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Design and Optimal Tuning of Nonlinear PI Compensators

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Abstract. In this paper, linear time-invariant single-input single-output (SISO) systems that are stabilizable by linear proportional and integral (PI) compensators are considered. For such systems, a five-parameter nonlinear PI compensator is proposed. The parameters of the proposed compensator are tuned by solving an optimization problem. The optimization problem always has a solution.

Additionally, a general nonlinear PI compensator is proposed and is approximated by easy-to-compute compensators, for instance, a six-parameter nonlinear PI compensator. The parameters of the approximate compensators are tuned to satisfy an optimality condition. The superiority of the proposed nonlinear PI compensators over linear PI compensators is discussed and is demonstrated for two feedback systems.

Key Words. Linear SISO systems, nonlinear PI compensators, tracking of step inputs, optimal tuning of compensators, rational approximations of functions, exponential approximation of functions.

1. Introduction

The tuning of linear proportional, integral, and derivative (PID) compensators has received considerable attention by researchers and process control designers. There are numerous tuning techniques for single-input single-output (SISO) and to a lesser extent for multi-input multi-output (MIMO) PID compensators. For extensive literature on tuning and auto-tuning techniques of PID compensators, the reader is referred to Refs. 1–6.

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Nonlinear PID compensators (PID compensators with nonconstant gains) have been considered by some researchers as a means of improving the performance of systems. There are, however, few references considering nonlinear PID compensators. In Refs. 7–9, there are design procedures for nonlinear PID compensators mostly based on heuristic rules. In Ref. 10, an intelligent integrator is proposed in order to improve the performance of linear systems and to avoid the wind-up problem. The proposed integrator has a feedback loop around it, which incorporates a dead-zone nonlinearity. In Ref. 11, the performance of different PI-type compensators with nonlinear gains is examined. A sampled-data PI controller is designed in Ref. 12, in which the integrator is similar to that proposed in Ref. 10. In Ref. 13, a nonlinear PID compensator is designed by the extended linearization technique, in which the three gains of the compensator are functions of the compensator state. More recently, in Ref. 14, a stabilizing nonlinear PI compensator is designed for DC-to-DC power converters by the extended linearization technique.

In this paper, we propose and optimally tune nonlinear PI compensators for linear time-invariant SISO systems. The organization of the paper is as follows. In Section 2, we propose a five-parameter nonlinear PI compensator which is a generalization of the linear PI compensator. In Section 3, we cast the problem of tuning the parameters of the proposed PI compensator into an optimization problem. In Section 4, we propose a general nonlinear PI compensator and approximate it by easy-to-compute compensators, for instance, a six-parameter nonlinear PI compensator. The parameters of the approximate compensators are tuned to satisfy an optimality condition. In Section 5, we determine the optimal linear and nonlinear PI compensators for two systems, and demonstrate the superiority of our proposed nonlinear PI compensators over linear PI compensators by comparing the performance of the systems in tracking step inputs.

2. Problem Formulation

Consider the unity feedback system $S(P, H)$ in Fig. 1. The plant P is a strictly proper linear time-invariant SISO system. A minimal state-space representation of P is

$$\dot{x}(t) = Ax(t) + bu(t), \qquad x(0) = \theta_n, \tag{1a}$$

$$y(t) = cx(t), \tag{1b}$$

for all $t \geq 0$. In (1), the state vector $x(t) \in \mathbb{R}^n$, the input to the plant $u(t) \in \mathbb{R}$, and the output $y(t) \in \mathbb{R}$; the coefficient matrices $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and

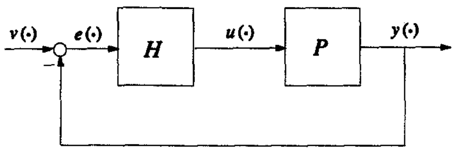

Fig. 1. Unity feedback system $S(P, H)$.

$c \in \mathbb{R}^{1 \times n}$; the vector θ_n denotes the zero vector in $\mathbb{R}^n$. The transfer function of the system (1) is denoted by $P(s)$. We assume that:

(A1) The plant P has no zeros at the origin, i.e., $P(0) \neq 0$.

(A2) There exists a linear PI compensator with the transfer function $H(s) = k_{pl} + k_{il}/s$ that places the poles of the closed-loop system $S(P, H)$ in a desired region D, given by

$$D := \{s = \mathrm{Re}(s) + j\,\mathrm{Im}(s) \in \mathbb{C}\colon \mathrm{Re}(s) \leq -\sigma_d,\ \mathrm{Re}(s) + |\mathrm{Im}(s)/\alpha| \leq 0\} \subset \mathbb{C}^{\circ}_{-}, \quad (2)$$

where $\sigma_d > 0$ and $\alpha > 0$ are constant real numbers, and $\mathbb{C}^{\circ}_{-}$ denotes the complex open left-half plane. The region D is depicted in Fig. 2.

The condition $P(0) \neq 0$ is necessary for the stabilizability of the system $S(P, H)$ by a linear PI compensator. The region D overlaps with the complex left-half plane as $\sigma_d \to 0$ and $\alpha \to \infty$. Thus, (A2) can be considered as an assumption on the stabilizability of $S(P, H)$ by a linear PI compensator. Some useful sufficient conditions for the stabilizability of linear systems

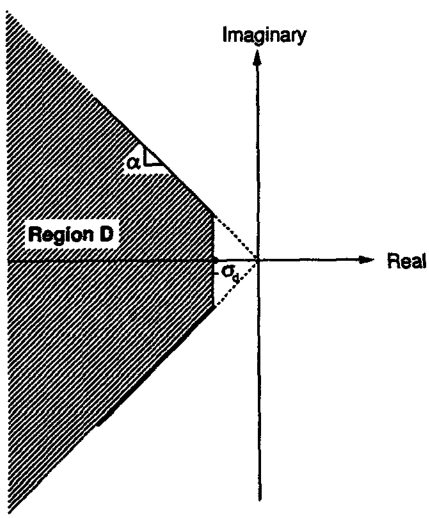

Fig. 2. Region D in the complex plane specified by (2).

by linear PI compensators are given in Ref. 15. It is not our intention to discuss these conditions here; we just assume that $S(P, H)$ is stabilizable by a linear PI compensator. Note that, if (A2) does not hold, then the linear PI compensator is not an appropriate compensator for controlling the system, and other compensators should be sought.

