\varphi \in \mathscr{H}^{\nu}(\Omega)$. Henceforth, we have the following Lemma.

LEMMA 2.1 $\mathscr{H}^{\nu}(\Omega)$ is a Banach space.

Thus, a direct consequence of Definition 2.1 and Lemma 2.1 is the following proposition.

PROPOSITION 2.1 Let $f \in \mathscr{H}^{\nu}(\Omega)$ and $g \in \mathscr{H}^{\eta}(\Omega)$, with Ω a bounded interval of $\mathbb{R}$ and $\nu, \eta \in (0,1)$ with $\nu \leqslant \eta$. Then, $f+g, fg \in \mathscr{H}^{\nu}(\Omega)$. Also, for $f \in \mathscr{H}^{\nu}(g[\Omega])$ and $g \in \mathscr{H}^{\eta}(f[\Omega])$, we have that $f\circ g, g\circ f \in \mathscr{H}^{\nu\eta}(\Omega)$, where $\circ$ is the composition operator.

It is of interest to notice that the fractional integral improves the topological properties of a locally Lebesgue integrable function as it is established in the following Lemma (Samko *et al.*, 1993).

LEMMA 2.2 For a function $f \in \mathscr{H}^{\lambda}(\Omega)$, with $0 \leqslant \lambda < 1$, and some $\nu \in (0, 1-\lambda)$, we have that

$${}_{t_0}I_t^{\nu} f(t) = \frac{f(t_0)}{\Gamma(1+\nu)}(t-t_0)^{\nu} + \frac{1}{\Gamma(\nu)}\int_{t_0}^{t} \frac{f(\tau)-f(t_0)}{(t-\tau)^{1-\nu}}\,\mathrm{d}\tau,$$

where $\int_{t_0}^{t}((f(\tau)-f(t_0))/(t-\tau)^{1-\nu})\,\mathrm{d}\tau \in \mathscr{H}^{\lambda+\nu}(\Omega)$.

An important consequence of Lemma 2.2 is the following Corollary (Samko *et al.*, 1993).

COROLLARY 2.1 The fractional integral operator of order $\nu \in (0,1)$ maps the set of bounded functions $\mathscr{H}^{0}(\Omega)$ into the set of Hölder continuous functions $\mathscr{H}^{\nu}(\Omega)$, with Ω a bounded interval of $\mathbb{R}$.

On the other hand, the Hölder condition is related intrinsically with the differentiability of fractional order. To see this, consider the following proposition.

PROPOSITION 2.2 Consider $\varphi : \Omega \subset \mathbb{R} \to \mathbb{R}$, with Ω a bounded interval. If $\sup |{}_{t_1}^{C}D_t^{\nu}\varphi(t)|_{t=t_2} = k_{\varphi} \in \mathbb{R}$, with $\nu \in (0,1)$, then $|\varphi(t_2)-\varphi(t_1)| \leqslant (k_{\varphi}/\Gamma(\nu+1))|t_2-t_1|^{\nu}$ and $\sup |{}_{t_1}^{C}D_t^{\nu}\varphi(t)|_{t=t_2} \leqslant \|\varphi(t)\|_{\mathscr{H}^{\nu}(\Omega)}$.

The proof of Proposition 2.2 is based on the monotonicity of the fractional integral and the Hölder space norm. Henceforth, the Hölder continuity is a necessary condition for fractional order differentiability, which leads to claim that Hölder continuity is intrinsically related with the fractional differentiability of a function.

In case of the Riemann–Liouville derivative given in definition (2.1), the fractional differentiability of a function throughout the Hölder condition can be analyzed in connection with the Marchaud fractional derivative as follows (Samko *et al.*, 1993; Ross *et al.*, 1994):

$${}_{t_0}^{M}D_t^{\nu}\varphi(t) = \frac{\varphi(t)}{\Gamma(1-\nu)(t-t_0)^{\nu}} + \frac{\nu}{\Gamma(1-\nu)}\int_{t_0}^{t} \frac{\varphi(t)-\varphi(\tau)}{(t-\tau)^{\nu+1}}\,\mathrm{d}\tau. \tag{2.7}$$

where the second term in the right hand side of (2.7) is a Hölder continuous function with a critical exponent $\lambda-\nu$, for a Hölder continuous function φ of critical exponent $\lambda \in (\nu, 1)$ (Samko *et al.*, 1993). Equation (2.7) indeed extends the formulation of Riemann–Liouville (2.1) for Hölder continuous functions of critical exponents greater than ν (Corollary of Theorem 13.1 of Samko *et al.*, 1993, 228 pp.).

