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Conference Paper · August 2017

DOI: 10.1115/DETC2017-68270

Proceedings of the ASME 2017 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
August 6-9, 2017, Cleveland, Ohio, USA

DRAFT_DETC2017-68270

A NOTE ON THE LYAPUNOV STABILITY OF FRACTIONAL-ORDER NONLINEAR SYSTEMS

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ABSTRACT

In this paper, stability of fractional order (FO) systems is investigated in the sense of the Lyapunov stability theory. A new definition for exponential stability of the fractional order systems is given and sufficient conditions are obtained for the exponential stability of the FO systems using the notion of Lyapunov stability. Besides, a less conservative sufficient condition is derived for asymptotical stability of FO systems. The stability analysis is done in the time domain. Numerical examples are given to show that the obtained conditions are effective and applicable in practice.

INTRODUCTION

Although the idea of differentiation and integration of arbitrary (fractional) order faces difficulty to find a real-world application for more than 300 years, recently, these operators have gained interests among engineering scientist and researchers for their superior results in control and modeling of physical systems [1-5]. With the increasing trend of introduced FO models for electrical, mechanical and chemical processes, in depth study of these systems from different points of view such as control [6-9], dynamical behavior analysis [10-12], and stability analysis [13-16] is noticeably growing.

One of the fundamental topics, which should be taken into consideration in all dynamic systems, is the stability analysis. There are limited published works in the area of FO systems [14-20] that are mainly concentrated on the stability analysis of FO linear systems [17-20]. From literature, the main approach for stability analysis of FO LTI systems mostly depends on calculating eigenvalues of state equations. However, Lyapunov stability of linear fractional systems based on an energy balance approach has been studied in [21], [22]. Nonetheless, seeking a direct systematic approach for stability analysis of FO nonlinear systems is still under development and investigation [23], [24].

The most well-known method to analyze the stability of nonlinear integer order systems is the Lyapunov stability technique. Very recently, the Lyapunov stability problem for the FO systems has been investigated in literature [25-28]. Fractional Lyapunov direct method for checking the stability problem in FO systems has been introduced in [25], [26]. Furthermore, FO systems have been studied from the aspect of Mittag-Leffler stability problem in [25]. In [26], introducing the class-K functions to the fractional Lyapunov direct method, asymptotical stability of the FO systems is discussed in the sense of fractional Lyapunov direct method. In [27], uniform stability of fractional order systems is studied proposing a complement theorem for [25].

This paper deals with the problem of stability, i.e. asymptotical stability and, in particular, exponential stability of nonlinear FO systems utilizing the extension of the Lyapunov stability notion. To the authors' best knowledge, the notion of exponential stability is not extended for fractional order systems, yet. Using the concept of fractional integration operator and Grownwall-Bellman lemma, different stability conditions are obtained for the FO systems. All the analyses are done in the time domain and the conditions are derived for asymptotical and exponential stability of the FO systems. In the case of asymptotical stability, the condition presented in this paper is less conservative than the one in [25].

The rest of the paper is organized as follows. In Section 2, basic definitions in fractional calculus and some useful lemmas are presented as the preliminaries. Section 3 is devoted to obtaining the stability criteria for FO systems. An illustrative example to show the applicability of the results of Section 3 is presented in Section 4. Finally, some concluding remarks are given in Section 5.

PRELIMINARIES

In the following section, we introduce some useful lemmas that will be used for proving the stability theorems. In this paper, for an arbitrary order α and $t\geq 0$, ${}_0D_t^\alpha y = D^\alpha y$ and ${}_0I_t^\alpha y = I^\alpha y$ represent the αth-order fractional derivative and the αth-order fractional integration, respectively.