In the system $S(P, H)$, we choose the compensator H to be the nonlinear SISO system represented by

$$\dot{\xi}(t) = e(t)/[1 + \mu^2 e^2(t)], \qquad \xi(0) = 0, \tag{3a}$$

$$u(t) = k_i \xi(t) + [k_p + g_p \exp(\lambda |e(t)|)]e(t), \tag{3b}$$

for all $t \geq 0$. In (3), the state $\xi(t) \in \mathbb{R}$, the input $e(t) \in \mathbb{R}$, the output $u(t) \in \mathbb{R}$, and the parameters $k_p, k_i, g_p, \lambda, \mu$ are constant real numbers. The input to the compensator is

$$e(t) = v(t) - y(t),$$

for all $t \geq 0$, where $v(\cdot)$ denotes the exogenous input to the feedback system. We remark that the nonlinear functions on the right-hand sides of (3) are continuously differentiable functions of e; this fact will be used in linearizing the closed-loop system $S(P, H)$.

We consider step inputs $v(t) = \bar{v}U(t)$, $t \geq 0$, where $U(t)$ denotes the unit step function, and $\bar{v} \in \mathbb{R}$ is the amplitude of the input. Our goal is to choose the parameters $k_p, k_i, g_p, \lambda, \mu$ of the compensator H, so that the output $y(\cdot)$ of the closed-loop system $S(P, H)$ tracks the step input $v(\cdot)$, while having satisfactory transient behavior.

The nonlinear compensator H in (3) is a generalization of the linear PI compensators; this can be seen by setting μ and g_p equal to zero in (3). We call H the five-parameter nonlinear PI compensator. The motivation for choosing the compensator H is given below.

(i) Suppose that the plant P is heavily damped. If k_p, g_p, λ are positive, then the proportional gain $k_p + g_p \exp(\lambda |e|)$ of the compensator H is large for large error e. Thus, right after applying the step input $v(\cdot)$, when the error between $y(\cdot)$ and the desired set point $\bar{v}$ is large, the output $y(\cdot)$ is steered toward $\bar{v}$ at a fast rate. As $y(\cdot)$ gets closer to $\bar{v}$ and the error decreases, the proportional gain decreases, and $y(\cdot)$ is steered toward $\bar{v}$ at a slower rate. Thus, the nonlinear proportional gain provides a fast system response with minimal overshoot.

Alternatively, suppose that the plant P is lightly damped. If k_p, g_p are positive and λ is negative, then the proportional gain $k_p + g_p \exp(\lambda |e|)$ of the compensator H is small for large error e. Thus, right after applying the step input $v(\cdot)$, when the error between $y(\cdot)$ and the desired set point $\bar{v}$ is large, the proportional part of the compensator is inactive, so as not to contribute to overshoot. Since the system is lightly damped, the output $y(\cdot)$ increases to $\bar{v}$ at a fast rate on its own. As $y(\cdot)$ gets closer to $\bar{v}$ and the error decreases, the proportional gain increases to increase the damping of the closed-loop poles. Thus again, the nonlinear proportional gain provides a fast system response with minimal overshoot.

(ii) The integral gain $k_i/(1+\mu^2 e^2)$ is small for large error e and is large for small error. Thus, right after applying the step input $v(\cdot)$, when the error between $y(\cdot)$ and the desired set point $\bar{v}$ is large, the integrator is inactive; this helps to reduce the wind-up effect. As $y(\cdot)$ gets closer to $\bar{v}$ and the error decreases, the integral gain increases to compensate for small errors. Thus, the nonlinear integral gain can result in a shorter settling time.

The heuristic arguments above imply that the performance of the closed-loop system with the nonlinear compensator H in (3) can be superior to that with linear PI compensators, when the parameters $k_p, k_i, g_p, \lambda, \mu$ are chosen appropriately. We obtain the parameters $k_p, k_i, g_p, \lambda, \mu$ by solving an optimization problem whose solution is the optimal values of these parameters.

3. Optimal Compensators

Consider the closed-loop system $S(P, H)$ in Fig. 1. The state-space representation of $S(P, H)$ is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} Ax(t) + b[k_p + g_p \exp(\lambda|\bar{v} - cx(t)|)](\bar{v} - cx(t)) + bk_i\xi(t) \\ [\bar{v} - cx(t)]/[1 + \mu^2(\bar{v} - cx(t))^2] \end{bmatrix}, \tag{4a}$$

$$y(t) = [c, 0]\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \tag{4b}$$

for all $t \geq 0$, with the initial conditions $x(0) = \theta_n$ and $\xi(0) = 0$.