These both operators coincide almost everywhere not just for differentiable functions but also for locally integrable and Hölder functions of critical exponents $\lambda > \nu$ (Samko *et al.*, 1993).

Despite (2.7) is a well-defined operator for Hölder continuous functions, dynamical systems described by this operator lead to initial conditions without an evident physical meaning, similar to the Riemann–Liouville derivative. Thus, we propose using

$$ {}_{t_0}^{C}D_t^{\nu}\varphi(t) = {}_{t_0}D_t^{\nu}\varphi(t) - \frac{\varphi(t_0)}{\Gamma(1-\nu)(t-t_0)^{\nu}} \tag{2.8}$$

to relate the Riemann–Liouville and Caputo operators and the Marchaud operator for locally integrable Hölder functions that leads to the following operator

$$ {}_{t_0}^{C}D_t^{\nu}\varphi(t) = \frac{\varphi(t)-\varphi(t_0)}{\Gamma(1-\nu)(t-t_0)^{\nu}} + \frac{\nu}{\Gamma(1-\nu)}\int_{t_0}^{t}\frac{\varphi(t)-\varphi(\tau)}{(t-\tau)^{\nu+1}}\,\mathrm{d}\tau, \tag{2.9}$$

which establishes the extension of the Caputo fractional derivative from the space of differentiable functions to the space of those functions without a well-posed integer order derivative (similar operator (2.9) can be obtained by integration by parts of (2.3)). Notice that operators (2.3) and (2.9) coincide for Hölder continuous and locally integrable functions with critical orders greater than ν. Thus, a real valued function $f(t)$ with a critical exponent $\lambda \in (0,1)$ has well-posed Caputo derivatives ${}_{t_0}^{C}D_t^{\nu}f(t)$ for all orders $\nu < \lambda$. However, initial conditions of (2.9) has a physical meaning and (2.9) maps $f(t)$ to a continuous function on t_0, then it is advantageous over (2.7); henceforth, we have the following lemma.

Lemma 2.3 The real valued function $f(t) \in \mathscr{H}^{\lambda}(\Omega)$, with $\lambda \in (0,1]$, has well-posed Caputo derivatives ${}_{t_0}^{C}D_t^{\nu}f(t)$ on Ω, for all orders $\nu < \lambda$.

From Lemma 2.3, we have that Hölder continuity also provides a sufficiency condition for an integrable function to be fractional order differentiable.

3. Continuous fractional order sliding mode control design

3.1 *Motivation and justification*

When the plant is of fractional order it is reasonable to propose fractional order control schemes. In Pisano *et al.* (2010), a fractional sliding mode scheme for fractional order systems subject to Lipschitz disturbances is proposed, and Kamal *et al.* (2013) addresses a discontinuous controller for fractional order plants with bounded disturbances. However, if the plant is of integer order subject to non-differentiable disturbances, it is unclear, and still an open problem, how to stabilize the plant with a chattering-free sliding mode controller. To address this problem, we firstly introduce some assumptions and the specific problem formulation, then the proposed solution is presented, and finally the main result is given with its stability analysis.

3.2 *Assumptions and problem statement*

Consider the following disturbed affine non-linear system

$$ \dot{x}(t) = f(x) + g(x)u + \xi(t,x), \tag{3.1}$$

with $x \in \mathbb{R}^n$ the state, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^{n\times m}$ sufficiently smooth functions, and $u \in \mathbb{R}^m$, with $m < n$, the control input; $\xi(x,t) \in \mathbb{R}^n$ models matched uncertainties and Hölder disturbances. We also assume the following over system (3.1).

ASSUMPTION 3.1 The disturbance-free system $\dot{x}(t) = f(x) + g(x)u$ is globally controllable.

ASSUMPTION 3.2 There exists an ideal controller $u_0 \in \mathbb{R}^m$ such that $\dot{x}(t) = f(x) + g(x)u_0$ yields $x \to 0$ as $t \to \infty$.

ASSUMPTION 3.3 There exists a function $s(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that $s(x)$ is at least two times differentiable with respect to x and $\det((\partial s/\partial x)g(x)) \neq 0\ \forall x$.

ASSUMPTION 3.4 There exists a real valued vector function $\varphi(t,x)$, not necessarily differentiable in the common sense, such that $\xi(t,x) = g(x)\varphi(t,x)$; this is, $\varphi(t,x)$ stands for an additive matched disturbance.

ASSUMPTION 3.5 The Euclidean norm of ${}_a^C D_t^\nu \varphi(t)|_{t=b}$ is uniformly bounded for all $a,\ b \in \mathbb{R}$ and $0 < \nu < \eta$, for some fractional number $\eta \in (0,1)$.