Property 1. [1] If the fractional derivative ${}_0D_t^\alpha y(t),\ (k-1\leq\alpha<k)$ of a function $y(t)$ is integrable,

$$ {}_0I_t^\alpha\left({}_0D_t^\alpha y(t)\right) = y(t) - \sum_{j=1}^{k}\left[{}_0D_t^{\alpha-j}y(t)\right]_{t=0}\frac{t^{\alpha-j}}{\Gamma(\alpha-j+1)}. \tag{1} $$

Definition 1. [29] The equilibrium point x=0 of

$$ \dot{x} = f(t,x) \tag{2} $$

is exponentially stable if there exist positive constants c, k and λ such that

$$ \|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall\|x(t_0)\|<c. \tag{3} $$

Lemma 1. [30] (Hardy-Littlewood theorem) The fractional integration operator ${}_aI_t^\alpha$ with $\lfloor\alpha\rfloor>0$ and $-\infty<\alpha<+\infty$ is bounded in $L_p(a,b),\ 1\leq p\leq\infty$,

$$ \left\|I^\alpha y\right\|_p \leq K\|y\|_p. \tag{4} $$

Lemma 2. [31] (Gronwall-Bellman lemma) Assume that $u(t)$ and $f(t)$ are real-valued piecewise-continuous functions defined on the real interval $[a,b]$ and $K(t)$ is also real-valued and $K(t)\in L(a,b)$. Also, $u(t)$ and $K(t)$ are nonnegative on this interval. If for all $t\in[a,\ b]$,

$$ u(t) \leq f(t) + \int_a^t K(\tau)u(\tau)d\tau, \tag{5} $$

then for all $t\in[a,\ b]$, we have

$$ u(t) \leq f(t) + \int_a^t f(\tau)K(\tau)\exp\left\{\int_\tau^t K(r)dr\right\}d\tau. \tag{6} $$

FRACTIONAL-ORDER SYSTEM

Let the FO system be presented by the following differential equation:

$$ D^q x(t) = f(t,x) \tag{7} $$

with initial condition $x(t_0)$, where $q\in(0,1)$ is the fractional derivative order and the system's dynamic is piecewise continuous in t and locally Lipschitz in x.

It is proved that the equilibrium points of system (7) are asymptotically stable if condition

$$ \left|\arg(eig(J))\right| = \left|\arg(\lambda_i)\right| > q\frac{\pi}{2},\ \ i=1,2,...,n \tag{8} $$

is satisfied for all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of Jacobian matrix $J=\partial f/\partial x$ where $f=[f_1 \quad f_2 \quad \cdots \quad f_n]^T$, evaluated at the equilibrium E^* [13].

Theorem 1. [25] Let x=0 be an equilibrium point for the nonautonomous fractional-order system (7). Assume that there exist a Lyapunov function $V(t,x(t))$ and class-K functions α_i (i=1, 2, 3) satisfying

$$ \alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \tag{9} $$

and

$$ D^\beta V(t,x(t)) \leq -\alpha_3(\|x\|) \tag{10} $$

where $\beta\in(0,1)$. Then the system (7) is asymptotically stable.

MAIN RESULTS

In what follows, some stability theorems for FO systems and their detailed proof are given out. The stability conditions are derived based on the Lyapunov stability theorem.

Theorem 2. Let x=0 be an equilibrium point for the system (7) and $D\subset R^n$ be a domain containing the origin. Let $V(t,x(t)):[0,\infty)\times D\to R$ be a continuously differentiable function and locally Lipschitz with respect to x such that

$$\alpha_1\|x\| \le V(t,x(t)) \le \alpha_2\|x\|, \tag{11}$$

$$D^q V(t,x(t)) \le 0, \tag{12}$$

where 0<q<1, α_1 and α_2 are arbitrary positive constants. The x=0 is asymptotically stable for any $\|x_0\| \le \gamma$.

