An Improved Frequency-domain Method for the Fractional Order $PI^\lambda D^\mu$ Controller Optimal Design

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Abstract: An improved frequency-domain design method is proposed to design the fractional order $PI^\lambda D^\mu$ controller. Using this improved method, the parameters of the fractional order $PI^\lambda D^\mu$ controllers can be obtained immediately according to the model characteristics and design specifications. A proportional relation between the integral gain and derivative gain is built, while the derivative order is set to be equal to the integral order. The proportional coefficient between integral gain and derivative gain is studied and modeled based on priori knowledge and data fitting, and then the estimation model for the optimal proportional coefficient is built. The proposed tuning method is applied to design a fractional order $PI^\lambda D^\mu$ controller for a permanent magnet synchronous motor servo system. Motor speed control simulations are performed to verify the proposed method. Simulation results show that the obtained control system can achieve robustness and the optimized step response performance.

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1. INTRODUCTION

The proportional integral derivative (PID) control is the most widely used control method in the industrial control area. In recent years, fractional calculus has aroused interest and attention of scholars (Podlubny (1999a), Monje et al. (2010), Luo et al. (2010), Luo and Chen (2009)). The fractional order proportional integral derivative ($PI^\lambda D^\mu$) controller has the potential to achieve better control performance over the traditional PID controller because the adjustable integral order $\lambda$ and derivative order $\mu$ are introduced, expanding the control scope of the controller (Podlubny (1999b)). However, on the other hand, the tuning of the $PI^\lambda D^\mu$ controller is more complicated.

The tuning methods of fractional order $PI^\lambda D^\mu$ controller can mainly be divided into two kinds, the frequency-domain design method (Luo et al. (2010), Luo and Chen (2009)) and the optimization methods (Biswas et al. (2009), Zheng and Pi (2016)). The frequency-domain method is often applied to design the fractional order $PI^\lambda$ or $PD^\mu$ controller. Based on the given gain crossover frequency and phase margin, the controller parameters are calculated according to the gain robustness specification. The obtained control system achieves reliable stability and the robustness to gain variations. However, as discussed in this paper, this method cannot be directly applied to tune the fractional order $PI^\lambda D^\mu$ controller.

A tuning method based on the differential evolution (DE) algorithm is proposed (Zheng et al. (2017)), satisfying the specifications in both frequency-domain and time-domain simultaneously. The obtained control system achieves the optimal dynamic performance, while the frequency-domain design requirements are also satisfied. However, applying this method, large amount of space and time are spent in the numerical optimization. Therefore, this method may not be suitable for engineering application.

An improved frequency-domain design method is proposed to design the fractional order $PI^\lambda D^\mu$ controller in this paper. In order to reduce the pending parameters of the controller, the proportional relation between the integral gain $K_i$ and derivative gain $K_d$ is built, while the derivative order $\mu$ is set to be equal to the integral order $\lambda$. Based on this modification, the number of the pending parameters is reduced from five to three. Therefore, the current frequency-domain design method can be applied to tune the fractional order $PI^\lambda D^\mu$ controller. The proportional coefficient between $K_i$ and $K_d$ is studied and modeled based on priori knowledge and data fitting, and then the estimation model for the optimal proportional coefficient
To ensure the stability of the control system, the phase-frequency curve is zero, namely, the phase Bode with the least ITAE is selected to be $\phi_m$. The fractional order PI$^\lambda$D$^\mu$ controller has five parameters to be tuned, $K_p$, $K_i$, $K_d$, $\lambda$ and $\mu$. However, only three equations are derived from the design specifications. Therefore, the current frequency-domain design method cannot be applied to tune the fractional order PI$^\lambda$D$^\mu$ controller directly.