We denote the equilibrium point of the system (4) by $(x_e, \xi_e) \in \mathbb{R}^n \times \mathbb{R}$. Clearly, x_e satisfies $cx_e = \bar{v}$, and the output at the equilibrium is $y_e := cx_e = \bar{v}$. We denote the constant input to the plant when the system is at the equilibrium by u_e. The input u_e generates the desired set point $\bar{v}$, and hence is given by

$$u_e := (P(0))^{-1}\bar{v}, \tag{5}$$

where by (A1), $P(0) \neq 0$. Assuming that $k_i \neq 0$, from (3b) and (5) we obtain

$$(x_e, \xi_e) = (x_e, k_i^{-1}u_e) = (x_e, k_e^{-1}(P(0))^{-1}\bar{v}). \tag{6}$$

Suppose that the states of the system $S(P, H)$ are in a small neighborhood of the equilibrium point of the system. Then, the system output $y(\cdot) = cx(\cdot)$ is close to the desired set point $\bar{v}$, i.e., $e(\cdot) = \bar{v} - cx(\cdot) \approx 0$. In this case, the dynamics of the closed-loop system (4) can be approximated by the dynamics of the linear system obtained by the Jacobian linearization of the system (4) at the equilibrium point (x_e, ξ_e), at which

$$e = \bar{v} - cx_e = 0.$$

The linearized closed-loop system is

$$\begin{bmatrix} \delta\dot{x}(t) \\ \delta\dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A - b(k_p + g_p)c & bk_i \\ -c & 0 \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta\xi(t) \end{bmatrix} + \begin{bmatrix} b(k_p + g_p) \\ 1 \end{bmatrix} \bar{v}, \tag{7a}$$

$$\delta y(t) = [c, 0] \begin{bmatrix} \delta x(t) \\ \delta\xi(t) \end{bmatrix}, \tag{7b}$$

for all $t \geq 0$, where

$$\delta x(t) := x(t) - x_e, \qquad \delta\xi(t) := \xi(t) - \xi_e, \qquad \delta y(t) := c\delta x(t).$$

It is well known that (see, e.g., Ref. 16, pp. 209–219) in a neighborhood of the equilibrium point, the stability of the linearized closed-loop system (7) implies the exponential stability of the system (4). Thus, the stability of the system (4) is determined by the eigenvalues of the matrix

$$A_c := \begin{bmatrix} A - b(k_p + g_p)c & bk_i \\ -c & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{8}$$

We denote the eigenvalues of A_c by

$$\lambda_i(A_c) = \mathrm{Re}(\lambda_i(A_c)) + j\,\mathrm{Im}(\lambda_i(A_c)), \qquad i = 1, 2, \ldots, n+1.$$

With this setup, we cast the problem of determining the parameters $k_p, k_i, g_p, \lambda, \mu$ in (4) into an optimization problem.

Problem 3.1. Consider the closed-loop system $S(P, H)$ in (4) whose solution is $t \mapsto [x^T(t), \xi(t)]^T$. Let a scalar-valued cost function J be defined as

$$J := J_T + \gamma J_s, \tag{9}$$

where

$$J_T := \int_0^T [q|e(t)/\bar{v}| + r[(u(t) - u_e)/\bar{v}]^2]\, dt, \tag{10a}$$

$$J_s := \max_{1 \leq i \leq n+1} \max\{0, \rho(\mathrm{Re}(\lambda_i(A_c)) + \sigma_d), \\ [\mathrm{Re}(\lambda_i(A_c)) + |\mathrm{Im}(\lambda_i(A_c))/\alpha|]/[|\mathrm{Re}(\lambda_i(A_c))| + \delta]\}, \tag{10b}$$

and $\gamma \geq 0$ is a weighting factor. In J_T, the integration is carried out over $[0, T]$ where $T < \infty$, the constants $q > 0$ and $r > 0$ are weighting factors, $e(\cdot)$ and $u(\cdot)$ are respectively the tracking error and the input to the plant P, given by

$$e(t) = v(t) - y(t) = \bar{v} - cx(t), \tag{11a}$$

$$u(t) = k_i\xi(t) + [k_p + g_p \exp(\lambda|\bar{v} - cx(t)|)](\bar{v} - cx(t)), \tag{11b}$$

for all $t \geq 0$, and u_e is that in (5). In J_s, the matrix A_c is that in (8), the constant $\rho > 0$ is a weighting factor, $\sigma_d > 0$ and $\alpha > 0$ are the same as those in (2), and $0 < \delta \ll 1$ is a constant real number.

With the above setup, the optimization problem is as follows: determine the parameters $k_p, k_i, g_p, \lambda, \mu$, such that J is minimized.

Remark 3.1. The cost J_T has two terms. The first term is the weighted L_1-norm of $e(\cdot)/\bar{v}$ over $[0, T]$. We have chosen this norm, and not the L_2-norm, in order to take small tracking errors into account, and hence achieve higher tracking accuracy. By penalizing the tracking error $e(\cdot)/\bar{v}$ substantially, i.e., choosing the weighting factor q large, we expect small error, and hence fast tracking of desired step inputs. The second term is the weighted L_2-norm of $(u(\cdot) - u_e)/\bar{v}$ over $[0, T]$. We have chosen this norm in order to avoid large control energy. By penalizing the input $(u(\cdot) - u_e)/\bar{v}$ substantially, i.e., by choosing the weighting factor r large, we expect small control effort.

Remark 3.2. The cost J_s has two nonzero terms. By penalizing the term $(\mathrm{Re}(\lambda_i(A_c)) + \sigma_d)$ substantially (large ρ), we expect the poles of the linearized closed-loop system not to be far to the right of the vertical line $\mathrm{Re}(s) = -\sigma_d$ in the complex plane. By penalizing the other nonzero term in J_s substantially (small ρ), we expect the ratios $|\mathrm{Im}(\lambda_i(A_c))|/|\mathrm{Re}(\lambda_i(A_c))|$ not to be much smaller than $-\alpha$, when $\mathrm{Re}(\lambda_i(A_c)) < 0$ and $\mathrm{Im}(\lambda_i(A_c)) \geq 0$, and not to be much larger than α, when $\mathrm{Re}(\lambda_i(A_c)) < 0$ and $\mathrm{Im}(\lambda_i(A_c)) \leq 0$.

