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Robust dissipativity and dissipation of a class of fractional-order uncertain linear systems

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Abstract: The paper addresses robust QSR-dissipativity and feedback dissipation of a class of fractional-order (FO) uncertain linear systems. Both the state and controlled output matrices are with time-varying norm-bounded parameter uncertainties. Firstly, some new notions of QSR-dissipativity and passivity for FO systems are introduced, the relationship between QSR-dissipativity and asymptotic stability and input-output stability are discussed, respectively. Then, a sufficient condition in the form of linear matrix inequality (LMI) is proposed to ensure that such system is robustly QSR-dissipative. According to this condition, a state feedback controller is proposed when the full states can be measured. Secondly, by employing LMI techniques and matrices singular value decomposition, sufficient conditions for the existence and a robust dissipation synthesis method are derived, respectively. Thirdly, a design method of dynamic output feedback controller is developed in order to guarantee that the closed-loop system is dissipative. Finally, some numerical examples are provided to show the application of the proposed methods.

1 Introduction

The fractional order (arbitrary order) calculus, as a non-standard operator, solves the problem that the constitutive model of classical differential equations can not accurately characterize the dynamic behavior of complex systems. It provides a powerful tool for describing practical models with memory properties and historical dependence [1–5]. With the help of fractional operators, more and more complex systems have been modeled in engineering, nature and society, such as fractional Schrodinger equation [6], fractional Langevin equation [7], fractional economic systems [8], fractional order biological equations [9], and so on. So it is extremely significant to study the dynamic characteristics of FO systems and the possible method of changing the dynamics, which will provide the necessary theoretical tools for the construction of high-performance FO automatic control systems [10–16]. Recently, analysis and synthesis of FO system have been attracted much attention and a lot of research achievements on FO systems and control are obtained both in time domain and frequency domain. For example, [17–19] proposed stability conditions in time domain for FO nonlinear systems, interval FO nonlinear systems with time-delay and FO delayed linear systems, respectively. Refs. [20–22] presented stability criteria claimed in frequency/complex-domains for FO (delayed) systems.

Analysis and synthesis of control systems based on dissipative theory are important topics in modern control theory. Dissipativity theory was firstly proposed by Willems in 1972 [23], which aims to characterize the energy attenuation of the systems. The core idea is derived from the physical circuit systems. Since then, it has gradually become a research hotspot in the fields of systems, circuits, networks and control theory [24–28]. Dissipativity theory provides a new tool to design and analyze systems by employing an input-output relationship based on energy-related consideration. This idea not only simplifies the analysis and design of the systems, but also plays a vital role in many aspects of system control. It has been widely used in multi-agent system [29], vehicle system [30], power system [31], physics system [32], energy storage system [33], artificial neural network [34] and so on. The main idea behind this is that many important physical systems have certain input-output properties related to the conservation, dissipation and transport of energy. Models for use in controller design and analysis are usually derived from the basic laws of physics (electrical systems, dynamics, thermodynamics). For a system where the basic laws of physics imply dissipative properties, it may make sense to define the model so that it possesses the same dissipative properties regardless of the numerical values of the physical parameters. Then if a controller is designed so that stability relies on the dissipative properties only, the closed-loop system will be stable whatever the values of the physical parameters. Even a change of the system order will be tolerated provided it does not destroy the dissipativity. There is another aspect of dissipativity which is very useful in practical applications. It turns out that dissipativity considerations are helpful as a guide for the choice of a suitable variable for output feedback. This is helpful for selecting where to place sensors for feedback control [35]. On the other hand, as pointed out in [36, 37], dissipativity has attracted many researchers' attention because it not only unifies the H_∞ and passivity performance but also provides a more flexible robust control design in practical engineering.

However, these theoretical results can not be applied directly to FO systems, the reasons are as follows. First, the dissipative inequality in the definition of dissipativity of integer order systems can not characterize the memory property of fractional energy dissipation of FO systems. Second, fractional systems usually have polynomial convergence speed, rather than the exponential convergence speed that integer order systems generally have, so output of FO systems may diverge. Third, most classical processes observed in the physical world are nonconservative with frictional or dissipative processes, which are well characterized by FO system. Some analysis methods such as the variational principle and passivity theorem described by classical calculus cannot be directly applied to nonconservative systems with energy dissipation. Fourth, although the diffusion properties of fractional operators determine that the intrinsic variables have dissipative properties, the relationship between the initial conditions of FO systems and their diffusivity is not clear. Therefore, to provide a new dissipative theoretical framework for FO systems, it is important and significant to study the dissipative and passive dynamic characteristics of FO systems [38, 39].

Motivated by the above discussions, in this paper we generalize notions of QSR-dissipativity to FO systems, asymptotic stability and input-output stability can be derived from QSR-dissipativity properties. The main contributions of this paper are summarized as follows: (1) A sufficient LMI condition for the robust QSR-dissipativity for uncertain FO linear systems is presented. (2) State feedback controller, static output feedback controller and dynamic output feedback controller are proposed such that the closed system is QSR-dissipative, respectively. (3) The relationship between stability and QSR-dissipativity of FO systems are revealed.

The content of this paper can be listed as follows. Section 2 describes the fundamental concepts and the problem to be addressed. Section 3 presents the main results and discusses the most relevant details. For illustration of the correctness of theoretical results, two numerical examples are presented in Section 4. Finally, some conclusions are given in Section 5.

2 Preliminaries and model description

Standard symbols and notations are used throughout the paper. The following symbols stand for: $AC[a, b]$ the space of function f which is absolutely continuous on $[a, b]$. I identity matrix of appropriate order, and $*$ the elements below the main diagonal of a symmetric block matrix. The superscript T the transpose, $diag\{\cdot\}$ the diagonal matrix, respectively. $L_2[0, \infty)$ the space of square integrable functions on $[0, \infty)$. $X^T = X > 0(< 0)$ a symmetric positive definite (negative definite) matrix. X matrix, if not explicitly stated, is assumed to have the compatible dimension. $\|y\|_\alpha^2 := I^\alpha(y^Ty)$, where I^α denotes α order fractional integral as follows.

Definition 1. *[40] The fractional integral with non-integer order $\alpha > 0$ of function $x(t)$ is defined as follows:*

$$I_{t_0,t}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function, $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$.

Definition 2. *[40] The Caputo derivative of fractional order α of function $x(t)$ is defined as follows:*

$$\begin{aligned} {}_C D_{t_0,t}^{\alpha}x(t) &= I_{t_0,t}^{n-\alpha}\frac{d^n}{dt^n}x(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau, \end{aligned}$$

where $n-1 < \alpha < n \in Z^+$.

Some basic concepts of passivity and dissipativity of FO systems are firstly introduced. Let us consider the following FO nonlinear systems.

$$\begin{cases} D^\alpha x(t) = f(x(t), u(t)), \\ \quad y(t) = h(x(t), u(t)), \end{cases} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^p$ are the state, input and output of the system respectively.

Storage function (energy stored in the system) and a supply function (externally supplied energy) are core concepts in the definition of a dissipative system. In general, dissipativity implies that the increment of the stored energy is bounded by the supplied energy.

Definition 3. *If there exists a storage function $V(x) \geq 0$ such that the (integral) dissipation inequality*

$$\begin{aligned} V &\quad (x(t)) - V(x(t_0)) \leq I_{t_0,t}^{\alpha}w(t) \\ &= \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}w(x(s), u(s), y(s))ds, \end{aligned} \tag{2}$$

holds for all t_0, t_1 with $t_0 \leq t_1$ and all solutions $x = x(t), y = y(t), u = u(t)$, $t \in [t_0, t_1]$, then system (1) is said to be dissipative with respect to the supply rate $w(x, u, y)$, Moreover, assume that the storage function is differentiable, the integral dissipation inequality (2) can be rewritten as

$$D^\alpha V(x(t)) \leq w(x(t), u(t), y(t)). \tag{3}$$

QSR-dissipativity, as a particular case of dissipativity, was presented in [41]. On this occasion the supply rate is selected to be

$$w(x, u, y) = y^TQy + 2y^TSu + u^TRu, \tag{4}$$

where Q, S, and R are real symmetric matrices with appropriate dimensions.