Proof. Taking the concept of fractional integral operator into account and using (12), one obtains

$$V(t,x(t)) - \left[D^{q-1}V(t,x(t))\right]_{t=0}\frac{t^{q-1}}{\Gamma(q)} \le 0 \tag{13}$$

Substituting (11) into (13), one has

$$\alpha_1\|x\| \le \left[D^{q-1}V(t,x(t))\right]_{t=0}\frac{t^{q-1}}{\Gamma(q)} \tag{14}$$

It follows from Lemma (1) that

$$\begin{aligned}\left[D^{q-1}V(t,x(t))\right]_{t=0} &\le \left\|D^{q-1}V(t,x(t))\right\|_{t=0} \\ &= \left\|I^{1-q}V(t,x(t))\right\|_{t=0} \\ &\le M\left\|V(t,x(t))\right\|_{t=0} \\ &\le M\alpha_2\|x(t)\|_{t=0} \le M\alpha_2\gamma = \Psi\end{aligned} \tag{15}$$

Now, replacing (15) into (14), it can be concluded that

$$\|x\| \le \left[D^{q-1}V(t,x(t))\right]_{t=0}\frac{t^{q-1}}{\alpha_1\Gamma(q)} \le \frac{\Psi}{\alpha_1\Gamma(q)}t^{q-1} \tag{16}$$

Therefore, $\|x\|$ tends to zero as $t\to\infty$ from which the asymptotical stability of system (7) can be inferred.

Remark 1. In Theorem 2, the condition for the fractional derivative of the Lyapunov candidate take a more general form when compared with the work in [25]; i.e. the limitation on the derivative of the Lyapunov candidate in Eq. (12) is less conservative that the one in Eq. (10).

Theorem 3. Let x=0 be an equilibrium point for the system (7) and $D\subset\Re^n$ be a domain containing the origin. Let $V(t,x(t)):[0,\infty)\times D\to\Re$ be a continuously differentiable function and locally Lipschitz with respect to x such that

$$\alpha_1\|x\| \le V(t,x(t)) \le \alpha_2\|x\| \tag{17}$$

$$D^q V(t,x(t)) \le -\alpha_3\|x\| \tag{18}$$

where 0<q<1, α_1, α_2 and α_3 are arbitrary positive constants. The x=0 is asymptotically stable for any $\|x_0\| \le \gamma$.

Proof. Integrating both sides of (18), one has

$$\begin{aligned}&V(t,x(t)) - [D^{q-1}V(t,x(t))]_{t=0}\frac{t^{q-1}}{\Gamma(q)} \le \\ &\qquad -\frac{1}{\Gamma(q)}\int_0^t \frac{\alpha_3\|x\|}{(t-\tau)^{1-q}}d\tau\end{aligned} \tag{19}$$

Substituting (19) into (17) yields

$$\begin{aligned}&\alpha_1\|x\| \le [D^{q-1}V(t,x(t))]_{t=0}\frac{t^{q-1}}{\Gamma(q)} \le \\ &\qquad -\frac{1}{\Gamma(q)}\int_0^t \frac{\alpha_3\|x\|}{(t-\tau)^{1-q}}d\tau\end{aligned} \tag{20}$$

Now, let us define

$$\begin{aligned}&u = \|x\| \\ &f = \left[D^{q-1}V(t,x(t))\right]_{t=0}\frac{t^{q-1}}{\alpha_1\Gamma(q)} \\ &K = -\frac{\alpha_3}{\alpha_1\Gamma(q)}(t-\tau)^{q-1}\end{aligned} \tag{21}$$

So, according to Lemma (2) and from (15), one can obtain

$$\begin{aligned}&\|x\| \le \frac{\Psi t^{q-1}}{\alpha_1\Gamma(q)} \\ &\quad -\frac{\alpha_3\Psi}{\left(\alpha_1\Gamma(q)\right)^2}\int_0^t \frac{t^{q-1}}{(t-\tau)^{1-q}}\left(e^{\int_\tau^t -\frac{\alpha_3}{\alpha_1\Gamma(q)}(t-r)^{q-1}dr}\right)d\tau\end{aligned} \tag{22}$$

The inner integral in (22) is equal to

$$\int_\tau^t -\frac{\alpha_3}{\alpha_1\Gamma(q)}(t-r)^{q-1}dr = -\frac{\alpha_3}{\alpha_1 q\Gamma(q)}(t-\tau)^q \tag{23}$$