With these assumptions at hand, the control problem can be phrased as follows: '*Design a chatterless controller u for system* (3.1) *such that for a given finite-time $t_s \in \mathbb{R}$, the invariant $\dot{x}(t) - f(x) - g(x)u_0 = 0$ implies $x \to 0$ asymptotically*'.

It is worth to mention that this control problem has not been solved for a continuous sliding mode controller that proves robustness against non-differentiable disturbances $\varphi(x,t)$.

3.3 *Proposed solution and control design*

Let flow $f(x)$ and input matrix $g(x)$ be known functions such that there exists an ideal control u_0 that guarantees $x \to 0$ asymptotically for the system without disturbances,

$$\dot{x} = f(x) + g(x)u_0. \tag{3.2}$$

Similar to Utkin & Shi (1996), consider

$$u = u_0 + u_s, \tag{3.3}$$

with u_0 the nominal controller and u_s the sliding mode control that sustains a sliding mode regime in finite-time with the following sliding manifold

$$\sigma(t) = s(x(t)) - s(x(t_0)) - \int_{t_0}^{t} \frac{\partial s}{\partial x}[f(x) + g(x)u_0]\,\mathrm{d}\tau. \tag{3.4}$$

Differentiating (3.4) with respect to the time, we obtain

$$\dot{\sigma} = \frac{\partial s}{\partial x} g(x)(u_s + \varphi). \tag{3.5}$$

Also, consider the following continuous fractional order control law

$$u_s(t) = \left[\frac{\partial s}{\partial x} g(x)\right]^{-1} z(a) - \left[\frac{\partial s}{\partial x} g(x)\right]^{-1} k_a I_t^\nu \operatorname{sign}(\sigma(t)), \tag{3.6}$$

with ${}_a^C D_t^\nu z(t) = -k \operatorname{sign}(t)$, fractional order integration of the discontinuous function provides chattering alleviation still preserving robustness to Hölder not necessarily differentiable disturbances, properties not performed by any integer order sliding mode controller; $k > 0$ is a feedback gain and $a > 0$ is a lower terminal for the differintegral operator (to be defined). The substitution of (3.6) into (3.5) produces the following reaching law

$$\dot{\sigma} = z(a) - k\, {}_aI_t^{\nu}\, \mathrm{sign}(\sigma) + \frac{\partial s}{\partial x} g(x)\varphi, \tag{3.7}$$

or equivalently, using the Caputo differintegral operator, for $\nu < \eta$ and using Assumption 3.4, (3.7) becomes

$${}_a^C D_t^{1+\nu}\sigma = -k\, \mathrm{sign}(\sigma) + {}_a^C D_t^{\nu}\left[\frac{\partial s}{\partial x} g(x)\varphi\right]. \tag{3.8}$$

Before providing the stability properties of (3.8), the following discussion on finite-time converge is in order.

3.4 *On finite-time convergence of fractional order systems*

An extension of the Filipov's regularization method (Filipov, 1988) has been established (Cernea, 2010; Danca, 2011; Garrapa, 2013) in the realm of differential inclusions, which applies for fractional order discontinuous systems (a rigorous study on it is out of the scope of this paper).

Stability, finite-time convergence, and robustness for a general class of fractional order reaching phases, such as (3.8) are still open problems (Önder-Efe, 2010), essentially because the memory associated with the differintegral operators is difficult to handle. Thus, the general finite-time stability conditions for the fractional order system (3.8) is also an open problem in virtue of such system is of order $1 + \nu \in (1, 2)$. To deal with that issue due to memory, we propose a dynamic lower terminal in (3.8) throughout a *resetting memory principle*, leading to the fractional order system

$${}_a^C D_t^{1+\nu}\sigma_i(t) = -k\, \mathrm{sign}(\sigma_i(t)) + {}_a^C D_t^{\nu}\left[\frac{\partial s}{\partial x} g(x)\varphi\right]_i \tag{3.9}$$

with $\sigma_i(t)$ the ith component of vector function $\sigma(t)$, this establishes that memory can be reseted each time $t = a$ when $\sigma_i(a) = 0$, i.e. when $\sigma_i(t)$ crosses zero. Thus, it provides the basis of the main result given in the following theorem.