Definition 4. *Assume that system (1) is dissipative. It is called:*
(1) passive if system (2) is satisfied with $w(u, y) = u^Ty$.
(2) input feedforward strictly passive (IFP), if (2) is satisfied for $w(u, y) = u^Ty - \nu u^Tu$, where $\nu > 0$.
(3) output feedback strictly passive (OFP) if (2) is satisfied for $w(u, y) = u^Ty - \rho y^Ty$, where $\rho > 0$.

Considering the following n-dimensional FO uncertain linear system that is controllable and observable

$$\begin{cases} D^\alpha x(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) \\ \qquad + (B_\varpi + \Delta B_\varpi)\varpi(t), \\ \quad y(t) = (C + \Delta C)x(t) + (D + \Delta D)u(t) \\ \qquad + (D_\varpi + \Delta D_\varpi)\varpi(t), \\ \quad z(t) = C_1x(t), \end{cases} \tag{5}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T \in R^n$ represents system state vector, $\varpi(t) \in R^q$ denotes the external disturbance belonging to $L_2[0, \infty]$, $y(t)$ is the controlled output, $u(t) \in R^p$ denotes the control input, $z(t) \in R^l$ represents the measured output, the FO α belongs to the interval $(0, 1)$. A, B, C, D, C_1 are some nominal constant matrices with appropriate dimensions. Time-varying uncertain matrices $\Delta A, \Delta B, \Delta C, \Delta D$ are with appropriate dimensions and the following form:

$$\begin{bmatrix} \Delta A & \Delta B & \Delta B_\varpi \\ \Delta C & \Delta D & \Delta D_\varpi \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} H(t) \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix},$$

where E_1, E_2, and F_1, F_2, F_3 are known real constant matrices and unknown time-varying matrices $H(t)$ satisfies

$$H^T(t)H(t) \leq I.$$

Lemma 2.1. *[42] (Schur complement): For a real matrix $\Sigma = \Sigma^T$, the following assertions are equivalent*

$$\begin{aligned} \Sigma : &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ * & \Sigma_{22} \end{bmatrix} < 0; \\ \Sigma_{11} &< 0, \quad \Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12} < 0; \\ \Sigma_{22} &< 0, \quad \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T < 0. \end{aligned}$$

Lemma 2.2. *[43] Given matrices $Q = Q^T$, H, E and $R = R^T > 0$ of appropriate dimension,*

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfying $F^T F \leq R$, if and only if there exists some $\lambda > 0$ such that

$$Q + \lambda HH^T + \lambda^{-1} E^T RE < 0.$$

Lemma 2.3. *[44] Let $x(t) \in R^n$ be a differentiable vector-value function. Then, for any time instant $t \geq t_0$*

$$D^\alpha(x^T(t)Px(t)) \leq (x^T(t)P)D^\alpha x(t) + (D^\alpha x(t))^T Px(t),$$

where $P \in R^{n\times n}$ is a symmetric positive definite matrix, $\alpha \in (0, 1)$.

Lemma 2.4. *[45] For any matrix $\Pi \in R^{q\times n}$ with $q < n$ and full row rank (rank$(\Pi) = q$), there exists an SVD of Π as follows*

$$\Pi = U \begin{bmatrix} S & 0 \end{bmatrix} V^T,$$

where $S \in R^{q\times q}$ is a diagonal matrix with non-negative diagonal elements in decreasing order, $U \in R^{q\times q}, V \in R^{n\times n}$ are the unitary matrices.

Lemma 2.5. *[45] Given matrix $\Pi \in R^{q\times n}$ with $q < n$ and rank(Π)=q, assume that $X \in R^{n\times n}$ is a symmetric matrix, then there exists a matrix $X \in R^{q\times q}$ satisfying $\Pi X = X\Pi$ if and only if X can be described as*

$$X = V \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix} V^T,$$

where $X_{11} \in R^{q\times q}, X_{22} \in R^{(n-q)\times(n-q)}$ and $V \in R^{n\times n}$ is the unitary matrix of SVD of Π.

The following hypothesis shall be made without loss of generality.

Assumption 2.1.

$$\begin{aligned} Q &< 0, \\ R &+ D_\varpi^T S + SD_\varpi > 0. \end{aligned}$$

In light of Theorem 5 in [46], one can get the following relationship between QSR-dissipativity and Lyapunov stability.

Theorem 2.1. *If system (1) is QSR-dissipative with $Q < 0$ and there exists a continuously differentiable and locally Lipschitz storage function $V(x) > 0$ such that $\alpha_1\|x\|^a \leq V(x) \leq \alpha_2\|x\|^{ab}$, where a, b, α_1, α_2 are arbitrary positive constants, then the system is stable in the sense of Lyapunov with the origin being the equilibrium point.*

Proof. By setting $u = 0$ in (4) and using Definition 3, $D^\alpha V(x) \leq w(0, y) = y^T Qy$. It follows from Assumption 2.1 that $D^\alpha V(x) \leq w(0, y) = y^T Qy < 0$, by virtue of Theorem 5 in [46], the system (1) is Mittag-Leffler stable (asymptotically stable).

Theorem 2.2. *If system (1) is QSR-dissipative with $Q < 0$ and a positive semi-definite storage function $V = V(x)$, then the system is input-output stable.*

Proof. Denote $-a_1 = \lambda_{\max}(Q) < 0$ $(a_1 > 0), a_2 = \lambda_{\max}(S) > 0$, $a_3 = \lambda_{\max}(R) > 0$, it follows from Definition 3 that

$$\begin{aligned} D^\alpha V(x) &\leq -a_1\|y\|^2 + a_2\|y\|\|u\| + a_3\|u\|^2 \\ &\leq -a_1\|y\|^2 + a_2\|y\|\|u\| \\ &+ a_3\|u\|^2 + 2a_2^2 a_1^{-1}\|u\|^2 \\ &= -\frac{1}{2}a_1(\|y\| - 2a_2 a_1^{-1}\|u\|)^2 \\ &+ (2a_2^2 a_1^{-1} + a_3)\|u\|^2 - \frac{1}{2}a_1\|y\|^2 \\ &\leq (2a_2^2 a_1^{-1} + a_3)\|u\|^2 - \frac{1}{2}a_1\|y\|^2. \end{aligned}$$

Take the fractional-order integral,

$$\frac{1}{2}a_1 I^\alpha y^T y \leq (2a_2^2 a_1^{-1} + a_3)I^\alpha u^T u - V(x) + V(0).$$

Since $V(x) > 0$, one can derive

$$I^\alpha y^T y \leq (4a_2^2 a_1^{-2} + 2a_1^{-1}a_3)I^\alpha u^T u + 2a_1^{-1}V(0).$$

which is equivalent to

$$\|y\|_\alpha^2 \leq (4a_2^2 a_1^{-2} + 2a_1^{-1}a_3)\|u\|_\alpha^2 + 2a_1^{-1}V(0),$$

which means that system (1) is input-output stable.

Remark 2.1. *Since Lyapunov method has not been well developed for FO systems, stability or stabilization of many kinds of FO nonlinear systems or uncertain systems remains a formidable problem. It is an undeniable fact that estimation of FO derivative of quadratic form Lyapunov function can address some problems, yet the technology still has major limitations. Theorems 2.1 and 2.2 offer a new insight and method to study stability and stabilization of FO systems, including FO nonlinear systems, uncertain systems and delayed systems.*

3 Main results

3.1 Dissipativity analysis

In this subsection, a QSR-dissipative criterion for system (5) will be presented.

Theorem 3.1. *Under Assumption 2.1, for the given real symmetric matrices Q, S and R, if there exists a symmetric positive-definite matrix $\bar{P}$, such that the following inequality is satisfied,*

$$\Omega = \begin{bmatrix} A\bar{P} + \bar{P}A^T & B_\varpi - \bar{P}C^T S & \bar{P}C^T & \bar{P}F_1^T & \lambda E_1 \\ * & -D_\varpi^T S - SD_\varpi - R & D_\varpi^T & F_3^T & -\lambda SE_2 \\ * & * & Q^{-1} & 0 & E_2 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0, \quad (6)$$

then FO uncertain system (5) is QSR-dissipative.