We note that, if the poles of the linearized closed-loop system $\lambda_i(A_c) \in D$, for all $i = 1, 2, \ldots, n+1$, then $J_s = 0$.

Remark 3.3. The cost J_s provides a measure of the stability of the closed-loop system. We have incorporated J_s in J in order to guarantee the stability of the closed-loop system. If J_s were not considered, then the solution of Problem 3.1 can be a set of parameters for which $J = J_T$ is minimum, but yet the closed-loop system is unstable. The minimum of $J = J_T$ can be achieved while the closed-loop system is unstable, because J_T

is computed over a finite interval of time and is always finite, even when the closed-loop system is unstable.

Remark 3.4. When a linear PI compensator is used in the system $S(P, H)$, the magnitudes of $e(\cdot)$ and $u(\cdot) - u_e$ are proportional to the amplitude $\bar{v}$ of the step input. Since $e(\cdot)$ and $u(\cdot) - u_e$ are normalized by $\bar{v}$ in J_T, and since J_s is independent of $\bar{v}$, the optimal parameters of linear PI compensators obtained by minimizing J are independent of $\bar{v}$.

By (A2), there exists a linear PI compensator with the transfer function $H(s) = k_{pl} + k_{il}/s$ that stabilizes the equilibrium point of the system $S(P, H)$. We can solve Problem 3.1 with $g_p = \mu = 0$, in order to obtain the optimal k_{pl} and k_{il}, denoted by k_{pl}^* and k_{il}^*, respectively. The optimal parameters achieve the minimum value of the cost J in (9), denoted by J_l^*. Our goal, however, is to solve Problem 3.1: we are to determine the set of optimal parameters $\pi_n^* := \{k_p = k_p^*, k_i = k_i^*, g_p = g_p^*, \lambda = \lambda^*, \mu = \mu^*\}$ of the nonlinear PI compensator, for which g_p^* and μ^* are not necessarily zero. The set of optimal parameters π_n^* achieves the minimum of J, denoted by J_n^*. Since the set of nonlinear PI compensators includes that of linear PI compensators, $J_n^* \leq J_l^*$. Thus, the search for the set of optimal parameters π_n^* can do no worse than to return to $k_p = k_{pl}^*$, $k_i = k_{il}^*$, and $g_p = \mu = 0$, which corresponds to the optimal linear PI compensator. That is, Problem 3.1 always has a solution.

Problem 3.1 can be solved efficiently when standard numerical packages are used. We use the fact that $J_s = 0$, when $\lambda_i(A_c) \in D$, for all $i = 1, 2, \ldots, n+1$, to devise an efficient algorithm for solving Problem 3.1. In the remainder of this section, we designate the dependence of J, J_T, J_s on the set of parameters $\pi = \{k_p, k_i, g_p, \lambda, \mu\}$ by $J(\pi)$, $J_T(\pi)$, $J_s(\pi)$, respectively. We first establish the following proposition.

Proposition 3.1. Consider the closed-loop system $S(P, H)$ in (4) and the cost function J in (9). Let $\epsilon > 0$ be given. There exists a $\gamma' \geq 0$ such that, for all $\gamma \geq \gamma'$, the set of optimal parameters π^* minimizing the cost function J results in $J_s(\pi^*) \leq \epsilon$.

Proof. By (A2), there exist a linear PI compensator with the transfer function $H(s) = k_{pl} + k_{il}/s$ that places the poles of the system $S(P, H)$ in the region D in (2). That is, for $k_p = k_{pl}$, $k_i = k_{il}$, and $g_p = 0$, all the eigenvalues of the matrix A_c in (8) are in D. Using the theory of perturbation of the eigenvalues of a matrix due to the perturbation in its elements (see, e.g., Refs. 17 and 18), we conclude that, for $k_p = k_{pl}$, $k_i = k_{il}$, and

$g_p = \epsilon \neq 0$, when $|\epsilon|$ is sufficiently small, all the eigenvalues of the matrix A_c are in D. Thus, there exists a set of parameters π_s such that $J_s(\pi_s) = 0$, and hence $J(\pi_s) = J_T(\pi_s)$. Let $J^* = J_T^*$ be the minimum of J when $\gamma = 0$. Then, for any $\gamma \geq 0$, we have

$$J^* = J_T^* = \min_{\pi} J_T(\pi) \leq J_T(\pi_s) = J(\pi_s). \tag{12}$$

Thus, $J(\pi_s) - J_T^* \geq 0$. Let

$$\gamma' := (J(\pi_s) - J_T^*)/\epsilon \geq 0. \tag{13}$$

For any π and $\gamma \geq \gamma'$, we have

$$J(\pi) = J_T(\pi) + \gamma J_s(\pi) \geq J_T^* + (J(\pi_s) - J_T^*) J_s(\pi)/\epsilon. \tag{14}$$

Now, suppose that $\pi = \pi^*$ is the set of optimal parameters minimizing J. For the sake of contradiction, suppose that $J_s(\pi^*) > \epsilon$. Then, from (14), we obtain $J(\pi^*) > J(\pi_s)$, which is a contradiction to the optimality of π^*. Thus, $J_s(\pi^*) \leq \epsilon$. □

Remark 3.5. Proposition 3.1 implies that regardless of the values of T, q, r, the poles of the linearized closed-loop system can be placed inside and/or arbitrarily close to the region D by choosing γ in (9) sufficiently large.

Using the result in Proposition 3.1, we devise the following efficient algorithm for solving Problem 3.1.

Algorithm 3.1. Computing the Optimal Parameters of Compensators.

Step 1. Choose a cost function as J in (9).

Step 2. Set $\gamma = 0$ in J.

Step 3. Start with the initial guesses $k_p \neq 0$, $k_i \neq 0$, which corresponds to a stabilizing linear PI compensator, and $g_p = \mu = 0$.