A modification on the current design method is proposed to solve this problem. The proportional relation between the integral gain $K_i$ and derivative gain $K_d$ is built, as described by (13),

$$K_d = aK_i,$$

where $a$ is the proportional coefficient. Besides, the derivative order $\mu$ is set to be equal to the integral order $\lambda$. Therefore, the modified fractional order PI$^\lambda$D$^\mu$ controller is described by (14),

$$C(s) = K_p \left(1 + \frac{K_i}{s^{\lambda}} + aK_i s^{\lambda}\right).$$

Substituting (14) into (12), $K_i$ can be represented as (15),

$$K_i = \frac{M}{\omega_c \cos(\frac{\pi}{2} \lambda) + \omega_c \sin(\frac{\pi}{2} \lambda)} + \frac{B(\omega_c)}{A(\omega_c)}.$$

Similarly, substituting (14) into (10), the equation about $K_i$ is obtained, as described by (18),

$$Q_2 K_i^2 + Q_1 K_i + Q_0 = 0,$$

where

$$Q_2 = \frac{2a\lambda}{\omega_c} \sin(\lambda \pi) + 2aQ_0 \cos(\lambda \pi) + a^2Q_0 \omega_c^{-2\lambda} + \frac{2a}{\omega_c} \cos(\frac{\pi}{2} \lambda),$$

$$Q_1 = \frac{2a\lambda}{\omega_c} \sin(\frac{\pi}{2} \lambda) + \frac{2a\lambda}{\omega_c} \cos(\frac{\pi}{2} \lambda) + \frac{2a}{\omega_c} \cos(\frac{\pi}{2} \lambda),$$

$$Q_0 = \frac{d[\arg(G(s))]}{d\omega} \bigg|_{\omega = \omega_c}.$$
of the optimal $a$ corresponding to each $(\omega_c, \varphi_m)$ setting can be approximated by a model related to the $(\omega_c, \varphi_m)$ setting and the plant model characteristics. Furthermore, the approximated model of the optimal $a$ is continuously defined in the hyperspace of $\omega_c$, $\varphi_m$ and the related model characteristic parameters.

The estimation model of the optimal $a$ is built according to the following steps. First, several test models are built based on the interested model parameter ranges. Similarly, several $(\omega_c, \varphi_m)$ pairs are selected according to the design requirements. Second, the optimal values of $a$ corresponding to different $(\omega_c, \varphi_m)$ pairs and test models are collected. Third, based on the collected data, the distribution rules of the optimal $a$ for different $(\omega_c, \varphi_m)$ settings and model characteristics are studied. Finally, the estimation model of $a$ is built according to the summarized distribution rules.

3.1 Optimal Data Collection

In this paper, the estimation model of the optimal $a$ is built for the PMSM servo system having the form described by (6). According to the commonly used PMSM models, the range of parameter $\tau_1$ is set to be 100 to 140, while that of $\tau_2$ is set to be 8000 to 11000. Besides, in order to satisfy the general design requirement, the range of the given gain crossover frequency $\omega_c$ is set to be 35rad/s to 60rad/s, while that of the given phase margin $\varphi_m$ is set to be 45° to 60° (Ruan et al. (2016)).

Based on the range of $\tau_1$, three values of $\tau_1$: 100, 120 and 140 are selected to build the test models. Similarly, three values of $\tau_2$: 8000, 9500 and 11000 are also selected. The gain $K$ has no influence on the value of $a$, it is fixed to be 30000. Therefore, nine test models are built by combining the values of $\tau_1$ and $\tau_2$, as described from (22) to (30),

\[
\begin{align*}
G_1(s) &= \frac{30000}{s^3 + 100s^2 + 8000s}, \\
G_2(s) &= \frac{30000}{s^3 + 120s^2 + 8000s}, \\
G_3(s) &= \frac{30000}{s^3 + 140s^2 + 8000s}, \\
G_4(s) &= \frac{30000}{s^3 + 100s^2 + 11000s}, \\
G_5(s) &= \frac{30000}{s^3 + 120s^2 + 11000s}, \\
G_6(s) &= \frac{30000}{s^3 + 140s^2 + 11000s}, \\
G_7(s) &= \frac{30000}{s^3 + 100s^2 + 9500s}, \\
G_8(s) &= \frac{30000}{s^3 + 120s^2 + 9500s}, \\
G_9(s) &= \frac{30000}{s^3 + 140s^2 + 9500s}.
\end{align*}
\]

Based on the range of $\omega_c$, seven values of $\omega_c$: 35rad/s, 37rad/s, 40rad/s, 45rad/s, 50rad/s, 55rad/s, 60rad/s are selected to tune the controllers. Similarly, four values of $\varphi_m$: 45°, 50°, 55°, 60° are also selected.