Proof. Constructing the storage candidate function $V(x(t)) = x^T(t)Px(t)$ and calculating the derivative of $V(x(t))$ along system (5) with $u(t) = 0$, one yields,

$$D^\alpha V(x(t)) \leq (x^T(t)P)D^\alpha x(t) + (D^\alpha x(t))^T Px(t)$$

$$
\begin{aligned}
&= x^T(t)(PA + A^TP + P\Delta A + \Delta A^TP)x(t)\\
&+ x^T(t)PB_{\varpi}\varpi(t) + \varpi^T(t)B_{\varpi}^TPx(t)\\
&+ x^T(t)P\Delta B_{\varpi}\varpi(t) + \varpi^T(t)\Delta B_{\varpi}^TPx(t).
\end{aligned}
$$

According to the dissipativity definition, one has

$$
\begin{aligned}
D^{\alpha}V(x(t)) &- y^T(t)Qy(t) - 2y^T(t)S\varpi(t)\\
&- \varpi^T(t)R\varpi(t)\\
&\leq x^T(t)(PA + A^TP - C^TQC - C^TQ\Delta C\\
&- \Delta C^TQC - \Delta C^TQ\Delta C + P\Delta A\\
&+ \Delta A^TP)x(t) + x^T(t)(2PB_{\varpi} - CQD_{\varpi}\\
&- C^TQ\Delta D - \Delta C^TQD_{\varpi} - \Delta C^TQ\Delta D_{\varpi}\\
&- C^TS - \Delta C^TS + 2P\Delta B_{\varpi})\varpi(t)\\
&- \varpi^T(t)(D_{\varpi}^TQD_{\varpi} + D_{\varpi}^TQ\Delta D_{\varpi}\\
&+ \Delta D_{\varpi}^TQD_{\varpi} + \Delta D_{\varpi}^TQ\Delta D_{\varpi} + D_{\varpi}^TS\\
&+ SD_{\varpi}^T + \Delta D^TS + S\Delta D^T + R)\varpi(t)\\
&=: \xi^T(t)\Pi\xi(t),
\end{aligned}
$$

where $\xi(t) = [x^T(t), \varpi^T(t)]^T$,

$$
\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ * & \Gamma_{22} \end{bmatrix}. \tag{7}
$$

where

$$
\begin{aligned}
\Gamma_{11} &= PA + A^TP + P\Delta A + \Delta A^TP\\
&- (C^T + \Delta C^T)Q(C + \Delta C),\\
\Gamma_{12} &= PB_{\varpi} + P\Delta B_{\varpi} - (C + \Delta C)S\\
&- (C^T + \Delta C^T)Q(D_{\varpi}^T + \Delta^TD_{\varpi}),\\
\Gamma_{22} &= -(D_{\varpi}^T + \Delta D_{\varpi}^T)S - S(D_{\varpi} + \Delta D_{\varpi}) - R\\
&- (D_{\varpi}^T + \Delta D_{\varpi}^T)Q(D_{\varpi}^T + \Delta D_{\varpi}^T).
\end{aligned}
$$

It follows from Lemma 2.1 that Eq. (7) < 0 is equivalent to

$$
\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & (C^T + \Delta C^T) \\ * & -(D_{\varpi}^T + \Delta D_{\varpi}^T)S - S(D_{\varpi} + \Delta D_{\varpi}) - R & (D_{\varpi}^T + \Delta D_{\varpi}^T) \\ * & * & Q^{-1} \end{bmatrix} < 0, \tag{8}
$$

where

$$
\begin{aligned}
\Gamma_{11} &= PA + A^TP + P\Delta A + \Delta A^TP,\\
\Gamma_{12} &= PB_{\varpi} + P\Delta B_{\varpi} - (C^T + \Delta C^T)S.
\end{aligned}
$$

In fact, (8) can be rewritten as follows

$$
\begin{aligned}
\Pi &= \begin{bmatrix} PA + A^TP & PB_{\varpi} - C^TS & C^T \\ * & -D_{\varpi}^TS - SD_{\varpi} - R & D_{\varpi}^T \\ * & * & Q^{-1} \end{bmatrix}\\
&+ \begin{bmatrix} PE_1 \\ -SE_2 \\ E_2 \end{bmatrix} H(t) \begin{bmatrix} F_1 & F_3 & 0 \end{bmatrix}\\
&+ \begin{bmatrix} F_1^T \\ F_3^T \\ 0 \end{bmatrix} H(t) \begin{bmatrix} E_1^TP & -E_2^TS & E_2^T \end{bmatrix} < 0.
\end{aligned} \tag{9}
$$

In light of Lemma 2.2, there exists a scalar λ such that

$$
\begin{aligned}
&\begin{bmatrix} PA + A^TP & PB_{\varpi} - C^TS & C^T \\ * & -D_{\varpi}^TS - SD_{\varpi} - R & D_{\varpi}^T \\ * & * & Q^{-1} \end{bmatrix}\\
&+ \lambda \begin{bmatrix} PE_1 \\ -SE_2 \\ E_2 \end{bmatrix} \begin{bmatrix} E_1^TP & -E_2^TS & E_2^T \end{bmatrix}\\
&+ \lambda^{-1} \begin{bmatrix} F_1^T \\ F_3^T \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & F_3 & 0 \end{bmatrix} < 0,
\end{aligned}
$$

which can be rearranged as

$$
\Pi = \begin{bmatrix} \Pi_{11} & PB_{\varpi} - C^TS - \lambda PE_1E_2^TS + \lambda^{-1}F_1^TF_3 & \lambda PE_1E_2^T + C^T \\ * & -D_{\varpi}^TS - SD_{\varpi} - R + \lambda SE_2E_2^T + \lambda^{-1}F_3^TF_3 & D_{\varpi}^T - \lambda SE_2E_2^T \\ * & * & Q^{-1} + \lambda E_2E_2^T \end{bmatrix} < 0. \tag{10}
$$

where $\Pi_{11} = PA + A^TP + \lambda PE_1E_1^TP + \lambda^{-1}F_1^TF_1$.

By Schur complement, (10) is equivalent to

$$
\Omega = \begin{bmatrix} PA + A^TP & PB_{\varpi} - C^TS & C^T & F_1^T & \lambda PE_1 \\ * & -D_{\varpi}^TS - SD_{\varpi} - R & D_{\varpi}^T & F_3^T & -\lambda SE_2 \\ * & * & Q^{-1} & 0 & E_2 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0. \tag{11}
$$

By pre-multiplying and post-multiplying (11) with $\mathrm{diag}\,\{P^{-1}, I, I, I, I\}$, and let $P^{-1} = \bar{P}$, it follows that inequality (11) is equivalent to (6). The proof is completed.

3.2 *Dissipation via state feedback control*

This subsection will give a sufficient condition for system (5) to be robust QSR-dissipative by designing a state feedback control law.

$$
u(t) = Kx(t). \tag{12}
$$

Theorem 3.2. *Suppose that Assumption 2.1 is satisfied. For the given real symmetric matrices Q, S and R, the controlled system (5) with controller (12) is QSR dissipative if there exists a symmetric positive-definite matrix $\bar{P}$ such that*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \bar{P}C^T + X^T D^T & \bar{P}F_1^T + X^T F_2^T & \lambda E_1 \\ * & -D_\varpi^T S - SD_\varpi - R & D_\varpi^T & F_3^T & -\lambda SE_2 \\ * & * & Q^{-1} & 0 & E_2 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0. \quad (13)$$

where

$$\Omega_{11} = A\bar{P} + BX + \bar{P}A^T + X^T B^T,$$
$$\Omega_{12} = B_\varpi - \bar{P}C^T S - X^T D^T S.$$

Moreover, a dissipation state-feedback gain matrix is obtained by

$$K = X\bar{P}^{-1}.$$

Proof. Consider system (5) with control input (12) as follows:

$$\begin{cases} D^\alpha x(t) = (A + BK + \Delta A + \Delta BK)x(t) \\ \qquad + (B_\varpi + \Delta B_\varpi)\varpi(t), \\ y(t) = (C + DK + \Delta C + \Delta DK)x(t) \\ \qquad + (D_\varpi + \Delta D_\varpi)\varpi(t), \\ z(t) = C_1 x(t), \end{cases} \quad (14)$$

Since $\Delta A + \Delta BK = E_1 H(t)(F_1 + F_2 K)$, $\Delta C + \Delta DK = E_2 H(t)(F_1 + F_2 K)$, replacing A, C and F_1 with $A + BK$, $C + DK$ and $F_1 + F_2 K$ in (6), respectively, one yields

$$\Omega = \begin{bmatrix} \Omega_{11} & B_\varpi - \bar{P}(C+DK)^T S & \bar{P}(C^T + K^T D^T) & \bar{P}(F_1^T + K^T F_2^T) & \lambda E_1 \\ * & -D_\varpi^T S - SD_\varpi - R & D_\varpi^T & F_3^T & -\lambda SE_2 \\ * & * & Q^{-1} & 0 & E_2 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0. \quad (15)$$

where $\Omega_{11} = (A + BK)\bar{P} + \bar{P}(A^T + K^T B^T)$.