Step 4. Use a program that solves ordinary differential equations to compute $[x^T(t), \xi(t)]^T$, $e(t)$, $u(t)$ in (4), (11a), (11b), respectively, over $[0, T]$. Then, use an integration program to compute J_T in (10a).

Step 5. Use a program that computes the eigenvalues of matrices to compute J_s in (10b).

Step 6. Compute J in (9), and use a minimization program to compute the optimal parameters that minimize J.

Step 7. Use a program that computes the eigenvalues of matrices to compute J_s in (10b) for the optimal parameters computed in Step 6. If $0 \leq J_s < 1$, then stop. In this case, the poles of the linearized closed-loop system are inside and/or close to the region D, and the system has satisfactory tracking and stability. If $J_s \gg 1$, then increase γ, and go to Step 3. By Proposition 3.1, for a sufficiently large γ, the cost J_s will be smaller than 1.

Remark 3.6. For the optimal parameters the cost J in (9) is minimized, and the following is achieved:

(i) the cost J_T is small, and so is the tracking error $e(\cdot)$, while large control effort $u(\cdot)$ is avoided;

(ii) the cost J_s is small, and hence the poles of the linearized closed-loop system, $\lambda_i(A_c)$, $i = 1, 2, \ldots, n+1$, are placed inside and/or close to the region D in Fig. 2.

Remark 3.7. Suppose that the exogenous step input to the closed-loop system $v(t) = \bar{v}U(t)$, $t \geq 0$, can assume different amplitudes; more precisely, $\bar{v} \in [v_{\min}, v_{\max}] =: I \subset \mathbb{R}$. In this case, Problem 3.1 can be solved for the parameters $k_p, k_i, g_p, \lambda, \mu$ at a finite number of points in I. Then, the optimal parameters can be tabulated as functions of $\bar{v}$. Let $\bar{v}_i$ and $\bar{v}_{i+1}$ be two adjacent points in I at which the optimal parameters of the nonlinear PI compensator are computed. At a point $\bar{v} \in [\bar{v}_i, \bar{v}_{i+1}]$, the value of the parameters can be taken as the linear interpolation of those computed at $\bar{v}_i$ and $\bar{v}_{i+1}$. If this linear interpolation is carried out between all adjacent points in I at which the parameters $k_p, k_i, g_p, \lambda, \mu$ are computed, then these parameters will be piecewise linear functions of $\bar{v}$ on I.

4. Generalization and Other Nonlinear PI Compensators

In Section 2, we proposed a specific nonlinear PI compensator for controlling the system (1). In this section, we formulate the design of a general nonlinear PI compensator. We then propose an approximate technique for determining such a compensator.

Consider the feedback system $S(P, H)$ in Fig. 1, and recall that the plant P is represented by (1). We choose the compensator H to be the nonlinear SISO system represented by

$$\dot{\xi}(t) = f(e(t)), \qquad \xi(0) = 0, \tag{15a}$$

$$u(t) = g(\xi(t)) + h(e(t)), \tag{15b}$$

for all $t > 0$. In (15), the state $\xi(t) \in \mathbb{R}$, the input $e(t) \in \mathbb{R}$, and the output $u(t) \in \mathbb{R}$; the functions $f: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ are odd and continuously differentiable functions of e, hence $f(0) = h(0) = 0$, and $g: \mathbb{R} \to \mathbb{R}$ is an odd and continuously differentiable function of ξ whose inverse exists. The system H is a general nonlinear PI compensator. Note that, if $f(e) = e$, $g(\xi) = k_i \xi$, and $h(e) = k_p e$, then the system (15) represents a linear PI compensator.

The state-space representation of the system $S(P, H)$ with the nonlinear compensator H in (15) is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} Ax(t) + bh(\bar{v} - cx(t)) + bg(\xi(t)) \\ f(\bar{v} - cx(t)) \end{bmatrix}, \tag{16a}$$

$$y(t) = [c, 0] \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \tag{16b}$$

for all $t \geq 0$, with the initial conditions $x(0) = \theta_n$ and $\xi(0) = 0$. The equilibrium point of the system (16), denoted by $(x_e, \xi_e) \in \mathbb{R}^n \times \mathbb{R}$, is

$$(x_e, \xi_e) = (x_e, g^{-1}(u_e)) = (x_e, g^{-1}((P(0))^{-1}\bar{v})), \tag{17}$$

where x_e satisfies $cx_e = \bar{v}$ and u_e is that in (5). Therefore, the output of the closed-loop system at the equilibrium is $y_e := cx_e = \bar{v}$.

Our goal is to determine the optimal functions f, g, h, denoted respectively by f^*, g^*, h^*, that minimize a cost function such as J in (9), with A_c obtained by linearization of (16) at the equilibrium point, subject to (16). The task of determining the optimal functions f^*, g^*, h^* is difficult; these function, however, can be computed approximately.