3.2 Estimation Model Study

The optimal value of $a$ should be related to the given crossover frequency $\omega_c$, phase margin $\varphi_m$ and the phase-frequency characteristics of the plant model. First, the relation between the optimal $a$ and the given $\varphi_m$ is studied, under the condition that the given $\omega_c$ is fixed. Taking $G_1(s)$ as an example, the distributions of the optimal $a$ with regard to $\varphi_m$ are plotted in Fig. 1. Based on Fig. 1, the optimal $a$ satisfies the linear relationship with $\varphi_m$ when $\omega_c$ is fixed. Therefore, the optimal $a$ can be described by (32),

\[
a = A\varphi_m + B, \quad (32)
\]

where $A$ and $B$ are related to the given gain crossover frequency $\omega_c$ and the phase-frequency characteristics of the plant model.

The distributions of the optimal $a$ with regard to $\varphi_m$ are fitted applying the least square method. Thus, the values of $A$ and $B$ corresponding to different given $\omega_c$ for model $G_1(s)$ are obtained. The same process is performed on the
Fig. 2. The distributions of $A$ with regard to $\omega_c$

other eight test models and then the values of $A$ and $B$ for $G_1(s)$ to $G_9(s)$ are obtained. The distributions of $A$ with regard to $\omega_c$ for different test models are plotted in Fig. 2. Based on Fig. 2, $A$ approximately satisfies the linear relationship with $\omega_c$. Therefore, $A$ can be described by (33),

$$A = M_1 \omega_c + N_1,$$

where $M_1$ and $N_1$ are also related to the phase-frequency characteristics of the plant model. The values of $M_1$ and $N_1$ are obtained by fitting the distributions of $A$ with regard to $\omega_c$ using the least square method. Thus, the values of $M_1$ and $N_1$ corresponding to different test models are obtained.

In order to study the distributions of $M_1$ and $N_1$ with regard to the phase-frequency characteristics of the plant model, seven frequency points: $35rad/s$, $37rad/s$, $40rad/s$, $45rad/s$, $50rad/s$, $55rad/s$, $60rad/s$ are selected in the gain crossover frequency range. The phases at these frequency points are calculated and the mean phase $\varphi_m0$ within the frequency range is obtained by calculating the average of seven values. Performing such calculations on nine test models, the mean phases corresponding to nine models, $\varphi_m0(G_1(s))$, $\varphi_m0(G_2(s))$, ..., $\varphi_m0(G_9(s))$ are obtained.

The distributions of $M_1$ with regard to mean phase $\varphi_m0$ are plotted as data points in Fig. 3. Based on Fig. 3, $M_1$ satisfies the linear relationship with $\varphi_m0$, as described by (34),

$$M_1 = P_1 \varphi_m0 + Q_1,$$

where $P_1$ and $Q_1$ are pending constants. Applying the least square method to fit the distributions of $M_1$ with regard to $\varphi_m0$, the values of $P_1$ and $Q_1$ are obtained, $P_1 = 5.033 \times 10^{-8}$, $Q_1 = 5.627 \times 10^{-6}$, the fitting line is plotted in red in Fig. 3.

The distributions of $N_1$ with regard to mean phase $\varphi_m0$ are plotted as data points in Fig. 4. Based on Fig. 4, $N_1$ satisfies the linear relationship with $\varphi_m0$, as described by (35),

$$N_1 = P_2 \varphi_m0 + Q_2,$$

where $P_2$ and $Q_2$ are pending constants. Applying the least square method to fit the distributions of $N_1$ with regard to $\varphi_m0$, the values of $P_2$ and $Q_2$ are obtained, $P_2 = -2.838 \times 10^{-6}$, $Q_2 = -3.044 \times 10^{-4}$, the fitting line is plotted in red in Fig. 4.