Denote $K\bar{P} = X$ in (15), inequality (13) is equivalent to (15). This completes the proof.

3.3 Dissipation via static output-feedback control

This subsection aims to design a static output feedback controller in the form of

$$u(t) = Ly(t), \quad (16)$$

such that the closed-loop systems are robustly QSR-dissipative.

Theorem 3.3. *Assume that Hypothesis 2.1 holds and the singular value decomposition (SVD) of the output matrix C_1 is $\Pi = U\begin{bmatrix} S_1 & 0 \end{bmatrix}V^T$. Then, for the given real symmetric matrices Q, S and R, the controlled system (5) with the control input (16) is strictly (Q, S, R) dissipative if there exist symmetric positive-definite matrices $\tilde{P} > 0$, $\tilde{P}_1 > 0$, $\tilde{P}_2 > 0$ together with matrix X such that*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \tilde{P}C^T + (XC_1)^T D^T & \tilde{P}F_1^T + (XC_1)^T F_2^T & \lambda E_1 \\ * & -D_\varpi^T S - SD_\varpi - R & D_\varpi^T & F_3^T & -\lambda SE_2 \\ * & * & Q^{-1} & 0 & E_2 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0. \quad (17)$$

where

$$\Omega_{11} = A\tilde{P} + BXC_1 + \tilde{P}A^T + (XC_1)^T B^T,$$
$$\Omega_{12} = B_\varpi - \tilde{P}C^T S - C_1^T X^T D^T S,$$
$$\tilde{P} = V\begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix} V^T.$$

Moreover, dissipation output feedback gain matrix, L is provided by

$$L = XUS_1\tilde{P}_1^{-1}S_1^{-1}U^{-1}.$$

Proof By introducing (16) into system (5), the following closed-loop system is obtained

$$\begin{cases} D^\alpha x(t) = (A + BLC_1 + \Delta A + \Delta BLC_1)x(t) \\ \qquad + (B_\varpi + \Delta B_\varpi)\varpi(t), \\ y(t) = (C + DLC_1 + \Delta C + \Delta DLC_1)x(t) \\ \qquad + (D_\varpi + \Delta D_\varpi)\varpi(t), \\ z(t) = C_1 x(t), \end{cases} \quad (18)$$

Since

$$\tilde{P} = V\begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix} V^T.$$

From Lemma 2.5, there exists $\hat{\tilde{P}} = US\tilde{P}_1 S^{-1}U^{-1}$ such that $C_1\tilde{P} = \hat{\tilde{P}}C_1$, where $\hat{\tilde{P}}^{-1} = US\tilde{P}_1^{-1}S^{-1}U^{-1}$. Denote $X = L\tilde{P}_1^{-1}$, inequality (17) is equivalent to (6). The proof is completed.

3.4 Dissipation via dynamic output-feedback control

The aims of this subsection is to design the following dynamic output feedback controller

$$D^\alpha x_k(t) = A_k x_k(t) + B_k z(t),$$
$$u(t) = C_k x_k(t) + D_k z(t), \quad (19)$$

where $x_k(t) \in R_k^n$ is the state of the dynamic output feedback, A_k, B_k, C_k and D_k are appropriate matrices to be determined, such that the following closed-loop system constructed from (5) and (19) is robustly dissipative.

$$\begin{cases} D^\alpha \bar{x}(t) = (\bar{A} + \Delta\bar{A})\bar{x}(t) + (\bar{B}_\varpi + \Delta\bar{B}_\varpi)\varpi(t), \\ y(t) = (\bar{C} + \Delta\bar{C})x(t) + (\bar{D}_\varpi + \Delta\bar{D}_\varpi)\varpi(t), \end{cases} \quad (20)$$

where

$$\bar{A} = \begin{bmatrix} A + BD_k C_1 & BC_k \\ B_k C_1 & A_k \end{bmatrix},$$

$$
\begin{aligned}
\Delta\bar{A} &= \begin{bmatrix} \Delta A + \Delta B D_k C_1 & \Delta B C_k \\ 0 & 0 \end{bmatrix}, \\
\bar{x}(t) &= \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix}, \bar{B}_\varpi = \begin{bmatrix} B_\varpi \\ 0 \end{bmatrix}, \\
\Delta\bar{B}_\varpi &= \begin{bmatrix} \Delta B_\varpi \\ 0 \end{bmatrix}, \\
\bar{C} &= \begin{bmatrix} C_0 + D_0 D_k C_1 & D_0 C_k \end{bmatrix}, \\
\Delta\bar{C} &= \begin{bmatrix} \Delta C + \Delta D D_k C_1 & \Delta D C_k \end{bmatrix}, \\
\bar{D}_\varpi &= D_\varpi, \Delta\bar{D}_\varpi = \Delta D_\varpi.
\end{aligned}
$$

Theorem 3.4. *Under Assumption 2.1, if there exists a symmetrical matrix P, together with matrices A_k, B_k, C_k and D_k of appropriate dimensions and a real scalar λ such that*

$$
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & F_1^T + C_1^T D_k^T F_2^T & \lambda P E_1 \\
* & PA_k + A_k^T P & -C_k^T D^T S & C_k^T D^T & C_k^T F_2^T & 0 \\
* & * & -D_\varpi^T S - S D_\varpi - R & D_\varpi^T & F_3^T & -\lambda S E_2 \\
* & * & * & Q^{-1} & 0 & \lambda E_2 \\
* & * & * & * & -\lambda I & 0 \\
* & * & * & * & * & -\lambda I
\end{bmatrix} < 0. \quad (21)
$$

where

$$
\begin{aligned}
\Omega_{11} &= P(A + BD_kC_1) + (A^T + C_1^T D_k^T B^T)P, \\
\Omega_{12} &= PBC_k + C_1^T B_k^T P, \\
\Omega_{13} &= PB_\varpi - (C^T + C_1^T D_k^T D^T)S, \\
\Omega_{14} &= C^T + C_1^T D_k^T D^T,
\end{aligned}
$$

then the controlled closed-loop system (20) is robustly QSR-dissipative with the given real symmetric matrices Q, S and R.

Proof Let us define a storage function for the closed-loop system (20)

$$
V(\bar{x}(t)) = \bar{x}^T(t)\bar{P}\bar{x}(t),
$$

where $\bar{P} = diag(P, P)$.