Hence,

$$\begin{aligned}&\|x\| \le \frac{\Psi t^{q-1}}{\alpha_1\Gamma(q)} \\ &\quad -\frac{\alpha_3\Psi}{\left(\alpha_1\Gamma(q)\right)^2}\int_0^t \frac{t^{q-1}}{(t-\tau)^{1-q}}\left(e^{-\frac{\alpha_3}{\alpha_1 q\Gamma(q)}(t-\tau)^q}\right)d\tau\end{aligned} \tag{24}$$

Consequently, one has

$$
\begin{aligned}
\|x\| &\le \frac{\Psi t^{q-1}}{\alpha_1\Gamma(q)} \\
&-\frac{\Psi t^{q-1}}{\alpha_1\Gamma(q)}\int_0^t \frac{\alpha_3(t-\tau)^{q-1}}{\alpha_1\Gamma(q)}\left(e^{-\frac{\alpha_3}{\alpha_1 q\Gamma(q)}(t-\tau)^q}\right)d\tau \\
&=\frac{\Psi t^{q-1}}{\alpha_1\Gamma(q)}-\frac{\Psi t^{q-1}}{\alpha_1\Gamma(q)}\left(1-e^{-\frac{\alpha_3}{\alpha_1 q\Gamma(q)}t^q}\right) \\
&=\frac{\Psi}{\alpha_1\Gamma(q)}t^{q-1}e^{-\frac{\alpha_3}{\alpha_1 q\Gamma(q)}t^q}
\end{aligned}
\tag{25}
$$

It follows from the above inequality (25) and Definition (2) that system (7) is exponentially stable. It should be noticed that regarding to (25), the system satisfying the conditions of Theorem (4) shows a faster convergence speed than the exponential stability (3).

Remark 2. Based on the Theorem 3, the equilibrium point x=0 of

$$D^q x = f(t,x), \tag{26}$$

where $q\in(0,1)$, is exponentially stable if there exist positive constants c, k, and λ such that

$$\|x(t)\| \le k\|x(t_0)\|(t-t_0)^{q-1}e^{-\lambda(t-t_0)},\quad \forall\|x(t_0)\|<c\,. \tag{27}$$

Remark 3. For q=1, Eq. (27) will be the same as Eq. (3). In other words, the notion given for the exponential stability of the fractional order systems will be the same as the definition of exponential stability for integer order systems in the case q=1 (Definition 1).

Remark 4. Equation (27) implies that exponential stability for FO systems shows a faster convergence speed than exponential stability of integer order systems (Definition 1) near the origin; i.e.

$$
\begin{aligned}
&\left\{\frac{d}{dt}\left(e^{-\lambda t}\right)\right\}_{t=0} = \left[-\lambda e^{-\lambda t}\right]_{t=0} = -\lambda \\
&\left\{\frac{d}{dt}\left(t^{-\gamma}e^{-\lambda t}\right)\right\}_{t=0} = \\
&\qquad\left[-\gamma t^{-(\gamma+1)}e^{-\lambda t}-\lambda t^{-\gamma}e^{-\lambda t}\right]_{t=0} = \infty
\end{aligned}
\tag{28}
$$

where λ and γ positive constants.

NUMERICAL EXAMPLE

Example 1. Suppose the following system [25]

$$D^q|x| = -|x| \tag{29}$$

Let the Lyapunov candidate to be defined as

$$V(t,x(t)) = |x| \tag{30}$$

with α_1=0.5 and α_2= 2. So, we have

$$D^q V = D^q|x| \le -|x| \tag{31}$$

where α_3=1. Therefore, from Theorem (4), FO system (29) is exponentially stable.