THEOREM 3.1 Let $c = \max_i \|[(\partial s/\partial x)g(x)\varphi]_i\|_{\mathscr{H}^{\nu}}$ and $k > ((3+\nu)/(1-\nu))c$. Then, by using the resetting memory principle, the system (3.9) can be written as the following fractional order system

$${}_{t_n}^C D_t^{1+\nu}\sigma_i(t) = -k\, \mathrm{sign}(\sigma_i(t)) + {}_{t_n}^C D_t^{\nu}\left[\frac{\partial s}{\partial x} g(x)\varphi\right]_i$$

with $\nu \in (0, 1)$, and (t_n) and (t_n') strictly increasing sequences of non-negative real numbers such that $\sigma_i(t_n) = 0$ and $\dot{\sigma}_i(t_n') = 0$ since the lower terminal becomes $a = t_n$. Also, there exists $t_s \in \mathbb{R}$ such that $(\sigma_i(t), \dot{\sigma}_i(t)) = (0, 0)$ $\forall t \geqslant t_s$.

Proof. Without loss of generality, consider $\dot{\sigma}_i(t_0) > 0$, and then $\sigma_i(t_0^+) > 0$. We firstly analyze the time interval $[t_0, t_1]$ wherein sets of zero Lebesgue measure do not affect the solution of the system (Danca, 2011), supposing t_1 is not finite leads to a contradiction. Thus, for $t \in (t_0, t_1)$, with t_1 the time when

$\sigma_i(t_1) = 0$, we have that

$$ {}_{t_n}^{C}D_t^{1+\nu}\sigma_i(t) = -k + {}_{a}^{C}D_t^{\nu}\left[\frac{\partial s}{\partial x}g(x)\varphi\right]_i $$

since $\sigma_i(t) > 0$ for all t in (t_0, t_1), this in turns implies that

$$ -(k+c) \leqslant {}_{t_n}^{C}D_t^{1+\nu}\sigma_i(t) \leqslant -(k-c). \tag{3.10} $$

Thus, using the monotonicity of the fractional integral and integrating ν-times (3.10) over $[t_0, t] \subseteq [t_0, t_1]$, one obtains

$$ \dot{\psi}_1 \leqslant \dot{\sigma}_i(t) \leqslant \dot{\psi}_2. \tag{3.11} $$

Integrating (3.11) one obtains

$$ \psi_1(t) \leqslant \sigma_i(t) \leqslant \psi_2(t), \tag{3.12} $$

with

$$ \psi_1(t) = \dot{\sigma}_i(t_0)(t-t_0) - \frac{k+c}{\Gamma(2+\nu)}(t-t_0)^{1+\nu}, \tag{3.13} $$

$$ \psi_2(t) = \dot{\sigma}_i(t_0)(t-t_0) - \frac{k-c}{\Gamma(2+\nu)}(t-t_0)^{1+\nu}. \tag{3.14} $$

Then, consider $t_1 > t_0$ such that $\sigma_i(t_1) = 0$, and solving $\psi_1(t) = 0$ and $\psi_2(t) = 0$, by taking into account that $\psi_1(t)$ crosses zero before t_1 (at $\sigma_i(t_1) = 0$) but $\psi_2(t)$ crosses zero after t_1, we obtain

$$ \frac{\dot{\sigma}_i(t_0)\Gamma(2+\nu)}{k+c} \leqslant (t_1 - t_0)^{\nu} \leqslant \frac{\dot{\sigma}_i(t_0)\Gamma(2+\nu)}{k-c}. \tag{3.15} $$

Now, since $\dot{\sigma}_i(t_1)$ is lower bounded by $\dot{\psi}_1(t_{\psi_2})$, where t_{ψ_2} is the time when $\psi_2(t_{\psi_2}) = 0$, we obtain the following

$$ -\mu\dot{\sigma}_i(t_0) \leqslant \dot{\sigma}_i(t_1) \leqslant 0, $$

with $\mu = ((k+c)/(k-c))(1+\nu) - 1 < 1$, since $c < ((1-\nu)/(3+\nu))k$ by definition. Assuming for an arbitrary n, $|\sigma_i(t_n)| \leqslant \mu^n|\sigma_i(t_0)|$, by integrating again we obtain $|\dot{\sigma}_i(t_{n+1})| \leqslant \mu^{n+1}|\dot{\sigma}_i(t_0)|$. Thus, by mathematical induction,

$$ |\dot{\sigma}_i(t_n)| \leqslant \mu^n|\dot{\sigma}_i(t_0)|, \quad \forall n \in \mathbb{N}, $$

whence $\dot{\sigma}_i(t_n) \to 0$ as $n \to \infty$. Besides, the time t_1', for the first cross $\dot{\sigma}_i(t_0') = 0$, can be estimated from (3.11) in a similar fashion. Then we have that

$$ \frac{\dot{\sigma}_i(t_0)\Gamma(1+\nu)}{k+c} \leqslant (t_0' - t_0)^{\nu} \leqslant \frac{\dot{\sigma}_i(t_0)\Gamma(1+\nu)}{k-c}. $$