Taking the FO time derivative of $V(\bar{x}(t))$ along the trajectory of system (20) and applying the passivity definition, one has

$$
\begin{aligned}
D^\alpha V(\bar{x}(t)) &- y^T(t)Qy(t) - 2y^T(t)S\varpi(t) \\
&- \varpi^T(t)R\varpi(t) \\
&\leq x^T(t)\Big(P(A + BD_kC_1) + (A^T \\
&+ C_1^T D_k^T B^T)P + P(\Delta A + \Delta B D_k C_1) \\
&+ (\Delta A^T + C_1^T D_k^T \Delta B^T)P - (C^T \\
&+ C_1^T D_k^T D^T + \Delta C^T + C_1^T D_k^T \Delta D^T)Q \\
&\times (C + DD_kC_1 + \Delta C + \Delta D D_k C_1)\Big) \\
&\times x(t) + 2x^T(t)(PBC_k + P\Delta B C_k \\
&+ C_1^T B_k^T P)x_k(t) + 2x^T(t)(PB_\varpi \\
&+ P\Delta B_\varpi)\varpi(t) + x_k^T(t)\Big((PA_k + A_k^T P) \\
&- (C_k^T D^T + C_k^T \Delta D^T)Q(DC_k + \Delta DC_k)\Big) \\
&\times x_k(t) - x^T(t)2(C^T + C_1^T D_k^T D^T + \Delta C^T \\
&+ C_1^T D_k^T \Delta D^T)Q(DC_k + \Delta DC_k)x_k(t) \\
&- \varpi^T(t)2(D_\varpi^T + \Delta D_\varpi^T)Q(C + DD_kC_1 \\
&+ \Delta C + \Delta D D_k C_1)x(t) - \varpi^T(t)2(D_\varpi^T \\
&+ \Delta D_\varpi^T)Q(DC_k + \Delta DC_k)x_k(t) - 2x^T(t) \\
&\times (C^T + C_1^T D_k^T D^T + \Delta C^T + C_1^T D_k^T \Delta D^T) \\
&\times S\varpi(t) - 2x_k^T(t)(C_k^T D^T + C_k^T \Delta D^k) \\
&\times S\varpi(t) - \varpi^T(t)\Big(2(D_\varpi^T + \Delta D_\varpi^T)S \\
&+ (D_\varpi^T + \Delta D_\varpi^T)Q(D_\varpi + \Delta D_\varpi) + R\Big)\varpi(t) \\
&= \zeta^T(t)\Phi\zeta(t),
\end{aligned}
$$

where $\zeta(t) = [x^T(t), x_k^T(t), \varpi^T(t)]^T$,

$$
\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ * & \Phi_{22} & \Phi_{23} \\ * & * & \Phi_{33} \end{bmatrix}. \quad (22)
$$

where

$$
\begin{aligned}
\Phi_{11} &= P(A + BD_kC_1) + (A^T + C_1^T D_k^T B^T)P \\
&+ P(\Delta A + \Delta B D_k C_1) + (\Delta A^T + C_1^T D_k^T \Delta B^T)P \\
&- (C^T + C_1^T D_k^T D^T + \Delta C^T + C_1^T D_k^T \Delta D^T)Q \\
&\times (C + DD_kC_1 + \Delta C + \Delta D D_k C_1), \\
\Phi_{12} &= PBC_k + P\Delta BC_k + C_1^T B_k^T P - (C^T \\
&+ C_1^T D_k^T D^T + \Delta C^T + C_1^T D_k^T \Delta D^T) \\
&\times Q(DC_k + \Delta DC_k), \\
\Phi_{13} &= PB_\varpi + P\Delta B_\varpi - (C^T + C_1^T D_k^T D^T + \Delta C^T \\
&+ C_1^T D_k^T \Delta D^T)Q(D_\varpi + \Delta D_\varpi) - 2(C^T \\
&+ C_1^T D_k^T D^T + \Delta C^T + C_1^T D_k^T \Delta D^T)S, \\
\Phi_{22} &= (PA_k + A_k^T P) - (C_k^T D^T + C_k^T \Delta D^T) \\
&\times Q(DC_k + \Delta DC_k), \\
\Phi_{23} &= (C_k^T D^T + C_k^T \Delta D^T)Q(D_\varpi + \Delta D_\varpi) \\
&- (C_k^T D^T + C_k^T \Delta D^k)S, \\
\Phi_{33} &= -(D_\varpi^T + \Delta D_\varpi^T)S - S(D_\varpi + \Delta D_\varpi) \\
&- (D_\varpi^T + \Delta D_\varpi^T)Q(D_\varpi + \Delta D_\varpi) - R.
\end{aligned}
$$

In view of Lemma 2.1, Eq. (22) < 0 is equivalent to

$$
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} & \Sigma_{13} & C^T + C_1^T D_k^T D^T + \Delta C^T + C_1^T D_k^T \Delta D^T \\
* & PA_k + A_k^T P & \Sigma_{23} & C_k^T D^T + C_k^T \Delta D^T \\
* & * & \Sigma_{33} & D_\varpi^T + \Delta D_\varpi^T \\
* & * & * & Q^{-1}
\end{bmatrix} < 0. \quad (23)
$$

where

$$\begin{aligned}\Sigma_{11} &= P(A+BD_kC_1)+(A^T+C_1^TD_k^TB^T)P\\ &+ P(\Delta A+\Delta BD_kC_1)\\ &+ (\Delta A^T+C_1^TD_k^T\Delta B^T)P,\\ \Sigma_{12} &= PBC_k+P\Delta BC_k+C_1^TB_k^TP,\\ \Sigma_{13} &= PB_\varpi+P\Delta B_\varpi-(C^T+C_1^TD_k^TD^T\\ &+ \Delta C^T+C_1^TD_k^T\Delta D^T)S,\\ \Sigma_{23} &= -(C_k^TD^T+C_k^T\Delta D^k)S,\\ \Sigma_{33} &= -(D_\varpi^T+\Delta D_\varpi^T)S-S(D_\varpi+\Delta D_\varpi)-R.\end{aligned}$$

Note that (23) can be rewritten as

$$\begin{aligned}\Pi &= \Pi_1+\begin{bmatrix}PE_1\\0\\-SE_2\\E_2\end{bmatrix}H(t)\\ &\times\begin{bmatrix}F_1+F_2D_kC_1 & F_2C_k & F_3 & 0\end{bmatrix}\\ &+\begin{bmatrix}F_1^T+C_1^TD_k^TF_2^T\\C_k^TF_2^T\\F_3^T\\0\end{bmatrix}H^T(t)\\ &\times\begin{bmatrix}E_1^TP & 0 & -E_2^TS & E_2^T\end{bmatrix}<0. \qquad (24)\end{aligned}$$

where

$$\Pi_1=\begin{bmatrix}\Pi_{11} & \Pi_{12} & PB_\varpi-(C^T+C_1^TD_k^TD^T)S & C^T+C_1^TD_k^TD^T\\ * & PA_k+A_k^TP & -C_k^TD^TS & C_k^TD^T\\ * & * & -D_\varpi^TS-SD_\varpi-R & D_\varpi^T\\ * & * & * & Q^{-1}\end{bmatrix}$$

$$\begin{aligned}\Pi_{11} &= P(A+BD_kC_1)+(A^T+C_1^TD_k^TB^T)P,\\ \Pi_{12} &= PBC_k+C_1^TB_k^TP.\end{aligned}$$

In light of Lemma 2.2, there exists a scalar λ such that

$$\begin{aligned}\Pi_1 &+ \lambda\begin{bmatrix}PE_1\\0\\-SE_2\\E_2\end{bmatrix}\begin{bmatrix}E_1^TP & 0 & -E_2^TS & E_2^T\end{bmatrix}\\ &+ \lambda^{-1}\begin{bmatrix}F_1^T+C_1^TD_k^TF_2^T\\C_k^TF_2^T\\F_3^T\\0\end{bmatrix}\\ &\times\begin{bmatrix}F_1+F_2D_kC_1 & F_2C_k & F_3 & 0\end{bmatrix}<0,\end{aligned}$$

which can be rearranged as

$$\Phi=\begin{bmatrix}\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14}\\ * & \Phi_{22} & -C_k^TD^TS+\lambda^{-1}C_k^TF_2^TF_3 & C_k^TD^T\\ * & * & \Phi_{33} & D_\varpi^T\\ * & * & * & Q^{-1}+\lambda E_2E_2^T\end{bmatrix}<0. \qquad (25)$$

where

$$\begin{aligned}\Phi_{11} &= P(A+BD_kC_1)+(A^T+C_1^TD_k^TB^T)P\\ &+ \lambda PE_1E_1^TP+\lambda^{-1}(F_1+C_1^TD_k^TF_2^T)\\ &\times (F_1+F_2D_kC_1),\\ \Phi_{12} &= PBC_k+C_1^TB_k^TP+\lambda^{-1}(F_1^T\\ &+ C_1^TD_k^TF_2^T)F_2C_k,\\ \Phi_{13} &= PB_\varpi-(C^T+C_1^TD_k^TD^T)S-\lambda PE_1E_2^TS\\ &+ \lambda^{-1}(F_1^T+C_1^TD_k^TF_2^T)F_3,\\ \Phi_{14} &= \lambda PE_1E_2^T+(C^T+C_1^TD_k^TD^T),\\ \Phi_{22} &= PA_k+A_k^TP+\lambda^{-1}C_k^TF_2^TF_2C_k,\\ \Phi_{33} &= -D_\varpi^TS-SD_\varpi-R+\lambda SE_2E_2^TS\\ &+ \lambda^{-1}F_3^TF_3.\end{aligned}$$

By applying Schur complement to (25), one has that (25) is equivalent to (21).

