Monotone iterative method for a class of fractional differential equations

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1 Abstract

Problem considered:

\[ D_{0+}^\alpha u(t) = f(t, u(t)), \quad t \in (0, h), \]
\[ t^{1-\alpha} u(t) \mid_{t=0} = u_0, \]

where \( 0 < h < +\infty \), \( f \in C([0, h] \times R, R) \), \( D_{0+}^\alpha u(t) \) is the standard Riemann-Liouville fractional derivative, \( 0 < \alpha < 1 \).

1. Some mistakes in the literatures are corrected;
2. A new condition on the nonlinear term is given to guarantee the equivalence;
3. The existence of maximal and minimal solutions for the problem is given.
2 Introduction

Some recent contributions to the theory of fractional differential equations initial value problems can be seen in [1].

In [7], the lower and upper solution method was used firstly to study the IVP

\[ D^{\alpha}_{0+} u(t) = f(t, u(t)), \quad t \in (0, 1), \quad (0 < \alpha < 1), \]
\[ u(0) = 0, \]

where \( f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty) \) is continuous and \( f(t, \cdot) \) is nondecreasing.

In [3, 6], the existence and uniqueness of solution of the IVP

\[ D^{\alpha}_{0+} u(t) = f(t, u(t)), \quad (0 < \alpha < 1; \quad t > 0), \quad (2.1) \]
\[ D^{\alpha-1}_{0+} u(0+) = u_0. \quad (2.2) \]

was obtained under the assumption that \( f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) is Lipchitzian.
In [8], the existence and uniqueness of solution of the IVP

\[ D^\alpha u(t) = f(t, u(t)), \quad t \in (0, T], \]

\[ t^{1-\alpha}u(t) \big|_{t=0} = u_0 \]

was discussed by using the method of lower and upper solutions and its associated monotone iterative method.

However, the maximum principle (Lemma 2.1 of [8]) is proved incorrectly. The author used the relation

\[ I_{0+}^1 D_{0+}^\alpha = I_{0+}^{1-\alpha} \]

although the conditions were not meet.
Motivated by the above references, we will focus our attention in this paper on the following problem

\[ D_{0+}^\alpha u(t) = f(t, u(t)), \quad t \in (0, h), \quad (2.3) \]

\[ t^{1-\alpha} u(t) |_{t=0} = u_0, \quad (2.4) \]

where \( f \in C([0, h] \times R, R) \), \( D_{0+}^\alpha u(t) \) is the standard Riemann-Liouville fractional derivative, \( 0 < \alpha < 1 \). The existence of the blow-up solution, that is to say \( u \in C(0, h] \) and \( \lim_{t \to 0^+} u(t) = \infty \), is obtained by the use of the lower and upper solution method.
3 Preliminaries

Given $0 \leq a < b < +\infty$ and $r > 0$, define a set

$$C_r[a, b] = \{ u \mid u \in C(a, b), (t - a)^r u(t) \in C[a, b] \}.$$

Clearly, $C_r[a, b]$ is a linear space with the normal multiplication and addition.

Given $u \in C_r[a, b]$, define

$$\|u\| = \max_{t \in [a, b]} (t - a)^r |u(t)|,$$

then $(C_r[a, b], \| \cdot \|)$ is a Banach space.

\[ D_{0+}^{\alpha}u(t) + \lambda u(t) = q(t), \]
\[ t^{1-\alpha}u(t) \big|_{t=0} = u_0, \]

where \( \lambda \geq 0 \) is a constant and \( q \in L(0, h) \), the solution

\[ u(t) = \Gamma(\alpha)u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t - s)^{\alpha}) q(s) ds. \]

Here, \( E_{\alpha,\alpha}(t) \) is a Mittag-Leffler function.
Lemma 3.2. For $0 < \alpha \leq 1$, the Mittag-Leffler type function $E_{\alpha,\alpha}(-\lambda t^\alpha)$ satisfies

$$0 \leq E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)}, \quad t \in [0, \infty), \quad \lambda \geq 0.$$ 