Therefore, by considering that $\sigma_i(t) \leqslant \psi_2(t) \leqslant \sup_{t\in[t_0,t_{\psi_2}]}\psi_2(t)$, we obtain

$$ \sigma_i(t_0') \leqslant \dot{\sigma}_i(t_0)^{1+1/\nu}\frac{\nu}{1+\nu}\left[\frac{\Gamma(1+\nu)}{k-c}\right]^{1/\nu} \tag{3.16} $$

that establishes the upper bound of $\sigma_i(t)$. Additionally, proceeding again by mathematical induction and using the fact $|\dot{\sigma}_i(t_n)| \leqslant \mu|\dot{\sigma}_i(t_{n-1})|$, it follows that

$$|\sigma_i(t_n')| \leqslant \mu^{n(1+1/\nu)}|\sigma_i(t_0')|,$$

with t_n' a time when $\dot{\sigma}_i(t_n') = 0$, which leads to

$$\sigma_i(t_n') \to 0 \quad \text{as } n \to \infty.$$

Now, for each interval $[t_n, t_{n+1}]$ we can find that the following relation holds

$$(t_{n+1} - t_n)^\nu \leqslant \mu^n \frac{\dot{\sigma}_i(t_0)\Gamma(2+\nu)}{k-c}.$$

meaning that the time of convergence t_s is

$$\begin{aligned} t_s &= t_0 + \sum_{n=0}^{\infty}(t_{n+1} - t_n) \\ &\leqslant t_0 + \left[\frac{\dot{\sigma}_i(t_0)\Gamma(\nu+2)}{k-c}\right]^{1/\nu} \sum_{n=0}^{\infty}\left(\mu^{1/\nu}\right)^n \\ &= t_0 + \frac{1}{\left(1-\mu^{1/\nu}\right)}\left[\frac{\dot{\sigma}_i(t_0)\Gamma(\nu+2)}{k-c}\right]^{1/\nu} \in \mathbb{R} \end{aligned}$$

since $\mu^{1/\nu} < 1$. Moreover, from $t_n = t_0 + \sum_{k=0}^{n}(t_{k+1} - t_k)$ and $(t_n' - t_n)^\nu \leqslant \mu^n(\dot{y}(t_0)\Gamma(1+\nu)/(k-c))$, we have $\lim_{n\to\infty} t_n' = \lim_{n\to\infty} t_n = t_s$, then $(\dot{\sigma}_i(t), \sigma_i(t)) \to (0,0)$ as $t \to t_s$.

Finally, to demonstrate that $(\sigma_i(t), \dot{\sigma}_i(t)) = (0,0)\ \forall t \geqslant t_s$, it is shown that $\dot{\sigma}_i(t) = 0$ for $t \geqslant t_s$ which suggests that $\sigma_i(t) = \sigma_i(t_s) + \int_{t_s}^{t} \dot{\sigma}_i(\tau)\,\mathrm{d}\tau = 0$ for all $t \geqslant t_s$. Let

$$t^* = \sup\{T > t_s : \dot{\sigma}_i(t) = 0\ \forall t \in [t_s, T]\},$$

and assume t^* is finite. In virtue of $\dot{\sigma}_i^{-1}[(-\infty, 0)]$ and $\dot{\sigma}_i^{-1}[(0,\infty)]$ are open sets by the continuity of $\dot{\sigma}_i(t)$, these constitute unions of countable collections of disjoint open intervals, one of these of the form $\mathscr{O} = (t^*, t^{**})$. Suppose $\dot{\sigma}_i(t) > 0$ on $t \in \mathscr{O}$ (analogously for $\dot{\sigma}(t) < 0$), then, it results $\sigma_i(t) > 0$ on $\mathscr{O}$, and consequently,

$$\dot{\sigma}_i(t) = -k\, {}_{t^*}I_t^\nu \operatorname{sign}(\sigma_i(t)) + {}_{t^*}^{C}D_t^\nu \left[\frac{\partial s}{\partial x} g(x)\varphi\right]_i \leqslant -\frac{k-c}{\Gamma(\nu+1)}(t-t^*)^\nu < 0$$

on $\mathscr{O}$ which is absurd. Therefore, by this contradiction, $\dot{\sigma}_i(t) = 0\ \forall t \geqslant t_s$. □

4. Remarks

The intuitive notions of integer order differentiation take a more abstract and general meaning when we consider fractional orders which we find convenient to discuss in remarks.