It is obvious that the matrix inequality (21) in Theorem 3.4 is not an LMI because some cross terms of these determined parameters appear in (21) in nonlinear fashion, such as PBD_kC_1, PBC_k. However, it can be transformed into an LMI by employing Lemma 2.4 and Lemma 2.5, which will be shown below.

Theorem 3.5. *Under Assumption 2.1, if there exist matrices $\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k$, X_1 and Y_1 of appropriate dimensions and a real scalar λ such that*

$$\Xi=\begin{bmatrix}\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & X_1^T(F_1^T+C_1^TD_k^TF_2^T)+\tilde{C}_k^T & E_1\\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & F_1^T+C_1^TD_k^TF_2^T & Y_1^TE_1\\ * & * & -D_\varpi^TS-SD_\varpi-R & D_\varpi^T & F_3^T & -SE_2\\ * & * & * & Q^{-1} & 0 & E_2\\ * & * & * & * & -I & 0\\ * & * & * & * & * & -I\end{bmatrix}<0. \qquad (26)$$

where

$$\begin{aligned}\Xi_{11} &= (A+B\tilde{D}_kC_1)X_1+B\tilde{C}_k+X_1^T(A^T\\ &+ C_1^T\tilde{D}_k^TB^T)+\tilde{C}_k^TB^T,\\ \Xi_{12} &= A+B\tilde{D}_kC_1+\tilde{A}_k,\\ \Xi_{13} &= B_w-(X_1^T(C^T+C_1^T\tilde{D}_k^TD^T)+\tilde{C}_k^TD^T)S,\\ \Xi_{14} &= (X_1^T(C^T+C_1^TD_k^TD^T)+\tilde{C}_k^TD^T),\\ \Xi_{22} &= Y_1^T(A+B\tilde{D}_kC_1)+\tilde{B}_kC_1+(A^T\\ &+ C_1^T\tilde{D}_k^TB^T)Y_1+C_1^T\tilde{B}_k^T,\\ \Xi_{23} &= Y_1^TB_w-(C^T+C_1^T\tilde{D}_k^TD^T)S,\\ \Xi_{24} &= C^T+C_1^T\tilde{D}_k^TD^T.\end{aligned}$$

then the controlled closed-loop system (20) is robustly QSR-dissipative with the given real symmetric matrices Q, S and R.

Moreover, a desired dissipation output-feedback controller in the form of (19) with the following parameters is given:

$$\begin{aligned} A_k &= Y_3^{-1}(\tilde{A}_k^T - Y_1^T BD_kC_1X_1 - Y_1^T BD_kC_1X_1 \\ &- Y_1^T B\tilde{C}_k - \tilde{B}_kC_1X_1)X_3^{-1}, \\ B_k &= Y_3^{-T}\tilde{B}_k, \\ C_k &= \tilde{C}_kX_3^{-1}, \\ D_k &= \tilde{D}_k. \end{aligned}$$

where X_3 and Y_3 are any nonsingular matrices satisfying

$$X_1Y_1 + X_3Y_3 = I.$$

Proof. Since

$$\begin{aligned} \Delta\bar{C} &= \begin{bmatrix} \Delta C + \Delta DD_kC_1 & \Delta DC_k \end{bmatrix} \\ &= E_2H(t)\begin{bmatrix} F_1 + F_2D_kC_1 & F_2C_k \end{bmatrix} \\ \Delta\bar{B}_\varpi &= \begin{bmatrix} E_1 & 0 \end{bmatrix}^T HF_3. \end{aligned}$$

It follows from Theorem 3.1 that the controlled closed-loop system is dissipative if there exist a symmetric positive-definite matrix $\bar{P}$, such that the following inequality holds:

$$\Omega = \begin{bmatrix} \bar{A}\bar{P} + \bar{P}\bar{A}^T & \bar{B}_\varpi - \bar{P}\bar{C}^TS & \bar{P}\bar{C}^T & \bar{P}\bar{F}_1^T & \lambda\bar{E}_1 \\ * & -D_\varpi^TS - SD_\varpi - R & D_\varpi^T & F_3^T & -\lambda SE_2 \\ * & * & Q^{-1} & 0 & E_2 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0. \quad (27)$$

where $\bar{F}_1 = \begin{bmatrix} F_1 + F_2D_kC_1 & F_2C_k \end{bmatrix}, \bar{E}_1 = \begin{bmatrix} E_1 & 0 \end{bmatrix}^T$,

Matrix $\bar{P}$ and its inverse can be written as

$$\bar{P} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \bar{P}^{-1} = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix},$$

where $X_i, Y_i (i = 1, 2, 3, 4)$ are nonsingular.

From $\bar{P}\bar{P}^{-1} = I$, one has

$$\begin{aligned} X_1Y_1 + X_2Y_3 = I, \\ X_3Y_1 + X_4Y_3 = 0. \end{aligned}$$

Since

$$\bar{P}\begin{bmatrix} I & Y_1 \\ 0 & Y_3 \end{bmatrix} = \begin{bmatrix} X_1 & I \\ X_3 & 0 \end{bmatrix},$$

then,

$$\bar{P} = \begin{bmatrix} X_1 & I \\ X_3 & 0 \end{bmatrix}\begin{bmatrix} I & Y_1 \\ 0 & Y_3 \end{bmatrix}^{-1}.$$

Denote

$$\Theta_1 = \begin{bmatrix} X_1 & I \\ X_3 & 0 \end{bmatrix}, \Theta_2 = \begin{bmatrix} I & Y_1 \\ 0 & Y_3 \end{bmatrix}. \quad (28)$$

Then

$$\bar{P} = \Theta_1\Theta_2^{-1} = \begin{bmatrix} X_1 & X_3 \\ X_3 & \Upsilon \end{bmatrix}, \quad (29)$$

where $\Upsilon = -X_3Y_1Y_3^{-1}$. Form Lemma 2.1, $\bar{P} = \Theta_1\Theta_2^{-1} > 0$ is equivalent to $X_1Y_1 + X_3Y_3 > 0$.

Submitting (29) into (27), one yields

$$\Omega = \begin{bmatrix} \Omega_{11} & \bar{B}_\varpi - \Theta_2^{-T}\Theta_1^T\bar{C}^TS & \Theta_2^{-T}\Theta_1^T\bar{C}^T & \Theta_2^{-T}\Theta_1^T\bar{F}_1^T & \lambda\bar{E}_1 \\ * & -D_\varpi^TS - SD_\varpi - R & D_\varpi^T & F_3^T & -\lambda SE_2 \\ * & * & Q^{-1} & 0 & E_2 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0. \quad (30)$$

where $\Omega_{11} = \bar{A}\Theta_1\Theta_2^{-1} + \Theta_2^{-T}\Theta_1^T\bar{A}^T$.