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$ 

The function $g(t) := E_{\alpha,\alpha}(-\lambda t^\alpha)$, $t \in (0, +\infty)$ is completely monotonic, that is to say that $g(t)$ possesses of derivatives $g^{(n)}(t)$ for all $n = 0, 1, 2, \cdots$, and $(-1)^n f^{(n)}(t) \geq 0$ for all $t \in (0, \infty)$. 
Lemma 3.3. ([2]) Suppose that \( E \) is an ordered Banach space, \( x_0, y_0 \in E, \ x_0 \leq y_0, \ D = [x_0, y_0] \), \( T : D \to E \) is an increasing completely continuous operator and \( x_0 \leq Tx_0, \ y_0 \geq Ty_0 \). Then the operator \( T \) has a minimal fixed point \( x^* \) and a maximal fixed point \( y^* \). If we let

\[
x_n = Tx_{n-1}, \quad y_n = Ty_{n-1}, \quad n = 1, 2, 3, \ldots,
\]

then

\[
x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_2 \leq y_1 \leq y_0,
\]

\[
x_n \to x^*, \quad y_n \to y^*.
\]

Zero-point theorem

\[
x^5 - 20x^3 + 3x^2 - 1 = f(x) = 0, \quad f(-1) > 0; \quad f(0) < 0; \quad f(10) > 0.
\]
**Definition 3.4.** A function $v(t) \in C_{1-\alpha}[0, h]$ is called as a lower solution of problem (2.3), (2.4), if it satisfies

\[ D_{0+}^\alpha v(t) \leq f(t, v(t)), \quad t \in (0, h), \] (3.1)

\[ t^{1-\alpha}v(t) \mid_{t=0} \leq u_0. \] (3.2)

**Definition 3.5.** A function $w(t) \in C_{1-\alpha}[0, h]$ is called as an upper solution of problem (2.3), (2.4), if it satisfies

\[ D_{0+}^\alpha w(t) \geq f(t, w(t)), \quad t \in (0, h), \] (3.3)

\[ t^{1-\alpha}w(t) \mid_{t=0} \geq u_0. \] (3.4)
4 The existence of solutions

The following assumptions will be used in our main results:

[S1] \( f : [0, h] \times \mathbb{R} \rightarrow \mathbb{R} \) and there exist constants \( A, B \geq 0 \) and \( 0 < r_1 \leq 1 < r_2 < 1/(1 - \alpha) \) such that for \( t \in [0, h] \)

\[
|f(t, u) - f(t, v)| \leq A|u - v|^{r_1} + B|u - v|^{r_2}, \quad u, v \in \mathbb{R}. \tag{3.1}
\]

[S2] Assume that \( f : [0, h] \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies

\[
f(t, u) - f(t, v) + \lambda(u - v) \geq 0, \quad \text{for} \quad \hat{u} \leq v \leq u \leq \tilde{u}, \tag{3.2}
\]

where \( \lambda \geq 0 \) is a constant and \( \hat{u}, \tilde{u} \) are lower and upper solutions of Problem (2.3), (2.4) respectively.
**Theorem 3.1.** Suppose \([S1]\) holds. The function \(u\) solves the problem (2.3), (2.4) if and only if it is a fixed-point of the operator \(T : C_{1-\alpha}[0, h] \rightarrow C_{1-\alpha}[0, h]\) defined by

\[
(Tu)(t) = \Gamma(\alpha)u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \\
+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)[f(s, u(s)) + \lambda u(s)]ds.
\]

**Theorem 3.2.** Suppose \([S1]\) holds. Then the operator \(T\) is a completely continuous operator.
Theorem 3.3. Assume \([S1]\) and \([S2]\) hold and \(v, w \in C_{1-\alpha}[0, h]\) be lower and upper solutions of problem (2.3), (2.4) respectively such that

\[ \nu(t) < \omega(t), \quad 0 \leq t \leq h. \]  

(3.3)

Then, the fractional IVP (2.3), (2.4) has a minimal solution \(x^*\) and a maximal solution \(y^*\) such that

\[ x^* = \lim_{n \to \infty} T^n v, \quad y^* = \lim_{n \to \infty} T^n w. \]
References


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Thank you for your time and patience.