REMARK 4.1 (Comparison with respect to integer order reaching phases:) Sliding mode based controllers inducing integer order reaching phases have been proposed for both integer and fractional order systems (Pisano *et al.*, 2010; Dadras & Momeni, 2013; Kamal *et al.*, 2013). For these integer order reaching phases, we can consider two cases, first order reaching laws which provide robustness against bounded measurable disturbances based on discontinuous control definitions with harmful chattering (Utkin, 1992), and absolutely continuous controllers (Pisano *et al.*, 2010) inducing two or higher order reaching phases without chattering but requiring differentiability of matched disturbances. Thus, in contrast to conventional integer order schemes, our proposed fractional order controller provides robustness to non-differentiable Hölder disturbances by means of a continuous control signal alleviating chattering phenomena and exploiting topological properties of differintegral operators.

REMARK 4.2 (Regularity of control signal:) Theorem 3.1 guarantees that the closed-loop system enforces a sustained sliding regime at $\dot{\sigma}(t) = \sigma(t) = 0$, for $t \geqslant t_s$ and some finite-time $t_s \in \mathbb{R}$, without any chattering due to this controller u_s is continuous. The effect of increasing the order ν induces a large overshoot of the sliding variable; nevertheless, the regularity of the control signals is improved for a higher order ν. For practical purposes, this establishes a compromise on the regularity of the control signal and the robustness with respect to Hölder disturbances since for a lower ν the controller is robuster but less regular.

REMARK 4.3 (Exact disturbance observer:) Considering that $(\partial s/\partial x)g(x)$ is invertible by Assumption 3.3, we have an important by-product of this formulation, that for $t \geqslant t_s$, $u_s(t)$ is an exact disturbance observer in virtue of the uniform continuity of $\dot{\sigma}$, without depending on the equivalent control method, that is, the disturbance observation is not in the mean sense but exact, i.e. $u_s(t) = -\varphi(t)$ for $t \geqslant t_s$, inducing the ideal system (3.2), which in turn provides $x \to 0$ via u_0. It is straightforward to show that conventional classical schemes, including high-order sliding mode controllers cannot reject this class of disturbances by means of a continuous control definition.

5. Simulations

A representative simulation study, considering an integer order non-linear system subject to a nowhere differentiable but Hölder continuous disturbance, is presented to show the viability of the proposed control scheme.

5.1 *The simulator*

The simulation was programmed in Matlab 2013 in m-files running on a PC with an Intel processor of 1.90 GHz and 4 GB of RAM. The Euler integrator is considered to solve the integer order dynamics and the Grünwald–Letnikov operator (2.4) for computing numerical differintegrals. The integration step was established at 0.1 ms.

5.2 *Plant*

Consider a simplified, yet representative, one-dimensional non-linear model of an underwater autonomous vehicle,

$$m\ddot{\xi} + \beta\dot{\xi}|\dot{\xi}| = u + \phi,$$

where ξ is the position of the vehicle, $m = 11\,\mathrm{kg}$ the mass, $\beta = 15 + 5\sin(5t)\,\mathrm{Ns^2/m^2}$ the drag coefficient and u the control input provided by the thrusters, and $\phi(t) = 5\sum_{r=1}^{\infty} 5^{-r\eta}\sin(5^r t) + 10$ models a nowhere differentiable disturbance given by a Celleriér function, which is a Weierstrass-Mandelbrot-type function, see Fig. 1. This disturbance has been associated to turbulence fluid effects

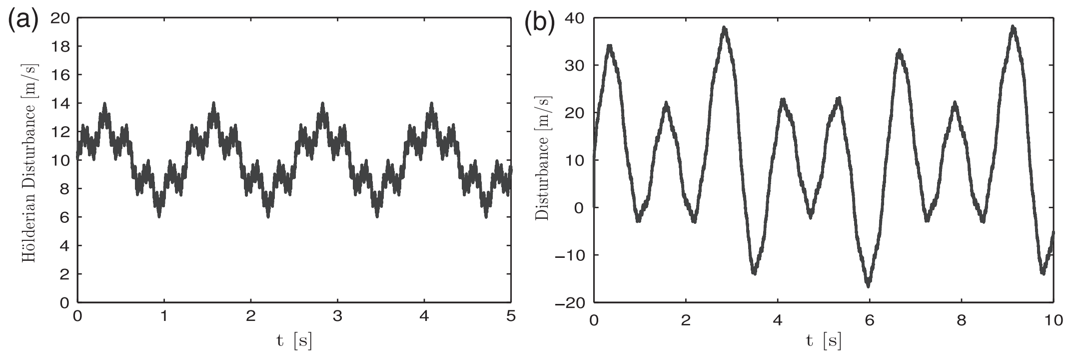

FIG. 1. Hölderian disturbance and combined effects of disturbance and unmodelled effects. (a) Nowhere differentiable disturbance $\phi(t)$ and (b) disturbance and uncertainties effects $\varphi(t)$.