Let us multiply the left sides of (30) by $\mathrm{diag}\{\Theta_2^T, I, I, I, I\}$, and the right sides of (30) by diag $\{\Theta_2, I, I, I, I\}$, it yields

$$\Omega = \begin{bmatrix} \Theta_2^T\bar{A}\Theta_1 + \Theta_1^T\bar{A}^T\Theta_2 & \Theta_2^T\bar{B}_\varpi - \Theta_1^T\bar{C}^TS & \Theta_1^T\bar{C}^T & \Theta_1^T\bar{F}_1^T & \lambda\Theta_2^T\bar{E}_1 \\ * & -D_\varpi^TS - SD_\varpi - R & D_\varpi^T & F_3^T & -\lambda SE_2 \\ * & * & Q^{-1} & 0 & E_2 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0. \quad (31)$$

Combining (31) with (28), one obtains

$$\Omega = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & X_1^T(F_1^T + C_1^TD_k^TF_2^T) + X_3^TC_k^T & \lambda E_1 \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & F_1^T + C_1^TD_k^TF_2^T & \lambda Y_1^TE_1 \\ * & * & \Xi_{33} & D_\varpi^T & F_3^T & -\lambda SE_2 \\ * & * & * & Q^{-1} & 0 & E_2 \\ * & * & * & * & -\lambda I & 0 \\ * & * & * & * & * & -\lambda I \end{bmatrix} < 0. \quad (32)$$

where

$$\begin{aligned} \Xi_{11} &= (A + BD_kC_1)X_1 + BC_kX_3 + X_1^T(A^T \\ &+ C_1^TD_k^TB^T) + X_3^TC_k^TB^T, \\ \Xi_{12} &= A + BD_kC_1 + X_1^T(A^T + C_1^TD_k^TB^T)Y_1 \\ &+ X_3^TC_k^TB^TY_1 + (X_1^TC_1^TB_k^T + X_3^TA_k^T)Y_3, \\ \Xi_{13} &= B_w - (X_1^T(C^T + C_1^TD_k^TD^T) \\ &+ X_3^TC_k^TD^T)S, \\ \Xi_{14} &= (X_1^T(C^T + C_1^TD_k^TD^T) + X_3^TC_k^TD^T), \end{aligned}$$

IET Research Journals, pp. 1–12

$$
\begin{aligned}
\Xi_{22} &= Y_1^T(A + BD_kC_1) + Y_3^T B_k C_1 + (A^T \\
&+ C_1^T D_k^T B^T)Y_1 + C_1^T B_k^T Y_3, \\
\Xi_{23} &= Y_1^T B_w - (C^T + C_1^T D_k^T D^T)S, \\
\Xi_{33} &= -D_\varpi^T S - SD_\varpi - R, \\
\Xi_{24} &= (C^T + C_1^T D_k^T D^T).
\end{aligned}
$$

Let $D_k = \tilde{D}_k$, $C_kX_3 = \tilde{C}_k$, $Y_3^T B_k = \tilde{B}_k$, $\tilde{A}_k = X_1^T(A^T + C_1^T D_k^T B^T)Y_1 + X_3^T C_k^T B^T Y_1 + (X_1^T C_1^T B_k^T + X_3^T A_k^T)Y_3$, and $\lambda = 1$ in (32), one readily obtains (26). Therefore, from Definition 3, one can observe that the closed-loop system is robustly dissipative. The proof is completed.

Remark 3.1. *According to Theorem 2.1, Theorem 3.1 can act as the asymptotic stability criterion for system (5) with $\varpi(t) = 0$. Theorems 3.2, 3.3 and 3.5 provide design methods of stabilization state-feedback controller, static output feedback controller and dynamic output feedback controller for system (5) with $\varpi(t) = 0$, respectively.*

Remark 3.2. *It follows from Theorem 2.2 that Theorem 3.1 can ensure that system (5) is input-output stable. Similarly, Theorems 3.2, 3.3 and 3.5 can offer design methods of BIBO stabilization state-feedback controller, static output feedback controller and dynamic output feedback controller for system (5), respectively.*

Remark 3.3. *Refs.[47–50] were concerned with stability, static output feedback control design and state feedback control design for FO uncertain systems described by the following form*

$$
\begin{cases}
D^\alpha x(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))u(t), \\
\quad y(t) = Cx(t).
\end{cases}
\tag{33}
$$

Obviously, system (33) is a special case of system (5). Theorems 3.2-3.5 can be also applied to system (33). However, these results in Refs.[47–50] can not be used to determine the dissipativity and dissipation of the system (5)

Remark 3.4. *Theorem 3.1 does just apply to system with measurable system states, if system states are all unknown or partially known, the problem of design a controller based on state observer will be considered, which is our future work. In addition, inequality analysis techniques are used to derive LMI QSR-dissipativity criterion and feedback dissipation controller design method, which will result in somewhat conservative. Therefore, how to select suitable inequality scaling technique to derive less conservative result is also our future work. On the other hand, for integer-order systems, we can construct various types of the improved Lyapunov function to reduce the conservatism. However, constructing a Lyapunov function and calculating its fractional derivative poses difficulties, to reduce the conservatism by using FO Lyapunov method is still formidable.*

Remark 3.5. *The direct Lyapunov method provides a very effective approach to analyze the stability of integer order nonlinear systems without explicitly finding their solution. However, the Leibniz rule of derivative of two functions f and g, $D^\alpha(fg)$=$(D^\alpha f)g + (D^\alpha g)f$, does not hold for the fractional case, which poses difficulties in constructing a Lyapunov function and calculating its fractional derivative. In fact, this is the main reason why there are limited results on stability of fractional nonlinear (delayed) systems. And because of that, there are few results about control design methods for system (5). This paper proposed the new definition of QSR-dissipativity and feedback dissipation, by revealing the relationship of QSR-dissipativity and stability, stabilization controller design of system (5) is presented.*

Remark 3.6. *One can see that QSR-dissipativity criterion and all feedback dissipation controller design methods are in form of LMI. There are two advantages: first, it needs no tuning of parameters and/or matrices; second, it can be efficiently verified via solving the LMI numerically by MATLAB toolbox*

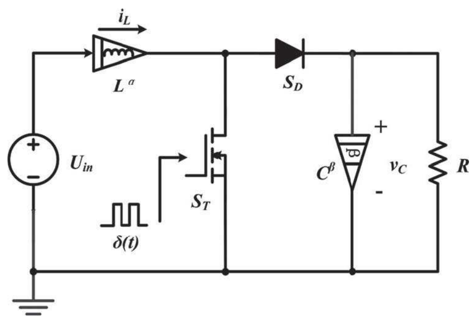

Fig. 1: Fractional-order DC-DC boost converter.

4 Illustrative example

In this section, to demonstrate the applicability of the proposed approach, two illustrative examples are provided.

Example 1 The boost converter, sometimes called a step-up/down power stage, is an inverting power stage topology. Schematic diagram of a DC-DC boost converter is shown in Fig. 1. When S_T OFF and S_D ON, the expression of the FO mathematical model proposed in [51] is described by

$$
\begin{cases}
D^\alpha i_L = \dfrac{1}{L}U_{in} - \dfrac{1}{L}v_C, \\
D^\alpha v_C = \dfrac{1}{C}i_L - \dfrac{v_C}{RC}.
\end{cases}
\tag{34}
$$

Taking i_L and v_C as state variable and selecting the input voltage $U_{in} = 12V$, the load resistance $R = 40\Omega$, the $L = 477\mu H$, $C = 10\mu F/(s)^{1-\alpha}$. Note that Theorem 3.3 is also still valid for nominal model (34). By using LMI Matlab toolbox, one could see that the LMI (6) in Theorem 3.3 with $Q = S = R = I$ is feasible. The feasible solution is given by

$$
\bar{P} = \begin{bmatrix} 0.0052 & 0.0043 \\ 0.0043 & 0.0196 \end{bmatrix}, \lambda = 11.0146.
$$

Therefore, it follow from Theorem 3.3 that fractional-order DC-DC converter (34) is QSR-dissipative.