We propose to approximate g by

$$g(\xi) \approx \hat{g}(\xi) = k_i \xi, \tag{18}$$

and f and h by their rational approximations (see, e.g., Ref. 19, p. 107)

$$f(e) \approx \hat{f}(e) = \left[\sum_{j=0}^{m_f} a_{f,j} |e|^j \Big/ \sum_{j=0}^{n_f} b_{f,j} |e|^j \right] e, \tag{19a}$$

$$h(e) \approx \hat{h}(e) = \left[\sum_{j=0}^{m_h} a_{h,j} |e|^j \Big/ \sum_{j=0}^{n_h} b_{h,j} |e|^j \right] e, \tag{19b}$$

where the coefficients k_i and $a_{f,j}$, $j = 0, \ldots, m_f$, $b_{f,j} > 0$, $j = 0, \ldots, n_f$, $a_{h,j}$, $j = 0, \ldots, m_h$, and $b_{h,j} > 0$, $j = 0, \ldots, n_h$, are constant real numbers. Note that $\hat{f}$ and $\hat{h}$ are odd and continuously differentiable functions of e. The function $\hat{g}$ is obviously an odd and continuously differentiable function of ξ whose inverse exists. We substitute the approximate representations of f, g, h from (18) and (19) into (16). Then, we determine the optimal parameters k_i, $a_{f,j}$, $b_{f,j}$, $a_{h,j}$, $b_{h,j}$, denoted respectively by k_i^*, $a_{f,j}^*$, $b_{f,j}^*$, $a_{h,j}^*$, $b_{h,j}^*$, for which the minimum value of a cost function such as J in (9) with the appropriate A_c is achieved. The optimal parameters in (18) and (19) result in the optimal functions $\hat{f}^*$, $\hat{g}^*$, $\hat{h}^*$, which are approximations of the optimal functions f^*, g^*, h^*, respectively.

It is clear that, if more terms are considered in (19), then the functions $\hat{f}$ and $\hat{h}$ are better approximations of f and h, respectively. However, including more terms increases the computation time of the optimal parameters in $\hat{f}$ and $\hat{h}$. The following procedure can be used to compute the optimal parameters efficiently:

Algorithm 4.1. Computing the Optimal Parameters in $\hat{f}, \hat{g}, \hat{h}$.

Step 1. Let $m_f = n_f = m_h = n_h = 1$.

Step 2. Use Algorithm 3.1 to compute the optimal parameters k_i^*, $a_{f,j}^*$, $b_{f,j}^*$, $a_{h,j}^*$, $b_{h,j}^*$, $j = 0, 1, \dots, m_f$, for which the minimum value of a cost function such as J in (9) with

$$A_c = \begin{bmatrix} A - b(k_p + a_{h,0}/b_{h,0})c & bk_i \\ -(a_{f,0}/b_{f,0})c & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \tag{20}$$

is achieved.

Step 3. Increase m_f, n_f, m_h, n_h by one; repeat Step 2. If the minimum value of J does not change appreciably by increasing m_f, then stop; otherwise, repeat Step 3.

Remark 4.1. Our experiments with the functions $\hat{f}$ and $\hat{h}$ indicate that the following nonlinear PI compensator can result in satisfactory step responses:

$$\dot{\xi}(t) = e(t)/[1 + \mu^2 e^2(t)], \qquad \xi(0) = 0, \tag{21a}$$

$$u(t) = k_i \xi(t) + [(a_0 + a_1|e(t)|)/(b_0 + b_1|e(t)|)]e(t). \tag{21b}$$

The nonlinear compensator in (21) has six parameters μ, k_i, a_0, a_1, $b_0 > 0$, $b_1 > 0$ to be computed. This compensator is derived from the general nonlinear PI compensator in (15) by letting $f(e) \approx e/(1 + \mu^2 e^2)$, $g(\xi) \approx k_i \xi$, and h as that in (19b) with $m_h = n_h = 1$.

When the six-parameter nonlinear compensator in (21) is used in the system $S(P, H)$, the coefficient matrix A_c whose eigenvalues determine the stability of the linearized closed-loop system is

$$A_c = \begin{bmatrix} A - b(k_p + a_0/b_0)c & bk_i \\ -c & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{22}$$

Remark 4.2. Another technique to approximate the functions f, g, h is by exponential sums (see, e.g., Ref. 19, p. 167). For instance, h in (15) can be approximated by

$$h(e) \approx \hat{h}(e) = \left[k_p + g_p \exp(\lambda |e|) + \sum_{j=1}^{m_h} h_j \exp(\lambda_j |e|) \right] e, \tag{23}$$

where k_p, g_p, λ, h_j, λ_j, $j = 1, \dots, m_h$, are constant real numbers. Clearly, the five-parameter nonlinear PI compensator in (3), which was constructed based on a heuristic argument, is an approximation to the general nonlinear PI compensator in (15): in order to obtain the compensator in (3) from that in (15), approximate f by the rational function $e/(1 + \mu^2 e^2)$, replace g by $k_i \xi$, and approximate h by the function in (23) while keeping only the first two terms.

5. Examples

In this section, we consider the unity feedback system $S(P, H)$ for two different plants P. For each of these systems, we determine the optimal linear PI compensator as well as the optimal nonlinear PI compensators in (3) and (21) via Algorithm 3.1. We demonstrate the superiority of our nonlinear PI compensators over the optimal linear PI compensator by comparing the performance of the closed-loop systems in tracking step inputs.

Example 5.1. We consider a lightly damped linear SISO plant whose transfer function is

$$P(s) = (s + 1)/(s^2 + 0.01s + 1). \tag{24}$$

Our goal is to determine the optimal linear and nonlinear PI compensators that achieve satisfactory tracking and stability for the closed-loop system $S(P, H)$.

We chose the cost functions J_T in (10a) with $T = 10$, $q = 30$, $r = 9$, and J_s in (10b) with $\rho = 1000$, $\sigma_d = 0.1$, $\alpha = 1$, $\delta = 0.001$.

First, we computed the optimal parameters of the linear PI compensator $H(s) = k_{pl} + k_{il}/s$ via Algorithm 3.1. The optimal parameters are

$$k_{pl}^* = 3.15, \qquad k_{il}^* = 3.38, \tag{25}$$

when $\gamma = 0$ in J given in (9). For the parameters in (25), $J_T = 32.77$ and $J_s = 0.61$, where in computing J_s we used A_c in (8) with $g_p = 0$. Small J_s implies that the closed-loop system has satisfactory stability.