In order to study the relations between $B$ in (32) and the given $\omega_c$, the distributions of $B$ with regard to $\omega_c$ for different test models are plotted in Fig. 5. Based on Fig. 5, $B$ approximately satisfies the linear relationship with $\omega_c$, as described by (36),

$$B = M_2 \omega_c + N_2,$$

where $M_2$ and $N_2$ are related to the phase-frequency characteristics of the plant model. The values of $M_2$ and $N_2$ are obtained by fitting the distributions of $B$ with regard to $\omega_c$ using the least square method. Thus, the values of $M_2$ and $N_2$ corresponding to different test models are obtained.
In order to study the relations between least square method to fit the distributions of \(N\) phase-frequency characteristics of the plant model, the distributions of \(X\) are plotted as data points in Fig. 6. Based on Fig. 6, \(M\) satisfies the linear relationship with \(\phi\) as described by (37),

\[
M = X_1\phi m_0 + Y_1,
\]

where \(X_1\) and \(Y_1\) are pending constants. Applying the least square method to fit the distributions of \(M\) with regard to \(\phi m_0\), the values of \(X_1\) and \(Y_1\) are obtained, \(X_1 = -1.707 \times 10^{-6}\), \(Y_1 = -1.782 \times 10^{-4}\), the fitting line is plotted in red in Fig. 6.

In order to study the relations between \(N\) and the phase-frequency characteristics of the plant model, the distributions of \(N\) with regard to mean phase \(\phi m_0\) are plotted as data points in Fig. 7. Based on Fig. 7, \(N\) satisfies the linear relationship with \(\phi m_0\), as described by (38),

\[
N = X_2\phi m_0 + Y_2,
\]

where \(X_2\) and \(Y_2\) are pending constants. Applying the least square method to fit the distributions of \(N\) with regard to \(\phi m_0\), the values of \(X_2\) and \(Y_2\) are obtained, \(X_2 = 8.425 \times 10^{-5}\), \(Y_2 = 0.0083\), the fitting line is plotted in red in Fig. 7.

All the parameters of the estimation model are obtained. The estimation model of \(a\) is described by (39),

\[
a = [(P_1\phi m_0 + Q_1)\omega_1 + P_2\phi m_0 + Q_2]\phi m_0 + (X_1\phi m_0 + Y_1)\omega_2 + X_2\phi m_0 + X_2
\]

\[
\begin{align*}
&= (0.00000005533\phi m_0 + 0.0000005267)\omega_1
&+ 0.0000000628\phi m_0 + 0.000001707\omega_2
\end{align*}
\]

\[
= (0.00000005533\phi m_0 + 0.0000005267)\omega_1 + 0.0000000628\phi m_0 + 0.000001707\omega_2 + 0.000008425\phi m_0 + 0.00000835
\]

Fig. 6. The distributions of \(M\) with regard to \(\phi m_0\)

Fig. 7. The distributions of \(N\) with regard to \(\phi m_0\)

The distributions of \(M\) and \(N\) are plotted as data points in Fig. 6. Based on Fig. 6, \(M\) satisfies the linear relationship with \(\phi m_0\), as described by (37),

\[
M = X_1\phi m_0 + Y_1,
\]

where \(X_1\) and \(Y_1\) are pending constants. Applying the least square method to fit the distributions of \(M\) with regard to \(\phi m_0\), the values of \(X_1\) and \(Y_1\) are obtained, \(X_1 = -1.707 \times 10^{-6}\), \(Y_1 = -1.782 \times 10^{-4}\), the fitting line is plotted in red in Fig. 6.

In order to study the relations between \(N\) and the phase-frequency characteristics of the plant model, the distributions of \(N\) with regard to mean phase \(\phi m_0\) are plotted as data points in Fig. 7. Based on Fig. 7, \(N\) satisfies the linear relationship with \(\phi m_0\), as described by (38),

\[
N = X_2\phi m_0 + Y_2,
\]

where \(X_2\) and \(Y_2\) are pending constants. Applying the least square method to fit the distributions of \(N\) with regard to \(\phi m_0\), the values of \(X_2\) and \(Y_2\) are obtained, \(X_2 = 8.425 \times 10^{-5}\), \(Y_2 = 0.0083\), the fitting line is plotted in red in Fig. 7.