Based on the results presented in [25], solution of system (29) satisfies the following condition

$$|x(t)| \le |x(0)|E_q(-t^q) \tag{32}$$

On the other hand, applying the result of Theorem 3 to system (29) gives

$$|x(t)| \le \frac{k|x(0)|}{\Gamma(q)}t^{q-1}e^{-\frac{2}{q\Gamma(q)}t^q} \tag{33}$$

where $q\in(0,1)$ and k>0. Comparing the right hand side of Eq. (32) and (33), it can be concluded that the second inequality decreases more rapidly when $t\to\infty$.

Example 2. Consider the following FO system

$$D^q x = -x \tag{34}$$

where 0<q<1 and $x(0)\neq 0$. Here, D^q represents the Caputo fractional derivative. Choosing the Lyapunov candidate as $V(x)=x^2(t)$, we have

$$D^q V = D^q x^2 = D^{q-1}(2x\dot{x}) \tag{35}$$

On the other hand, applying the fractional integral operator to the system (34) yields

$$
\begin{aligned}
\dot{x}(t) &= D^{1-q}(-x(t)) = -D^{1-q}(x(t)) \\
&= \frac{-1}{\Gamma(q)}\frac{d}{dt}\int_0^t (t-\tau)^{q-1}x(\tau)d\tau \\
&= \frac{1-q}{\Gamma(q)}\int_0^t (t-\tau)^{q-2}x(\tau)d\tau
\end{aligned}
\tag{36}
$$

Since $x(0)x(t)>0$ for all $t>0$, one can easily conclude that $x(0)x(t) \le x^2(0)$. Using this inequality, it follows from (36) that

$$x(0)\dot{x}(t) \le -\frac{x^2(0)t^{q-1}}{\Gamma(q)} < 0 \quad (37)$$

Substituting (37) in (35) and assuming that there exists a positive constant δ which satisfies $x(0)x(t) \ge \delta$ gives

$$D^q V \le \frac{2\delta}{x^2(0)} D^{q-1} x(0)\dot{x} \le -\frac{2\delta}{\Gamma(q)} D^{q-1} t^{q-1} = -2\delta \le 0 \quad (38)$$

However, from Eq. (38), it can be concluded that $V(x) = x^2(t)$ is a decreasing function and $\lim_{t\to\infty} x^2(t) = 0$ which contradicts the assumption $x(0)x(t) \ge \delta$. Hence, the equilibrium point $x=0$ is asymptotically stable. The solutions of FO system (34) for different values of q are depicted in Fig. 1.

Remark 5. Asymptotic stability of the origin ($x=0$) for a class of fractional systems noted as $D^q x = -x^m, m = 2n-1, n \in N$ can be easily verified using the same procedure as example 2.

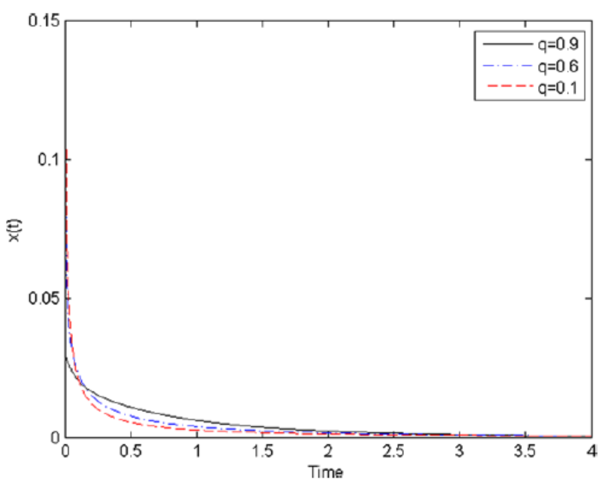

Figure 1. The solution of FO system (34) for q = 0.1, 0.6, and 0.9

CONCLUSION

This paper has introduced a new concept for exponential stability of fractional order systems and presented sufficient conditions to guarantee the asymptotical and exponential stability of FO dynamic systems utilizing Lyapunov stability theorem. Our results for asymptotical stability are less conservative than those existed in the literature and consequently, using them in practice is easier.

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