(Humphrey *et al.*, 1992), the proof of $\phi \in \mathscr{H}^{\nu}(\Omega)$ for all $\nu < \eta \leqslant 1$ is discussed in Ross *et al.* (1994) and Hardy (1916) and in the references therein. For simulations, only the first two hundred terms of ϕ are considered and $\eta = 0.5$. Also, the drag effect is considered known just for simulation purposes and not for the control design, that is, the effects are regrouped into an endogenous uncertainty term, i.e. $\varphi = \phi - \beta\dot{\xi}|\dot{\xi}| - m\ddot{\xi}_d$, where ξ_d is the desired trajectory.

5.3 *The task and control gains*

The task is to track the trajectory $\xi_d(t) = 0.25\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t)\right]$. For the design of the control law, both the disturbance ϕ and the drag effect are considered unknown. Then, it is convenient to define the state variables as $x_1 = \xi - \xi_d$ and $x_2 = \dot{x}_1$. Thus, we obtain the state-space representation

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \frac{1}{m}(u + \varphi),$$

with $\varphi = \phi - \beta\dot{\xi}|\dot{\xi}| - m\ddot{\xi}_d$ the term of disturbance and uncertainties, see Fig. 1.

It is desired to get an ideal dynamics without disturbances $\dot{x}_2 = -4x_1 - 4x_2$, which is associated to a critically damped regime of convergence. Since $\dot{x}_1$ does not depend explicitly of the disturbance, we can design $s(x) = x_2$, and the sliding variable becomes

$$\sigma(t) = x_2(t) - x_2(t_0) - \frac{1}{m}\int_{t_0}^{t} u_0(\tau)\,\mathrm{d}\tau,$$

which in turn produces $\dot{\sigma} = (1/m)(u_s + \varphi)$, and accordingly with Theorem 3.1, the sliding controller (3.6) and the ideal controller $u_0 = -m(4x_1 + 4x_2)$ induce the required sliding phase for all instant after some finite-time t_s. The control parameters are designed as $k = 8$ and $\nu = 0.4$, providing a uniformly continuous definition which guarantee in finite-time the exact observation of the Hölderian disturbance and uncertainties effects.

A comparison with respect to the conventional integer order scheme, corresponding to $\nu = 0.4$, is performed using the feedback gain $k = 5$; the other control parameters are the same as those used in the fractional order control design.