All the parameters of the estimation model are obtained. The estimation model of \(a\) is described by (39),

\[
\begin{align*}
a &= [(P_1\phi m_0 + Q_1)\omega_1 + P_2\phi m_0 + Q_2]\phi m_0 + (X_1\phi m_0 + Y_1)\omega_2 + X_2\phi m_0 + X_2
\end{align*}
\]

\[
\begin{align*}
&= (0.00000005533\phi m_0 + 0.0000005267)\omega_1
&+ 0.0000000628\phi m_0 + 0.000001707\omega_2
\end{align*}
\]

\[
= (0.00000005533\phi m_0 + 0.0000005267)\omega_1 + 0.0000000628\phi m_0 + 0.000001707\omega_2 + 0.000008425\phi m_0 + 0.00000835
\]

Fig. 8. The PMSM speed closed-loop control system

Fig. 9. The open Bode plot of the control system

4. ESTIMATION MODEL APPLICATION

In this section, in order to verify the obtained estimation model, the coefficient \(a\) is calculated and the fractional order \(PF^D\) controller is designed for a real PMSM servo system. Besides, given the same \(\omega_c\) and \(\phi m\), a fractional order \(PL^A\) controller is also obtained using the current frequency-domain method. Step response simulations are performed and the dynamic performance of two control systems are compared. The Oustaloup method (Oustaloup (1995), Oustaloup et al. (2000)) is applied to approximate the fractional order operator in the simulation models.

The PMSM speed closed-loop control system is shown in Fig. 8, where \(G(s)\) represents the PMSM plant model, \(C(s)\) represents the speed controller, \(n_r\) represents the reference speed and \(n\) represents the speed output.

The transfer function of the PMSM plant model is described by (40),

\[
G(s) = \frac{47979.257}{s^3 + 127.38s^2 + 9995.678s}.
\]

Given the gain crossover frequency \(\omega_c = 35\text{rad/s}\), the phase margin \(\phi m = 45^\circ\), the optimal coefficient \(a\) is obtained using (39), \(a = 3.185 \times 10^{-4}\). Then the fractional order \(PF^D\) controller is obtained applying the frequency-domain method, as described by (41),

\[
C_1(s) = 6.5754 \left(1 + \frac{14.7083}{s^{0.9615}} + 0.00478s^{0.9615}\right).
\]

The open-loop Bode plot of the control system is shown in Fig. 9. The gain crossover frequency \(\omega_c = 35\text{rad/s}\) and the phase margin \(\phi m = 45^\circ\). Besides, the phase characteristic curve is flat at \(\omega_c\). Therefore, the design requirements on stability and robustness are satisfied.

In order to check the robustness of the obtained control system, the loop-gain of \(C_1(s)\) is set to be 100%, 120% and 80% of its nominal value to simulate the plant model uncertainty. Setting the reference speed \(n_r\) to be 1000rpm, the motor speed step response simulations are performed,
PI response performance than that of the system using the coefficient \( \lambda \).

Based on the simulation results, the proposed tuning method is valid for the fractional order PI\(^\lambda\) controller design. For any plant model having the form represented by (6), whose parameters are located in the specified ranges, the value of \( a \) can be calculated applying the estimation model and then the controller parameters can be obtained applying the frequency-domain method.

The obtained control system achieves robustness and the optimized step response performance. Besides, compared with the optimization methods, the tuning procedure of the improved frequency-domain method is straightforward and timesaving, suitable for engineering application.

5. CONCLUSION

An improved frequency-domain design method for fractional order PI\(^\lambda\)D\(^\mu\) controller is proposed. The proportional relation between \( K_i \) and \( K_d \) is built, while \( \mu \) is set to be equal to \( \lambda \). The estimation model of the proportional coefficient between \( K_i \) and \( K_d \) is built for the commonly used PMSM servo systems. Motor speed step response simulations are performed to verify the estimation model and the proposed tuning method. Simulation results show that the improved frequency-domain design method is valid and the obtained control system can achieve robustness and the optimized step response performance.

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