Next, we computed the optimal parameters of the five-parameter nonlinear PI compensator in (3) via Algorithm 3.1. The optimal parameters, when the amplitude of the step input $\bar{v} = 3$, are

$$k_p^* = 2.36, \qquad g_p^* = 171.00, \qquad \lambda^* = -90.99, \tag{26a}$$

$$k_i^* = 267.39, \qquad \mu^* = 37.01, \tag{26b}$$

when $\gamma = 0$ in J. For the parameters in (26), $J_T = 18.91$ and $J_s = 0$, where in computing J_s we used A_c in (8). Zero J_s implies that all poles of the linearized closed-loop system are in the region D.

Finally, we computed the optimal parameters of the six-parameter nonlinear PI compensator in (21) via Algorithm 3.1. The optimal parameters, when the amplitude of the step input $\bar{v} = 3$, are

$$a_0^* = 19.36, \qquad a_1^* = 19.04, \qquad b_0^* = 0.5748, \qquad b_1^* = 13.01, \tag{27a}$$

$$k_i^* = 270.00, \qquad \mu^* = 30.17, \tag{27b}$$

when $\gamma = 0$ in J. For the parameters in (27), $J_T = 19.12$ and $J_s = 0$, where in computing J_s we used A_c in (22). Zero J_s implies that all poles of the linearized closed-loop system are in the region D.

Responses of the closed-loop system to the step input of amplitude $\bar{v} = 3$, when the optimal linear and nonlinear compensators are used, are depicted in Fig. 3a. The control inputs to the plant, generated by the optimal linear and nonlinear compensators, are shown in Fig. 3b. The superior performance of the system controlled by the optimal nonlinear PI compensators while applying smaller control inputs is evident from Figs. 3a and 3b.

Example 5.2. We consider a nonminimum phase linear SISO plant whose transfer function is

$$P(s) = 2(s^2 - 1.2s + 0.48)/(s^2 + 4s + 2)(s^2 + 1.2s + 0.48). \tag{28}$$

The transfer function $P(s)$ is an approximate representation of the delayed system $2\exp(-0.2s)/(s^2 + 4s + 2)$.

We set the same goals as those in Example 5.1 for the system (28). We chose the same cost functions as those in Example 5.1, except that we set $r = 0.9$ in J_T.

First, we computed the optimal parameters of the linear PI compensator $H(s) = k_{pl} + k_{il}/s$ via Algorithm 3.1. The optimal parameters are

$$k_{pl}^* = 2.313, \qquad k_{il}^* = 1.181, \tag{29}$$

when $\gamma = 1$ in J. For the parameters in (29), $J_T = 32.07$ and $J_s = 0.465$,

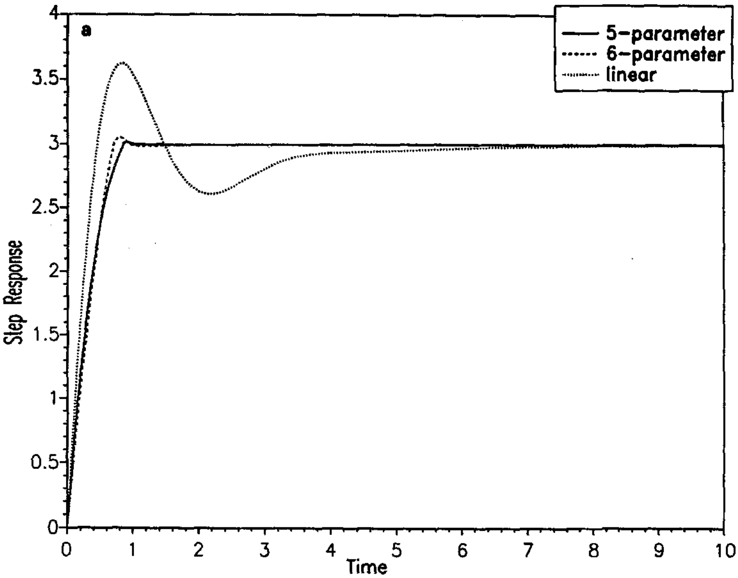

Fig. 3a. Responses of the closed-loop system $S(P, H)$ with the lightly damped plant P in (23) to the step input of amplitude 3, when the compensator H is the optimal linear PI compensator, the optimal five-parameter nonlinear compensator in (3), and the optimal six-parameter nonlinear compensator in (21).

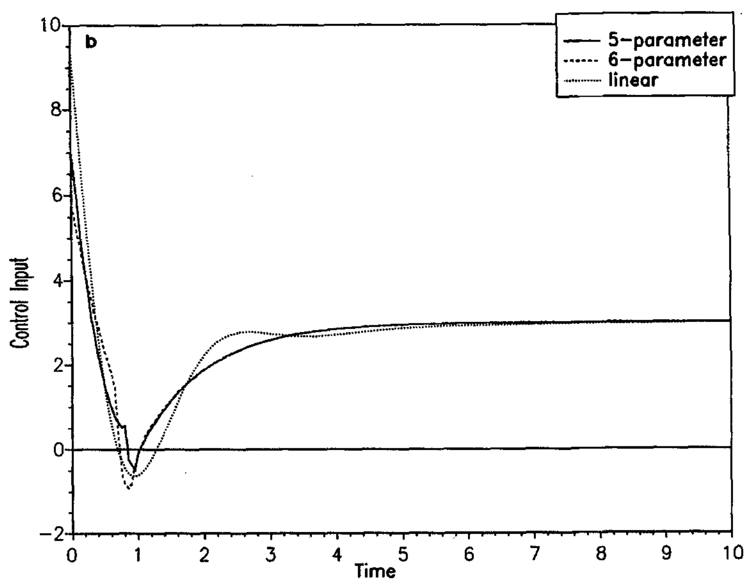

Fig. 3b. Control inputs to the plant P in (23), when the compensator is the optimal linear PI and the optimal five- and six-parameter nonlinear PI compensators.