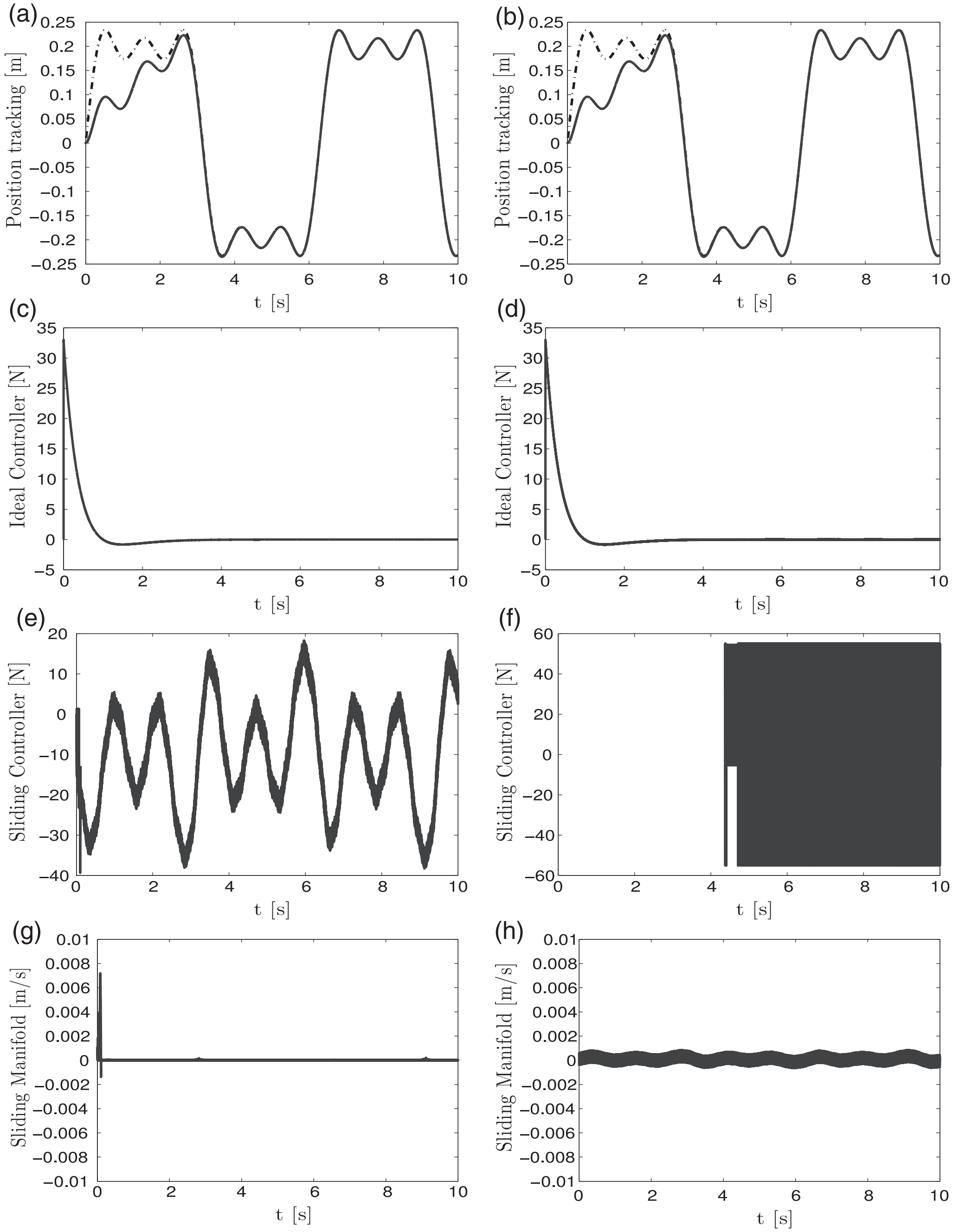

FIG. 2. Comparison between the proposed fractional order controller and the classical integer order controller: Tracking of position ξ_d in dotted line, ξ in solid line; nominal controller u_0, sliding controller u_s; and sliding manifold σ. (a) Fractional order control: Position tracking, (b) Integer Order Control: Position tracking, (c) Fractional Order Control: Nominal control u_0, (d) Integer Order Control: Nominal control u_0, (e) Fractional Order Control: Sliding control u_s, (f) Integer Order Control: Sliding control u_s, (g) Fractional Order Control: Sliding surface σ and (h) Integer Order Control: Sliding surface σ.

5.4 *Results*

Figure 2 highlights the asymptotic tracking of position, even when drag forces are considered unknown and anomalous non-differentiable disturbances are present. After a finite-time, the sliding mode controller rejects the effects caused by the Hölderian disturbance, inducing the nominal system that is free of disturbances for a critically damped regime. Thus, in the fractional order case the sliding control term u_s also stands for an exact and finite-time disturbance observer that compensates the effects of disturbances and uncertainties, inducing an ideal system after this finite-time. We can appreciate a clear advantage of the fractional order control signal showing a chattering-free performance by means of the fractional integral of the discontinuous signum function, preserving robustness against Hölder disturbances. Notice that, by increasing the order of integration, a more regular control signal is obtained, but robustness against less regular disturbances cannot be obtained. In addition, we can see that in the integer order case, the reaching phase of σ is eliminated and the invariance is provided by any initial condition, nevertheless at expenses of chattering; the reaching phase is present in the fractional order case to alleviate chattering, which also improves the performance of the sliding manifold after the finite time of convergence.

6. Conclusions

In contrast to other works that address the control design of fractional order systems by means of standard integer order sliding modes, we propose a fractional order controller for a general class of integer order non-linear systems subject to Hölder disturbances. It is shown that the notion of fractional derivative is more suitable to design robust and continuous sliding mode controllers by exploiting structural properties of differintegral operators. To handle the additional complexities that introduce these operators, those are memory and heritage, the principle of dynamic memory reseting is proposed so as to the closed-loop system enforces in finite-time the sliding motion, which provides the establishment of an ideal system without disturbances. This scheme provides a robust and a chatter-less control signal that is uniformly continuous and robust against Hölder continuous but not necessarily differentiable disturbances. Our proposal illustrates that the fractional sliding mode stands indeed for a viable control scheme to stabilize uncertain dynamical systems subject to a wide variety of anomalous disturbances.

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