Example 2 Consider a 3-dimensional uncertain FO linear system (5) with the following parameters:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 2 & -0.5 \\ 2 & -5 & 1 \\ 3 & 1 & -2.5 \end{bmatrix}, B_\varpi = \begin{bmatrix} -2 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix}, \\
B &= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, D_\varpi = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, \\
D &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}, E_1 = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.2 \end{bmatrix}, E_2 = \begin{bmatrix} 0.3 \\ -0.2 \end{bmatrix}, \\
\alpha &= 0.8, F_1 = \begin{bmatrix} 0.2 & -0.3 & 0.1 \end{bmatrix},
\end{aligned}
$$

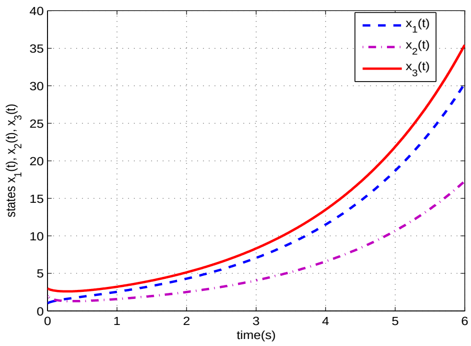

Fig. 2: Time response of the selected systems without the control input.

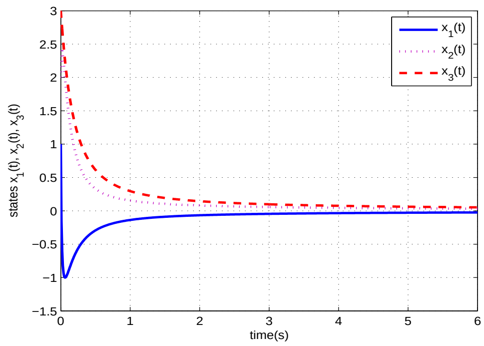

Fig. 3: Time response of the selected systems with state feedback controller.

$$F_2 = 0.2, F_3 = \begin{bmatrix} -0.1 & 0.2 \end{bmatrix}.$$

Select $Q^{-1} = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$. Applying matrix SVD for C_1, one can obtain

$$U = 1, S = 1.4142, V = \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ 0.7071 & -0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In the simulation, choose $h(t) = 0.1\cos t$ and the initial state $x_0 = [1,\ 2,\ 3]^T$. When the external disturbance $\varpi(t) = 0$, the time response of selected systems without the control input is depicted in Fig.2. One can see that the open-loop system is divergent.

Case 1. Assume that the state variables are measurable. By utilizing packages Yalmip in Matlab, one could find that the LMI (13) in Theorem 3.2 is feasible. The feasible solution with the parameter: $\lambda = 10.5821$ is

$$\bar{P} = \begin{bmatrix} 0.8504 & -0.6558 & -1.4285 \\ -0.6558 & 2.9440 & -0.2103 \\ -1.4285 & -0.2103 & 3.5268 \end{bmatrix},$$
$$X = \begin{bmatrix} -2.4876 & -0.1217 & 3.1361 \end{bmatrix}.$$

Furthermore, the dissipation feedback controller gain is given by

$$K = \begin{bmatrix} -14.2030 & -3.5679 & -5.0763 \end{bmatrix}.$$

Therefore, in light of Theorem 3.2, the controlled system in the example is robustly dissipative. Fig.3 shows that the controlled system with state feedback controller is asymptotically stable.

Case 2. Assume that the states of controlled system are unmeasurable and only the output is available. In this case, the static output feedback controller is designed to make the closed-loop system robustly dissipative. It follows from LMI condition (17) that the following feasible solutions are obtained.

$$\tilde{P} = \begin{bmatrix} 0.8473 & -0.1938 & -0.9032 \\ -0.1938 & 0.8473 & 0.9032 \\ -0.9032 & 0.9032 & 1.7690 \end{bmatrix},$$
$$\tilde{P}_2 = \begin{bmatrix} 1.0411 & -1.2773 \\ -1.2773 & 1.7690 \end{bmatrix},$$
$$\tilde{P}_1 = 0.6535, X = -0.7967.$$

Moreover, a static output feedback gain matrix is given by $L = -1.2191$. Based on Theorem 3.3, the closed-loop system with static output feedback controller (16) is robustly dissipative. The time response of the closed-loop systems with the static output feedback controller is illustrated in Fig.4.

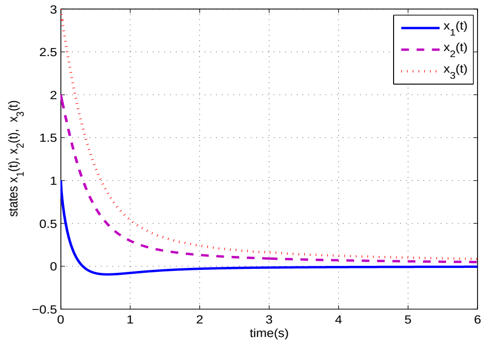

Fig. 4: Time response of the closed-loop systems based on the static output feedback controller

Case 3. If the static output feedback controller does not satisfy the design requirement, better dynamic performance can be obtained via dynamic output feedback control. According to Theorem 3.5, one can use Yalmip in the Matlab to solve the LMIs (26), and obtain the solution as

$$\tilde{A}_k = \begin{bmatrix} 0.5260 & -2.3311 & -0.8849 \\ -2.3311 & 5.1835 & -0.2638 \\ -0.8849 & -0.2638 & 2.9187 \end{bmatrix},$$
$$\tilde{B}_k = \begin{bmatrix} -2.9292 & 1.1786 & -2.0016 \end{bmatrix}^T,$$
$$\tilde{C}_k = \begin{bmatrix} -1.1183 & -0.1110 & 0.0034 \end{bmatrix},$$
$$\tilde{D}_k = -0.1287,$$
$$X_1 = \begin{bmatrix} -0.9351 & -0.4745 & -1.4799 \\ -0.4745 & 0.1854 & -0.5260 \\ -1.4799 & -0.5260 & -0.8474 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} 0.3261 & -0.1795 & -0.0312 \\ -0.1795 & 0.7066 & 0.0446 \\ -0.0312 & 0.0446 & 1.0903 \end{bmatrix},$$
$$X_3 = \begin{bmatrix} 1.1736 & 1.2335 & 2.6056 \\ 1.1716 & 0.8073 & 1.5504 \\ 1.3617 & 1.1438 & 1.9013 \end{bmatrix}, Y_3 = I.$$

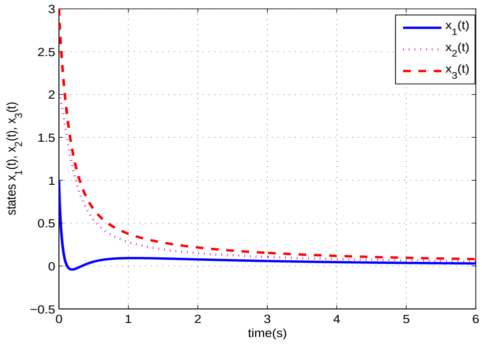

Fig. 5: Time response of the closed-loop systems based on the dynamic output feedback controller

Further, the dynamics output feedback controlled (19) is obtained as follows:

$$
\begin{aligned}
A_k &= \begin{bmatrix} -3.2167 & -0.8359 & 1.3069 \\ 1.7286 & -35.6034 & 28.0602 \\ 3.6816 & -8.0474 & 0.9696 \end{bmatrix}, \\
B_k &= \begin{bmatrix} -2.9292 & 1.1786 & -2.0016 \end{bmatrix}^T, \\
C_k &= \begin{bmatrix} 1.3745 & -2.7591 & 0.3681 \end{bmatrix}, \\
D_k &= -0.1287.
\end{aligned}
$$

Therefore, by Theorem 3.5, the controller system with dynamic output feedback controller (19) is robustly dissipative. Fig.5 shows that all states could converge to zero asymptotically under dynamic output feedback controller.

5 Conclusion

Robust dissipativity and dissipation for uncertain FO linear systems with time-varying norm-bounded parameter uncertainties are considered. By employing QSR-dissipativity methodology, asymptotic stability and input-output stability are drawn. An LMI sufficient condition for such a system to be robustly dissipative is given. Sufficient state feedback dissipation criterion is also proposed based on the available states. A static output feedback controller and dynamic output feedback controller are designed for the closed-loop system to be dissipative, respectively. Furthermore, an illustrative example is provided to show the usefulness and effectiveness of the presented results. It is well known that the time-delay phenomenon often appears in many dynamic systems, so it is our future work to discuss passivity and dissipativity issues of fractional-order delayed systems.

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