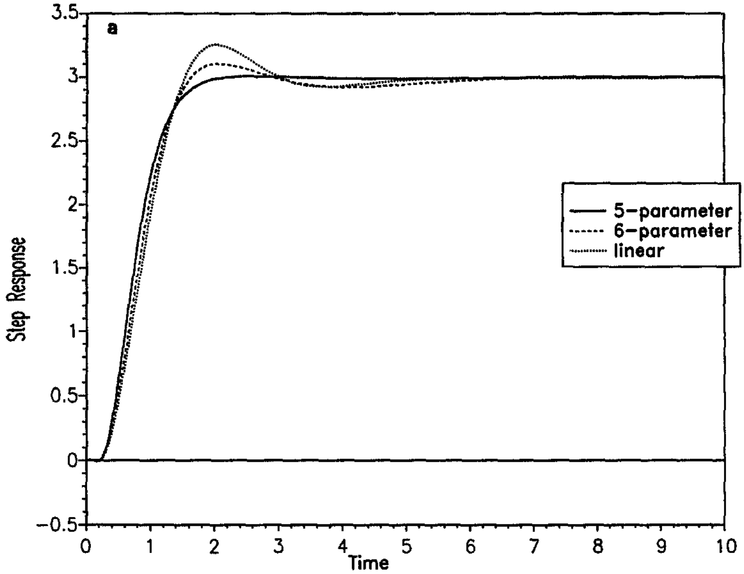

Fig. 4a. Responses of the closed-loop system $S(P, H)$ with the nonminimum phase plant P in (27) to the step input of amplitude 3, when the compensator H is the optimal linear PI compensator, the optimal five-parameter nonlinear compensator in (3), and the optimal six-parameter nonlinear compensator in (21).

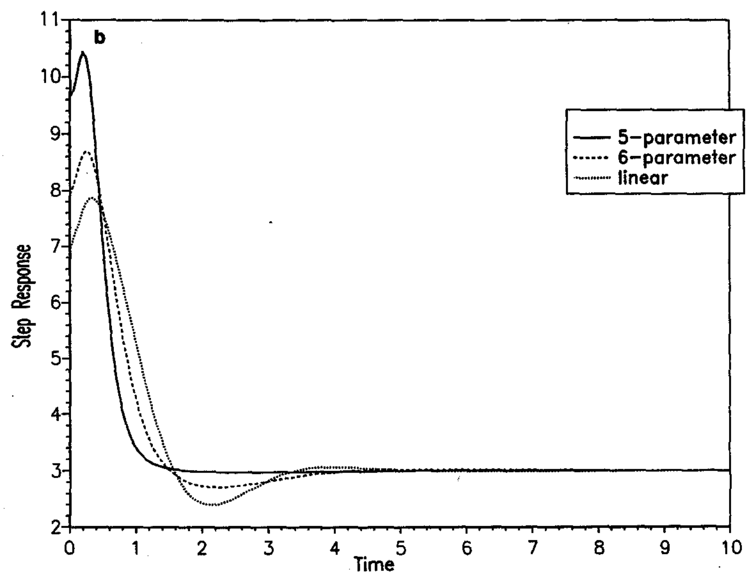

Fig. 4b. Control inputs to the plant P in (27), when the compensator is the optimal linear PI and the optimal five- and six-parameter nonlinear PI compensators.

where in computing J_s we used A_c in (8) with $g_p = 0$. Small J_s implies that the closed-loop system has satisfactory stability.

Next, we computed the optimal parameters of the five-parameter nonlinear PI compensator in (3) via Algorithm 3.1. The optimal parameters, when the amplitude of the step input $\bar{v} = 3$, are

$$k_p^* = 0.2309, \qquad g_p^* = 0.4366, \qquad \lambda^* = 0.6403, \tag{30a}$$

$$k_i^* = 1.2312, \qquad \mu^* = 0.0312, \tag{30b}$$

when $\gamma = 1$ in J. For the parameters in (30), $J_T = 28.656$ and $J_s = 1.527$, where in computing J_s we used A_c in (8). Small J_s implies that the linearized closed-loop system has reasonable stability.

Finally, we determined the optimal parameters of the six-parameter nonlinear PI compensator in (21) via Algorithm 3.1. The optimal parameters, when the amplitude of the step input $\bar{v} = 3$, are

$$a_0^* = 0.8924, \qquad a_1^* = 0.2925, \quad b_0^* = 0.6674, \quad b_1^* = 0.0008, \tag{31a}$$

$$k_i^* = 1.1442, \qquad \mu^* = 0.0016, \tag{31b}$$

when $\gamma = 1$ in J. For the parameters in (31), $J_T = 29.90$ and $J_s = 0$, where in computing J_s we used A_c in (22). Zero J_s implies that all poles of the linearized closed-loop system are in the region D.

Responses of the closed-loop system to the step input of amplitude $\bar{v} = 3$, when the optimal linear and nonlinear compensators are used, are depicted in Fig. 4a. The control inputs to the plant, generated by the optimal linear and nonlinear compensators, are shown in Fig. 4b.

6. Conclusions

In this paper, we provided a technique of designing and tuning high-performance nonlinear PI compensators for linear time-invariant SISO systems that are stabilizable by linear PI compensators. We proposed different nonlinear PI compensators. We tuned the parameters of the proposed compensators by solving an optimization problem via an easy-to-implement algorithm. Our design methodology can be viewed as a computer-aided design technique, by which optimal nonlinear PI compensators are designed and tuned. The optimal nonlinear compensators achieve superior tracking and stability for closed-loop systems as compared to what is achieved by the optimal linear PI compensators; this is evident from the examples provided in the paper.

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